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UNCERTAINTY PRINCIPLE FOR WIGNER–YANASE–DYSON INFORMATION IN SEMIFINITE VON NEUMANN ALGEBRAS

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In Ref. 9, Kosaki proved an uncertainty principle for matrices, related to Wigner–Yanase–Dyson information, and asked if a similar inequality could be proved in the von Neumann algebra setting. In this paper we prove such an uncertainty principle in the semifinite case.

Keywords: Uncertainty principle; Wigner–Yanase–Dyson information.

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1. Introduction

Let $M_n := M_n(\mathbb{C})$ (resp. $M_{n,sa} := M_n(\mathbb{C})_{sa}$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices). Let $\mathcal{D}_n^1$ be the set of strictly positive density matrices namely

$$\mathcal{D}_n^1 = \{\rho \in M_n : \mathrm{Tr}\,\rho = 1, \rho > 0\}\,.$$

Definition 1.1. For A, $B \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$, define covariance and variance as

$$\mathrm{Cov}_\rho(A,B) := \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho A) \cdot \mathrm{Tr}(\rho B)\,,$$
$$\mathrm{Var}_\rho(A) := \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho A)^2\,.$$

Then the well-known Schrödinger and Heisenberg uncertainty principles are given in the following:

Theorem 1.2. (Refs. 8 and 14) *For A, $B \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$ one has*

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) - |\mathrm{ReCov}_\rho(A,B)|^2 \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2\,,$$

that implies

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2\,.$$

Recently, a different uncertainty principle has been found.[9–12,15]

Definition 1.3. For A, $B \in M_{n,sa}$, $\beta \in (0,1)$, and $\rho \in \mathcal{D}_n^1$ define β-correlation and β-information as

$$\mathrm{Corr}_{\rho,\beta}(A,B) := \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho^\beta A \rho^{1-\beta} B)$$
$$I_{\rho,\beta}(A) := \mathrm{Corr}_{\rho,\beta}(A,A) \equiv \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho^\beta A \rho^{1-\beta} A)\,.$$

The latter coincides with the Wigner–Yanase–Dyson information.

Theorem 1.4.

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) - |\mathrm{ReCov}_\rho(A,B)|^2 \geq I_{\rho,\beta}(A)I_{\rho,\beta}(B) - |\mathrm{ReCorr}_{\rho,\beta}(A,B)|^2\,.$$

Kosaki[9] asked if the previous inequality, which makes perfect sense in a von Neumann algebra setting, could indeed be proved. In the sequel, we provide such a proof in the semifinite case.

In closing, we mention that different generalizations of Theorem 1.4 have been recently obtained by the authors.[2–7]

2. Auxiliary Lemmas

In all this Section we let $(\mathcal{M},\tau)$ be a semifinite von Neumann algebra with a n.s.f. trace, and denote by $\mathrm{Proj}(\mathcal{M})$ the set of orthogonal projections in $\mathcal{M}$, and by $\bar{\mathcal{M}}$ the topological *-algebra of τ-measurable operators. We fix ρ, $\sigma \in \bar{\mathcal{M}}_{sa}$, with spectral decompositions $\rho = \int_{-\infty}^{+\infty} \lambda de_\rho(\lambda)$, and $\sigma = \int_{-\infty}^{+\infty} \lambda de_\sigma(\lambda)$.

Finally, we denote by $\mathcal{A}$ the algebra generated by the sets $\Omega_1 \times \Omega_2$, for Ω_1, Ω_2 Borel subsets of $\mathbb{R}$, and observe that $\sigma(\mathcal{A})$, the σ-algebra generated by $\mathcal{A}$, coincides with the Borel subsets of $\mathbb{R}^2$.

Lemma 2.1. *Let a, $b \in \mathcal{M} \cap L^2(\mathcal{M},\tau)$. Let $\mu_{ab}(\Omega_1 \times \Omega_2) := \tau(e_\rho(\Omega_1)a^* e_\sigma(\Omega_2)b)$, for Ω_1, Ω_2 Borel subsets of $\mathbb{R}$. Then μ_{ab} extends uniquely to a bounded Borel measure on $\mathbb{R}^2$.*

Proof. For $\Omega \subset \mathbb{R}$ Borel subset, $x \in L^2(\mathcal{M},\tau)$, let $P(\Omega)x := e_\rho(\Omega)x$, $Q(\Omega)x := xe_\sigma(\Omega)$. Then, P, Q are commuting Borel spectral measures on $L^2(\mathcal{M},\tau)$, and their product $P \otimes Q(\Omega_1 \times \Omega_2) := P(\Omega_1)Q(\Omega_2)$ extends uniquely to a Borel spectral measure on $\mathbb{R}^2$ (Chap. 5 of Ref. 1). Observe that $\mu_{ab}(\Omega_1 \times \Omega_2) = \tau(P \otimes Q(\Omega_1 \times \Omega_2)(a^*)\cdot b)$, and, if $\{A_n\}$ is a sequence of disjoint Borel sets, then $P \otimes Q(\cup A_n)(a^*) =$

$\sum_n P \otimes Q(A_n)(a^*)$ converges in $L^2(\mathcal{M}, \tau)$, so that $\tau(P \otimes Q(\cup A_n)(a^*) \cdot b)$ is well defined. So $\mu_{ab} = \tau(P \otimes Q(\cdot)(a^*) \cdot b)$ is the desired extension.

Observe now that μ_{ab} is a bounded Borel (complex) measure on $\mathcal{A}$. Indeed, with $A \in \mathcal{A}$,

$$|\mu_{ab}(A)|^2 = |\tau(P \otimes Q(A)(a^*) \cdot b)|^2 \leq \|P \otimes Q(A)(a^*)\|_{L^2} \|b\|_{L^2} \leq \|a\|_{L^2} \|b\|_{L^2} \,.$$

Therefore, by Corollary 4.4.6 of Ref. 13, there is a unique extension of μ_{ab} to a bounded (complex) measure on $\sigma(\mathcal{A})$, the σ-algebra generated by $\mathcal{A}$, i.e. the Borel subsets of $\mathbb{R}^2$. $\square$

Lemma 2.2. *Let a, $b \in \mathcal{M} \cap L^2(\mathcal{M}, \tau)$. Then*

(i) $\mu_{ab} = \frac{1}{4} \sum_{k=1}^4 (-i)^k \mu_{a+i^k b, a+i^k b}$,
(ii) *if $\sigma = \rho$, μ_{aa} is a real positive measure,*
(iii) *if a, b are self-adjoint,* $\mathrm{Re}\mu_{ab} = \mathrm{Re}\mu_{ba}$.

Proof. (i) is standard.

(ii) Let Ω_1, Ω_2 be Borel sets in $\mathbb{R}$, and set $e_j := e_\rho(\Omega_j)$, $j = 1, 2$. Then $\mu_{aa}(\Omega_1 \times \Omega_2) = \tau(e_1 a^* e_2 a) = \tau((e_2 a e_1)^* e_2 a e_1) \geq 0$, and the thesis follows by uniqueness of the extension from $\mathcal{A}$ to $\sigma(\mathcal{A})$.

(iii) Let Ω_1, Ω_2 be Borel sets in $\mathbb{R}$, and set $e_1 := e_\rho(\Omega_1)$, $e_2 := e_\sigma(\Omega_2)$. Then $\mathrm{Re}\mu_{ab}(\Omega_1 \times \Omega_2) = \mathrm{Re}\tau(e_1 a e_2 b) = \mathrm{Re}\tau(b e_2 a e_1) = \mathrm{Re}\tau(e_1 b e_2 a) = \mathrm{Re}\mu_{ba}(\Omega_1 \times \Omega_2)$.$\square$

Lemma 2.3. *Let a, $b \in \mathcal{M} \cap L^2(\mathcal{M}, \tau)$. Let g, $h : \mathbb{R} \to \mathbb{C}$ be bounded Borel functions. Then*

$$\tau(g(\rho) a^* h(\sigma) b) = \iint g(x) h(y) d\mu_{ab}(x, y) \,.$$

Proof. We use notation as in the proof of Lemma 2.1. Let $s = \sum_{i=1}^h s_i \chi_{A_i}$, $t = \sum_{j=1}^k t_j \chi_{B_j}$ be simple Borel functions. Then

$$\begin{aligned} \tau(s(\rho) a^* t(\sigma) b) &= \sum_{i=1}^h \sum_{j=1}^k s_i t_j \tau(\chi_{A_i}(\rho) a^* \chi_{B_j}(\sigma) b) \\ &= \sum_{i=1}^h \sum_{j=1}^k s_i t_j \tau(P \otimes Q(A_i \times B_j)(a^*) \cdot b) \\ &= \sum_{i=1}^h \sum_{j=1}^k s_i t_j \iint \chi_{A_i \times B_j} d\mu_{ab} = \iint s(x) t(y) d\mu_{ab}(x, y) \,. \end{aligned}$$

Now let g, h be bounded Borel functions, and $\{s_m\}$, $\{t_n\}$ sequences of simple Borel functions such that $s_m \to g$, $t_n \to h$ and $|s_m| \leq |g|$, $|t_n| \leq |h|$. Denote $r_n(x, y) := s_n(x) t_n(y)$, $k(x, y) := g(x) h(y)$. Then, by (Theorem V.3.2 of Ref. 1), $s_n(\rho) a^* t_n(\sigma) = P \otimes Q(r_n)(a^*) \to P \otimes Q(k)(a^*) = g(\rho) a^* h(\sigma)$ in $L^2(\mathcal{M}, \tau)$, so that

$\tau(s_n(\rho)a^*t_n(\sigma)b) \to \tau(g(\rho)a^*h(\sigma)b)$. Moreover, $\iint r_n d\mu_{ab} \to \iint k d\mu_{ab}$, because μ_{ab} is a bounded measure. The thesis follows. $\square$

Lemma 2.4. *Let a, $b \in \mathcal{M} \cap L^2(\mathcal{M}, \tau)$, $\rho \in L^1(\mathcal{M}, \tau)_+$, $\beta \in (0,1)$. Then*

$$\tau(\rho^\beta a^* \rho^{1-\beta} b) = \iint_{[0,\infty)^2} x^\beta y^{1-\beta} d\mu_{ab}(x,y)\,.$$

Proof. Let $n \in \mathbb{N}$, and set

$$f_n(x) := \begin{cases} x\,, & 0 \le x \le n\,, \\ 0\,, & \text{otherwise} \end{cases} \qquad f(x) := \begin{cases} x\,, & x \ge 0\,, \\ 0\,, & x < 0\,. \end{cases}$$

Then

$$\tau(f_n(\rho)^\beta a^* f_n(\rho)^{1-\beta} b) = \int_{\mathbb{R}^2} f_n(x)^\beta f_n(y)^{1-\beta} d\mu_{ab}(x,y)\,.$$

Observe now that $f_n(\rho)^\beta \to f(\rho)^\beta = \rho^\beta$ in $L^{1/\beta}(\mathcal{M}, \tau)$, so that $f_n(\rho)^\beta a^* f_n(\rho)^{1-\beta} b \to \rho^\beta a^* \rho^{1-\beta} b$ in $L^1(\mathcal{M}, \tau)$, which implies

$$\tau(f_n(\rho)^\beta a^* f_n(\rho)^{1-\beta} b) \to \tau(\rho^\beta a^* \rho^{1-\beta} b)\,.$$

Moreover, in case $\sigma = \rho$, μ_{aa} is a positive measure, so that, by monotone convergence,

$$\int_{\mathbb{R}^2} f_n(x)^\beta f_n(y)^{1-\beta} d\mu_{aa}(x,y) \to \iint_{[0,\infty)^2} x^\beta y^{1-\beta} d\mu_{aa}(x,y)\,.$$

Therefore, the thesis holds for $a = b$. By polarization (Lemma 2.2(i)) the result is true in general. $\square$

Lemma 2.5. *Let a, $b \in \mathcal{M} \cap L^2(\mathcal{M}, \tau)$. Then,*

$$\mu := \mu_{aa} \otimes \mu_{bb} + \mu_{bb} \otimes \mu_{aa} - 2\mathrm{Re}\mu_{ab} \otimes \mathrm{Re}\mu_{ab}$$

is a real positive Borel measure on $\mathbb{R}^4$.

Proof. Indeed, if $\Omega_1, \dots, \Omega_4 \subset \mathbb{R}$ are measurable subsets, and $E_j := e_\rho(\Omega_j) \in \mathrm{Proj}(\mathcal{M})$, $j = 1, 3$, $E_j := e_\sigma(\Omega_j) \in \mathrm{Proj}(\mathcal{M})$, $j = 2, 4$, then

$$\begin{aligned} \mu(\Omega_1 \times \cdots \times \Omega_4) &= \tau(E_1 a^* E_2 a) \cdot \tau(E_3 b^* E_4 b) + \tau(E_3 a^* E_4 a) \cdot \tau(E_1 b^* E_2 b) \\ &\quad - 2\mathrm{Re}\tau(E_1 a^* E_2 b) \cdot \mathrm{Re}\tau(E_3 a^* E_4 b) \\ &\ge \tau(E_1 a^* E_2 a) \cdot \tau(E_3 b^* E_4 b) + \tau(E_3 a^* E_4 a) \cdot \tau(E_1 b^* E_2 b) \\ &\quad - 2|\tau(E_1 a^* E_2 b)| \cdot |\tau(E_3 a^* E_4 b)|\,. \end{aligned}$$

Moreover,

$$\begin{aligned} |\tau(E_1 a^* E_2 b)| &= |\tau((E_2 a E_1)^* E_2 b E_1)| \\ &\le \tau((E_2 a E_1)^* E_2 a E_1)^{1/2} \tau((E_2 b E_1)^* E_2 b E_1)^{1/2} \\ &= \tau(E_1 a^* E_2 a)^{1/2} \cdot \tau(E_1 b^* E_2 b)^{1/2}\,. \end{aligned}$$

Therefore, setting $\alpha_1 := \tau(E_1 a^* E_2 a)^{1/2}$, $\beta_1 := \tau(E_1 b^* E_2 b)^{1/2}$, $\alpha_2 := \tau(E_3 a^* E_4 a)^{1/2}$, $\beta_2 := \tau(E_3 b^* E_4 b)^{1/2}$, we have $\mu(\Omega_1 \times \cdots \times \Omega_4) \geq \alpha_1^2\beta_2^2 + \alpha_2^2\beta_1^2 - 2\alpha_1\beta_1\alpha_2\beta_2 \geq 0$, and the thesis follows by standard measure theoretic arguments. □

3. The Main Result

Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra with a n.s.f. trace. Let ω be a normal state on $\mathcal{M}$, and $\rho_\omega \in L^1(\mathcal{M}, \tau)_+$ be such that $\omega(x) = \tau(\rho_\omega x)$, for $x \in \mathcal{M}$. Then, for any A, $B \in \mathcal{M}_{sa}$, $\beta \in (0, 1)$, we set

Definition 3.1.

$$\begin{aligned}
\mathrm{Cov}_\omega(A, B) &:= \omega(AB) - \omega(A)\omega(B) \equiv \tau(\rho_\omega AB) - \tau(\rho_\omega A)\tau(\rho_\omega B)\,,\\
\mathrm{Var}_\omega(A) &:= \mathrm{Cov}_\omega(A, A) \equiv \omega(A^2) - \omega(A)^2 \equiv \tau(\rho_\omega A^2) - \tau(\rho_\omega A)^2\,,\\
\mathrm{Corr}_{\omega,\beta}(A, B) &:= \tau(\rho_\omega AB) - \tau(\rho_\omega^\beta A \rho_\omega^{1-\beta} B)\,,\\
I_{\omega,\beta}(A) &:= \mathrm{Corr}_{\omega,\beta}(A, A) \equiv \tau(\rho_\omega A^2) - \tau(\rho_\omega^\beta A \rho_\omega^{1-\beta} A)\,.
\end{aligned}$$

Proposition 3.2. *Let $A_0 := A - \omega(A)I$, $B_0 := B - \omega(B)I$. Then*

$$\begin{aligned}
\mathrm{Cov}_\omega(A, B) &= \tau(\rho_\omega A_0 B_0)\,,\\
\mathrm{Corr}_{\omega,\beta}(A, B) &= \tau(\rho_\omega A_0 B_0) - \tau(\rho_\omega^\beta A_0 \rho_\omega^{1-\beta} B_0)\,.
\end{aligned}$$

Theorem 3.3. *For any A, $B \in \mathcal{M}_{sa}$, $\beta \in (0, 1)$, we have*

$$\mathrm{Var}_\omega(A)\mathrm{Var}_\omega(B) - |\mathrm{Re}\mathrm{Cov}_\omega(A, B)|^2 \geq I_{\omega,\beta}(A) I_{\omega,\beta}(B) - |\mathrm{Re}\mathrm{Corr}_{\omega,\beta}(A, B)|^2\,.$$

Proof. To start with, let us assume that A, $B \in \mathcal{M} \cap L^2(\mathcal{M}, \tau)$. Set

$$\begin{aligned}
\mathcal{F} &:= \mathrm{Var}_\omega(A)\mathrm{Var}_\omega(B) - |\mathrm{Re}\mathrm{Cov}_\omega(A, B)|^2 - I_{\omega,\beta}(A) I_{\omega,\beta}(B) + |\mathrm{Re}\mathrm{Corr}_{\omega,\beta}(A, B)|^2\\
&= \tau(\rho_\omega A_0^2) \cdot \tau(\rho_\omega^\beta B_0 \rho_\omega^{1-\beta} B_0) + \tau(\rho_\omega B_0^2) \cdot \tau(\rho_\omega^\beta A_0 \rho_\omega^{1-\beta} A_0)\\
&\quad - \tau(\rho_\omega^\beta A_0 \rho_\omega^{1-\beta} A_0) \cdot \tau(\rho_\omega^\beta B_0 \rho_\omega^{1-\beta} B_0) - 2\mathrm{Re}\,\tau(\rho_\omega A_0 B_0) \cdot \mathrm{Re}\,\tau(\rho_\omega^\beta A_0 \rho_\omega^{1-\beta} B_0)\\
&\quad + (\mathrm{Re}\,\tau(\rho_\omega^\beta A_0 \rho_\omega^{1-\beta} B_0))^2\,.
\end{aligned}$$

Then, using Lemma 2.4 and symmetries of the integrands, we obtain

$$\begin{aligned}
\mathcal{F}_1 &:= \tau(\rho_\omega A_0^2) \cdot \tau(\rho_\omega^\beta B_0 \rho_\omega^{1-\beta} B_0) + \tau(\rho_\omega B_0^2) \cdot \tau(\rho_\omega^\beta A_0 \rho_\omega^{1-\beta} A_0)\\
&\quad - \tau(\rho_\omega^\beta A_0 \rho_\omega^{1-\beta} A_0) \cdot \tau(\rho_\omega^\beta B_0 \rho_\omega^{1-\beta} B_0)\\
&= \int_{[0,\infty)^4} \lambda_1 \lambda_3^\beta \lambda_4^{1-\beta} d\mu_{A_0 A_0} \otimes \mu_{B_0 B_0}(\lambda_1, \ldots, \lambda_4)\\
&\quad + \int_{[0,\infty)^4} \lambda_3 \lambda_1^\beta \lambda_2^{1-\beta} d\mu_{A_0 A_0} \otimes \mu_{B_0 B_0}(\lambda_1, \ldots, \lambda_4)\\
&\quad - \int_{[0,\infty)^4} \lambda_1^\beta \lambda_2^{1-\beta} \lambda_3^\beta \lambda_4^{1-\beta} d\mu_{A_0 A_0} \otimes \mu_{B_0 B_0}(\lambda_1, \ldots, \lambda_4)
\end{aligned}$$

$$= \frac{1}{2}\int_{[0,\infty)^4} ((\lambda_1+\lambda_2)\lambda_3^{\beta}\lambda_4^{1-\beta} + \lambda_1^{\beta}\lambda_2^{1-\beta}(\lambda_3+\lambda_4)$$
$$- 2\lambda_1^{\beta}\lambda_2^{1-\beta}\lambda_3^{\beta}\lambda_4^{1-\beta})d\mu_{A_0A_0}\otimes\mu_{B_0B_0}(\lambda_1,\dots,\lambda_4)\,,$$

$$\mathcal{F}_2 := 2\mathrm{Re}\,\tau(\rho_\omega A_0B_0)\cdot\mathrm{Re}\,\tau(\rho_\omega^{\beta}A_0\rho_\omega^{1-\beta}B_0) - (\mathrm{Re}\,\tau(\rho_\omega^{\beta}A_0\rho_\omega^{1-\beta}B_0))^2$$
$$= 2\int_{[0,\infty)^4} \lambda_1\lambda_3^{\beta}\lambda_4^{1-\beta} d\ \mathrm{Re}\,\mu_{A_0B_0}\otimes\mathrm{Re}\,\mu_{A_0B_0}(\lambda_1,\dots,\lambda_4)$$
$$- \int_{[0,\infty)^4} \lambda_1^{\beta}\lambda_2^{1-\beta}\lambda_3^{\beta}\lambda_4^{1-\beta} d\ \mathrm{Re}\,\mu_{A_0B_0}\otimes\mathrm{Re}\,\mu_{A_0B_0}(\lambda_1,\dots,\lambda_4)$$
$$= \frac{1}{2}\int_{[0,\infty)^4} ((\lambda_1+\lambda_2)\lambda_3^{\beta}\lambda_4^{1-\beta} + \lambda_1^{\beta}\lambda_2^{1-\beta}(\lambda_3+\lambda_4)$$
$$- 2\lambda_1^{\beta}\lambda_2^{1-\beta}\lambda_3^{\beta}\lambda_4^{1-\beta})d\ \mathrm{Re}\,\mu_{A_0B_0}\otimes\mathrm{Re}\,\mu_{A_0B_0}(\lambda_1,\dots,\lambda_4)\,.$$

So that, using the notation of Lemma 2.5,

$$\mathcal{F} = \mathcal{F}_1 - \mathcal{F}_2 = \frac{1}{4}\int_{[0,\infty)^4} ((\lambda_1+\lambda_2)\lambda_3^{\beta}\lambda_4^{1-\beta} + \lambda_1^{\beta}\lambda_2^{1-\beta}(\lambda_3+\lambda_4)$$
$$- 2\lambda_1^{\beta}\lambda_2^{1-\beta}\lambda_3^{\beta}\lambda_4^{1-\beta})d\mu(\lambda_1,\dots,\lambda_4)\,.$$

Since μ is a real positive measure on $[0,\infty)^4$, because of Lemma 2.5, and

$$(\lambda_1+\lambda_2)\lambda_3^{\beta}\lambda_4^{1-\beta} + \lambda_1^{\beta}\lambda_2^{1-\beta}(\lambda_3+\lambda_4) - 2\lambda_1^{\beta}\lambda_2^{1-\beta}\lambda_3^{\beta}\lambda_4^{1-\beta}$$
$$= (\lambda_1+\lambda_2-\lambda_1^{\beta}\lambda_2^{1-\beta})\lambda_3^{\beta}\lambda_4^{1-\beta} + \lambda_1^{\beta}\lambda_2^{1-\beta}(\lambda_3+\lambda_4-\lambda_3^{\beta}\lambda_4^{1-\beta}) \geq 0\,,$$

we obtain $\mathcal{F}\geq 0$, which is what we wanted to prove.

Finally, to extend the validity of the inequality from $\mathcal{M}_{sa}\cap L^2(\mathcal{M},\tau)$ to $\mathcal{M}_{sa}$, let us observe that $\mathcal{M}_{sa}\cap L^2(\mathcal{M},\tau)$ is σ-weakly dense in $\mathcal{M}_{sa}$, and $a\in\mathcal{M}\mapsto\tau(\rho_\omega ab)$, $b\in\mathcal{M}\mapsto\tau(\rho_\omega ab)$, $a\in\mathcal{M}\mapsto\tau(\rho^{\beta}a\rho^{1-\beta}b)$, and $b\in\mathcal{M}\mapsto\tau(\rho^{\beta}a\rho^{1-\beta}b)$ are σ-weakly continuous. □

Remark 3.4. Observe that, reasoning as in Theorem 5 of Ref. 9, one can prove that the function

$$g(\beta) := \mathrm{Var}_\omega(A)\mathrm{Var}_\omega(B) - |\mathrm{ReCov}_\omega(A,B)|^2 - I_{\omega,\beta}(A)I_{\omega,\beta}(B) + |\mathrm{ReCorr}_{\omega,\beta}(A,B)|^2$$

is monotone increasing on the interval $[\frac{1}{2},1)$. Therefore, the best bound in Theorem 3.3 is given by $\beta=\frac{1}{2}$, i.e. by the Wigner–Yanase information.

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