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ON THE CENTRAL EXTENSIONS OF THE HEISENBERG ALGEBRA

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Abstract. We describe the nontrivial central extensions $CE(Heis)$ of the Heisenberg algebra and their representation as sub–algebras of the Schroedinger algebra. We also present the characteristic and moment generating functions of the random variable corresponding to the self-adjoint sum of the generators of $CE(Heis)$.

1. Central extensions of Lie algebras

In the applications of Lie algebras to physical systems the symmetries of the system are frequently described at the level of classical mechanics by some Lie algebra L, and in the quantum theoretic description by L plus some extra, constant, not arbitrary terms which are interpreted as the eigenvalues of some new operators K^i which have constant eigenvalue on any irreducible module of L (by Schur's lemma the K^i must commute with all elements of L). The new generators K^i extend L to a new Lie algebra $\hat{L}$.

In general, given a Lie algebra L with basis $\{T^a \ ; \ a = 1, 2, ..., d\}$, by attaching additional generators $\{K^i \ ; \ i = 1, 2, .., l\}$ such that

$$[K^i, K^j] = [T^a, K^j] = 0 \tag{1.1}$$

we obtain an l–dimensional central extension $\hat{L}$ of L with Lie brackets

$$[T^a, T^b] = \sum_{c=1}^{d} f_c^{ab}\, T^c + \sum_{i=1}^{l} g_i^{ab}\, K^i \tag{1.2}$$

where f_c^{ab} are the structure constants of L in the basis $\{T^a \ ; \ a = 1, 2, ..., d\}$. If through a constant redefinition of the generators $\{T^a \ ; \ a = 1, 2, ..., d\}$ (i.e. if $\hat{L}$ is the direct sum of L and an Abelian algebra) the commutation relations of $\hat{L}$ reduce to those of L then the central extension is trivial.

A basis independent (or cocycle) definition of an one-dimensional (i.e. having only one central generator) central extension can be given as follows:

If L and $\hat{L}$ are two complex Lie algebras, we say that $\hat{L}$ is an one-dimensional central extension of L with central element E if

$$[l_1, l_2]_{\hat{L}} = [l_1, l_2]_L + \phi(l_1, l_2)\, E \ ; \ [l_1, E]_{\hat{L}} = 0 \tag{1.3}$$

for all $l_1, l_2 \in L$, where $[\cdot, \cdot]_{\hat{L}}$ and $[\cdot, \cdot]_L$ are the Lie brackets in $\hat{L}$ and L respectively, and $\phi : L \times L \mapsto C$ is a bilinear form (2-cocycle) on L satisfying the skew-symmetry condition

$$\phi(l_1, l_2) = -\phi(l_2, l_1) \tag{1.4}$$

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and the Jacobi identity

$$\phi([l_1, l_2]_L, l_3) + \phi([l_2, l_3]_L, l_1) + \phi([l_3, l_1]_L, l_2) = 0 \tag{1.5}$$

A central extension is trivial if there exists a linear function $f : L \mapsto \mathbb{C}$ satisfying for all $l_1, l_2 \in L$

$$\phi(l_1, l_2) = f([l_1, l_2]_L) \tag{1.6}$$

For more information on central extensions we refer to [3]. Detailed proofs of the material presented in sections 2–5 below will appear in [1]

2. Central extensions of the Heisenberg algebra

The Heisenberg $*$-Lie algebra $Heis$ is the 3-dimensional Lie algebra with generators $\{a^\dagger, a, h\}$, commutation relations

$$[a, a^\dagger]_{Heis} = h \;\; ; \;\; [a, h^\dagger]_{Heis} = [h, a]_{Heis} = 0 \tag{2.1}$$

and involution

$$\left(a^\dagger\right)^* = a \quad ; \quad (a)^* = a^\dagger \quad ; \quad (h)^* = h \tag{2.2}$$

All 2-cocycles ϕ corresponding to a central extension $CE(Heis)$ of $Heis$ are of the form

$$\phi(a, a^\dagger) = \lambda \tag{2.3}$$
$$\phi(h, a^\dagger) = z \tag{2.4}$$
$$\phi(a, h) = \bar{z} \tag{2.5}$$
$$\phi(h, h) = \phi(a^\dagger, a^\dagger) = \phi(a, a) = 0 \tag{2.6}$$

where $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$. To see that, let $l_i = a_i\, a^\dagger + b_i\, a + c_i\, h$ where $a_i, b_i, c_i \in \mathbb{C}$ for all $i \in \{1, 2, 3\}$, be three elements of $Heis$. Then

$$[l_1, l_2]_{Heis} = (b_1\, a_2 - a_1\, b_2)\, h \tag{2.7}$$
$$[l_2, l_3]_{Heis} = (b_2\, a_3 - a_2\, b_3)\, h \tag{2.8}$$
$$[l_3, l_1]_{Heis} = (b_3\, a_1 - a_3\, b_1)\, h \tag{2.9}$$

and

$$\phi([l_1, l_2]_{Heis}, l_3) = (b_1\, a_2\, a_3 - a_1\, b_2\, a_3)\, \phi(h, a^\dagger) + (b_1\, a_2\, b_3 - a_1\, b_2\, b_3)\, \phi(h, a) \tag{2.10}$$
$$\phi([l_2, l_3]_{Heis}, l_1) = (b_2\, a_3\, a_1 - a_2\, b_3\, a_1)\, \phi(h, a^\dagger) + (b_2\, a_3\, b_1 - a_2\, b_3\, b_1)\, \phi(h, a) \tag{2.11}$$
$$\phi([l_3, l_1]_{Heis}, l_2) = (b_3\, a_1\, a_2 - a_3\, b_1\, a_2)\, \phi(h, a^\dagger) + (b_3\, a_1\, b_2 - a_3\, b_1\, b_2)\, \phi(h, a) \tag{2.12}$$

and the Jacobi identity (1.5) for ϕ reduces to

$$0 \cdot \phi(h, a^\dagger) + 0 \cdot \phi(h, a) = 0 \tag{2.13}$$

which implies that $\phi(h\,,a^\dagger\,)$ and $\phi(h\,,a\,)$ are arbitrary complex numbers. Since it does not appear in (2.13), $\phi(a\,,a^\dagger\,)$ is also an arbitrary complex number. Therefore the, non–zero among generators, $CE(Heis)$ commutation relations have the form

$$[a\,,a^\dagger\,]_{CE(Heis)} = h + \phi(a\,,a^\dagger\,)\,E \tag{2.14}$$
$$[a\,,h\,]_{CE(Heis)} = \phi(a\,,h\,)\,E \tag{2.15}$$
$$[h\,,a^\dagger\,]_{CE(Heis)} = \phi(h\,,a^\dagger)\,E \tag{2.16}$$

where E is the, non–zero, central element. By skew-symmetry

$$\phi(a^\dagger\,,h\,) = -\phi(h\,,a^\dagger\,)\ ;\ \phi(a^\dagger\,,a\,) = -\phi(a\,,a^\dagger\,)\ ;\ \phi(a\,,h\,) = -\phi(h\,,a\,) \tag{2.17}$$

and

$$\phi(a\,,a\,) = \phi(a^\dagger\,,a^\dagger\,) = \phi(h\,,h\,) = 0 \tag{2.18}$$

By taking the adjoints of (2.14)-(2.16), assuming the involution conditions

$$\left(a^\dagger\,\right)^* = a \quad ; \quad (a)^* = a^\dagger \quad ; \quad (h\,)^* = h\ ;\ (E\,)^* = E \tag{2.19}$$

we find that

$$\phi(a\,,a^\dagger\,) = \overline{\phi(a\,,a^\dagger\,)} = \lambda \in \mathbb{R} \tag{2.20}$$

and

$$\phi(a\,,h) = \overline{\phi(h\,,a^\dagger)} = \bar{z} \tag{2.21}$$

where

$$z = \phi(h\,,a^\dagger) \in \mathbb{C} \tag{2.22}$$

If a central extension $CE(Heis)$ of $Heis$ is trivial then there exists a linear complex-valued function f defined on $Heis$ such that

$$f([a\,,a^\dagger\,]_{Heis}) = \lambda \tag{2.23}$$
$$f([a\,,h\,]_{Heis}) = \bar{z} \tag{2.24}$$
$$f([h\,,a^\dagger\,]_{Heis}) = z \tag{2.25}$$

Since $[h\,,a^\dagger\,]_{Heis} = 0$ and (for a linear f) $f(0) = 0$, by (2.25) we conclude that $z = 0$.

Conversely, suppose that $z = 0$. Define a linear complex-valued function f on $Heis$ by

$$f\left(z_1\,h + z_2\,a^\dagger + z_3\,a\right) = z_1\,\lambda \tag{2.26}$$

where λ is as above and $z_1, z_2, z_3 \in \mathbb{C}$. Then

$$f([a\,,a^\dagger\,]_{Heis}) = f(1\,h\,+0\,a^\dagger\,+0\,a\,) = \lambda = \phi(a\,,a^\dagger\,) \tag{2.27}$$
$$f([a\,,h\,]_{Heis}) = f(0\,h\,+0\,a^\dagger\,+0\,a\,) = 0 = \bar{z} = \phi(a\,,h) \tag{2.28}$$
$$f([h\,,a^\dagger\,]_{Heis}) = f(0\,h\,+0\,a^\dagger\,+0\,a\,) = 0 = z = \phi(h\,,a^\dagger\,) \tag{2.29}$$

which, by (1.6), implies that the central extension is trivial.

Thus, a central extension of $Heis$ is trivial if and only if $z = 0$.

The centrally extended Heisenberg commutation relations (2.14)-(2.16) now have the form

$$[a, a^\dagger]_{CE(Heis)} = h + \lambda\,E \;\;;\;\; [h, a^\dagger\,]_{CE(Heis)} = z\,E \;\;;\;\; [a, h]_{Heis} = \bar{z}\,E \tag{2.30}$$

Renaming $h + \lambda\,E$ to just h we obtain the equivalent (canonical) $CE(Heis)$ commutation relations

$$[a, a^\dagger]_{CE(Heis)} = h \;;\; [h, a^\dagger]_{CE(Heis)} = z\,E \;;\; [a, h]_{CE(Heis)} = \bar{z}\,E \tag{2.31}$$

For $z = 0$ we recover the Heisenberg commutation relations (2.1). Commutation relations (2.31) define a nilpotent (thus solvable) four–dimensional $*$–Lie algebra $CE(Heis)$ with generators $a, a^\dagger, h$ and E. Moreover, if we define p, q and H by

$$a^\dagger = p + i\,q \qquad ; \qquad a = p - i\,q \qquad ; \qquad H = -ih/2 \tag{2.32}$$

then p, q, E are self-adjoint, H is skew-adjoint, and $\{p, q, E, H\}$ are the generators of a real four-dimensional solvable $*$–Lie algebra with central element E and commutation relations

$$[p, q] = H \;;\; [q, H] = c\,E \;;\; [H, p] = b\,E \tag{2.33}$$

where b, c are (not simultaneously zero) real numbers given by

$$c = \frac{Re\,z}{2}\;,\; b = \frac{Im\,z}{2} \tag{2.34}$$

Conversely, if p, q, H, E are the generators (with p, q, E self-adjoint and H skew-adjoint) of a real four-dimensional solvable $*$–Lie algebra with central element E and commutation relations (2.33) with $b, c \in \mathbb{R}$ not simultaneously zero, then, defining z by (2.34), the operators $a, a^\dagger, h$ defined by (2.32) and E are the generators of the nontrivial central extension $CE(Heis)$ of the Heisenberg algebra defined by (2.31) and (2.19).

The real four–dimensional solvable Lie algebra generated by $\{p, q, E, H\}$ can be identified to the Lie algebra η_4 (one of the fifteen classified real four–dimensional solvable Lie algebras, see for example [4]) with generators e_1, e_2, e_3, e_4 and (non-zero) commutation relations among generators

$$[e_4, e_1] = e_2 \qquad ; \qquad [e_4, e_2] = e_3 \tag{2.35}$$

This algebra has been studied by Feinsilver and Schott in [2].

3. Representations of $CE(Heis)$

The non–trivial central extensions of $CE(Heis)$ (corresponding to $z \neq 0$) can be realized as proper sub–algebras of the Schroedinger algebra, i.e. the six–dimensional $*$–Lie algebra generated by b, $b^\dagger$, b^2, ${b^\dagger}^2$, $b^\dagger\, b$ and 1 where $b^\dagger$, b and 1 are the generators of a Boson Heisenberg algebra with

$$[b, b^\dagger] = 1 \qquad ; \qquad (b^\dagger)^* = b \tag{3.1}$$

More precisely,

(i) If $z \in \mathbb{C}$ with $Re\, z \neq 0$, then for arbitrary $\rho, r \in \mathbb{R}$ with $r \neq 0$, letting

$$a = \left(\frac{4\,\rho\, Im\, z - r^2}{4\, Re\, z} + i\,\rho\right)(b - b^\dagger)^2 - \frac{i\,\bar{z}}{2\,r}\,(b + b^\dagger) \tag{3.2}$$

$$a^\dagger = \left(\frac{4\,\rho\, Im\, z - r^2}{4\, Re\, z} - i\,\rho\right)(b - b^\dagger)^2 + \frac{i\,z}{2\,r}\,(b + b^\dagger) \tag{3.3}$$

and

$$h = i\,r\,(b^\dagger - b) \tag{3.4}$$

we find that the quadruple $\{a^+, a, h, E = 1\}$ satisfies commutation relations (2.31) and duality relations (2.19) of $CE(Heis)$.

(ii) If $z \in \mathbb{C}$ with $Re\, z = 0$, then for arbitrary $\rho, r \in \mathbb{R}$ with $r \neq 0$, letting

$$a = \left(\rho - \frac{i\, Im\, z}{16\, r^2}\right)(b - b^\dagger)^2 + r\,(b + b^\dagger) \tag{3.5}$$

$$a^\dagger = \left(\rho + \frac{i\, Im\, z}{16\, r^2}\right)(b - b^\dagger)^2 + r\,(b + b^\dagger) \tag{3.6}$$

and

$$h = \frac{i\, Im\, z}{2\, r}\,(b^\dagger - b) \tag{3.7}$$

we find that the quadruple $\{a^+, a, h, E = 1\}$ satisfies commutation relations (2.31) and duality relations (2.19) of $CE(Heis)$.

Using the fact that for non-negative integers n, k

$${b^\dagger}^n\, b^k\, y(\xi) = \xi^k\, \frac{\partial^n}{\partial\, \epsilon^n}|_{\epsilon=0}\, y(\xi + \epsilon) \tag{3.8}$$

where, for $\xi \in \mathbb{C}$, $y(\xi) = e^{\xi\, b}$ we may represent $CE(Heis)$ on the Heisenberg Fock space $\mathcal{F}$ defined as the Hilbert space completion of the linear span of the exponential vectors $\{y(\xi)\,;\, \xi \in \mathbb{C}\}$ with respect to the inner product

$$\langle y(\xi), y(\mu)\rangle = e^{\bar{\xi}\,\mu} \tag{3.9}$$

We have that:

(i) If $z \in \mathbb{C}$ with $Rez \neq 0$ then

$$a\,y(\xi)=\left(\left(\tfrac{4\,\rho\,Im\,z-r^2}{4\,Re\,z}+i\,\rho\right)\,(\xi^2-1)-\tfrac{i\,\bar{z}}{2\,r}\,\xi\right)\,y(\xi)$$
$$+\left(\left(\tfrac{4\,\rho\,Im\,z-r^2}{4\,Re\,z}+i\,\rho\right)\,\tfrac{\partial^2}{\partial\,\epsilon^2}|_{\epsilon=0}-\left(\left(\tfrac{4\,\rho\,Im\,z-r^2}{4\,Re\,z}+i\,\rho\right)\,2\,\xi+\tfrac{i\,\bar{z}}{2\,r}\right)\,\tfrac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\right)\,y(\xi+\epsilon) \tag{3.10}$$

$$a^\dagger\,y(\xi)=\left(\left(\tfrac{4\,\rho\,Im\,z-r^2}{4\,Re\,z}-i\,\rho\right)\,(\xi^2-1)+\tfrac{i\,z}{2\,r}\,\xi\right)\,y(\xi)$$
$$+\left(\left(\tfrac{4\,\rho\,Im\,z-r^2}{4\,Re\,z}-i\,\rho\right)\,\tfrac{\partial^2}{\partial\,\epsilon^2}|_{\epsilon=0}-\left(\left(\tfrac{4\,\rho\,Im\,z-r^2}{4\,Re\,z}-i\,\rho\right)\,2\,\xi-\tfrac{i\,z}{2\,r}\right)\,\tfrac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\right)\,y(\xi+\epsilon) \tag{3.11}$$

$$h\,y(\xi)=i\,r\,\left(\frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,y(\xi+\epsilon)-\xi\,y(\xi)\right) \tag{3.12}$$

and

$$E\,y(\xi)=y(\xi) \tag{3.13}$$

(ii) If $z\in\mathbb{C}$ with $Re\,z=0$ then

$$a\,y(\xi)=\left(\left(\rho-\tfrac{i\,Im\,z}{16\,r^2}\right)\,(\xi^2-1)+r\,\xi\right)\,y(\xi)$$
$$+\left(\left(\rho-\tfrac{i\,Im\,z}{16\,r^2}\right)\,\tfrac{\partial^2}{\partial\,\epsilon^2}|_{\epsilon=0}+\left(r-\left(\rho-\tfrac{i\,Im\,z}{16\,r^2}\right)\,2\,\xi\right)\,\tfrac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\right)\,y(\xi+\epsilon) \tag{3.14}$$

$$a^\dagger\,y(\xi)=\left(\left(\rho+\tfrac{i\,Im\,z}{16\,r^2}\right)\,(\xi^2-1)+r\,\xi\right)\,y(\xi)$$
$$+\left(\left(\rho+\tfrac{i\,Im\,z}{16\,r^2}\right)\,\tfrac{\partial^2}{\partial\,\epsilon^2}|_{\epsilon=0}+\left(r-\left(\rho+\tfrac{i\,Im\,z}{16\,r^2}\right)\,2\,\xi\right)\,\tfrac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\right)\,y(\xi+\epsilon) \tag{3.15}$$

$$h\,y(\xi)=\frac{i\,Im\,z}{2\,r}\,\left(\frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,y(\xi+\epsilon)-\xi\,y(\xi)\right) \tag{3.16}$$

and

$$E\,y(\xi)=y(\xi) \tag{3.17}$$

4. Random variables associated with $CE(Heis)$

Self-adjoint operators X on the Heisenberg Fock space $\mathcal{F}$ correspond to classical random variables with moment generating function $\langle\Phi,e^{s\,X}\,\Phi\rangle$ and characteristic function $\langle\Phi,e^{i\,s\,X}\,\Phi\rangle$, where $s\in\mathbb{R}$ and Φ is the Heisenberg Fock space cyclic vacuum vector such that $b\,\Phi=0$.

Using the splitting (or disentanglement) formula

$$e^{s\,(L\,b^2+L\,b^{\dagger\,2}-2\,L\,b^\dagger\,b-L+M\,b+N\,b^\dagger)}\,\Phi=e^{w_1(s)\,b^{\dagger 2}}\,e^{w_2(s)\,b^\dagger}\,e^{w_3(s)}\,\Phi \tag{4.1}$$

where $L\in\mathbb{R}$, $M,N\in\mathbb{C}$, $s\in\mathbb{R}$,

$$w_1(s)\;=\;\frac{L\,s}{2\,L\,s+1} \tag{4.2}$$

$$w_2(s)\;=\;\frac{L\,(M+N)\,s^2+N\,s}{2\,L\,s+1} \tag{4.3}$$

and

$$w_3(s) = \frac{(M+N)^2\,(L^2\,s^4 + 2\,L\,s^3) + 3\,M\,N\,s^2}{6\,(2\,L\,s+1)} - \frac{\ln(2\,L\,s+1)}{2} \tag{4.4}$$

we find that the moment generating function MGF_X of the self-adjoint operator

$$X = a + a^\dagger + h \tag{4.5}$$

where $a, a^\dagger, h$ are three of the generators of $CE(Heis)$, is

$$MGF_X(s) = \langle \Phi, e^{s\,(a+a^\dagger+h)}\,\Phi\rangle = (2\,L\,s+1)^{-1/2}\,e^{\frac{(M+N)^2\,(L^2\,s^4+2\,L\,s^3)+3\,M\,N\,s^2}{6\,(2\,L\,s+1)}} \tag{4.6}$$

where $s \in \mathbb{R}$ is such that $2\,L\,s+1>0$.

Similarly, the characteristic function of X is

$$CF_X(s) = \langle \Phi, e^{i\,s\,(a+a^\dagger+h)}\,\Phi\rangle = (2\,i\,L\,s+1)^{-1/2}\,e^{\frac{(M+N)^2\,(L^2\,s^4-2\,i\,L\,s^3)-3\,M\,N\,s^2}{6\,(2\,i\,L\,s+1)}} \tag{4.7}$$

In both MGF_X and CF_X, in the notation of section 3:

(i) if $Re\,z \neq 0$ then

$$L = \frac{4\,\rho\,Im\,z - r^2}{2\,Re\,z} \tag{4.8}$$

$$M = -\left(\frac{Im\,z}{r} + i\,r\right) \tag{4.9}$$

$$N = -\left(\frac{Im\,z}{r} - i\,r\right) \tag{4.10}$$

(ii) if $Re\,z = 0$ then

$$L = 2\,\rho \tag{4.11}$$

$$M = 2\,r - i\,\frac{Im\,z}{2\,r} \tag{4.12}$$

$$N = 2\,r + i\,\frac{Im\,z}{2\,r} \tag{4.13}$$

Notice that, if $L=0$ (corresponding to $\rho\,Im\,z>0$ and $r^2 = 4\,\rho\,Im\,z$ in the case when $Re\,z \neq 0$ and to $\rho = 0$ in the case when $Re\,z = 0$) then

$$MGF_X(s) = e^{\frac{M\,N\,s^2}{2}} = \begin{cases} e^{\left(\frac{(Im\,z)^2}{2\,r^2}+\frac{r^2}{2}\right)\,s^2} & \text{if } Re\,z \neq 0 \\ e^{\left(2\,r^2+\frac{(Im\,z)^2}{8\,r^2}\right)\,s^2} & \text{if } Re\,z = 0 \end{cases} \tag{4.14}$$

which means that X is a Gaussian random variable.

For $L \neq 0$ the term $(2\,L\,s+1)^{-1/2}$ is the moment generating function of a gamma random variable.

We may also represent $CE(Heis)$ in terms of two independent CCR copies as follows:

For $j,k\in\{1,2\}$ let $[q_j,p_k]=\frac{i}{2}\,\delta_{j,k}$ and $[q_j,q_k]=[p_j,p_k]=0$ with $p_j^*=p_j$, $q_j^*=q_j$ and $i^2=-1$.

(i) If $z\in\mathbb{C}$ with $Re\,z\neq 0$ and $Im\,z\neq 0$ then

$$a = i\,Re\,z\,q_1+\frac{1}{Re\,z}\,p_1^2-Im\,z\,p_2-\frac{i}{Im\,z}\,q_2^2 \tag{4.15}$$

$$a^\dagger = -i\,Re\,z\,q_1+\frac{1}{Re\,z}\,p_1^2-Im\,z\,p_2+\frac{i}{Im\,z}\,q_2^2 \tag{4.16}$$

$$h = -2\,(p_1+q_2) \tag{4.17}$$

and $E=1$ satisfy the commutation relations (2.31) and the duality relations (2.19) of $CE(Heis)$.

(ii) If $z\in\mathbb{C}$ with $Re\,z=0$ and $Im\,z\neq 0$ then for arbitrary $r\in\mathbb{R}$ and $c\in\mathbb{C}$

$$a = c\,p_1^2-Im\,z\,p_2+\left(r-\frac{i}{Im\,z}\right)\,q_2^2 \tag{4.18}$$

$$a^\dagger = \bar{c}\,p_1^2-Im\,z\,p_2+\left(r+\frac{i}{Im\,z}\right)\,q_2^2 \tag{4.19}$$

$$h = -2\,q_2 \tag{4.20}$$

and $E=1$ satisfy the commutation relations (2.31) and the duality relations (2.19) of $CE(Heis)$.

(iii) If $z\in\mathbb{C}$ with $Re\,z\neq 0$ and $Im\,z=0$ then for arbitrary $r\in\mathbb{R}$ and $c\in\mathbb{C}$

$$a = i\,Re\,z\,q_1+\left(\frac{1}{Re\,z}+i\,r\right)\,p_1^2+c\,q_2^2 \tag{4.21}$$

$$a^\dagger = -i\,Re\,z\,q_1+\left(\frac{1}{Re\,z}-i\,r\right)\,p_1^2+\bar{c}\,q_2^2 \tag{4.22}$$

$$h = -2\,p_1 \tag{4.23}$$

and $E=1$ satisfy the commutation relations (2.31) and the duality relations (2.19) of $CE(Heis)$.

We may take

$$q_1=\frac{b_1+b_1^\dagger}{2}\ \ ;\ \ p_1=\frac{i\,(b_1^\dagger-b_1)}{2} \tag{4.24}$$

and

$$q_2=\frac{b_2+b_2^\dagger}{2}\ \ ;\ \ p_2=\frac{i\,(b_2^\dagger-b_2)}{2} \tag{4.25}$$

where

$$[b_1,b_1^\dagger]=[b_2,b_2^\dagger]=1 \tag{4.26}$$

and

$$[b_1^\dagger,b_2^\dagger]=[b_1,b_2]=[b_1,b_2^\dagger]=[b_1^\dagger,b_2]=0 \tag{4.27}$$

In that case, MGF_X would be the product of the moment generating functions of two independent random variables defined in terms of the generators of two mutually commuting Schroedinger algebras.

5. The centrally extended Heisenberg group

For $u, v, w, y \in \mathbb{C}$ define

$$g(u, v, w, y) = e^{u\, a^{\dagger}}\, e^{v\, h}\, e^{w\, a} e^{y\, E} \tag{5.1}$$

The family of operators of the form (5.1) is a group with group law given by

$$g(\alpha, \beta, \gamma, \delta)\, g(A, B, C, D) = \tag{5.2}$$

$$= g(\alpha + A, \beta + B + \gamma\, A, \gamma + C, \left(\frac{\gamma\, A^2}{2} + \beta\, A\right) z + \left(\frac{\gamma^2\, A}{2} + \gamma\, B\right) \bar{z} + \delta + D)$$

Restricting to $u, v, w \in \mathbb{R}$ and $y \in \mathbb{C}$ we obtain the centrally extended Heisenberg group $\mathbb{R}^3 \times \mathbb{C}$ endowed with the composition law:

$$(\alpha, \beta, \gamma, \delta)\, (A, B, C, D) = \tag{5.3}$$

$$\left(\alpha + A, \beta + B + \gamma\, A, \gamma + C, \left(\frac{\gamma\, A^2}{2} + \beta\, A\right) z + \left(\frac{\gamma^2\, A}{2} + \gamma\, B\right) \bar{z} + \delta + D\right)$$

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