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UNIQUENESS OF THE EPR–CHAMELEON MODEL

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A classical deterministic, reversible dynamical systems, reproducing the Einstein–Podolsky–Rosen (EPR) correlations in full respect of causality and locality and without the introduction of any *ad hoc* selection procedure, was constructed in Ref. 3.

In this paper we prove that the above-mentioned model is unique (see Theorem 3.1) in the sense that any local causal probability measure which reproduces the EPR correlations must coincide, under natural and generic assumptions, with the one constructed in Ref. 3.

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1. Introduction

It is now understood:

(i) that the common mathematical root of the apparent paradoxes arising in connection with 2-slit type or EPR type experiments is that certain statistical data (conditional probabilities, correlations, ...) cannot be reproduced by a single Kolmogorovian probability space.[1]

(ii) that there exist classical deterministic, reversible dynamical systems, reproducing the singlet correlations of spins pairs (or of polarizations of a pair of entangled photons), called EPR correlations in the following.[3,4]

The construction of such dynamical systems was made possible by a new physical idea (the chameleon effect) and a new mathematical tool (the notion of notrivial local causal measure).

The chameleon effect is in the statement that the local dynamics of some systems (adaptive systems) may depend on the observable that one measures. The purpose of the EPR-chameleon model is a simple realization of this general idea.

The striking feature of the EPR-chameleon model is that the dynamics of each spin as well as the structure of the state (i.e. the probability measure defining the statisitics) is local and causal, i.e. there is no action at distance between the spins in the pair or between the two measurement apparata and no previous knowledge of the future measurements. Everything is completely pre-determined at the source through an *if-then* scheme which is typical of adaptive systems and which justifies the chameleon metaphora (*if I meet a leaf I will become green, if I meet a piece of wood I will become brown*). In the mathematical model, the *if-then* scheme is entirely coded in an intrinsic dynamics and an initial state and no artificial selection or rejection procedures are introduced by hand.

Even if the models described in this paper are inspired to the EPR–Bohm type experiments,[7,8] we emphasize that all our constructions will be entirely within the classical theory of dynamical systems.

The organization of the paper is as follows:

Section 2 introduces the notion of triviality of a LC measure and shows that such measures cannot violate Bell's inequality. Thus if we want to reproduce the EPR-type correlations, then we must investigate nontrivial LC measures.

The main result of Sec. 2 is the proof of the fact that the class of trivial LC measures and the class of nontrivial LC measures cannot be connected by any local and reversible dynamics (Corollary 2.1).

Section 3 contains the main result of the present paper, i.e. the proof (see Theorem 3.1) of the fact that any LC probability measure which reproduces the EPR correlations must coincide, under natural and generic assumptions, with the one proposed in Ref. 3.

Section 4 makes explicit the mathematical differences between passive and adaptive dynamical systems (see also Ref. 4).

Section 5 shows how the difference between standard and distant particles empirical correlations is reflected in the corresponding mathematical models.

The generic assumptions used in the proof of our uniqueness theorem (Theorem 3.1) are the following:

(i) The condition of statistical pre-determination (see Definition 3.3).
(ii) The rotation invariance of the densities describing the local apparata (see condition (3.8)).
(iii) The twice continuous differentiability of these densities (see Theorem 3.1).
(iv) The absolute continuity of the source measure with respect to the Lebesgue measure (see Proposition 3.1).

While conditions (i) and (ii) have a natural physical interpretation, we do not see any natural physical justification for conditions (iii) and (iv).

For example, at the moment we have no reasons to exclude the possibility of reproducing the EPR correlations with a source measure having a fractal support.

Therefore it would be interesting to know if, by dropping some of these assumptions, the uniqueness result continues to be true. This problem will be the object of further investigations.

2. Trivial LC Measures

We consider a composite system made up of two subsystems, often called "particles" and denoted with the symbols 1 and 2 respectively. Their "configuration" (or "phase") spaces will be denoted by S_1 and S_2, respectively. The two systems are spatially separated so that the mutual interactions between them can be neglected. Each system interacts locally with a measurement apparatus, i.e. system 1 with apparatus m_1 and system 2 with apparatus m_2. The configuration spaces of the measurement apparata will be denoted by M_1 and M_2 respectively. We use the indices $a, b, \ldots \in I$ to represent settings of the measurement apparata. In the second part of the paper from Sec. 3 on, we specialize the set of indices I to be the interval $[0, 2\pi]$.

The notion of "local and causal probability measure" is crucial for EPR-chameleon models.

Definition 2.1. (Ref. 4, Definition 6) A probability measure $P_{a,b}$ on $S_1 \times S_2 \times M_1 \times M_2$ is called local and causal (LC, shortly) if it has the form

$$dP_{a,b}(s_1, s_2, \lambda_1, \lambda_2) = dP_S(s_1, s_2)P_{1,a}(d\lambda_1; s_1)P_{2,b}(d\lambda_2; s_2)\,, \qquad (2.1)$$

where P_S is a probability measure on $S_1 \times S_2$; for all $s_1 \in S_1$, $P_{1,a}(\cdot; s_1)$ is a positive measure on M_1; for all $s_2 \in S_2$, $P_{2,b}(\cdot; s_2)$ is a positive measure on M_2.

Notice that the requirement that P_S is a probability measure on $S_1 \times S_2$ is not essential: if P_S is any finite measure, by multiplying P_S, $P_{1,a}(\cdot; s_1)$ and $P_{2,b}(\cdot; s_2)$ by positive constants whose product is equal to 1, one can always reduce oneself to the case that P_S is a probability measure.

This multiplication and division by the same constant is trivial from the mathematical point of view, but it may be essential for the purpose of a local simulation of a LC measure (see the discussion in Sec. 5 below). This is precisely the case for the measure constructed in Ref. 4.

Let us assume that all the following are compact Hausdorff spaces:

— the configuration space S_1 of the subsystem 1,
— the configuration space S_2 of the subsystem 2,
— the configuration space M_1 of the measurement apparatus for the subsystem 1,
— the configuration space M_2 of the measurement apparatus for the subsystem 2.

In terms of these, we define the configuration spaces for the composite systems:

$$S := S_1 \times S_2\,, \qquad M := M_1 \times M_2\,, \qquad \Omega_1 := S_1 \times M_1\,, \qquad \Omega_2 := S_2 \times M_2\,,$$
$$\Omega := \Omega_1 \times \Omega_2 = S_1 \times M_1 \times S_2 \times M_2 = S_1 \times S_2 \times M_1 \times M_2\,. \tag{2.2}$$

Let $\mathrm{Meas}(\Omega)$ denote the set of all regular, signed, finite Borel measures on $(\Omega, \mathcal{B})$. $\langle \mathrm{Meas}(\Omega), C(\Omega)\rangle$ denotes the duality $\mathrm{Meas}(\Omega) = C(\Omega)^*$. $\mathrm{Meas}_+(\Omega)$ and $\mathrm{Prob}(\Omega)$ denote the set of all positive measures and the set of all probability measures in $\mathrm{Meas}(\Omega)$, respectively.

Then, since P_S is a probability measure on $S_1 \times S_2$, $P_{a,b}$, given by (2.1), is a LC measure on $S_1 \times S_2 \times M_1 \times M_2$ which can be written in the following functional form:

$$P_{a,b} := P_S \circ (\bar{P}_{1,a} \otimes \bar{P}_{2,b}) \in (C(\Omega_1) \otimes C(\Omega_2))^* = C(\Omega_1 \times \Omega_2)^*\,, \tag{2.3}$$

where, for $j = 1$, 2 and $x = a$, b, the linear maps

$$\bar{P}_{j,x} : C(\Omega_j) = C(S_j \times M_j) \to C(S_j) \subseteq C(\Omega_j)$$

are defined by

$$\bar{P}_{j,x}(f)(s_j) := \int_{M_j} f(s_j, \lambda_j) dP_{j,x}(\lambda_j; s_j) \tag{2.4}$$

for each $f \in C(S_j \times M_j)$.

Definition 2.2. (Ref. 4, Definition 7) A LC probability measure on the space $S_1 \times S_2 \times M_1 \times M_2$

$$dP_{a,b}(s_1, s_2, \lambda_1, \lambda_2) = dP_S(s_1, s_2) dP_{1,a}(\lambda_1; s_1) dP_{2,b}(\lambda_2; s_2)$$

is called trivial if, in the notation (2.4), $\forall\ a,\ b \in I$ the map

$$\bar{P}_{1,a} \otimes \bar{P}_{2,b} : C(\Omega_1 \times \Omega_2) \to C(S_1 \times S_2)$$

is a P_S-conditional expectation, i.e.

$$\bar{P}_{1,a}(1_1)(s_1)\bar{P}_{2,b}(1_2)(s_2) \equiv 1\,, \qquad P_S\text{-a.e.} \tag{2.5}$$

Denoting

$$p_{1,a}(s_1) := \bar{P}_{1,a}(1_1)(s_1) = \int_{M_1} dP_{1,a}(\lambda_1; s_1)\,, \tag{2.6}$$

$$p_{2,b}(s_2) := \bar{P}_{2,b}(1_2)(s_2) = \int_{M_2} dP_{2,b}(\lambda_2; s_2)\,, \tag{2.7}$$

condition (2.5) becomes equivalent to:

$$p_{1,a}(s_1) p_{2,b}(s_2) = 1\,, \qquad P_S\text{-a.e.} \tag{2.8}$$

Remark 2.1. If a LC measure is trivial, then from

$$p_{1,a}(s_1) = \frac{1}{p_{2,b}(s_2)}\,, \qquad P_S\text{-a.e.}\,,$$

there exists a positive real number c such that

$$p_{1,a}(s_1) = c\,, \qquad p_{2,b}(s_2) = \frac{1}{c}\,, \qquad P_S\text{-a.e.}$$

By redefining $P'_{1,a} := (1/c)P_{1,a}$, $P'_{2,b} := cP_{2,b}$, we can assume without loss of generality that

$$p_{1,a}(s_1) = 1\,, \qquad p_{2,b}(s_2) = 1\,, \qquad P_S\text{-a.e.}$$

The following result shows why contextuality alone is not sufficient to account for the violation of Bell's inequality.

Proposition 2.1. (*Ref.* 3) *Let I be any index set and let $P_{a,b}$ $(a, b \in I)$ be a family of trivial LC probability measures on the space Ω defined by* (2.2). *Then the pair correlations of any family of random variables $S_a^{(1)}$, $S_b^{(2)} : \Omega \to [-1, 1]$ $(a, b \in I)$ satisfying the locality condition*

$$S_a^{(1)}(\omega_1, \omega_2) = S_a^{(1)}(\omega_1)\,, \qquad S_b^{(2)}(\omega_1, \omega_2) = S_b^{(2)}(\omega_2)\,, \qquad (\omega_1, \omega_2) \in \Omega = \Omega_1 \times \Omega_2$$

cannot violate Bell's inequality.

Proof. The pair correlations of the random variables $S_a^{(1)}$, $S_b^{(2)}$ are defined by

$$C(a, b) := \langle P_{a,b}, S_a^{(1)} \otimes S_b^{(2)} \rangle = \langle P_S, barP_{1,a}(S_a^{(1)}) \otimes \bar{P}_{2,b}(S_b^{(2)}) \rangle_{S_1 \times S_2}\,.$$

Using the functional form (2.3) of the trivial measures $P_{a,b}$, one finds

$$\begin{aligned} &|C(a, b) - C(a, b')| + |C(a', b) + C(a', b')| \\ &\quad \le \langle P_S, |\bar{P}_{1,a}(S_a^{(1)}) \otimes [\bar{P}_{2,b}(S_b^{(2)}) - \bar{P}_{2,b'}(S_{b'}^{(2)})]| \rangle_{S_1 \times S_2} \\ &\qquad + \langle P_S, |\bar{P}_{1,a'}(S_{a'}^{(1)}) \otimes [\bar{P}_{2,b}(S_b^{(2)}) + \bar{P}_{2,b'}(S_{b'}^{(2)})]| \rangle_{S_1 \times S_2} \\ &\quad \le \langle P_S, |\bar{P}_{2,b}(S_b^{(2)}) - \bar{P}_{2,b'}(S_{b'}^{(2)})| + |\bar{P}_{2,b}(S_b^{(2)}) + \bar{P}_{2,b'}(S_{b'}^{(2)})| \rangle_{S_1 \times S_2} \le 2\,, \end{aligned}$$

where in the last inequality we have used the fact that Bell's inequality (in CHSH form) is satisfied by any quadruple of random variables, on a single probability space, with values in the interval $[-1, 1]$ (for a proof of this statement see Refs. 2 and 9). □

Remark 2.2. To be a trivial LC measure is a sufficient, but not necessary contition to satisfy Bell's inequality. There are nontrivial LC measures which are essentially trivial and do not violate Bell's inequality. For example let $P_{a,b} = P_S \circ (\bar{P}_{1,a} \otimes \bar{P}_{2,b})$ be a trivial LC measure. Let q_1 and q_2 be nonzero measurable functions on S_1 and S_2 respectively, such that

$$\int_{S_1 \times S_2} dP_S(s_1, s_2) q_1(s_1) q_2(s_2) = 1\,.$$

Define $Q \in \mathrm{Prob}(S_1 \times S_2)$ by $dQ(s_1, s_2) := q_1(s_1)q_2(s_2)dP_S(s_1, s_2)$. Since

$$P_{a,b} = Q \circ (((1/q_1)\bar{P}_{1,a}) \otimes ((1/q_2)\bar{P}_{2,b}))\,,$$

if $q_1 \otimes q_2$ is not constant on $\operatorname{supp} Q$, then $P_{a,b}$ becomes nontrivial.

Recall that, for any pair of compact topological spaces Ω, S, a linear map

$$\mathcal{T}^* : C(\Omega) \to C(S)$$

is called a Markov operator if it is positivity preserving ($f \geq 0 \Rightarrow \mathcal{T}^*(f) \geq 0$, $f \in \mathcal{C}(\Omega)$) and

$$\mathcal{T}^*(1_\Omega) = 1_S \,.$$

If on S there is a probability measure P_S and $\mathcal{T}^*$ satisfies the weaker conditions

$$f \geq 0 \Rightarrow \mathcal{T}^*(f) \geq 0\,; \qquad P_S\text{-a.e.} f \in \mathcal{C}(\Omega)$$
$$\mathcal{T}^*(1_\Omega) = 1_S\,, \qquad P_S\text{-a.e.}$$

we call it a P_S-Markov operator. Now let

$$\Omega = \Omega_1 \times \Omega_2\,, \qquad S = S_1 \times S_2\,.$$

The identifications:

$$s_1 \equiv s_1 \times S_2\,, \qquad s_2 \equiv S_1 \times s_2\,, \qquad s_1 \in S_1\,, \qquad s_2 \in S_2$$

allows us to consider both S_1 and S_2 as subsets of $S_1 \times S_2$.

Lemma 2.1. *For $j = 1, 2$, let $\mathcal{T}_j^* : C(\Omega_j) \to C(\Omega_j)$ be a positivity preserving linear operator. The following conditions are equivalent*:

$$\bar{P}_{1,a}(\mathcal{T}_1^*(1))\bar{P}_{1,b}(\mathcal{T}_2^*(1)) = 1\,, \qquad P_S\text{-a.e.} \tag{2.9}$$

there exists a constant $c > 0$ such that

$$\bar{P}_{1,a}(c\mathcal{T}_1^*(1)) = \bar{P}_{1,b}(\mathcal{T}_2^*(1)/c) = 1\,, \qquad P_S\text{-a.e.} \tag{2.10}$$

Proof. It is clear that (2.10) $\Rightarrow$ (2.9). Let us prove the converse implication. If (2.9) holds, then

$$P_S \circ ([\bar{P}_{1,a} \circ \mathcal{T}_1^*] \otimes [\bar{P}_{2,b} \circ \mathcal{T}_2^*])$$

is a trivial measure. Therefore, by Remark 2.1 there exists a constant $c > 0$ such that

$$c\bar{P}_{1,a}(\mathcal{T}_1^*(1))(s_1) = \frac{1}{c}\bar{P}_{2,b}(\mathcal{T}_2^*(1))(s_2) = 1\,, \qquad P_S - \forall\, (s_1, s_2) \in S_1 \times S_2$$

and this is (2.10). □

Definition 2.3. A linear positive operator $\mathcal{T}_1^* \otimes \mathcal{T}_2^* : C(\Omega_1 \times \Omega_2) \to C(\Omega_1 \times \Omega_2)$ (or equivalently its dual $\mathcal{T}_1 \otimes \mathcal{T}_2$, acting on measures), which satisfies the conditions of Lemma 2.1 will be called a $P_{a,b}$-Markovian operator. In such a case, by absorbing the constants c, $1/c$ in the definition of $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$, one can always assume that they are equal to 1.

Remark 2.3. Notice that any Markovian operator is $P_{a,b}$-Markovian for any $P_{a,b}$.

Theorem 2.1. *Let, for* $j = 1, 2$, $\mathcal{T}_j$ *be a linear mapping of* $\text{Meas}_+(\Omega_j)$ *into* $\text{Meas}_+(\Omega_j)$ *such that* $\mathcal{T}_j^* : C(\Omega_j) \to C(\Omega_j)$ *and let*

$$P_{a,b} = P_S \circ (\bar{P}_{1,a} \otimes \bar{P}_{2,b}) \in \text{Prob}(\Omega_1 \times \Omega_2)$$

be any trivial LC measure. Then if $\mathcal{T}_{1,a} \otimes \mathcal{T}_{2,b}$ *is a* $P_{a,b}$*-Markovian operator,* $(\mathcal{T}_{1,a} \otimes \mathcal{T}_{2,b})(P_{a,b})$ *is a trivial LC measure. In particular, if* $\mathcal{T}_{1,a} \otimes \mathcal{T}_{2,b}$ *is a Markov operator, it maps trivial LC measures into trivial LC measures.*

Proof. The functional form of $(\mathcal{T}_{1,a} \otimes \mathcal{T}_{2,b})(P_{a,b})$ is:

$$(\mathcal{T}_{1,a} \otimes \mathcal{T}_{2,b})(P_{a,b}) = P_S \circ (\bar{P}_{1,a} \circ \mathcal{T}_{1,a}^* \otimes \bar{P}_{2,b} \circ \mathcal{T}_{2,b}^*) . \tag{2.11}$$

Condition (2.10) (with $c = 1$) is equivalent to

$$\bar{P}_{1,a}(\mathcal{T}_1^*(1)) = \bar{P}_{2,b}(\mathcal{T}_2^*(1)) = 1 , \qquad P_S\text{-a.e.}$$

which is equivalent to the triviality of $(\mathcal{T}_{1,a} \otimes \mathcal{T}_{2,b})(P_{a,b})$. □

Corollary 2.1. *Any local reversible dynamics induces a mapping which maps a nontrivial* (*resp. trivial*) *LC measure into a nontrivial* (*resp. trivial*) *LC measure.*

Proof. The statement about trivial LC measures follows from Theorem 2.1.

Let μ be a nontrivial LC measure and T be a reversible measurable transformation of $S_1 \times M_1 \times S_2 \times M_2$ into itself. Suppose by contradiction that $\nu := \mu \circ T$ is trivial.

The linear mapping $\mathcal{T}$ induced by T is a Markov operator satisfying $\mu = \mathcal{T}(\nu) := \nu \circ T^{-1}$. Its inverse is also a Markov operator satisfying $\nu = \mathcal{T}^{-1}(\mu) := \mu \circ T$.

But if T is local, i.e. of the form $T = T_1 \times T_2$ for some $T_1 : S_1 \times M_1 \to S_1 \times M_1$ and $T_2 : S_2 \times M_2 \to S_2 \times M_2$, then $\mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_2$ where $\mathcal{T}_1$ and $\mathcal{T}_2$ are Markov operators. By the remark after Definition 2.3 this contradicts Theorem 2.1. □

3. AIR Models

In the EPR-chameleon model constructed in Refs. 3 and 4 (hereinafter AIR model), which reproduces the EPR–Bohm correlations, the configuration space of the single particle is chosen to be the unit circle, i.e.

$$S_1 = S_2 = S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

and the observables to be functions $f : S^1 \to \mathbb{R}$. It is convenient, in order to calculate easily the integrals expressing the correlations, to identify S^1 with the quotient space $\mathbb{R}/(2\pi\mathbb{Z}) \equiv [0, 2\pi)$, i.e. the real numbers defined modulo 2π and the observables with periodic functions $f : \mathbb{R} \to \mathbb{R}$ with period 2π. We will freely use this identification in the following. $S_1 \times S_2$ is a two-dimensional torus $T^2 := S^1 \times S^1$. Define

$$I_a := \left[-\frac{\pi}{2} + a, a + \frac{\pi}{2}\right) ,$$

$$J_a := \left[a + \frac{\pi}{2}, a + \frac{3\pi}{2}\right) .$$

Under our convenction of identifying numbers modulo 2π, one has

$$I_{a+\pi} = J_a\,, \qquad J_{a+\pi} = I_a\,.$$

The random variables $S_a^{(1)}$ and $S_b^{(2)}$, representing outcomes of measurements of spins, are parametrized by a, $b \in [0, 2\pi)$ and are defined by

$$S_a^{(1)}(s_1) := \chi_{I_a}(s_1) - \chi_{J_a}(s_1)\,, \qquad s_1 \in S_1\,, \tag{3.1}$$

$$S_b^{(2)}(s_2) := -\chi_{I_b}(s_2) + \chi_{J_b}(s_2)\,, \qquad s_2 \in S_2\,, \tag{3.2}$$

thus they depend only on the final configurations of the particles, $s_1 \in S_1$ and $s_2 \in S_2$ respectively and are independent of the (final) configurations of the measurement apparata (the reason why we interpret these points as final rather than as initial configurations is discussed in Secs. 4 and 5).

In this section we study the most general family of local causal probability measures on T^2 which reproduce the EPR–Bohm correlations and we prove that, under natural generic conditions, they must have the form used in the AIR model.

If $P_{a,b}$ is a local causal probability measure on $S_1 \times S_2 \times M_1 \times M_2$ of the form (2.1), we denote $R_{a,b}$ its marginal probability on $T^2 = S^1 \times S^1$. Using the notations (2.6), (2.7), we can write $R_{a,b}$ in the following form:

$$dR_{a,b}(s_1, s_2) = dP_S(s_1, s_2) p_{1,a}(s_1) p_{2,b}(s_2)\,, \tag{3.3}$$

where s_1, $s_2 \in [0, 2\pi)$ are fixed parametrizations of $S_1 = S^1$ and $S_2 = S^1$ respectively, P_S is a probability measure on T^2 and $p_{1,a}(s_1)$, $p_{2,b}(s_2) \geq 0$.

We say that the family of probability measures (3.3) reproduces the statistics of the EPR–Bohm experiment if, for any a, $b \in [0, 2\pi)$ one has:

$$\begin{aligned}
R_{a,b}(I_a \times I_b) &= \frac{1}{2}\cos^2\left(\frac{b-a}{2}\right) =: P_{a,b}^{+-}\,,\\
R_{a,b}(J_a \times J_b) &= \frac{1}{2}\cos^2\left(\frac{b-a}{2}\right) =: P_{a,b}^{-+}\,,\\
R_{a,b}(I_a \times J_b) &= \frac{1}{2}\sin^2\left(\frac{b-a}{2}\right) =: P_{a,b}^{++}\,,\\
R_{a,b}(J_a \times I_b) &= \frac{1}{2}\sin^2\left(\frac{b-a}{2}\right) =: P_{a,b}^{--}\,.
\end{aligned} \tag{3.4}$$

Remark 3.1. Let us fix (arbitrarily) a single oriented reference framework for the whole experiment, determined by three orthogonal axes x, y, z. We assume that the trajectories of all particles lay entirely in the (x, y)-plane and that the parameters a and b represent the angles of the orientation of the spin analyzers with the x-axis.

The identities (3.4) show that the experimental probabilities do not depend on the arbitrarily chosen global reference frame but, as one would expect intuitively, only on the relative orientation of the spin analyzers. Given our assumptions, this invariance of (3.4) expresses the invariance of the experimental probabilities under rotations around the z-axis, i.e. under transformations of the form $a \mapsto a + c$ and $b \mapsto b + c$ for any real number c : $P^{++}_{a,b} = P^{++}_{a+c,b+c}$, etc. Choosing $c = -a$ or $-b$, this implies that $P^{++}_{a,b} = P^{++}_{a-b,0} = P^{++}_{0,b-a}$, etc. This suggests the following:

Definition 3.1. Two probability measures $R_{a,b}$, $R_{a',b'}$, of the family (16), are called empirically equivalent if thy reproduce exactly the same empirical data, i.e. if:

$$R_{a,b}(I_a \times I_b) = R_{a',b'}(I_{a'} \times I_{b'}) \,,$$

$$R_{a,b}(J_a \times I_b) = R_{a',b'}(J_{a'} \times I_{b'}) \,,$$

$$R_{a,b}(I_a \times J_b) = R_{a',b'}(I_{a'} \times J_{b'}) \,,$$

$$R_{a,b}(J_a \times J_b) = R_{a',b'}(J_{a'} \times J_{b'}) \,.$$

Denoting $\sim$ the relation of empirical equivalence among probability measures and using the terminology of Definition 3.1, the rotation invariance property of the family of probability measures (3.3), can be reformulated as follows:

$$R_{a,b} \sim R_{a-b,0} \sim R_{0,b-a} \,; \qquad \forall\, a, b \in [0, 2\pi) \,. \tag{3.5}$$

Notice, however, that the rotation invariance of the experimentally measured probabilities is a weaker condition than the rotation invariance of the full probability measures.

3.1. *The support of $R_{a,b}$*

Let us consider a measurable space $(\Omega, \mathcal{B})$ consisting of a compact Hausdorff space Ω and its Borel σ-algebra $\mathcal{B}$ generated by the open sets of Ω.

Definition 3.2. For $P \in \mathrm{Prob}(\Omega)$ (the set of all probability measures on Ω), put $\mathcal{F} := \{A \in \mathcal{B}\colon A \text{ is open and } P(A) = 0\}$ and define $\operatorname{supp} P := (\bigcup_{A\in\mathcal{F}} A)^c$. We call $\operatorname{supp} P$ the support of P.

Define the diagonal subset Δ of T^2 by

$$\Delta := \{(s_1, s_2) \in T^2 : s_1 = s_2 (\mathrm{mod}\ 2\pi)\} \,. \tag{3.6}$$

Definition 3.3. The family (3.3) of probability measures satisfies the condition of *statistical pre-determination* if $\forall\ (s_1, s_2) \in T^2\backslash\Delta$ there exist $a \in S^1$ and a neighborhood G of (s_1, s_2), contained in $(I_a \times J_a) \cup (J_a \times I_a)$ such that

$$p_{1,a}(s_1')p_{2,a}(s_2') > 0 \,; \qquad \forall\ (s_1', s_2') \in G \,.$$

Remark. If $S_1 = S_2$ were a discrete space, the condition $R_{a,a}(s_1, s_2) = 0$ would define the forbidden configurations for the pair of observables $S^{(1)}_a(s_1)$, $S^{(2)}_a(s_2)$, i.e. those configurations which give zero contribution to the correlation of these observables.

Statistical pre-determination means that, the fact that a configuration is statistically forbidden for all such measurements that the outcomes are precisely (anti-) correlated cannot depend on the local measurements, but it is defined at the source.

Since our configuration space is not discrete, we introduce the neighborhood G, of (s_1, s_2), to express this idea.

Proposition 3.1. *Suppose that the family of probability measures* (3.3) *satisfies* (3.4) (*agreement with the empirical data*) *and the condition of statistical pre-determination. Then*

$$\operatorname{supp} P_S \subseteq \Delta .$$

In particular, if the restriction of P_S *to* Δ *is absolutely continuous with respect to the Lebesgue measure on* Δ, *then there exists a nonnegative function* $\rho(s_1)$ *on* $\Delta \equiv S^1$ *such that:*

$$dP_S(s_1, s_2) = \rho(s_1)\delta(s_1 - s_2)ds_1 ds_2 . \tag{3.7}$$

Proof. By assumption, for each $(s_1, s_2) \in T^2 \backslash \Delta$, there exist $a \in [0, 2\pi)$ and a neighborhood G of (s_1, s_2) contained in $(I_a \times J_a) \cup (J_a \times I_a)$ such that

$$\int_{S_1 \times S_2} dP_S p_{1,a} \otimes p_{2,a} \cdot \chi_G = R_{a,a}(G) \leq R_{a,a}((I_a \times J_a) \cup (J_a \times I_a))$$

$$= P^{++}_{a,a} + P^{--}_{a,a} = 0 .$$

Since $p_{1,a} \otimes p_{2,a} > 0$ on G, it follows that $P_S(G) = 0$, i.e. $G \subseteq (\operatorname{supp} P_S)^c$. Thus any point in $T^2 \backslash \Delta$ has a neighborhood contained in $(\operatorname{supp} P_S)^c$. This means that $T^2 \backslash \Delta \subseteq (\operatorname{supp} P_S)^c$ or equivalently that $\operatorname{supp} P_S \subseteq \Delta$.

In view of this property, the existence of ρ is equivalent to the absolute continuity of the restriction of P_S on Δ. □

Theorem 3.1. *Under the assumptions of Proposition* 3.1, *if* $p_{1,a}$ *and* $p_{2,b}$ *are rotation invariant, i.e.*

$$p_{1,a+\delta}(s_1 + \delta) = p_{1,a}(s_1) , \qquad p_{2,b+\delta}(s_2 + \delta) = p_{2,b}(s_2) , \qquad \forall\ \delta \in \mathbb{R} \tag{3.8}$$

and twice continuously differentiable, then the probability measure $dR_{a,b}(s_1, s_2)$, *defined by* (3.3), *must have either the form*

$$dR_{a,b}(s_1, s_2) = \delta(s_1 - s_2)ds_1 ds_2 \frac{1}{4}|\cos(s_1 - a)| \tag{3.9}$$

or the form

$$dR_{a,b}(s_1, s_2) = \delta(s_1 - s_2)ds_1 ds_2 \frac{1}{4}|\cos(s_2 - b)| . \tag{3.10}$$

Proof. Because of rotation invariance

$$p_{1,a}(s_1) = p_{1,0}(s_1 - a) =: p_1(s_1 - a) ,$$

$$p_{2,b}(s_2) = p_{2,0}(s_2 - b) =: p_2(s_2 - b) .$$

Using the result of Proposition 3.1, we have

$$dR_{a,b}(s_1, s_2) = \rho(s_1)p_1(s_1 - a)p_2(s_2 - b)\delta(s_1 - s_2)ds_1 ds_2 \,.$$

For a and b satisfying $0 \le b - a \le \pi$, $I_a \cap I_b = [-\pi/2 + b, a + \pi/2)$, and therefore

$$R_{a,b}(I_a \times I_b) = \int_{-\pi/2+b}^{a+\pi/2} ds_1 \rho(s_1)p_1(s_1 - a)p_2(s_1 - b) \,.$$

By (3.4),

$$\frac{1}{4}(1 + \cos(b - a)) = R_{a,b}(I_a \times I_b) = \int_{-\pi/2+b}^{a+\pi/2} ds_1 \rho(s_1)p_1(s_1 - a)p_2(s_1 - b) \,.$$

Differentiating this with respect to b, we have

$$\begin{aligned} -\frac{1}{4}\sin(b - a) &= -\rho(b - \pi/2)p_1(b - a - \pi/2)p_2(-\pi/2) \\ &\quad + \int_{b-\pi/2}^{a+\pi/2} ds_1 \rho(s_1)p_1(s_1 - a)p_2'(s_1 - b) \,. \end{aligned} \tag{3.11}$$

Putting $b = a + \pi$, we obtain

$$0 = \rho(a + \pi/2)p_1(\pi/2)p_2(-\pi/2) \,.$$

Since a is arbitrary and ρ is a probability density, $\rho(a + \pi/2)$ cannot vanish. Hence

$$p_1(\pi/2) = 0 \text{ or } p_2(-\pi/2) = 0 \,.$$

Let us assume that $p_1(\pi/2) = 0$. Differentiating (3.11) with respect to b and putting $b = a + \pi$, we obtain

$$\frac{1}{4} = -\rho(a + \pi/2)p_1'(\pi/2)p_2(-\pi/2) \,.$$

From this we can see

$$p_1'(\pi/2) \neq 0 \text{ and } p_2(-\pi/2) \neq 0$$

and $\rho(a + \pi/2) = 1/(4p_1'(\pi/2)p_2(-\pi/2)) = \text{const.}$, since a is arbitrary. Thus we write $\rho(s_1) = c$ hereinafter.

Since $I_a \cap J_b = [-\pi/2 + a, -\pi/2 + b)$, by (3.4)

$$\frac{1}{4}(1 - \cos(b - a)) = R_{a,b}(I_a \times J_b) = c\int_{-\pi/2+b}^{-\pi/2+a} ds_1 p_1(s_1 - a)p_2(s_1 - b) \,.$$

Differentiating this with respect to b, we have

$$\begin{aligned} \frac{1}{4}\sin(b - a) &= -cp_1(b - a - \pi/2)p_2(-\pi/2) \\ &\quad + c\int_{-\pi/2+b}^{-\pi/2+a} ds_1 p_1(s_1 - a)p_2'(s_1 - b) \,. \end{aligned}$$

Putting $b = a$, we obtain

$$0 = -cp_1(-\pi/2)p_2(-\pi/2) \,.$$

Since $p_2(-\pi/2) \neq 0$,

$$p_1(-\pi/2) = 0\,.$$

Since $(I_a \cap J_b) \cup (I_a \cap I_b) = [-\pi/2 + a, a + \pi/2)$, by (3.4) we have

$$\frac{1}{2} = R_{a,b}(I_a \times J_b \cup I_a \times I_b) = c\int_{-\pi/2+a}^{a+\pi/2} ds_1 p_1(s_1 - a)p_2(s_1 - b)\,.$$

Changing variable with $s = s_1 - a$, we obtain

$$\frac{1}{2} = c\int_{-\pi/2}^{\pi/2} dsp_1(s)p_2(s - b + a)\,.$$

In the same way, for $\pi \leq b - a \leq 2\pi$, $I_a \cap I_b = [-\pi/2 + a, -3\pi/2 + b)$ and $I_a \cap J_b = [-3\pi/2 + b, a + \pi/2)$, we have

$$\frac{1}{2} = R_{a,b}(I_a \times I_b \cup I_a \times J_b) = c\int_{-\pi/2+a}^{a+\pi/2} ds_1 p_1(s_1 - a)p_2(s_1 - b)$$

$$= c\int_{-\pi/2}^{\pi/2} dsp_1(s)p_2(s - b + a)\,.$$

Since p_1 is continuous, $p_1(\pi/2) = p_1(-\pi/2) = 0$, and a and b are arbitrary, we can see that $p_2(s) = \text{const.} =: c_2$. Thus by renaming

$$\tilde{p}_1(s_1) := cp_1(s_1)c_2$$

we find

$$dR_{a,b}(s_1, s_2) = \tilde{p}_1(s_1 - a)\delta(s_1 - s_2)ds_1 ds_2\,.$$

Our remaining task is to determine the form of $\tilde{p}_1$. For a and b satisfying $0 \leq b - a \leq \pi$, (3.11) becomes

$$-\frac{1}{4}\sin(b - a) = -\tilde{p}_1(-\pi/2 + b - a)\,.$$

By putting $s = b - \pi/2$, $\tilde{p}_1(s - a) = \frac{1}{4}\cos(s - a)$ for $-\pi/2 \leq s - a \leq \pi/2$. Therefore

$$\tilde{p}_1(s - a) = \frac{1}{4}|\cos(s - a)|\,, \qquad -\pi/2 \leq s - a \leq \pi/2\,.$$

Since $J_a \cap I_b = [a + \pi/2, b + \pi/2)$,

$$\frac{1}{4}(1 - \cos(b - a)) = R_{a,b}(J_a \times I_b) = \int_{a+\pi/2}^{b+\pi/2} ds_1 \tilde{p}_1(s_1 - a)\,.$$

By differentiating this with respect to b we have

$$\frac{1}{4}\sin(b - a) = \tilde{p}_1(b + \pi/2 - a)\,.$$

By putting $s = b + \pi/2$, $\tilde{p}_1(s-a) = \frac{1}{4}\sin(s-a-\pi/2) = -\frac{1}{4}\cos(s-a)$ for $\pi/2 \leq s-a \leq 3\pi/2$. Therefore

$$\tilde{p}_1(s-a) = \frac{1}{4}|\cos(s-a)|\,, \qquad \pi/2 \leq s-a \leq 3\pi/2\,.$$

Accordingly,

$$dR_{a,b}(s_1,s_2) = \delta(s_1-s_2)ds_1ds_2\frac{1}{4}|\cos(s_1-a)|\,.$$

If we assume that $p_2(-\pi/2)=0$ instead of $p_1(\pi/2)=0$, then in the same way we obtain

$$dR_{a,b}(s_1,s_2) = \delta(s_1-s_2)ds_1ds_2\frac{1}{4}|\cos(s_2-b)|\,. \qquad \square$$

4. Two Experimental Settings for Determinism

In classical statistical mechanics the dynamical evolution is deterministic but the initial information is incomplete and is represented by a probability measure which describes the preparation of the experiment.

In the case of adaptive systems however the experimental setup is not fully determined at the initial time in the sense that many measurements are *a priori* possible and the particles do not know which one will be actually performed. This means that part of the dices are cast at the source, where the particles are emitted, and part of the dices are cast at the final time, when each particle interacts with the measurement apparatus.

It is clear that the two experimental situations must correspond to different mathematical models. In this section we try to make these differences explicit.

Standard determinism can be summed up in the statement: the state at any time $t = t_0$ uniquely determines the states at any later time ($t > t_0$). For reversible determinism also the converse is true: the state at any time T uniquely determines the state at any time $t_0 < T$. In exact deterministic theories states are characterized by the values of some observables, like position and momentum in classical mechanics.

We call "configuration (or phase) space" the state space of an exact deterministic theory.

In statistical deterministic theories, one postulates the existence of an underlying exact theory and the states are probability measures on the configuration space of this theory. The prototype example is classical statistical mechanics and the models considered in this paper fall into this category, i.e. a statistical, reversible deterministic theory.

The mathematical model of such a theory is defined by

— a configuration space Ω,
— a deterministic, reversible dynamics $T^t : \Omega \to \Omega$,
— a probability measure P on Ω.

The interpretation of P depends on the experimental setting. We distinguish two cases:

(i) P condensates the experimental information available at an initial time t_0,
(ii) P condensates the experimental information available at a final time t_{f}, i.e. the time when the experiment is actually performed.

According to von Neumann measurement theory, a mathematical description of a measurement process must take into account the interaction of the measured system with the measurement apparatus.

This means that, for adaptive systems (like chameleons) the meaning of the probability measure P must be understood in the sense of (ii) above.

More precisely, von Neumann measurement scheme requires the specification of:

— a configuration space M of the apparatus,
— a joint dynamics

$$T^t_{S,M} : S \times M \to S \times M$$

describing the evolution of the composite system (system, apparatus).

In the case of adaptive systems, at the initial time t_0 one has a whole family of possible measurements and the one which will be performed will be known only at the final time t_{f}.

Therefore, a von Neumann type description of an adaptive system should consist of a multiplicity of triples

$$(S \times M, T^t_{S,M}, P_{S,M})\,,$$

i.e. on triple for each of the possible measurements.

Moreover, since the choice of the measurement, and therefore all the available experimental data, occur at a final time t_{f}, the identity

$$P_{t_0} = T^{-(t_f - t_0)}_{S,M} P_{S,M}$$

which expresses the unknown initial distribution (P_{t_0}) in terms of the experimentally found distribution ($P_{S,M}$), shows that the initial distribution depends on the measurement. This circumstance does not violate the causality principle because such an initial distribution should be interpreted as the conditional distribution at time t_0 of the composite system (S, M) given the knowledge of the results of the experiment M, performed at time $t_{\mathrm{f}} > t_0$.

The local causal measures discussed in this paper correspond to the final measures $P_{S,M}$ described here.

5. Empirical Correlations of Systems of Distant Particles

In this section we argue that the same term "pair correlation" is used to describe two completely different experimental procedures and that a good mathematical model should take into account these experimental differences.

If S is the configuration space of a classical system, then by definition a trajectory of this system is a map

$$\sigma : t \in [t_\sigma, +\infty) \mapsto \sigma_t \in S\,.$$

For each $t \in [t_\sigma, +\infty)$, σ_t is interpreted as the configuration of the system at time t. In the following we fix the interval $[t_\sigma, +\infty)$ and we will often not mention it.

If (1, 2) denotes a composite system made of two particles, a trajectory of the pair is by definition a pair (σ_1, σ_2), where σ_1 is a trajectory of particle 1 and σ_2 is a trajectory of particle 2.

We assume that all the particles 1_j (resp. 2_j), $j \in \{1, \ldots, N\}$, $N \in \mathbb{N}$, have the same configuration space S_1 (resp. S_2) so that all the $\sigma_{1,j}$ (resp. $\sigma_{2,j}$) are functions

$$\sigma_{1,j} : [t_{\sigma_1}, +\infty) \to S_1[\text{resp. } \sigma_{2,j} : [t_{\sigma_2}, +\infty) \to S_2]\,.$$

Let (f_1, f_2) be an observable of the pairs $(1_j, 2_j)$. The term *empirical correlation between* f_1 *and* f_2 has a multiplicity of meanings depending on the experimental procedure employed to measure this quantity. In the following we shall describe these possibilities which are frequently met.

By definition of classical system, if a configuration space of a system is S, an observable of the system is a real-valued function f defined on S, i.e. $f : S \to \mathbb{R}$. An observable of a pair of systems (1, 2) is a pair (f_1, f_2), where f_1 is an observable of system 1 and f_2 is an observable of system 2.

If it is given an ensemble of pairs

$$(1_j, 2_j)\,, \qquad j \in \{1, \ldots, N\}\,, \tag{5.1}$$

$(\sigma_{1,j}, \sigma_{2,j})$ denotes the trajectory of the jth pair ($j = 1, \ldots, N$). If this ensemble of pairs is obtained by repeating measurements with the same measurement apparata on successively emitted particles from a source, then $t_{\sigma_{1,1}} < \cdots < t_{\sigma_{1,N}}$, $t_{\sigma_{2,1}} < \cdots < t_{\sigma_{2,N}}$.

To fix the ideas, from now on we shall think of a source which emits pairs of particles and particles of a pair are emitted simultaneously, i.e.

$$t_{\sigma_{1,j}} = t_{\sigma_{2,j}} = t_j$$

for each trajectory $(\sigma_{1j}, \sigma_{2j})$ in concrete experimental situations.

5.1. *Standard correlations*

The term *standard correlation* is used when the following physical conditions are verified:

(a) The total number N of pairs is known.

(b) The trajectory of each pair can be followed without disturbance so that, at each time t, the experimenters know exactly to which of the pairs (5.1) their measurement is referred. This property will be called *distinguishability*.

(c) The observable (f_1, f_2) is measured on each pair of the ensemble. The result of the measurement of (f_1, f_2) on the jth pair will be denoted by

$$(f_{1,j}, f_{2,j})\,;$$

the measurement itself will be denoted by M_j.

Under these conditions the following definition makes sense.

Definition 5.1. The empirical correlation between the pair of observables (f_1, f_2), relative to the sequence of measurements $M = (M_j)$ on the ensemble $\{(1_j, 2_j): j = 1, \ldots, N\}$ is

$$\langle f_1 \cdot f_2 \rangle_M := \frac{1}{N} \sum_{j=1}^{N} f_{1,j} f_{2,j}\,. \tag{5.2}$$

We further specify our context of standard correlations as follows.

(d) Each measurement M_j is specified by a time

$$t'_j := t_j + T\,,$$

where T is independent of j (recall that t_j is the emission time for the pair $(1_j, 2_j)$).

(e) The result of the jth measurement does not depend on the interval $[t_j, t_j + T]$ but only on T (time homogeneity).

Under these conditions the correlations (5.2) are interpreted as the correlations of (f_1, f_2) at time T and T is interpreted as the final time of the single measurement.

5.2. *Correlations of distant pairs*

Suppose that the measurement protocol is the following.

(DP1) It is known that each pair is emitted simultaneously, but the experimenters do not know precisely when, i.e. $t_{\sigma,j}$ is not known.

(DP2) The experimenters cannot follow the trajectory of each particle, but only register the result of a measurement at time t (*indistinguishability*).

(DP3) The experimenters have synchronized clocks, so the time t is the same for both.

(DP4) The experimenters do not know the total number of emitted particles.

(DP5) The experimenters cannot postulate that, if a particle of a pair reaches one of them, then the other particle reaches the other experimenters.

Conditions (d) and (e) of the previous section are still meaningful because they are referred to single particles. However, condition (c) is meaningless because of indistinguishability. Moreover the N, in formula (5.2) is unknown. In a situation described by the above conditions we speak of *correlations of distant particles.*

In conclusion, under the above described physical conditions, the definition of standard correlations is meaningless and a new one is needed.

Definition 5.2. The protocol to define correlations of distant particles is the following:

(CDP1) The experimenter X, $X \in \{1,2\}$ performs measurements on M_X particles and records

— the time $t'_{X,j}$ of the jth measurement,
— the value $f_{X,j}$ of the measured observable f_X

$\forall\, j \in \{1,\dots,M_X\}$.

(CDP2) The two experimenters exchange the sequences

$$((t'_{1,j}, f_{1,j}) : j = 1,\dots,M_1) \text{ and } ((t'_{2,j}, f_{2,j}) : j = 1,\dots,M_2)\,.$$

(CDP3) Each experimenter extracts the sequences

$$(f'_{1,h} : h = 1,\dots,M_{f_1 f_2}) \text{ and } (f'_{2,h} : h = 1,\dots,M_{f_1 f_2})\,,$$

where

$$\{s_h : h \in \{1,\dots,M_{f_1 f_2}\}\} := \{t'_{1,j} : j \in \{1,\dots,M_1\}\} \cap \{t'_{2,j} : j \in \{1,\dots,M_2\}\}$$

and

$$f'_{X,h} := f_{X,j}\,, \quad \text{if } s_h = t'_{X,j} \qquad (X = 1,2)\,.$$

(CDP4) The empirical correlations of distant pairs are defined by

$$\langle f_1 f_2\rangle_{\mathrm{DP}} := \frac{1}{M_{f_1,f_2}} \sum_{h=1}^{M_{f_1,f_2}} f'_{1,h} f'_{2,h}\,.$$

In other words, by definition, correlation of distant pairs means conditioned correlations on coincidences.

Remark 5.1. Practically the totality of the EPR type experiments follow the protocol described in Definition 5.2.

5.3. *Mathematical Models of Empirical Correlations*

We keep the notations introduced in the previous sections. Instead of considering a single observable for each particle of a pair, we consider now two families of observables: $\hat{\mathcal{A}}_1$ — of particles of type 1, $\hat{\mathcal{A}}_2$ — of particles of type 2. We suppose that, for each pair

$$\hat{S}_{1,a} \in \hat{\mathcal{A}}_1\,, \qquad \hat{S}_{2,b} \in \hat{\mathcal{A}}_2$$

one has performed experiments leading to estimates of all the empirical correlations

$$\kappa_{a,b} := \langle \hat{S}_{1,a} \hat{S}_{2,b} \rangle_{\mathrm{EMP}} .$$

These numbers are experimental data.

We assume moreover, that the experimental protocols to determine these correlations have been homogeneous, e.g. always standard correlations or always distant pair correlations.

Definition 5.3. A mathematical model for the empirical correlations $\{\kappa_{ab}\}$ is defined by:

— a family of probability spaces $(\Omega, \mathcal{F}, P_{a,b})$ where the pairs (a, b) label the *a priori* possible experimental settings
— two families $\mathcal{A}_1, \mathcal{A}_2$ of real-valued functions on Ω with the property that $\forall\, S_{1,a} \in \mathcal{A}_1$, $\forall\, S_{2,b} \in \mathcal{A}_2$, one has

$$\kappa_{a,b} = \int_\Omega S_{1,a} S_{2,b} dP_{a,b} . \tag{5.3}$$

Such a model is called *local* if there exists a computer program which allows to simulate the protocol of the experiment in such a way that:

— the program must run on three noncommunicating computers: Computer S, Computer 1, Computer 2.
— Computer S should produce a family of pairs $(\sigma_{1,j}, \sigma_{2,j})$, $j \in \{1, \ldots, N\}$ without using any information on what Computers 1 and 2 will do. Then Computer S sends $(\sigma_{1,j})$ to Computer 1 and $(\sigma_{2,j})$ to Computer 2;
— Computer 1 (resp. 2) should for each $j \in \{1, \ldots, M\}$

(i) choose one observable

$$S_{1,a} \in \mathcal{A}_1 (\text{resp. } S_{2,b} \in \mathcal{A}_2) ,$$

(ii) compute the configuration $\sigma_{1,j,a}(T)$ of particle 1_j at time T using only informations on the trajectory $\sigma_{1,j}$ and the observable $S_{1,a}$ (resp. $\sigma_{2,j,b}(T); \sigma_{2,j}, S_{2,a}$),

(iii) check if $\sigma_{1,j,a}(T) \in W$ where $W \subseteq S_1 = S$ is window of the configuration space (resp. $\sigma_{2,j,b}(T) \in W$).
This simulates the physical phenomenon that certain local trajectories of the particles may end up outside the phase space window defining the coincidence.

(iv) In case $\sigma_{1,j,a}(T) \in W$ (resp. $\sigma_{2,j,b}(T) \in W$), compute the value $S_{1,a}(\sigma_{1,j}(T))$ (resp. $S_{2,b}(\sigma_{2,j}(T))$).

(v) The procedure to compute the correlations must reproduce exactly the procedure used in the corresponding experimental protocol and described by Definition 5.2.

The model in Ref. 10 can be considered as a local mathematical model for the empirical correlations in Definition 5.3, if the protocol for distant pairs is adopted, although this model reproduces the EPR correlations only approximately.

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