SCHRÖDINGER EQUATION, L^p-DUALITY AND THE GEOMETRY OF WIGNER-YANASE-DYSON INFORMATION

P. GIBILISCO

Dipartimento SEFEMEQ,
Facoltà di Economia, Università di Roma "Tor Vergata",
Via Columbia 2, 00133 Rome, Italy
E-mail: gibilisco@volterra.uniroma2.it
URL: http://www.economia.uniroma2.it/sefemeq/professori/gibilisco

D. IMPARATO

Dipartimento di Matematica,
Politecnico di Torino,
Corso Duca degli Abruzzi, 10129 Turin, Italy
E-mail: daniele.imparato@polito.it

T. ISOLA

Dipartimento di Matematica,
Università di Roma "Tor Vergata",
Via della Ricerca Scientifica, 00133 Rome, Italy
E-mail: isola@mat.uniroma2.it
URL: http://www.mat.uniroma2.it/~isola

We discuss the geometry of Wigner-Yanase-Dyson information via the so-called Amari-Nagaoka embeddings in L^p-spaces of quantum trajectories.

1. Introduction

The Wigner-Yanase-Dyson information was introduced in 1963.[28] Wigner and Yanase observed that "According to quantum mechanical theory, some observables can be measured much more easily than others: the observables which commute with the additive conserved quantities ... can be measured with microscopic apparatuses; those which do not commute with these quantities need for their measurements macroscopic systems. Hence the problem of defining a measure of our knowledge with respect to the latter quantities arises ...". After the discussion of the requirements such a measure should satisfy (convexity, ...) they proposed, tentatively, the following formula and called it *skew information*:

$$I_\rho(A) := -\frac{1}{2}\mathrm{Tr}([\rho^{\frac{1}{2}}, A]^2).$$

More generally they defined (following a suggestion by Dyson)

$$I_\rho^\beta(A) := -\frac{1}{2}\mathrm{Tr}([\rho^\beta, A]\cdot[\rho^{1-\beta}, A]), \qquad \beta \in [0,1].$$

The latter is known as WYD-information. The skew information should be considered as a measure of information contained in a state ρ with respect to a conserved observable A.

From that fundamental work WYD-information has found applications in a manifold of different fields. A possibly incomplete list should mention: i) strong subadditivity of entropy;[22,23] ii) homogeneity of the state space of factors (of type III_1);[6] hypothesis testing[3] iii) measures for quantum entanglement;[4,19] iv) uncertainty relations.[7,10–13,21,24,25,27]

Such a variety should be not surprising at the light of the result showing that WYD-information is just an example of monotone metric, namely it is a member of the vast family of quantum Fisher informations.[9] On the other hand one can prove that, among the family of all the quantum Fisher informations, the geometry of WYD-information is rather special.[8,16]

In this paper we want to discuss the particular features of WYD-information emphasizing the relation with the embedding of quantum dynamics in L^p-spaces.

2. Preliminary notions of matrix analysis

Let $M_n := M_n(\mathbb{C})$ (resp.$M_{n,sa} := M_n(\mathbb{C})_{sa}$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices). We shall denote general matrices by $X, Y, \ldots$ while letters $A, B, \ldots$ (or H) will be used for self-adjoint matrices. Let D_n be the set of strictly positive elements of M_n while $D_n^1 \subset D_n$ is the set of density matrices namely

$$D_n^1 = \{\rho \in M_n | \mathrm{Tr}\rho = 1,\ \rho > 0\}.$$

The tangent space to D_n^1 at ρ is given by $T_\rho D_n^1 \equiv \{A \in M_{n,sa} : \mathrm{Tr}(A) = 0\}$, and can be decomposed as $T_\rho D_n^1 = (T_\rho D_n^1)^c \oplus (T_\rho D_n^1)^o$, where $(T_\rho D_n^1)^c := \{A \in T_\rho D_n^1 : [A, \rho] = 0\}$, and $(T_\rho D_n^1)^o$ is the orthogonal complement of $(T_\rho D_n^1)^c$, with respect to the Hilbert-Schmidt scalar product $\langle A, B\rangle := \langle A, B\rangle_{HS} := \mathrm{Tr}(A^*B)$ (the Hilbert-Schmidt norm will be denoted by $||\cdot||$).

A typical element of $(T_\rho D_n)^o$ has the form $A = i[\rho, H]$, where H is self-adjoint.

In what follows we shall need the following result (pag. 124 in[2]).

Proposition 2.1. *Let $A \in M_{n,sa}$ be decomposed as $A = A^c + i[q, H]$ where $q \in D_n$, $[A^c, q] = 0$ and $H \in M_{n,sa}$. Suppose $\varphi \in C^1(0, +\infty)$. Then*

$$(D_q\varphi)(A) = \varphi'(q)A^c + i[\varphi(q), H].$$

3. Schrödinger equation and quantum dynamics

Let $\rho(t)$ be a curve in D_n^1 and let $H \in M_{n,sa}$ We say that $\rho(t)$ satisfy the Schrödinger equation w.r.t. H if $\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = i[\rho(t), H]$. This equation is also known in the literature as the Landau-von Neumann equation.

The solution of the above evolution equation (please note that H is time independent) is given by

$$\rho_H(t) := e^{-itH}\rho e^{itH}. \tag{1}$$

Therefore the commutator $i[\rho, H]$ appears as the tangent vector to the quantum trajectory (1) (at the initial point $\rho = \rho_H(0)$) generated by H. Suppose we are considering two different evolutions determined, through the Schrödinger equation, by H and K. If we want to quantify how "different" the trajectories $\rho_H(t), \rho_K(t)$ are, then it would be natural to measure the "area" spanned by the tangent vectors $i[\rho, H], i[\rho, K]$ (with respect to some scalar product[10]).

4. L^p-embedding for states and trajectories

The functions

$$\rho \to \frac{\rho^\beta}{\beta}, \qquad \beta \in (0, 1)$$

are known as Amari-Nagaoka embeddings.[1,14] They can be considered as an immersion of the state manifold into L^p-spheres.

Proposition 4.1. *Let $\rho(t)$ be a curve in D_n^1, let $H \in M_{n,sa}$ and let $\beta \in (0, 1)$. The following differential equations are equivalent*

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = i[\rho(t), H], \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho(t)^\beta\right) = i[\rho(t)^\beta, H]. \tag{2}$$

Proof. Let $\phi_\beta(\rho) := \rho^\beta$. By Proposition 2.1 we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho(t)^\beta\right) &= D_\rho\phi_\beta \circ \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = D_\rho\phi_\beta(i[\rho(t),H]) \\ &= (i[\phi_\beta(\rho(t)),H]) = i[\rho(t)^\beta,H].\end{aligned}$$

Therefore, Equation (1) implies Equation (2). Analogously, again using Proposition 2.1, Equation (2) implies Equation (1) because we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\rho(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\rho(t)^\beta\right)^{\frac{1}{\beta}}\right) = D_{(\rho(t)^\beta)}\phi_\beta^{-1} \circ \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho(t)^\beta\right) \\ &= D_{(\rho(t)^\beta)}\phi_\beta^{-1} \circ i[\rho(t)^\beta,H] = D_{(g(t))}\phi_\beta^{-1} \circ i[g(t),H] \\ &= i[\phi_\beta^{-1}(g(t)),H] = i[\rho(t),H].\end{aligned}$$

□

5. WYD-information by pairing of dual trajectories

The Wigner-Yanase-Dyson information is defined as

$$I_\rho^\beta(H) := -\frac{1}{2}\mathrm{Tr}([\rho^\beta,H]\cdot[\rho^{1-\beta},H]), \qquad \beta \in (0,1).$$

Let us explain the link between L^p-embeddings and WYD-information. Let V, W be vector spaces over $\mathbb{R}$ (or $\mathbb{C}$). One says that there is a duality pairing if there exists a separating bilinear form

$$\langle\cdot,\cdot\rangle : V \times W \to \mathbb{R}\,(\mathbb{C}).$$

In the case of L^p spaces the pairing is given by the L^2 scalar product. In our case this is just the HS-scalar product.

Note that using the function $\rho \to \rho^\beta$ we may look at dynamics as a curve on a $L^{\frac{1}{\beta}}$-sphere. The function $\rho \to \rho^{1-\beta}$ does the same on the dual space $\left(L^{\frac{1}{\beta}}\right)^* = L^{\frac{1}{1-\beta}}$.

Proposition 5.1. *If $\rho(t)$ satisfies the Schrödinger equation w.r.t. H then*

$$\langle\frac{\mathrm{d}}{\mathrm{d}t}\rho(t)^\beta, \frac{\mathrm{d}}{\mathrm{d}t}\rho(t)^{1-\beta}\rangle = 2\cdot I_{\rho(t)}^\beta(H) \qquad \beta \in (0,1).$$

Proof. Apply Proposition 4.1 to obtain

$$\begin{aligned}\langle\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho(t)^\beta\right), \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho(t)^{1-\beta}\right)\rangle &= \langle i[\rho(t)^\beta,H], i[\rho(t)^{1-\beta},H]\rangle \\ &= -\mathrm{Tr}([\rho(t)^\beta,H]\cdot[\rho(t)^{1-\beta},H]).\end{aligned}$$

□

In this way WYD-information appears as the "pairing" of the dual L^p-embeddings of the same quantum trajectory.

6. Quantum Fisher informations

In the commutative case a Markov morphism is a stochastic map $T : \mathbb{R}^n \to \mathbb{R}^k$. In the noncommutative case a Markov morphism is a completely positive and trace preserving operator $T : M_n \to M_k$. Let

$$\mathcal{P}_n := \{\rho \in \mathbb{R}^n | \rho_i > 0\} \qquad \mathcal{P}_n^1 := \{\rho \in \mathbb{R}^n | \sum \rho_i = 1,\, \rho_i > 0\}.$$

In the commutative case a monotone metric is a family of Riemannian metrics $g = \{g^n\}$ on $\{\mathcal{P}_n^1\}$, $n \in \mathbb{N}$, such that

$$g^m_{T(\rho)}(TX, TX) \le g^n_\rho(X, X)$$

holds for every Markov morphism $T : \mathbb{R}^n \to \mathbb{R}^m$ and all $\rho \in \mathcal{P}_n^1$ and $X \in T_\rho \mathcal{P}_n^1$.

In perfect analogy, a monotone metric in the noncommutative case is a family of Riemannian metrics $g = \{g^n\}$ on $\{\mathcal{D}_n^1\}$, $n \in \mathbb{N}$, such that

$$g^m_{T(\rho)}(TX, TX) \le g^n_\rho(X, X)$$

holds for every Markov morphism $T : M_n \to M_m$ and all $\rho \in \mathcal{D}_n^1$ and $X \in T_\rho \mathcal{D}_n^1$.

Let us recall that a function $f : (0, \infty) \to \mathbb{R}$ is called operator monotone if, for any $n \in \mathbb{N}$, any A, $B \in M_n$ such that $0 \le A \le B$, the inequalities $0 \le f(A) \le f(B)$ hold. An operator monotone function is said symmetric if $f(x) := xf(x^{-1})$. With such operator monotone functions f one associates the so-called Chentsov–Morotzova functions

$$c_f(x, y) := \frac{1}{yf(xy^{-1})} \qquad \text{for} \qquad x, y > 0.$$

Define $L_\rho(A) := \rho A$, and $R_\rho(A) := A\rho$. Since L_ρ and R_ρ commute we may define $c(L_\rho, R_\rho)$ (this is just the inverse of the operator mean associated to f by Kubo-Ando theory[10]). Now we can state the fundamental theorems about monotone metrics. In what follows uniqueness and classification are stated up to scalars (for reference see[26]).

Theorem 6.1. *(Chentsov 1982) There exists a unique monotone metric on $\mathcal{P}_n^1$ given by the Fisher information.*

Theorem 6.2. *(Petz 1996) There exists a bijective correspondence between monotone metrics on $\mathcal{D}_n^1$ and symmetric operator monotone functions. For $\rho \in \mathcal{D}_n^1$, this correspondence is given by the formula*

$$g_f(A, B) := g_{f,\rho}(A, B) := \mathrm{Tr}(A \cdot c_f(L_\rho, R_\rho)(B)).$$

Because of these two theorems, the terms "Monotone Metrics" and "Quantum Fisher Informations" are used with the same meaning.

Note that usually monotone metrics are normalized so that $[A, \rho] = 0$ implies $g_{f,\rho}(A, A) = \mathrm{Tr}(\rho^{-1}A^2)$, that is equivalent to set $f(1) = 1$.

7. The WYD monotone metric

The following functions are symmetric, normalized and operator monotone (see[9,16]). Let

$$f_\beta(x) := \beta(1-\beta)\frac{(x-1)^2}{(x^\beta - 1)(x^{1-\beta} - 1)} \qquad \beta \in (0, 1).$$

Proposition 7.1. *For the QFI associated to f_β one has*

$$g_{f_\beta}(i[\rho, H], i[\rho, K]) = -\frac{1}{\beta(1-\beta)}\mathrm{Tr}([\rho^\beta, H] \cdot [\rho^{1-\beta}, K]) \qquad \beta \in (0, 1).$$

One can find a proof in.[9,16] Because of the above Proposition, g_β is known as $WYD(\beta)$ monotone metric.

Of course what we have seen about L^p-embedding of quantum dynamics applies to this example of quantum Fisher information. Indeed we can summarize everything into the following final result.

Proposition 7.2.
Let H, K be selfadjoint matrices and ρ be a density matrix. Choose two curves $\rho(t), \sigma(t) \subset D_n^1$ such that
i) $\rho(t)$ satisfies the Schrödinger equation w.r.t. H;
ii) $\sigma(t)$ satisfies the Schrödinger equation w.r.t. K;
iii) $\rho = \rho(0) = \sigma(0)$.
One has

$$g_{f_\beta}(i[\rho, H], i[\rho, K]) = \langle \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho(t)^\beta}{\beta}\right), \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma(t)^{1-\beta}}{1-\beta}\right)\rangle|_{t=0} \qquad \beta \in (0, 1)$$

Proof. From Proposition 7.1, one gets

$$\begin{aligned} g_{f_\beta}(i[\rho, H], i[\rho, K]) &= -\tfrac{1}{\beta(1-\beta)}\mathrm{Tr}([\rho^\beta, H] \cdot [\rho^{1-\beta}, K]) \\ &= -\tfrac{1}{\beta(1-\beta)}\mathrm{Tr}([\rho(t)^\beta, H] \cdot [\sigma(t)^{1-\beta}, K])|_{t=0} \\ &= \langle \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{\rho(t)^\beta}{\beta}\right), \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{\sigma(t)^{1-\beta}}{1-\beta}\right)\rangle|_{t=0} \end{aligned}$$

□

8. Conclusion

All the ingredients of the above construction make sense on a von Neumann algebra: WYD-information, quantum dynamics, L^p-spaces, Amari-Nagoka embeddings and so on.[14,20] Nevertheless we are not aware of any attempt to see geometry of WYD-information along the lines described in the present paper, in the infinite-dimensional context. We plan to address this problem in future work.

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