RENORMALIZED HIGHER POWERS OF WHITE NOISE AND THE VIRASORO–ZAMOLODCHIKOV–w_∞ ALGEBRA

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We have recently proved that the generators of the second quantized centerless Virasoro (or Witt)–Zamolodchikov–w_∞ algebra can be expressed in terms of the Renormalized Higher Powers of White Noise (RHPWN) and conjectured that this inclusion might in fact be an identity, in the sense that the converse is also true. In this paper we prove that this conjecture is true. We also explain the difference between this result and the boson representation of the centerless Virasoro algebra, which realizes, in the 1-mode case (in particular without renormalization), an inclusion of this algebra into the full oscillator algebra. This inclusion was known in the physics literature and some heuristic results were obtained in the direction of the extension of this inclusion to the 1-mode centerless Virasoro (or Witt)–Zamolodchikov–w_∞ algebra. However, the possibility of an identification of the second quantizations of these two algebras was not even conjectured in the physics literature.

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1. Introduction

DEFINITION 1. The standard *d-dimensional Fock scalar white noise* is defined by a quadruple $\{\mathcal{H}, b_t, b_t^+, \Phi\}$, where $t \in \mathbb{R}^d$, $\mathcal{H}$ is a Hilbert space, $\Phi \in \mathcal{H}$ is a unit vector called the Fock vacuum, and b_t, $b_t^\dagger$ are operator-valued Hida distributions satisfying the boson commutation relations $[b_t, b_s^\dagger] = \delta(t - s)$ and having the Fock property $b_t \Phi = 0$ and the adjoint property $(b_t^\dagger)^\dagger = b_t$, plus additional domain properties (not specified here).

THEOREM 1. *Denoting by* $\mathcal{S}_0$ *the space of right-continuous step functions* $f : \mathbb{R} \longrightarrow \mathbb{C}$ *with compact support and satisfying* $f(0) = 0$, *there exists a* $*$*-Lie algebra with generators* $\{B_k^n(f) : k, n \in \mathbb{N},\ f \in \mathcal{S}_0\}$ *(whose precise white noise form is given in Eq.* (1.2) *below), involution given by* $\left(B_k^n(f)\right)^* = B_n^k(\bar{f})$, *and brackets given by*

$$[B_k^n(g), B_K^N(f)]_{\mathrm{RHPWN}} := (k\,N - K\,n)\, B_{k+K-1}^{n+N-1}(gf).$$

Proof: The $*$-property is clear by construction. By direct calculations one shows that the brackets $[\cdot, \cdot]_{\mathrm{RHPWN}}$ satisfy the Jacobi relations. For details see [1] and [2]. □

The $*$-Lie algebra defined by the above theorem is called the *Renormalized Higher Powers of White Noise* (RHPWN) $*$-Lie algebra. The following problems arise: (i) Construct a concrete mathematical model for the abstractly defined RHPWN $*$-Lie algebra; (ii) Construct Hilbert space representations of the RHPWN $*$-Lie algebra; (iii) Prove exponentiability of the symmetric generators of the RHPWN $*$-Lie algebra in a given Hilbert space representation and identify the corresponding Lie group. Heuristic results in the physics literature (cf. [3]), and the results presented in this paper, suggest that a natural candidate for the corresponding Lie group is the group of area preserving diffeomorphisms on a (special) 2-manifold (there are classical realizations on the cylinder $\mathbb{R} \times S^1$). The above given definition of the RHPWN $*$-Lie algebra was motivated by the discovery that, if the powers of the Dirac delta function are renormalized by the prescription

$$\delta^l(t-s) = \delta(s)\,\delta(t-s), \qquad l = 2, 3, \ldots. \tag{1.1}$$

then, for $n, k \geq 0$, the resulting renormalized higher powers of white noise

$$B_k^n(f) := \int_{\mathbb{R}^d} f(t)\, b_t^{\dagger^n}\, b_t^k\, dt, \tag{1.2}$$

where $B_0^0(f) := \int_{\mathbb{R}^d} f(t)\, dt \cdot 1$ (multiple of the unique central element), which are well defined as sesquilinear forms (matrix elements) on the algebraic span of the number vectors $\prod_{i=1}^m B_0^1(f_i)\,\Phi$ of the usual boson Fock space, satisfy weakly on that domain, the defining relations of the RHPWN $*$-Lie algebra in the sense that the adjoint $B_k^n(f)^*$ of the sesquilinear form $B_k^n(f)$ is defined in the obvious way, and the bracket (commutator) $[B_k^n(f), B_K^N(g)]$ of the sesquilinear forms $B_k^n(f)$ and $B_K^N(g)$ is defined by bringing to normal order the products $B_k^n(f)\,B_K^N(g)$ and $B_K^N(g)\,B_k^n(f)$ by applying the RHPWN commutation relations, including the renormalization prescription, and letting the resulting normally ordered form act on the 1-st order number vectors through the usual prescriptions. The family of these sesquilinear forms is clearly a complex vector space with the pointwise operations and, with the above defined involution and brackets, it becomes a representation of the RHPWN $*$-Lie algebra introduced in Theorem 1.

LEMMA 1. *Let* $A_{n,k}$ *be an arbitrary double sequence of complex numbers and let* $f_{n,k}$ *be an arbitrary double sequence in* $\mathcal{S}_0$. *Then the sesquilinear form*

$$q := \sum_{n,k} A_{n,k} B^n_k(f_{n,k}) \tag{1.3}$$

is well defined (weakly) on the algebraic span of the number vectors of the usual boson Fock space.

Proof: For any pair of number vectors x, y, the expression

$$q(x, y) = \sum_{n,k} A_{n,k} \langle x, B^n_k(f_{n,k}) y\rangle$$

is a finite sum of complex numbers. □

DEFINITION 2. On the space of all sesquilinear forms on the algebraic span of the number vectors of the usual boson Fock space, we define a topology by the semi–norms

$$|q|_{x,y} := |q(x, y)| \tag{1.4}$$

where q is a sesquilinear form and x, y are number vectors.

It follows from Lemma 1 that the completion of the RHPWN $*$-Lie algebra with respect to the family of seminorms (1.4) is the family of formal series of the form (1.3). In the following when speaking of the RHPWN $*$-Lie algebra we will mean this larger family. Lemma 1 allows one to give a meaning to a rather general class of functions of the renormalized white noise.

DEFINITION 3. Let $F(b_t^\dagger, b_t)$ be a formal power series in the noncommutative indeterminates $b_t^\dagger, b_t$. If, by applying the RHPWN commutation relations, including the renormalization prescriptions, one can write this series in the form $\sum_{n,k} A_{n,k} b_t^{\dagger n} b_t^k$ where each coefficient $A_{n,k}$ is a complex number (in particular finite!), then we say that the formal power series $F(b_t^\dagger, b_t)$ defines *a function of* $b_t^\dagger, b_t$. The meaning of this function is that by multiplying by test functions $f_{n,k}(t)$ and integrating term by term in dt, we obtain the sesquilinear form $\sum_{n,k} A_{n,k} B^n_k(f_{n,k})$, which is well defined by Lemma 1.

In the following we will produce concrete examples of functions of $b_t^\dagger, b_t$.

2. The centerless Virasoro (or Witt)–Zamolodchikov–w_∞ $*$-Lie algebra

Following a completely different line of thought, people in conformal field theory and in string theory were led to introduce another $*$-Lie algebra (cf. [3, 5]).

THEOREM 2. *There exists a $*$-Lie algebra with generators $\{\hat{B}^n_k : n \in \mathbb{N},\ n \geq 2,\ k \in \mathbb{Z}\}$, involution given by $(\hat{B}^n_k)^* = \hat{B}^n_{-k}$, and brackets given by*

$$[\hat{B}^n_k, \hat{B}^N_K]_{w_\infty} = (k\,(N-1) - K\,(n-1))\,\hat{B}^{n+N-2}_{k+K}.$$

Proof: See [3]. □

DEFINITION 4. The $*$-Lie algebra defined in Theorem 2 is called the *centerless Virasoro (or Witt)–Zamolodchikov-w_∞* $*$-Lie algebra.

Notice that no test functions appear in the above definition. In our language, we can say that the above $*$-Lie algebra is the 1-mode version of the algebra we are interested in, or equivalently, that the algebra we are interested in, is a second quantization of the w_∞ $*$-Lie algebra. As usual, the existence of such an object has to be proved.

THEOREM 3. *There exists a $*$-Lie algebra with generators $\{\hat{B}^n_k(f) : n \in \mathbb{N},\ n \geq 2,\ k \in \mathbb{Z},\ f \in \mathcal{S}_0\}$, involution given by*

$$(\hat{B}^n_k(f))^* = \hat{B}^n_{-k}(\bar{f}) \tag{2.1}$$

and brackets given by

$$[\hat{B}^n_k(g), \hat{B}^N_K(f)]_{w_\infty} = (k\,(N-1) - K\,(n-1))\,\hat{B}^{n+N-2}_{k+K}(gf). \tag{2.2}$$

Proof: Clearly, for all test functions $f, g \in \mathcal{S}_0$ and $n, k, N, K \geq 0$,

$$[\hat{B}^N_K(g), \hat{B}^N_K(f)]_{w_\infty} = 0 \quad \text{and} \quad [\hat{B}^N_K(g), \hat{B}^n_k(f)]_{w_\infty} = -[\hat{B}^n_k(f), \hat{B}^N_K(g)]_{w_\infty}.$$

To show that commutation relations (2.2) satisfy the Jacobi identity we must show that for all test functions f, g, h and $n_i, k_i \geq 0$, where $i = 1, 2, 3$,

$$[\hat{B}^{n_1}_{k_1}(f), [\hat{B}^{n_2}_{k_2}(g), \hat{B}^{n_3}_{k_3}(h)]_{w_\infty}]_{w_\infty} + [\hat{B}^{n_3}_{k_3}(h), [\hat{B}^{n_1}_{k_1}(f), \hat{B}^{n_2}_{k_2}(g)]_{w_\infty}]_{w_\infty}$$
$$+ [\hat{B}^{n_2}_{k_2}(g), [\hat{B}^{n_3}_{k_3}(h), \hat{B}^{n_1}_{k_1}(f)]_{w_\infty}]_{w_\infty} = 0,$$

i.e. that

$$\{(k_2(n_3-1) - k_3(n_2-1))(k_1(n_2+n_3-3) - (k_2+k_3)(n_1-1))$$
$$+ (k_1(n_2-1) - k_2(n_1-1))(k_3(n_1+n_2-3) - (k_1+k_2)(n_3-1))$$
$$+ (k_3(n_1-1) - k_1(n_3-1))(k_2(n_3+n_1-3) - (k_3+k_1)(n_2-1))\}\,\hat{B}^{n_1+n_2+n_3-4}_{k_1+k_2+k_3}(fgh) = 0,$$

which is true, since

$$(k_2(n_3-1) - k_3(n_2-1))(k_1(n_2+n_3-3) - (k_2+k_3)(n_1-1))$$
$$+ (k_1(n_2-1) - k_2(n_1-1))(k_3(n_1+n_2-3) - (k_1+k_2)(n_3-1))$$
$$+ (k_3(n_1-1) - k_1(n_3-1))(k_2(n_3+n_1-3) - (k_3+k_1)(n_2-1)) = 0.$$

Finally, in order to show that, with involution defined by (2.1),

$$[\hat{B}^n_k(f), \hat{B}^N_K(g)]^*_{w_\infty} = \left[(\hat{B}^N_K(g))^*, (\hat{B}^n_k(f))^*\right]_{w_\infty},$$

i.e. that

$$[\hat{B}^n_k(f), \hat{B}^N_K(g)]^*_{w_\infty} = [\hat{B}^N_{-K}(\bar{g}), \hat{B}^n_{-k}(\bar{f})]_{w_\infty},$$

we notice that both sides of the above equation are equal to

$$(k(N-1) - K(n-1))\,\hat{B}^{n+N-2}_{-(k+K)}(\overline{fg}). \qquad \square$$

DEFINITION 5. The $*$-Lie algebra defined in Theorem 3 is called the *second quantized centerless Virasoro (or Witt)–Zamolodchikov–* w_∞ $*$-Lie algebra.

The term Zamolodchikov is due to the fact that w_∞ is a large N limit of Zamolodchikov's W_N algebra (cf. [5, 6]). The term Virasoro is justified by the following theorem.

THEOREM 4. *The family of operators* $\{\hat{B}^2_k : k \in \mathbb{Z}\}$ *forms a* $*$*-Lie subalgebra of the* w_∞ *Lie algebra, with involution* $(\hat{B}^2_k)^* = \hat{B}^2_{-k}$ *and brackets*

$$[\hat{B}^2_k, \hat{B}^2_K]_{\mathrm{Vir}} := (k - K)\, \hat{B}^2_{k+K}$$

which are precisely the defining relations of the centerless Virasoro (or Witt) algebra.

Proof: The proof follows directly from Theorem 2 for $n = N = 2$. □

The following second quantized version of the above theorem also holds.

THEOREM 5. *The family of operators* $\{\hat{B}^2_k(f) : f \in \mathcal{S}_0;\ k \in \mathbb{Z}\}$ *forms a* $*$*-Lie subalgebra of the second quantized* w_∞ *Lie algebra, with involution* $(\hat{B}^2_k(f))^* = \hat{B}^2_{-k}(\bar{f})$ *and brackets*

$$[\hat{B}^2_k(g), \hat{B}^2_K(f)]_{\mathrm{Vir}} := (k - K)\, \hat{B}^2_{k+K}(gf).$$

Proof: The proof follows directly from Theorem 3 for $n = N = 2$. □

DEFINITION 6. The $*$-Lie algebra defined in Theorem 5 is called *the second quantized centerless Virasoro (or Witt) algebra*.

3. The connection between the second quantized w_∞ and the RHPWN $*$-Lie algebras

The striking similarity between the brackets of the RHPWN $*$-Lie algebra

$$[B^n_k(g), B^N_K(f)]_{\mathrm{RHPWN}} := (k\,N - K\,n)\; B^{n+N-1}_{k+K-1}(gf)$$

and the brackets of the second quantized w_∞ $*$-Lie algebra

$$[\hat{B}^n_k(g), \hat{B}^N_K(f)]_{w_\infty} = (k\,(N-1) - K\,(n-1))\; \hat{B}^{n+N-2}_{k+K}(gf)$$

strongly suggests that there should be a connection between the two. However, there are also strong dissimilarities. The sets, indexing the generators of the two algebras, are different: $\{B^n_k(f) : k, n \in \mathbb{N}, , f \in \mathcal{S}_0\}$ for the RHPWN $*$-Lie algebra, and $\{\hat{B}^n_k(f) : n \in \mathbb{N},\ n \geq 2,\ k \in \mathbb{Z},\ f \in \mathcal{S}_0\}$ for the second quantized w_∞ $*$-Lie algebra . The following Theorem 6, obtained in [1], was the first definite result in the direction of establishing a connection between the RHPWN and w_∞ $*$-Lie algebras.

THEOREM 6. *In the sense of formal power series the following identity holds*

$$\hat{B}_k^n(f) = \int_{\mathbb{R}^d} f(t)\, e^{\frac{k}{2}(b_t - b_t^\dagger)} \left(\frac{b_t + b_t^\dagger}{2}\right)^{n-1} e^{\frac{k}{2}(b_t - b_t^\dagger)}\, dt \tag{3.1}$$

Proof: The proof can be found in [1]. □

The following Lemma (cf. [4]) on the generators of the Heisenberg–Weyl Lie algebra, is the basic tool used in the proofs of Theorems 6 above and 7, 9 below.

LEMMA 2. *Let x, D and h satisfy the Heisenberg commutation relations $[D, x] = h$ and $[D, h] = [x, h] = 0$. Then, for all $s, a, c \in \mathbb{C}$,*

$$e^{s(x+aD+ch)} = e^{sx} e^{saD} e^{(sc+\frac{s^2 a}{2})h}$$

and

$$e^{sD} e^{ax} = e^{ax} e^{sD} e^{ash}.$$

3.1. The inclusion: analytic continuation of the second quantized $w_\infty \subseteq$ RHPWN

The following theorem expresses the generators $\hat{B}_k^n(f)$ of the second quantized w_∞ $*$-Lie algebra as a series of the form $\sum_{n,k} A_{n,k} B_k^n(f_{n,k})$ of the generators $B_k^n(f)$ of the RHPWN $*$-Lie algebra. The considerations of the previous section show that this series has a meaning and that it is obtained from a function of b_t and $b_t^\dagger$, also defined in that section.

THEOREM 7. *Let $n \geq 2$ and $k \in \mathbb{Z}$. Then, for all $f \in \mathcal{S}_0$,*

$$\hat{B}_k^n(f) = \frac{1}{2^{n-1}} \sum_{m=0}^{n-1} \binom{n-1}{m} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^p \frac{k^{p+q}}{p!\,q!} B_{n-1-m+q}^{m+p}(f), \tag{3.2}$$

where convergence of infinite sums is understood in the topology introduced in Definition 2, *and the case $k = 0$ is interpreted as*

$$\hat{B}_0^n(f) = \frac{1}{2^{n-1}} \sum_{m=0}^{n-1} \binom{n-1}{m} B_{n-1-m}^{m}(f).$$

Proof: For fixed $t, s \in \mathbb{R}$, we will make repeated use of Lemma 2 with $D = b_t$, $x = b_s^\dagger$ and $h = \delta(t - s)$. We have

$$\begin{aligned}
\hat{B}_k^n(f) &= \int_{\mathbb{R}^d} f(s)\, e^{\frac{k}{2}(b_s - b_s^\dagger)} \left(\frac{b_s + b_s^\dagger}{2}\right)^{n-1} e^{\frac{k}{2}(b_s - b_s^\dagger)}\, ds \\
&= \frac{1}{2^{n-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t)\, e^{\frac{k}{2}(b_t - b_s^\dagger)} \left(b_t + b_s^\dagger\right)^{n-1} e^{\frac{k}{2}(b_t - b_s^\dagger)}\, \delta(t-s)\, dt\, ds \\
&= \frac{1}{2^{n-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t)\, e^{-\frac{k}{2}(b_s^\dagger - b_t)} \left(\frac{\partial^{n-1}}{\partial w^{n-1}}\Big|_{w=0} e^{w(b_t + b_s^\dagger)}\right) e^{-\frac{k}{2}(b_s^\dagger - b_t)}\, \delta(t-s)\, dt\, ds
\end{aligned}$$

$$= \frac{1}{2^{n-1}} \frac{\partial^{n-1}}{\partial w^{n-1}}|_{w=0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t)\, e^{-\frac{k}{2}(b_s^\dagger - b_t)} e^{w(b_t + b_s^\dagger)}\, e^{-\frac{k}{2}(b_s^\dagger - b_t)}\, \delta(t-s)\, dt\, ds$$

$$= \frac{1}{2^{n-1}} \frac{\partial^{n-1}}{\partial w^{n-1}}|_{w=0}$$

$$\times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t)\, e^{-\frac{k}{2} b_s^\dagger}\, e^{\frac{k}{2} b_t}\, e^{w\, b_s^\dagger}\, e^{w\, b_t}\, e^{-\frac{k}{2} b_s^\dagger}\, e^{\frac{k}{2} b_t}\, e^{(\frac{w^2}{2} - \frac{k^2}{4})\, \delta(t-s)}\, \delta(t-s)\, dt\, ds$$

$$= \frac{1}{2^{n-1}} \frac{\partial^{n-1}}{\partial w^{n-1}}|_{w=0}$$

$$\times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t)\, e^{-\frac{k}{2} b_s^\dagger}\, e^{w\, b_s^\dagger}\, e^{\frac{k}{2} b_t}\, e^{-\frac{k}{2} b_s^\dagger}\, e^{w\, b_t}\, e^{\frac{k}{2} b_t}\, e^{(\frac{w^2}{2} - \frac{k^2}{4}\, \delta(t-s)}\quad \delta(t-s)\, dt\, ds$$

$$= \frac{1}{2^{n-1}} \frac{\partial^{n-1}}{\partial w^{n-1}}|_{w=0}$$

$$\times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t)\, e^{(w - \frac{k}{2})\, b_s^\dagger}\, e^{\frac{k}{2} b_t}\, e^{-\frac{k}{2} b_s^\dagger}\, e^{(w + \frac{k}{2})\, b_t}\, e^{(\frac{w^2}{2} - \frac{k^2}{4})\, \delta(t-s)}\quad \delta(t-s)\, dt\, ds$$

$$= \frac{1}{2^{n-1}} \frac{\partial^{n-1}}{\partial w^{n-1}}|_{w=0}$$

$$\times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t)\, e^{\left(w - \frac{k}{2}\right) b_s^\dagger}\, e^{-\frac{k}{2} b_s^\dagger}\, e^{\frac{k}{2} b_t}\, e^{(w + \frac{k}{2})\, b_t}\, e^{(\frac{w^2}{2} - \frac{k^2}{2})\, \delta(t-s)}\quad \delta(t-s)\, dt\, ds$$

$$= \frac{1}{2^{n-1}} \frac{\partial^{n-1}}{\partial w^{n-1}}|_{w=0}$$

$$\times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t)\, e^{(w-k)\, b_s^\dagger}\, e^{(w+k)\, b_t}\, e^{(\frac{w^2}{2} - \frac{k^2}{2})\, \delta(t-s)}\quad \delta(t-s)\, dt\, ds$$

$$= \frac{1}{2^{n-1}} \frac{\partial^{n-1}}{\partial w^{n-1}}|_{w=0}$$

$$\times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t)\, e^{(w-k)\, b_s^\dagger}\, e^{(w+k)\, b_t} \sum_{m=0}^{\infty} \frac{\left(\frac{w^2}{2} - \frac{k^2}{2}\right)^m}{m!}\, \delta^m(t-s)\, \delta(t-s)\, dt\, ds$$

$$= \frac{1}{2^{n-1}} \frac{\partial^{n-1}}{\partial w^{n-1}}|_{w=0}$$

$$\times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t)\, e^{(w-k)\, b_s^\dagger}\, e^{(w+k)\, b_t} \sum_{m=0}^{\infty} \frac{\left(\frac{w^2}{2} - \frac{k^2}{2}\right)^m}{m!}\, \delta^{m+1}(t-s)\, dt\, ds$$

$$= \frac{1}{2^{n-1}} \frac{\partial^{n-1}}{\partial w^{n-1}}|_{w=0}$$

$$\times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t) e^{(w-k) b_s^\dagger} e^{(w+k) b_t} \left(\delta(t-s) + \sum_{m=1}^{\infty} \frac{\left(\frac{w^2}{2} - \frac{k^2}{2}\right)^m}{m!} \delta(s) \delta(t-s) \right) dt\, ds$$

$$= \frac{1}{2^{n-1}} \frac{\partial^{n-1}}{\partial w^{n-1}}|_{w=0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t)\, e^{(w-k)\, b_s^\dagger}\, e^{(w+k)\, b_t}\, \delta(t-s)\, dt\, ds$$

since, by the assumption $f(0) = 0$, all $\delta(s)\,\delta(t-s)$ terms vanish. Thus, using Leibniz's rule, we obtain

$$\hat{B}_k^n(f) = \frac{1}{2^{n-1}} \frac{\partial^{n-1}}{\partial w^{n-1}}|_{w=0} \int_{\mathbb{R}^d} f(s)\, e^{(w-k)\, b_s^\dagger}\, e^{(w+k)\, b_s}\, ds$$

$$= \frac{1}{2^{n-1}} \sum_{m=0}^{n-1} \binom{n-1}{m} \int_{\mathbb{R}^d} f(s)\, \frac{\partial^m}{\partial w^m}|_{w=0} (e^{(w-k)\, b_s^\dagger})\, \frac{\partial^{n-1-m}}{\partial w^{n-1-m}}|_{w=0} (e^{(w+k) b_s})\, ds$$

$$= \frac{1}{2^{n-1}} \sum_{m=0}^{n-1} \binom{n-1}{m} \int_{\mathbb{R}^d} f(s)\, b_s^{\dagger m}\, e^{-k\, b_s^\dagger}\, b_s{}^{n-1-m}\, e^{k\, b_s}\, ds$$

$$= \frac{1}{2^{n-1}} \sum_{m=0}^{n-1} \binom{n-1}{m} \int_{\mathbb{R}^d} f(s)\, b_s^{\dagger m} \sum_{p=0}^{\infty} \frac{(-k)^p}{p!}\, b_s^{\dagger p}\, b_s{}^{n-1-m} \sum_{q=0}^{\infty} \frac{k^q}{q!}\, b_s{}^q\, ds$$

$$= \frac{1}{2^{n-1}} \sum_{m=0}^{n-1} \binom{n-1}{m} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^p \frac{k^{p+q}}{p!\, q!} \int_{\mathbb{R}^d} f(s)\, b_s^{\dagger m+p}\, b_s{}^{n-1-m+q}\, ds$$

$$= \frac{1}{2^{n-1}} \sum_{m=0}^{n-1} \binom{n-1}{m} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^p \frac{k^{p+q}}{p!\, q!}\, B_{n-1-m+q}^{m+p}(f). \qquad \square$$

From the identity (3.2), it is clear that one can analytically continue the parameter k, in the definition of $\hat{B}_k^n(f)$, to an arbitrary complex number $k \in \mathbb{C}$ and $n \geq 1$. After this extension the identity $[\hat{B}_k^1(g), \hat{B}_K^1(f)]_{w_\infty} = 0$ still holds. Moreover, the proof of Theorem 7 immediately extends to $k \in \mathbb{C}$ and we have this result.

THEOREM 8. *Let $n \geq 2$ and $z \in \mathbb{C}$. Then, for all $f \in \mathcal{S}_0$,*

$$\hat{B}_z^n(f) = \frac{1}{2^{n-1}} \sum_{m=0}^{n-1} \binom{n-1}{m} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^p \frac{z^{p+q}}{p!\, q!}\, B_{n-1-m+q}^{m+p}(f), \tag{3.3}$$

where convergence of infinite sums is understood in the topology introduced in Definition 2, and the case $z = 0$ is interpreted as

$$\hat{B}_0^n(f) = \frac{1}{2^{n-1}} \sum_{m=0}^{n-1} \binom{n-1}{m} B_{n-1-m}^m(f)$$

COROLLARY 1. *For $z \in \mathbb{C}$ and $k \in \{0, 1, \dots\}$,*

$$\frac{\partial^k}{\partial z^k}|_{z=0} \hat{B}_z^n(f) := \sum_{m=0}^{k} \frac{\binom{k}{m}}{2^{k+n-1}} \int_{\mathbb{R}^d} f(t)(b_t - b_t^\dagger)^m (b_t + b_t^\dagger)^{n-1} (b_t - b_t^\dagger)^{k-m}\, dt \tag{3.4}$$

Moreover, for $k \geq 1$

$$\frac{\partial^k}{\partial z^k}|_{z=0}\hat{B}^n_z(f) = \frac{1}{2^{n-1}} \sum_{m=0}^{n-1} \binom{n-1}{m} \sum_{p=0}^{\infty} (-1)^p \binom{k}{p} B^{m+p}_{n+k-m-1-p}(f) \tag{3.5}$$

Proof: (3.4) is obtained from (3.1) with the use of Leibniz's rule. Finally, (3.5) follows from (3.3), by differentiating term-by-term k-times with respect to z, and noticing that only the terms $p+q=k$ make a nonzero contribution. □

3.2. The inclusion: RHPWN $\subseteq$ analytic continuation of the second quantized w_∞

In this subsection we will find the expression of the generators $B^n_k(f)$ of the RHPWN $*$-Lie–algebra in terms of the generators $\hat{B}^n_z(f)$ of the analytically continued w_∞-$*$-Lie–algebra. We will thus complete the identification of the two algebras.

THEOREM 9. *Let* $n, k \in \mathbb{N} \cup \{0\}$. *Then, for all* $f \in \mathcal{S}_0$,

$$B^n_k(f) = \sum_{\rho=0}^{k} \sum_{\sigma=0}^{n} \binom{k}{\rho} \binom{n}{\sigma} \frac{(-1)^\rho}{2^{\rho+\sigma}} \frac{\partial^{\rho+\sigma}}{\partial z^{\rho+\sigma}}|_{z=0} \hat{B}^{k+n+1-(\rho+\sigma)}_z(f), \tag{3.6}$$

where, as in Theorem 7, *convergence of infinite sums is understood in the topology introduced in Definition* 2. *Moreover, the right hand side of* (3.6) *is well defined in the sense of Definition* 3.

Proof: For $t, s \geq 0$, let $p_{t,s} := b_t - b^\dagger_s$ and $q_{t,s} := b_t + b^\dagger_s$. Then

$$\begin{aligned} B^n_k(f) &= \int_{\mathbb{R}^d} f(t)\, b^{\dagger n}_t\, b^k_t\, dt \\ &= \frac{1}{2^{n+k}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, (p_{t,s} + q_{t,s})^n\, (q_{t,s} - p_{t,s})^k\, \delta(s-t)\, ds\, dt \\ &= \frac{1}{2^{n+k}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, \frac{\partial^n}{\partial \lambda^n}|_{\lambda=0}(e^{\lambda\,(p_{t,s}+q_{t,s})})\, \frac{\partial^k}{\partial \mu^k}|_{\mu=0}(e^{\mu\,(q_{t,s}-p_{t,s})})\, \delta(s-t)\, ds\, dt \\ &= \frac{1}{2^{n+k}} \frac{\partial^{n+k}}{\partial \lambda^n\, \mu^k}|_{\lambda=\mu=0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, e^{\lambda\,(p_{t,s}+q_{t,s})}\, e^{\mu\,(q_{t,s}-p_{t,s})}\, \delta(s-t)\, ds\, dt \end{aligned}$$

which, using Lemma 2 with $D = p_{t,s}$, $x = q_{t,s}$ and $h = 2\,\delta(s-t)$, is equal to

$$\begin{aligned} &= \frac{1}{2^{n+k}} \frac{\partial^{n+k}}{\partial \lambda^n\, \mu^k}|_{\lambda=\mu=0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, e^{\lambda\, p_{t,s}}\, e^{\lambda\, q_{t,s}}\, e^{\mu\, q_{t,s}}\, e^{-\mu\, p_{t,s}}\, e^{(\lambda^2-\mu^2)\,\delta(s-t)}\, \delta(s-t)\, ds\, dt \\ &= \frac{1}{2^{n+k}} \frac{\partial^{n+k}}{\partial \lambda^n\, \mu^k}|_{\lambda=\mu=0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, e^{\lambda\, p_{t,s}}\, e^{(\lambda+\mu)\, q_{t,s}}\, e^{-\mu\, p_{t,s}}\, e^{(\lambda^2-\mu^2)\,\delta(s-t)}\, \delta(s-t)\, ds\, dt \end{aligned}$$

which, again by Lemma 2, is equal to

$$= \frac{1}{2^{n+k}} \frac{\partial^{n+k}}{\partial \lambda^n\, \mu^k}|_{\lambda=\mu=0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, e^{\lambda\, p_{t,s}}\, e^{-\mu\, p_{t,s}}\, e^{(\lambda+\mu)\, q_{t,s}}\, e^{(\lambda+\mu)^2\,\delta(s-t)}\, \delta(s-t)\, ds\, dt$$

$$= \frac{1}{2^{n+k}} \frac{\partial^{n+k}}{\partial \lambda^n \, \mu^k}|_{\lambda=\mu=0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, e^{(\lambda-\mu)\, p_{t,s}}\, e^{(\lambda+\mu)\, q_{t,s}}\, e^{(\lambda+\mu)^2\, \delta(s-t)}\, \delta(s-t)\, ds\, dt.$$

Since, as in the proof of Theorem 7, only the constant term 1 in the exponential series $e^{(\lambda+\mu)^2\, \delta(s-t)}$ will eventually make a nonzero contribution, the above is equal to

$$\frac{1}{2^{n+k}} \frac{\partial^{n+k}}{\partial \lambda^n \, \mu^k}|_{\lambda=\mu=0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, e^{(\lambda-\mu)\, p_{t,s}}\, e^{(\lambda+\mu)\, q_{t,s}}\, \delta(s-t)\, ds\, dt$$

which, using Leibniz's rule first for the derivative with respect to μ and then for the derivative with respect to λ, is equal to

$$= \frac{1}{2^{n+k}} \sum_{\rho=0}^{k} \binom{k}{\rho} (-1)^\rho \frac{\partial^n}{\partial \lambda^n}|_{\lambda=0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, p_{t,s}^\rho\, e^{\lambda\, p_{t,s}}\, q_{t,s}^{k-\rho}\, e^{\lambda\, q_{t,s}}\, \delta(s-t)\, ds\, dt$$

$$= \frac{1}{2^{n+k}} \sum_{\rho=0}^{k} \sum_{\sigma=0}^{n} \binom{k}{\rho} \binom{n}{\sigma} (-1)^\rho \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, p_{t,s}^\rho\, p_{t,s}^\sigma\, q_{t,s}^{k-\rho}\, q_{t,s}^{n-\sigma}\, \delta(s-t)\, ds\, dt$$

$$= \frac{1}{2^{n+k}} \sum_{\rho=0}^{k} \sum_{\sigma=0}^{n} \binom{k}{\rho} \binom{n}{\sigma} (-1)^\rho \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, p_{t,s}^{\rho+\sigma}\, q_{t,s}^{k+n-(\rho+\sigma)}\, \delta(s-t)\, ds\, dt$$

$$= \frac{1}{2^{n+k}} \sum_{\rho=0}^{k} \sum_{\sigma=0}^{n} \binom{k}{\rho} \binom{n}{\sigma} (-1)^\rho \frac{\partial^{\rho+\sigma}}{\partial z^{\rho+\sigma}}|_{z=0}$$

$$\times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, e^{z\, p_{t,s}}\, q_{t,s}^{k+n-(\rho+\sigma)}\, \delta(s-t)\, ds\, dt$$

$$= \frac{1}{2^{n+k}} \sum_{\rho=0}^{k} \sum_{\sigma=0}^{n} \binom{k}{\rho} \binom{n}{\sigma} (-1)^\rho \frac{\partial^{\rho+\sigma}}{\partial z^{\rho+\sigma}}|_{z=0}$$

$$\times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, e^{\frac{z}{2}\, p_{t,s}}\, e^{\frac{z}{2}\, p_{t,s}}\, q_{t,s}^{k+n-(\rho+\sigma)}\, \delta(s-t)\, ds\, dt$$

$$= \frac{1}{2^{n+k}} \sum_{\rho=0}^{k} \sum_{\sigma=0}^{n} \binom{k}{\rho} \binom{n}{\sigma} (-1)^\rho$$

$$\times \frac{\partial^{\rho+\sigma}}{\partial z^{\rho+\sigma}}|_{z=0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, e^{\frac{z}{2}\, p_{t,s}} \frac{\partial^{k+n-(\rho+\sigma)}}{\partial w^{k+n-(\rho+\sigma)}}|_{w=0} (e^{\frac{z}{2}\, p_{t,s}}\, e^{w\, q_{t,s}})\, \delta(s-t)\, ds\, dt.$$

This, by Lemma 2, is equal to

$$\frac{1}{2^{n+k}} \sum_{\rho=0}^{k} \sum_{\sigma=0}^{n} \binom{k}{\rho} \binom{n}{\sigma} (-1)^\rho \frac{\partial^{\rho+\sigma}}{\partial z^{\rho+\sigma}}|_{z=0}$$

$$\times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s)\, e^{\frac{z}{2}\, p_{t,s}} \frac{\partial^{k+n-(\rho+\sigma)}}{\partial w^{k+n-(\rho+\sigma)}}|_{w=0} (e^{w\, q_{t,s}}\, e^{\frac{z}{2}\, p_{t,s}}\, e^{w\, z\, \delta(s-t)})\, \delta(s-t)\, ds\, dt$$

. Since, only the constant term 1 in the exponential series $e^{w\,z\,\delta(s-t)}$ will eventually make a nonzero contribution, the above is equal to

$$\begin{aligned}
&\frac{1}{2^{n+k}}\sum_{\rho=0}^{k}\sum_{\sigma=0}^{n}\binom{k}{\rho}\binom{n}{\sigma}(-1)^{\rho}\frac{\partial^{\rho+\sigma}}{\partial z^{\rho+\sigma}}|_{z=0}\\
&\times\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(s)\, e^{\frac{z}{2}\,p_{t,s}}\,\frac{\partial^{k+n-(\rho+\sigma)}}{\partial w^{k+n-(\rho+\sigma)}}|_{w=0}\left(e^{w\,q_{t,s}}\,e^{\frac{z}{2}\,p_{t,s}}\right)\delta(s-t)\,ds\,dt\\
=&\frac{1}{2^{n+k}}\sum_{\rho=0}^{k}\sum_{\sigma=0}^{n}\binom{k}{\rho}\binom{n}{\sigma}(-1)^{\rho}\frac{\partial^{\rho+\sigma}}{\partial z^{\rho+\sigma}}|_{z=0}\\
&\times\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(s)\, e^{\frac{z}{2}\,p_{t,s}}\,q_{t,s}^{k+n-(\rho+\sigma)}\,e^{\frac{z}{2}\,p_{t,s}}\,\delta(s-t)\,ds\,dt\\
=&\frac{1}{2^{n+k}}\sum_{\rho=0}^{k}\sum_{\sigma=0}^{n}\binom{k}{\rho}\binom{n}{\sigma}(-1)^{\rho}\frac{\partial^{\rho+\sigma}}{\partial z^{\rho+\sigma}}|_{z=0}\int_{\mathbb{R}^d} f(t)\, e^{\frac{z}{2}\,p_{t,t}}\,q_{t,t}^{k+n-(\rho+\sigma)}\,e^{\frac{z}{2}\,p_{t,t}}\,dt\\
=&\sum_{\rho=0}^{k}\sum_{\sigma=0}^{n}\binom{k}{\rho}\binom{n}{\sigma}\frac{(-1)^{\rho}}{2^{\rho+\sigma}}\frac{\partial^{\rho+\sigma}}{\partial z^{\rho+\sigma}}|_{z=0}\,\hat{B}_{z}^{k+n+1-(\rho+\sigma)}(f).
\end{aligned}$$

Finally, using (3.5), (3.6) becomes

$$\begin{aligned}
B_k^n(f)=&\sum_{\rho=0}^{k}\sum_{\sigma=0}^{n}\binom{k}{\rho}\binom{n}{\sigma}\frac{(-1)^{\rho}}{2^{\rho+\sigma}}\frac{1}{2^{k+n-(\rho+\sigma)}}\sum_{m=0}^{k+n-(\rho+\sigma)}\binom{k+n-(\rho+\sigma)}{m}\\
&\times\sum_{p=0}^{\infty}(-1)^{p}\binom{\rho+\sigma}{p}B_{n+k-m-p}^{m+p}(f).
\end{aligned}\tag{3.7}$$

Since n,k are fixed, we notice that for each m and p the coefficient of $B_{n+k-m-p}^{m+p}(f)$ is finite. Thus (3.7) is meaningful in the sense of Definition 3. □

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