



Hydroelastic theory for offshore floating plates of variable flexural rigidity

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ABSTRACT

We present a theoretical model of the hydrodynamic behaviour of a floating flexible plate of variable flexural rigidity connected to the seabed by a spring/damper system. Decomposition of the response modes into rigid and bending elastic components allows us to investigate the hydroelastic behaviour of the plate subject to monochromatic incident free-surface waves of constant amplitude. We show that spatially dependent plate stiffness affects the eigenfrequencies and modal shapes, with direct consequences on plate dynamics and wave power extraction efficiency. We also examine how plate length and Power Take-Off (PTO) distribution affect the response of the system and its consequent absorbed energy. This work highlights the need to improve existing models of flexible floating energy platforms, especially given their importance in the Offshore Renewable Energy (ORE) sector.

1. Introduction

In recent years, there has been growing interest in the use of flexible marine structures (Renzi et al., 2021; Lamas-Pardo et al., 2015; Watanabe et al., 2004; Kashiwagi, 2000). Very Large Floating Structures (VLFS) have been proposed for floating airports, breakwaters, bridges, piers and docks, storage facilities, recreation parks, habitation, offshore wind foundations, etc., and are essentially large, almost horizontal plates that rest on the sea surface (Zhang et al., 2021). Certain other artificial structures, such as flexible wave energy converters (WECs) (Michele et al., 2022) and offshore solar farms (Bjørneklett, 2018), also comprise floating plates but are usually smaller than VLFS. Further marine examples of floating plates include sea ice sheets (Meylan and Squire, 1996) which have grown in importance given the ongoing heating effect on the oceans due to global climate change. The hydroelastic characteristics of flexible plates have also been studied for an array of elastic plates (Peter and Meylan, 2004; Bennetts et al., 2010), free floating and submerged porous elastic plates (Zheng et al., 2020b; Liang et al., 2022; Behera and Sahoo, 2015; Zheng et al., 2020a,c), time-dependent transient responses of elastic plates (Nugroho et al., 1999; Montiel et al., 2012) and multiple floating elastic plates of variable properties (Kohout et al., 2007). Recently (Singh et al., 2023; Singh and Gayen, 2023) developed a mathematical model based on Green's theorem for a thin elastic plate of non-uniform thickness, Korobkin et al. (2023) derived added-mass matrices of floating compound plates and plates with piecewise linear thickness. Porter and Porter (2004) derived an analytical solution for hydroelastic wave scattering by an ice sheet of variable thickness and uneven bottom topography under a mild-slope assumption, Williams and Squire (2004) developed a theoretical analysis of wave propagation beneath sea-ice where the

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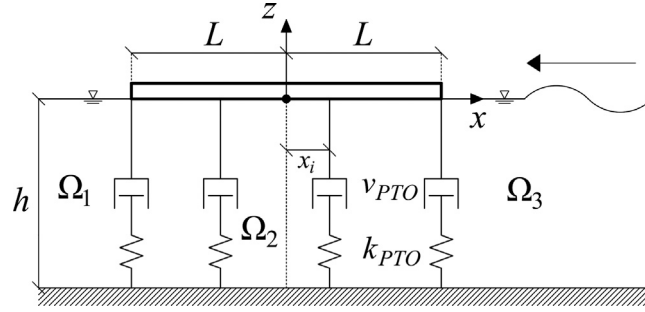


Fig. 1. Side view of floating flexible plate. Vertical cables are located at points $x_i, i = 1, \dots, I$.

ice is allowed to vary spatially, [Bennetts et al. \(2007\)](#) extended the work from [Porter and Porter \(2004\)](#) by adopting a Rayleigh–Ritz method in conjunction with a variational principle, [Sturova \(2009\)](#) investigated the time-domain problem of a heterogeneous elastic plate floating on shallow water of variable depth, whereas [Meylan et al. \(2021\)](#) have recently analysed flexural vibrations of thickening ice shelves.

Turning to the marine renewable energy sector, flexible WECs offer an effective way to increase wave power absorption ([Kurniawan et al., 2017](#); [Collins et al., 2021](#)). For example, [Renzi \(2016\)](#) analysed the coupled hydro-electromechanical response of a two-dimensional piezoelectric device and showed that the piezoelectric plate can extract sufficient energy for low-power devices. Hydrodynamic characteristics of piezoelectric-plate WECs in other circumstances have also been investigated, e.g., a piezoelectric plate WEC moored on a seabed-mounted/floating breakwater ([Zheng et al., 2021](#)). Recently, [Michele et al. \(2020, 2022\)](#) developed a dry-mode expansion technique to extend linear potential flow theory to the hydrodynamics of rectangular and circular flexible WECs combined with a series of power take-off units deployed under a free-floating elastic plate. [Michele et al. \(2023\)](#) experimentally validated a similar theory for regular and irregular waves. Meanwhile, [Zheng et al. \(2022\)](#) investigated wave power extraction by a circular elastic-plate WEC, employing a dispersion relation-based theoretical model ([Meylan et al., 2017](#); [Zheng et al., 2020c](#)).

This paper investigates the role of variable flexural rigidity in the hydrodynamic response of a floating plate typical of that encountered in hybrid wave/solar offshore energy systems. The vast majority of hydrodynamic models consider floater systems with constant properties, thereby narrowing the range of applications. In this paper, we analyse a floating elastic plate connected to the seabed by a series of vertical cables which are characterised by damping and elastic stiffness parameters. Hence, a single cable is capable of simulating the effects of PTO energy absorbers and mooring cables. To study the performance of our proposed flexible plate, we develop a mathematical model based on the free-edge dry mode expansions ([Newman, 1994](#)). This approach shows explicitly the effects of natural modes on plate motion ([Reddy, 2007](#)) and becomes particularly useful when we analyse plate response and energy extraction efficiency. Radiation and diffraction velocity potentials in the fluid domain are solved by matching their respective eigenfunctions at common boundaries ([Yeung, 1981](#)). The plate response is found by determining the complex amplitude of each natural mode. We show that plate flexural rigidity has a positive effect on power extraction efficiency because of the shift in bending modes towards lower frequencies. Conversely, rigid plates are much less efficient than flexible devices, and characterised by smaller efficiency bandwidth and lower resonant frequencies. We also analyse the effects of several discrete and continuous PTO distributions (such as piezoelectric systems) and variable plate lengths, and show that these can have both constructive and destructive impacts on power extraction. We finally investigate the effect of plate flexural rigidity on mooring displacements and related elastic forces. Our results indicate that the distribution of plate flexural rigidity has a significant effect on plate motion and so should be considered during the design of ORE platforms, such as hybrid floating solar farms.

2. Mathematical model

[Fig. 1](#) depicts a two-dimensional floating elastic plate of length $2L$ in open sea of constant depth h . A Cartesian reference system is defined with x -axis coincident with the undisturbed free-surface level and z -axis pointing vertically upward from still water level. The plate is connected to the seabed through vertical cables located at $x = x_i, i = 1, \dots, I$. Each cable has constant stiffness, k_{PTO} , and damping coefficient, v_{PTO} . The system may be treated as continuous when there are large numbers of discrete cables present, as discussed later in [Section 2.3](#). We assume the thickness of the plate to be much smaller than its length (thin-plate approximation), in which case the elastic vibration of the floater can be described by the following Euler–Bernoulli dynamic equation ([Reddy, 2007](#))

$$\partial_{xx}(D\partial_{xx}W) = q - \mu\partial_{tt}W, \quad x \in [-L, L], \quad (1)$$

where W is the plate vertical displacement, t denotes time, q is the transverse distributed load (positive in the z -direction), $D(x) = E(x)I$ is the spatially dependent flexural rigidity, I is the constant second moment of area, E is the Young’s modulus of the plate material, and μ is the constant mass per unit length of the plate. We point out that the assumptions above are valid for uniform plates of constant thickness and constant density but variable Young’s modulus; therefore they are suitable for describing a subclass of problems concerning plates of variable thickness, e.g., the case analysed by [Meylan et al. \(2021\)](#).

In defining the linearised hydrodynamic problem, inviscid fluid and irrotational flow are assumed, such that the velocity potential $\Phi(x, z, t)$ satisfies Laplace's equation in the fluid domain $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where Ω_1 is the subdomain for $x < -L$, Ω_2 is the subdomain below the plate confined by the surfaces $x = \pm L$, $z \in [-h, 0]$, and Ω_3 is the subdomain for $x > L$. At the free surface, we apply the following kinematic and mixed boundary conditions,

$$\partial_t \zeta = \partial_z \Phi, \quad \partial_{tt} \Phi + g \partial_z \Phi = 0, \quad z = 0, \quad |x| > L, \quad (2)$$

where ζ is free-surface elevation and $g = 9.81 \text{ m s}^{-2}$ is acceleration due to gravity. In addition, we require the fluid velocity to be tangential to the seabed, such that

$$\partial_z \Phi = 0, \quad z = -h. \quad (3)$$

Let us now assume the plate draft is much smaller than the plate length (Watanabe et al., 2004). This way the kinematic boundary condition at the wetted surface of the plate can be Taylor-expanded about $z = 0$, and effects such as dynamic wave forces on the vertical lateral plate walls in $x = \pm L$, and surge motion, become higher order contributions that can be neglected. As a consequence, at the leading order we obtain

$$\partial_z \Phi = \partial_t W, \quad z = 0, \quad x \in [-L, L]. \quad (4)$$

Let us neglect transient effects and assume the plate to be forced by monochromatic incident waves of frequency ω . The following harmonic expansion applies

$$\{\Phi, \zeta, W\} = \text{Re} \{(\phi, \eta, w)e^{-i\omega t}\}, \quad (5)$$

in which i is the imaginary unit. We are now in a position to write the governing equations in terms of spatial variables solely as follows:

$$\nabla^2 \phi = 0, \quad \text{in } \Omega, \quad (6)$$

$$\partial_z \phi = -i\omega \eta, \quad z = 0, \quad |x| > L, \quad (7)$$

$$\partial_z \phi = \frac{\omega^2}{g} \phi, \quad z = 0, \quad |x| > L, \quad (8)$$

$$\partial_z \phi = -i\omega w, \quad z = 0, \quad x \in [-L, L], \quad (9)$$

$$\partial_z \phi = 0 \quad z = -h. \quad (10)$$

Following Newman (1994) and Michele et al. (2020), we decompose the displacement of the plate into a set of rigid (heave and pitch) and elastic dry bending modes, i.e., in the absence of surrounding fluid. Hence,

$$w = \zeta_h w_h + \zeta_p w_p + \sum_{l=0}^{\infty} \zeta_l w_l, \quad (11)$$

where ζ_α represents the complex amplitude of each modal shape w_α , and the heave and pitch modal shapes are simply

$$w_h = 1, \quad w_p = x. \quad (12)$$

The elastic dry mode eigenfunctions w_l are given by the solution of the following boundary value problem (Reddy, 2007)

$$(Dw_l'')'' - \mu\omega_l^2 w_l = 0, \quad x \in (-L, L), \quad (13)$$

$$w_l'' = 0, \quad x = \pm L, \quad (14)$$

$$w_l''' = 0, \quad x = \pm L, \quad (15)$$

where primes indicate derivation with respect x , ω_l is the eigenfrequency of the l th dry bending mode, and conditions (14)–(15) represent zero moment and shear stress at the edges $x = \pm L$, respectively. The dry mode expansion approach facilitates an improved understanding of the resonance of natural modes and optimisation of power extraction efficiency. However, the fourth-order differential Eq. (13) has variable coefficients, and so is considerably more difficult to solve than the dynamic equation for homogeneous plates analysed by Michele et al. (2020). Closed-form solutions can be sought for a limited number of cases, such as plates of linearly varying or stepped flexural rigidity (Maurini et al., 2006; Korobkin et al., 2023). Alternatively, if D is slowly varying along the plate, then a weakly nonlinear perturbation approach similar to that by Michele and Renzi (2019) can be also used to find an analytical, or semi-analytical solution. This will highlight the effect of variable rigidity at higher orders whereas linear theory at leading order will be essentially the same as developed here.

This paper focuses on the solution of (1) for stepped variations in D . Natural modes are found using a matching method technique, and Fourier series expansion is used later to solve the hydrodynamic problem. We assume the following spatial dependence

$$D = \sum_{m=1}^M D_m H[x - x_{m-1}] H[x_m - x], \quad (16)$$

where H is the Heaviside step function, $D_m = E_m I$ is constant flexural rigidity along the m th plate segment, M is the total number of plate segments, and $x_0 = -L$ and $x_M = L$ define the edges of the plate. When $M = 2$ we obtain a problem similar to that analysed

by Kabakhpasheva and Korobkin (2002), i.e. a compound floating plate consisting of two segments with a connector. The boundary value problem (13)–(15) can then be expanded as follows

$$D_m w_{lm}'''' = \mu \omega_l^2 w_{lm}, \quad x \in (x_{m-1}, x_m), \quad m = 1, \dots, M, \quad (17)$$

$$w_{lm} = w_{lm+1}, \quad x = x_m, \quad m = 1, \dots, M-1, \quad (18)$$

$$w'_{lm} = w'_{lm+1}, \quad x = x_m, \quad m = 1, \dots, M-1, \quad (19)$$

$$E_m w''_{lm} = E_{m+1} w''_{lm+1}, \quad x = x_m, \quad m = 1, \dots, M-1, \quad (20)$$

$$E_m w'''_{lm} = E_{m+1} w'''_{lm+1}, \quad x = x_m, \quad m = 1, \dots, M-1, \quad (21)$$

$$w''_{l0} = 0, \quad x = x_0 = -L, \quad (22)$$

$$w'''_{l0} = 0, \quad x = x_0 = -L, \quad (23)$$

$$w''_{lM} = 0, \quad x = x_{M+1} = L, \quad (24)$$

$$w'''_{lM} = 0, \quad x = x_{M+1} = L, \quad (25)$$

where (17) is the displacement equation for the m th plate, and (18)–(21) represent continuity of displacement, rotation, moment, and shear between the m th and $(m+1)$ th segments. The general solution of (17) is given by Reddy (2007)

$$w_{lm} = a_{lm} \sinh [\sqrt{\omega_l} \beta_m x] + b_{lm} \cosh [\sqrt{\omega_l} \beta_m x] + c_{lm} \sin [\sqrt{\omega_l} \beta_m x] + d_{lm} \cos [\sqrt{\omega_l} \beta_m x], \quad (26)$$

where $\beta_m = (\mu/D_m)^{1/4}$. Substitution of the above into the boundary and matching conditions (18)–(25) yields the following homogeneous system in terms of real constants a_{lm} , b_{lm} , c_{lm} and d_{lm} , $m = 1, \dots, M$,

$$\mathbf{K}(\omega_l) \{c\}^T = 0, \quad (27)$$

where

$$\{c\} = \{a_{l1}, b_{l1}, c_{l1}, d_{l1}, \dots, a_{lm}, b_{lm}, c_{lm}, d_{lm}, \dots, a_{lM}, b_{lM}, c_{lM}, d_{lM}\}, \quad (28)$$

and $\mathbf{K}$ is a global stiffness matrix of dimension $4M \times 4M$. Eq. (27) can be used to determine the natural frequencies and mode shapes of a stepped plate with free edges (see Appendix). Specifically, ω_l can be found by solving the transcendental equation

$$\det \{\mathbf{K}(\omega_l)\} = 0, \quad (29)$$

whereas the natural mode shapes are given by the corresponding eigenvectors. The natural modes satisfy the orthogonality property, i.e.,

$$\int_{-L}^L w_l w_p dx \neq 0, \quad p = l, \quad (30)$$

$$\int_{-L}^L w_l w_p dx = 0, \quad p \neq l. \quad (31)$$

This property proves advantageous in solving the dynamic Eq. (1), as shown later. The roots of (29) have to be solved numerically because an explicit analytical solution does not exist. For example, in the simplest case of constant D we obtain the well-known eigenvalue condition for a homogeneous plate of length $2L$

$$\cosh [2L\sqrt{\omega_l}\beta_m] \cos [2L\sqrt{\omega_l}\beta_m] = 1. \quad (32)$$

Eq. (32), even though it appears simple, has to be solved numerically. The first four roots are

$$2L\sqrt{\omega_1}\beta_m = 1.50562\pi, \quad 2L\sqrt{\omega_2}\beta_m = 2.49975\pi, \quad (33)$$

$$2L\sqrt{\omega_3}\beta_m = 3.50001\pi, \quad 2L\sqrt{\omega_4}\beta_m = 4.50000\pi, \quad (34)$$

and the symmetric and antisymmetric mode shapes are given by

$$w_l = \cosh [(x+L)\sqrt{\omega_l}\beta_m] + \cos [(x+L)\sqrt{\omega_l}\beta_m] - \frac{(\sinh [(x+L)\sqrt{\omega_l}\beta_m] + \sin [(x+L)\sqrt{\omega_l}\beta_m]) (\cos [2L\sqrt{\omega_l}\beta_m] - \cosh [2L\sqrt{\omega_l}\beta_m])}{\sin [2L\sqrt{\omega_l}\beta_m] - \sinh [2L\sqrt{\omega_l}\beta_m]}. \quad (35)$$

Having completed the modal analysis we are now in a position to solve the hydrodynamic problem. The velocity potential ϕ is expanded in diffraction and radiation components (Mei et al., 2005) giving

$$\phi = \phi_D + \phi_R, \quad \phi_R = \zeta_h \phi_h + \zeta_p \phi_p + \sum_{l=0}^{\infty} \zeta_l \phi_l, \quad \phi_D = \phi_I + \phi_S, \quad (36)$$

where

$$\phi_I = -\frac{iAg}{\omega} \frac{\cosh[k_0(h+z)]}{\cosh[k_0h]} e^{-ik_0x}, \quad \text{in } \Omega_3, \quad (37)$$

is the velocity potential of monochromatic waves propagating in the negative direction along the x -axis, A is wave amplitude, k_0 is wavenumber satisfying the real root of the dispersion relation

$$\omega^2 = gk_0 \tanh(k_0h), \quad (38)$$

ϕ_S is scattering velocity potential, ϕ_D is diffraction velocity potential, ϕ_R is radiation velocity potential, ϕ_h is heaving velocity potential, ϕ_p is pitching velocity potential, and ϕ_l is the radiation velocity potential related to the l th bending mode.

We now derive the boundary value problem for each of the velocity potentials in the entire fluid domain. Let $\phi_D^{(2)}$ ($\phi_R^{(2)}$) be the diffraction (radiation) velocity potential in the fluid subdomain below the plate Ω_2 , $\phi_D^{(1)}$ ($\phi_R^{(1)}$) be the diffraction (radiation) velocity potential in the external subdomain Ω_1 , and $\phi_D^{(3)}$ ($\phi_R^{(3)}$) be the diffraction (radiation) velocity potential in Ω_3 . The corresponding boundary value problem within the fluid subdomains Ω_1 and Ω_3 is given by

$$\nabla^2 \phi_{D,R}^{(1,3)} = 0, \quad \text{in } \Omega_{1,3}, \quad (39)$$

$$g\partial_z \phi_{D,R}^{(1,3)} - \omega^2 \phi_{D,R}^{(1,3)} = 0, \quad z = 0, \quad |x| > L, \quad (40)$$

$$\partial_z \phi_{D,R}^{(1,3)} = 0, \quad z = -h. \quad (41)$$

Similarly, the boundary value problem within the subdomain Ω_2 is

$$\nabla^2 \phi_{D,R}^{(2)} = 0, \quad \text{in } \Omega_2, \quad (42)$$

$$\partial_z \phi_D^{(2)} = 0, \quad z = 0, \quad (43)$$

$$\partial_z \phi_R^{(2)} = -i\omega w, \quad z = 0, \quad (44)$$

$$\partial_z \phi_{D,R}^{(2)} = 0, \quad z = -h. \quad (45)$$

The following matching conditions are required at the common boundaries,

$$\phi_{D,R}^{(3)} = \phi_{D,R}^{(2)}, \quad \text{on } x = L, \quad (46)$$

$$\partial_x \phi_{D,R}^{(3)} = \partial_x \phi_{D,R}^{(2)}, \quad \text{on } x = L, \quad (47)$$

$$\phi_{D,R}^{(1)} = \phi_{D,R}^{(2)}, \quad \text{on } x = -L, \quad (48)$$

$$\partial_x \phi_{D,R}^{(1)} = \partial_x \phi_{D,R}^{(2)}, \quad \text{on } x = -L, \quad (49)$$

representing continuity of fluid pressure and velocity, respectively.

2.1. Diffraction velocity potential

A solution for the diffraction velocity potential throughout the entire fluid domain can be found by adopting a similar procedure to Michele et al. (2020). In subdomains Ω_3 and Ω_1 we get

$$\phi_D^{(3)} = C_0 e^{ik_0x} \cosh[k_0(h+z)] + \sum_{j=1}^{\infty} C_j e^{-k_jx} \cos[k_j(h+z)], \quad (50)$$

and

$$\phi_D^{(1)} = D_0 e^{-ik_0x} \cosh[k_0(h+z)] + \sum_{j=1}^{\infty} D_j e^{k_jx} \cos[k_j(h+z)], \quad (51)$$

where the first and second terms in (50)–(51) represent progressive and evanescent wave components, C_j and D_j are unknown complex constants, and k_j is the j th root of the dispersion relation given by Mei et al. (2005)

$$\omega^2 = -gk_j \tan(k_jh), \quad j = 1, \dots, \infty. \quad (52)$$

The solution in the fluid subdomain Ω_2 is slightly different from the above, and reads

$$\phi_D^{(2)} = \mathcal{A}_0 + B_0x + \sum_{j=1}^{\infty} \cos\left(\frac{j\pi z}{h}\right) \left[\mathcal{A}_j \cosh\left(\frac{j\pi x}{h}\right) + B_j \sinh\left(\frac{j\pi x}{h}\right) \right], \quad (53)$$

where $\mathcal{A}_j$ and B_j are unknown complex constants. Substituting the above velocity potentials into the matching conditions (46)–(49) and integrating over $z \in [-h, 0]$ in $x = \pm L$, yields an inhomogeneous linear system in the complex constants $\mathcal{A}_j$, B_j , C_j , and D_j which is solved numerically following Linton and McIver (2001) and Michele et al. (2020), but not reported here for brevity.

2.2. Radiation velocity potential

It is more difficult to solve for the radiation velocity potentials because the plate motion depends on a combination of rigid and bending modes. Furthermore, we expect the elastic displacement to differ within each plate segment due to the variable flexural rigidity of the plate. In subdomain $\Omega_{1,3}$ the general solution for each radiation velocity potential α is similar to (50)–(51), and may be written

$$\phi_{\alpha}^{(3)} = \mathcal{E}_{\alpha 0} e^{ik_0 x} \cosh[k_j (h + z)] + \sum_{j=1}^{\infty} \mathcal{E}_{\alpha j} e^{-k_j x} \cos[k_j (h + z)], \quad (54)$$

and

$$\phi_{\alpha}^{(1)} = \mathcal{F}_{\alpha 0} e^{-ik_j x} \cosh[k_j (h + z)] + \sum_{j=1}^{\infty} \mathcal{F}_{\alpha j} e^{k_j x} \cos[k_j (h + z)], \quad (55)$$

where $\mathcal{E}_{\alpha j}$ and $\mathcal{F}_{\alpha j}$ are unknown complex constants, and subscript α refers to the heave mode, pitching mode, or l th bending elastic mode. The radiation velocity potential solution in the fluid subdomain Ω_2 is given by

$$\phi_{\alpha}^{(2)} = \tilde{\phi}_{\alpha} + \mathcal{G}_{\alpha 0} + \mathcal{H}_{\alpha 0} x + \sum_{j=1}^{\infty} \cos\left(\frac{j\pi z}{h}\right) \left[\mathcal{G}_{\alpha j} \cosh\left(\frac{j\pi x}{h}\right) + \mathcal{H}_{\alpha j} \sinh\left(\frac{j\pi x}{h}\right) \right], \quad (56)$$

where $\mathcal{G}_{\alpha j}$ and $\mathcal{H}_{\alpha j}$ are unknown complex constants, $\tilde{\phi}_{\alpha}$ is a particular solution that accounts for the plate vibration in $z = 0$, $x \in [-L, L]$, and the remaining terms on the r.h.s. denote the velocity potential satisfying the homogeneous boundary condition on the plate $\partial_z w = 0$. The particular solution for the rigid heave mode is (Linton and McIver, 2001)

$$\tilde{\phi}_h = -\frac{i\omega}{2h} [z^2 - x^2 + 2zh], \quad (57)$$

and that for the pitching mode is (Drimer et al., 1992)

$$\tilde{\phi}_p = \frac{i\omega x}{6h} [x^2 - 3z^2 - 6zh]. \quad (58)$$

It is challenging to find a suitable particular solution for $\tilde{\phi}_l$ that satisfies the boundary condition (44) for each one of the bending modes, i.e.,

$$\partial_z \tilde{\phi}_l = -i\omega w_l, \quad z = 0, x \in [-L, L]. \quad (59)$$

In principle we can match the velocity potentials below each plate segment; however, this would lead to discontinuous behaviour of the particular solution in $x = x_m$, $m = 1, \dots, M - 1$, and require complicated algebraic manipulation and numerical computations. We propose a much simpler approach based on Fourier series expansion of natural shapes. We point out that a similar method was originally developed by Ohmatsu (1997, 2000) in the context of VLFS. This procedure enables the particular velocity potential solution to remain continuous throughout the entire subdomain Ω_2 without requiring matching below the plate. We assume the following expansion for the l th bending mode

$$w_l(x) = \frac{w_l|_{x=L} + w_l|_{x=-L}}{2} + x \frac{w_l|_{x=L} - w_l|_{x=-L}}{2L} + f_l(x), \quad (60)$$

where the first two terms in (60) represent heave and pitch motion components, and the last term represents a deformed shape with zero displacement and nonzero derivative in $x = \pm L$. This decomposition enables us to write f_l in terms of Fourier sine series

$$f_l = \sum_{n=0}^{\infty} a_{nl} \sin\left[\frac{n\pi(x+L)}{2L}\right], \quad (61)$$

where

$$a_{nl} = \frac{1}{L} \int_{-L}^L f_l \sin\left[\frac{n\pi(x+L)}{2L}\right] dx, \quad (62)$$

are Fourier coefficients. Using separation of variables, the corresponding velocity potential satisfying Laplace equation, boundary conditions (59)–(60), and zero-flux at the seabed, is given by

$$\begin{aligned} \tilde{\phi}_l = & \tilde{\phi}_h \frac{w_l|_{x=L} + w_l|_{x=-L}}{2} + \tilde{\phi}_p \frac{w_l|_{x=L} - w_l|_{x=-L}}{2L} \\ & - \frac{2iL\omega}{\pi} \sum_{n=0}^{\infty} \frac{a_{nl}}{n} \sin\left[\frac{n\pi(x+L)}{2L}\right] \frac{\cosh\left[\frac{n\pi(h+z)}{2L}\right]}{\sinh\left[\frac{n\pi h}{2L}\right]}, \end{aligned} \quad (63)$$

where $\tilde{\phi}_h$, $\tilde{\phi}_p$ are given by (57) and (58), respectively. A further advantage of this approach is that the x -dependence of the last term in (63) coincides with the sine series in (61). The foregoing method is completely general and can be extended to any free surface deformation provided the expansion is of the form (60).

Let us apply the orthogonality property (31) to the decomposition (60). Since the elastic modes are orthogonal to rigid heaving and pitching modes, we get the following properties

$$-\frac{1}{2L} \int_{-L}^L f_l dx = \frac{w_l|_{x=L} + w_l|_{x=-L}}{2}, \quad (64)$$

$$-\frac{3}{2L^3} \int_{-L}^L x f_l dx = \frac{w_l|_{x=L} - w_l|_{x=-L}}{2L}, \quad (65)$$

which relate the values of mode displacement at the edges $w_l|_{x=\pm L}$ to the shape f_l . For example, in the simple case of even modes we obtain

$$\int_{-L}^L f_l dx = -2L, \quad \int_{-L}^L x f_l dx = 0, \quad (66)$$

i.e. f_l has average value equal to -1 . If w_l is an odd elastic mode we obtain

$$\int_{-L}^L f_l dx = 0, \quad \int_{-L}^L x f_l dx = -\frac{2}{3L^2}. \quad (67)$$

We finally point out that decomposition (60) could be substituted into the original boundary value problem (13)–(15). Use of the orthogonality property of the sine eigenfunctions would give a homogeneous linear system in terms of a_{nl} , $w_l|_{x=L}$ and $w_l|_{x=-L}$, which can be numerically solved for the eigenfrequencies and natural mode shapes. In this paper we consider stepped variations of flexural rigidity and so prefer to continue to use the general solution in the form (26) because it has been well established that it is easier to manage. However, Fourier series decomposition (60) appears to be a promising method for more complicated variations of D , and possible applications will be considered in future works.

2.3. Dynamic response

The dynamic Eq. (1) in presence of all external forces is

$$\partial_{xx} (EI \partial_{xx} W) + \mu \partial_{tt} W = -\rho \partial_t \Phi - \rho g W - \sum_{i=1}^I (v_{PTO} \partial_t W + k_{PTO} W) \delta(x - x_i), \quad (68)$$

where ρ is fluid density. The first term on the r.h.s. represents the dynamic pressure exerted by the wave field, the second term represents the hydrostatic pressure due to plate displacements, and the third term denotes the stiffening and damping effects of localised forces due to the PTO and mooring system, with δ denoting the Dirac Delta function. By using both harmonic expansion (5) and dry mode decomposition (11), we obtain after some manipulation,

$$\begin{aligned} & \left(\zeta_h w_h + \zeta_p w_p + \sum_{l=0}^{\infty} \zeta_l w_l \right) \left[\rho g - \mu \omega^2 + \sum_{i=1}^I (-i\omega v_{PTO} + k_{PTO}) \delta(x - x_i) \right] \\ & + \sum_{l=0}^{\infty} \mu \omega_l w_l - i\omega \rho (\phi_R^{(2)} + \phi_D^{(2)}) = 0. \end{aligned} \quad (69)$$

The above equation is valid for any finite number of external springs and dampers. For very large I , the PTO system can be treated continuous in a similar fashion to the case of Winkler foundations (Reddy, 2007), such that

$$\begin{aligned} & \left(\zeta_h w_h + \zeta_p w_p + \sum_{l=0}^{\infty} \zeta_l w_l \right) (\rho g - \mu \omega^2 - i\omega v_{PTO} + k_{PTO}) \\ & + \sum_{l=0}^{\infty} \mu \omega_l w_l - i\omega \rho (\phi_R^{(2)} + \phi_D^{(2)}) = 0. \end{aligned} \quad (70)$$

The complex modal amplitudes ζ_h , ζ_p and ζ_l are found by multiplying both sides of (69) (or (70) depending on the problem of interest) by each of the modal shape functions w_a and then integrating over the wetted surface of the plate $x \in [-L, L]$. For additional details of the numerical procedure please refer to Michele et al. (2020, 2022, 2023) and Newman (1994). The resulting linear system is then written in matrix form as

$$[\mathbf{S} - \omega^2 (\mathbf{M} + \mathbf{I}) - i\omega (\mathbf{C} + v_{PTO} \mathbf{D})] \{\zeta_a\}^T = \{F\}^T, \quad (71)$$

where $\mathbf{S}$, $\mathbf{M}$, $\mathbf{I}$, $\mathbf{C}$ and $\mathbf{D}$ are the generalised stiffness matrix, mass matrix, added mass matrix, radiation damping matrix and PTO damping matrix, $\{F\}$ is the vector of exciting force components, and $\{\zeta_a\} = \{\zeta_h, \zeta_p, \zeta_1, \dots\}$ is the vector of unknown modal amplitudes. As in the case of arrays of WECs, the structure of (71) suggests that the continuous floating plate is equivalent to a system of linear coupled forced harmonic oscillators (Michele et al., 2016b). Consequently, the natural modes of the plate coupled with the surrounding fluid and mooring system are evaluated by equating to zero the determinant of the unforced and undamped system, i.e., Michele et al. (2016a)

$$\det \{\mathbf{S} - \omega^2 (\mathbf{M} + \mathbf{I})\} = 0. \quad (72)$$

Solution of this transcendental equation gives the numerical value of each eigenfrequency. The eigenvectors correspond to the natural mode shapes affected by the added mass of the surrounding fluid and mooring stiffness coefficient k_{PTO} .

2.4. Wave power extraction

The average power generated by the plate is simply

$$P = \frac{1}{2} \nu_{PTO} \omega^2 \sum_{i=1}^I \left| w_h \zeta_h + w_p \zeta_p + \sum_{l=0}^{\infty} w_l \zeta_l \right|_{x=x_i}^2. \quad (73)$$

Similarly, the generated power in the case of idealised continuous PTO system may be expressed

$$P = \frac{1}{2} \nu_{PTO} \omega^2 \int_{-L}^L \left| w_h \zeta_h + w_p \zeta_p + \sum_{l=0}^{\infty} w_l \zeta_l \right|^2 dx. \quad (74)$$

The system efficiency is given by the capture-width ratio (Mei et al., 2005)

$$C_W = \frac{P}{\frac{1}{2} \rho g A^2 C_g}, \quad (75)$$

where the denominator is energy flux, and C_g is group velocity defined in linear wave theory as

$$C_g = \frac{\omega}{2k_0} \left[1 + \frac{2k_0 h}{\sinh(2k_0 h)} \right]. \quad (76)$$

3. Hydrodynamic relations and maximum energy efficiency

The Haskind–Hanaoka formula valid for two-dimensional domains can be used to check numerical computations of diffraction and radiation velocity potentials. Its expression reads (Mei et al., 2005)

$$F_\alpha = -2i\rho g C_g A A_\alpha^+ \omega^{-1}, \quad (77)$$

where the term on the left hand side represents the exciting force given by

$$F_\alpha = i\omega\rho \int_{-L}^L w_\alpha \phi_D^{(2)} dx, \quad (78)$$

whereas A_α^+ is the amplitude of radiated progressive waves, for unit modal amplitude and in the direction opposite to that of the incident waves, i.e.

$$A_\alpha^+ = i\omega \cosh(k_0 h) \mathcal{E}_{\alpha 0} g^{-1}, \quad (79)$$

in which $\mathcal{E}_{\alpha 0}$ is the first complex coefficient of the radiation potential (54). Similarly, by applying Green's theorem we obtain the relation between radiation damping and radiated wave amplitude

$$\lambda_{\alpha\beta} = \rho g C_g \omega^{-2} \left(A_\alpha^+ A_\beta^{+*} + A_\alpha^- A_\beta^{-*} \right), \quad (80)$$

where subscript β represents the generic β th mode, A^- denotes the amplitude of radiated progressive waves in the domain Ω_1 , $(\cdot)^*$ indicates the complex conjugate of $(\cdot)$, and

$$\lambda_{\alpha\beta} = -\omega^{-1} \text{Im} \left\{ \int_{-L}^L w_\beta \phi_\alpha^{(2)} dx \right\}, \quad (81)$$

is the radiation damping coefficient. From (77) we can derive the most general expression for the capture-width ratio C_W of two-dimensional flexible floaters expressed in terms of radiation velocity potentials (Mei et al., 2005)

$$C_W = \text{Re} \left\{ 2 \sum_{\alpha=1}^{\infty} A_\alpha^+ \zeta_\alpha^* - \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \zeta_\alpha \zeta_\beta^* \left(A_\alpha^+ A_\beta^{+*} + A_\alpha^- A_\beta^{-*} \right) \right\}. \quad (82)$$

If plate motion is characterised by one symmetric or anti-symmetric mode, expressions (80) and (82) simplify into

$$\lambda_{\alpha\alpha} = 2\rho g C_g \omega^{-2} |A_\alpha^+|^2, \quad (83)$$

$$C_W = 2\text{Re} \{ A_\alpha^+ \zeta_\alpha^* \} - 2 |\zeta_\alpha A_\alpha^+|^2. \quad (84)$$

C_W results are maximised for $\text{Re} \{ A_\alpha^+ \zeta_\alpha^* \} = |\zeta_\alpha A_\alpha^+| = 1/2$. In this case we obtain the well-known maximum value $C_W = 0.5$. The same result can be also derived by solving the equation of motion (69) or (70) and assuming the resonance condition and that PTO damping is equal to $\lambda_{\alpha\alpha}$. For a discrete number of PTO cables the condition is

$$\lambda_{\alpha\alpha} = \nu_{PTO} \sum_{i=1}^I w_\alpha(x_i)^2, \quad (85)$$

whereas for continuous PTO system the condition becomes

$$\lambda_{\alpha\alpha} = \nu_{PTO} \sum_{i=1}^I \int_{-L}^L w_\alpha^2 dx. \quad (86)$$

Table 1
First five eigenfrequency values (in rad/s) for plate lengths $L = [5; 10; 15; 20]$ m.

Plate length	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$L = 5$ (m)	1.64	2.32	3.68	6.72	12.52
$L = 10$ (m)	1.14	1.60	2.21	2.73	3.62
$L = 15$ (m)	0.87	1.28	1.73	2.05	2.47
$L = 20$ (m)	0.70	1.08	1.48	1.72	2.02

We obtain (Mei et al., 2005)

$$C_W = \frac{|F_a|^2}{8EC_g \lambda_{aa}}, \quad (87)$$

which after substitution of (77) and (83) yields

$$C_W = \frac{|A_a^+|^2}{|A_a^+|^2 + |A_a^-|^2}. \quad (88)$$

This expression has a maximum that occurs when the amplitude of the radiated wave directed opposite to the incident wave is maximised. For symmetrical bodies $|A_a^+| = |A_a^-|$, and so the maximum capture-width ratio is capped at $C_W = 0.5$. Let us now consider two modes $\alpha = 1, 2$ that dominate the dynamics with respect to the others, one symmetric and the other anti-symmetric. This is representative of the case of a rigid plate characterised by heaving and pitching motion only. Expression (82) becomes

$$C_W = 2\text{Re} \left\{ \sum_{\alpha=1}^2 A_a^+ \zeta_\alpha^* \right\} - 2 \sum_{\alpha=1}^2 |\zeta_\alpha A_a^+|^2, \quad (89)$$

which gives a theoretical upper bound $C_W = 1$ for $\text{Re} \{A_1^+ \zeta_1^*\} = |\zeta_1 A_1^+| = \text{Re} \{A_2^+ \zeta_2^*\} = |\zeta_2 A_1^+| = 1/2$, independent of plate size. We remark that the theoretical maximum of the capture-width ratio for a two-dimensional WEC cannot be larger than 1 because of conservation of energy. Unfortunately, as also pointed out by Noad and Porter (2017), there are no cases that satisfy this exactly because the condition is fulfilled only if both natural modes have the same eigenfrequency. Please note that last result has been obtained using the simple assumption that plate motion is dominated by two modes. However, the flexible floater considered herein also includes rigid and infinite bending elastic mode. This implies that multiple objective optimisation may lead to increased efficiency with respect to standard rigid WECs.

4. Results and discussion

In this section we examine the effects of plate flexural rigidity, plate geometry, PTO, mooring system, and wave frequency on the hydrodynamic behaviour and energy extraction efficiency of the floating plate. For simplicity, let us consider a water-plate system with the following fixed parameters: $A = 1$ m, $h = 10$ m and $\rho = 1000$ kg m⁻³. Given that the expressions for the velocity potentials and elastic modes are infinite series, it is necessary to truncate the summations in order to perform the computations. Herein, we select $j = 20$ evanescent modes and consider the first $l = 12$ dry elastic modes, whereas the sine series (61) is truncated at $n = 100$.

4.1. Effect of plate length

We examine the effect of plate length on the response of the hydroelastic plate and its power extraction efficiency. Four plate lengths, $L = [5; 10; 15; 20]$ m, are considered. Here we assume the same plate properties as used by Michele et al. (2020), i.e., flexural rigidity $D = \bar{D} = 6.9 \times 10^4$ kg m³ s⁻² and $\mu = 44$ kg/m. Furthermore, the stiffness coefficient $k_{PTO} = 0$ and the PTO system comprises seven equally spaced PTO devices located at $x_i = \pm[0, L/3, 2L/3, L]$. Solution of (72) allows us to find the corresponding k eigenfrequencies affected by the added mass. Table 1 lists the first five numerical eigenfrequency values. Fig. 2 shows the surface plot of capture-width ratio (75) as a function of incident wave frequency and PTO coefficient for four different plate lengths. For the smallest plate length, C_W has a single visible peak at 2 rad s⁻¹, not far from the first corresponding eigenfrequencies. This highlights the beneficial effects of resonance of plate motion on power extraction efficiency discussed in Section 3. As the plate lengthens to 10 m, the contours exhibit a slight waver with a low amplitude secondary peak at $\omega \sim 5$ rad s⁻¹. (Fig. 2(b)). With further increase in plate length, additional secondary resonant peaks develop (Figs. 2(c)–2(d)). In general, the peak capture-width ratio increases with plate length, especially for wave frequencies exceeding 3 rad s⁻¹. This is mainly due to the increased number of eigenfrequencies in the range of interest which are located very close to each other. These results suggest that the plate length L plays an important role in power extraction efficiency. Even so, it is important for designers to consider the effect on plate structural resistance that could penalise the overall behaviour of an actual device in the open ocean. Larger plate dimensions undoubtedly incur greater wave loads and may render the device less durable.

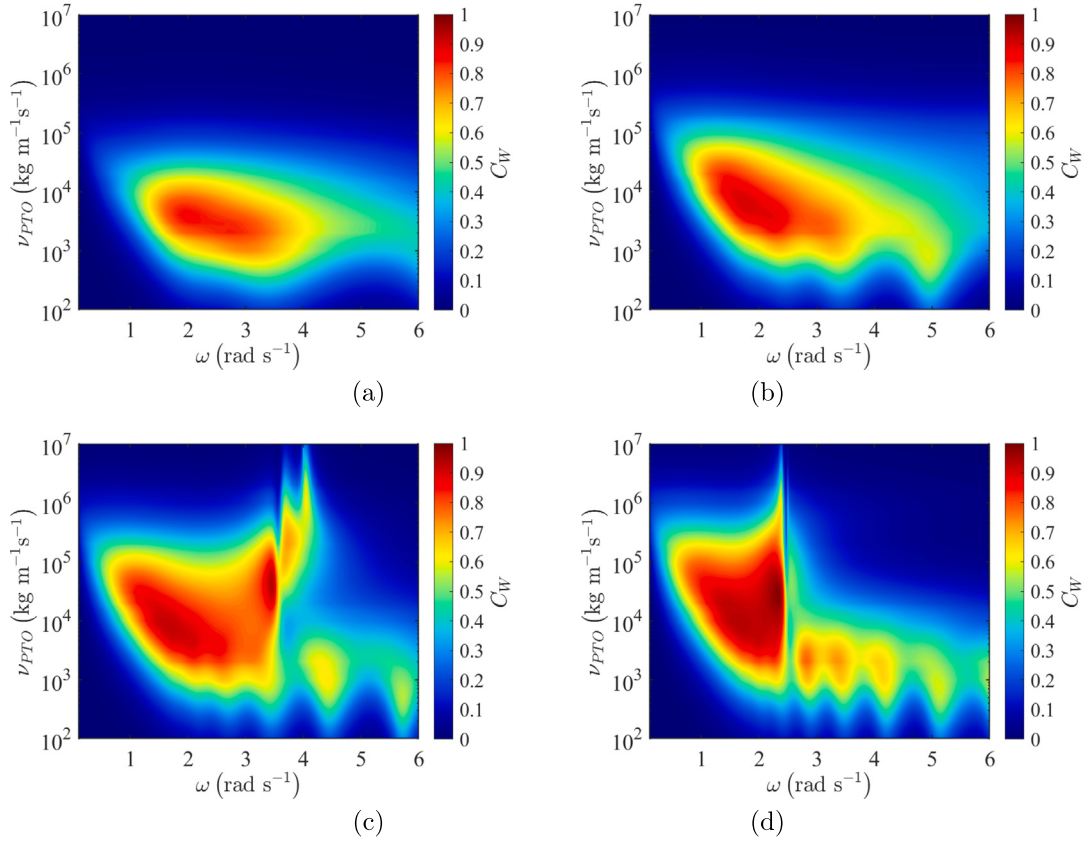


Fig. 2. Capture-width ratio C_W as a function of incident wave frequency ω and PTO coefficient ν_{PTO} for seven equally spaced PTO devices located at $x_i = \pm[0, L/3, 2L/3, L]$ and plate lengths of: (a) $L = 5$ m; (b) $L = 10$ m; (c) $L = 15$ m; and (d) $L = 20$ m.

As already shown in Section 3, radiation damping and the amplitude of radiated waves towards infinity have strong connection with the efficiency of the system. To this end we now discuss the behaviour of $\lambda_{\alpha\alpha}$ and capture-width ratio due to each mode component $C_{W\alpha}$. From (82), the modal capture-width ratio is given by

$$C_{W\alpha} = 2\text{Re} \left\{ A_{\alpha}^{+} \zeta_{\alpha}^{*} - \sum_{\beta=1}^{\infty} \zeta_{\alpha} \zeta_{\beta}^{*} \left(A_{\alpha}^{+} A_{\beta}^{+*} \right) \right\}. \quad (90)$$

Let us first consider the case when $L = 10$ m. Figs. 3(a)–3(c) depict the behaviour of C_W and $C_{W\alpha}$ versus wave frequency, for three values of PTO coefficient $\nu_{PTO} = [10^3, 10^4, 10^5] \text{ kg m}^{-1} \text{ s}^{-1}$. For brevity we only present curves relative to the first eight modes, i.e. heaving ($\alpha = 1$), pitching ($\alpha = 2$) and six bending modes ($\alpha = 3, \dots, 8$). When the PTO coefficient is small (Fig. 3(a)), C_W exhibits wave behaviour that is mainly related to the complex behaviour of rigid and bending modes. The contribution from bending and pitching is comparable with that given by each elastic bending mode. Note that $C_{W\alpha}$ can assume negative values whereas C_W is always positive; this means that certain modes may be more destructive in terms of power extraction efficiency. Figs. 3(b)–3(c) refer to greater $\nu_{PTO} = [10^4, 10^5] \text{ kg m}^{-1} \text{ s}^{-1}$. In these cases most of the efficiency is provided by pitching and heaving motion, especially at small frequency. In other words, $C_W \sim C_{W1} + C_{W2}$. At wave frequencies greater than $\omega \sim 2 \text{ rad s}^{-1}$ appreciate positive effects are likely to arise from the elastic modes. Figs. 3(d)–3(f) refer to a plate of length $L = 20$ m with same ν_{PTO} values as considered before. Fig. 3(d) refers to $\nu_{PTO} = 10^3 \text{ kg m}^{-1} \text{ s}^{-1}$. Again, the C_W exhibits complicated behaviour dominated by oscillation of all modes and the overall efficiency oscillates about $C_W \sim 0.6$ (as also represented in Fig. 2(d)). Fig. 3(e) refers to a larger coefficient such that $\nu_{PTO} = 10^4 \text{ kg m}^{-1} \text{ s}^{-1}$. The dominant effect is given by pitching and heaving motion for $\omega < 1.5 \text{ rad s}^{-1}$, with a sudden drop occurring close to the frequency $\omega = 2.46 \text{ rad s}^{-1}$ at which the discrete PTO cables do not move, i.e. $w(x_i) \sim 0, i = 1, \dots, I$. Hence radiated waves are enhanced and plate efficiency is considerably reduced. This is more evident for larger PTO coefficient, i.e. in the presence of stiffer cables. Fig. 3(f) shows the resulting behaviour where the effect of bending modes appears to be significantly reduced. We now consider analysis of the radiation damping by each mode. Figs. 4(a)–4(b) show the behaviour of normalised radiation damping

$$\tilde{\lambda}_{\alpha\alpha} = \frac{\lambda_{\alpha\alpha}}{\sum_{i=1}^I w_{\alpha}(x_i)^2}, \quad (91)$$

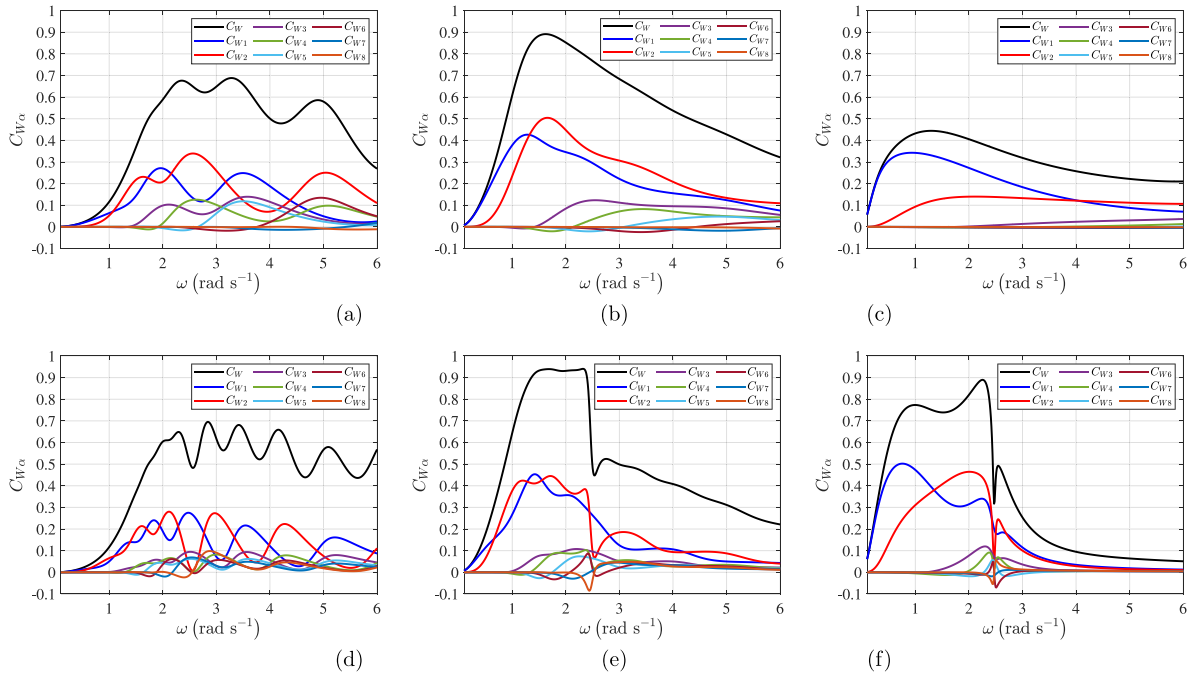


Fig. 3. Capture-width ratio component $C_{W\alpha}$ as a function of incident wave frequency ω , plate length L and PTO coefficient ν_{PTO} for the first eight natural modes: (a) $L = 10$ m, $\nu_{PTO} = 10^3$ kg m $^{-1}$ s $^{-1}$; (b) $L = 10$ m, $\nu_{PTO} = 10^4$ kg m $^{-1}$ s $^{-1}$; (c) $L = 10$ m, $\nu_{PTO} = 10^5$ kg m $^{-1}$ s $^{-1}$; (d) $L = 20$ m, $\nu_{PTO} = 10^3$ kg m $^{-1}$ s $^{-1}$; (e) $L = 20$ m, $\nu_{PTO} = 10^4$ kg m $^{-1}$ s $^{-1}$; and (f) $L = 20$ m, $\nu_{PTO} = 10^5$ kg m $^{-1}$ s $^{-1}$.

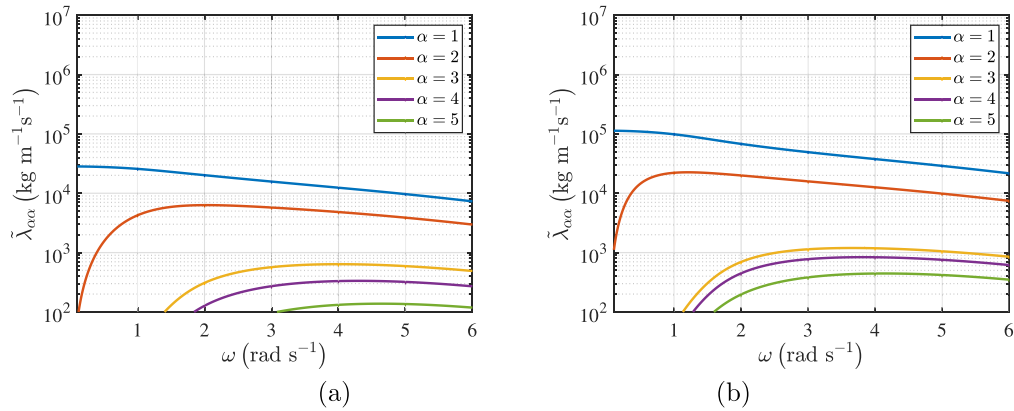


Fig. 4. Normalised radiation damping $\tilde{\lambda}_{\alpha\alpha}$ as a function of incident wave frequency ω and plate length L for heaving ($\alpha = 1$), pitching ($\alpha = 2$) and three bending modes ($\alpha = 3, 4, 5$): (a) $L = 10$ m and (b) $L = 20$ m.

versus frequency for plate lengths $L = [10, 20]$ m. We remark that according to (85), and for one degree of freedom system in resonance conditions, C_W can have a maximum when $\tilde{\lambda}_{\alpha\alpha} = \nu_{PTO}$. For brevity we show the curves associated with the first five modes $\alpha = 1, \dots, 5$, noting that higher modes are characterised by very small radiation damping. By comparing Figs. 2(a) and 4(a) it can be seen that maximum efficiency is attained in the range $10^3 < \nu_{PTO} < 10^4$ kg m $^{-1}$ s $^{-1}$. These results are surprisingly close to the curves $\tilde{\lambda}_{\alpha\alpha}$ for heaving and pitching motion $\alpha = 1, 2$. Therefore even for flexible structures of moderate length, the optimal PTO coefficient is very close to the radiation damping of dominant modes. Fig. 4(b) refers to $L = 20$ m and its analogy with Fig. 2(d) is more difficult to ascertain, primarily because of the increased number of participating bending modes. When the frequency is smaller than ~ 2 rad s $^{-1}$ the optimal PTO damping is $\sim 10^4$ kg m $^{-1}$ s $^{-1}$ which is very close to the peak normalised radiation damping in heaving and pitching motion given in Fig. 4(b). However when the frequency increases we obtain additional several

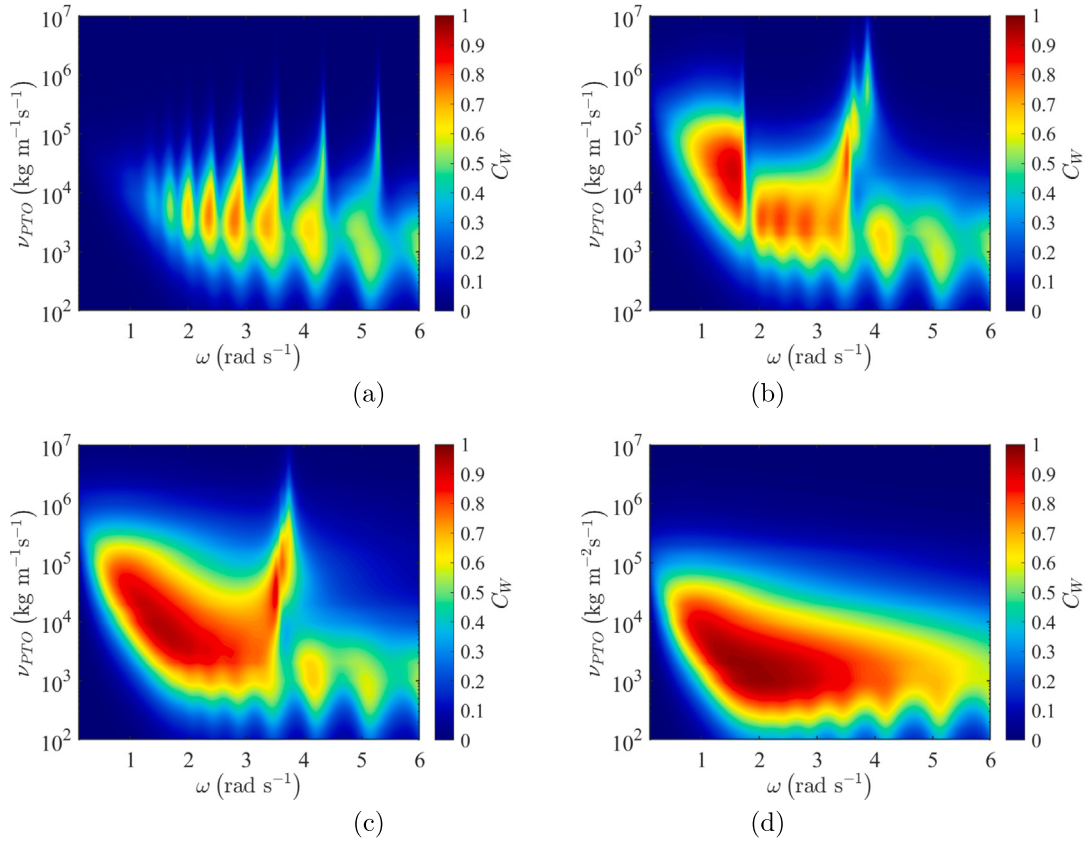


Fig. 5. Capture-width ratio C_W as a function of incident wave frequency ω and PTO coefficient ν_{PTO} for four distributed PTO systems: (a) $x_i = \pm L$; (b) $x_i = \pm[0, L/2, L]$; (c) $x_i = \pm[0, L/4, L/2, 3L/4, L]$; and (d) continuous PTO system.

peaks characterised by large efficiency. In this case $\nu_{PTO} \sim 10^3 \text{ kg m}^{-1} \text{ s}^{-1}$ which is very close to the radiation damping of the bending modes. Indeed, in this range of frequencies the effect of bending modes starts to grow, multiple resonance occurs, and optimal PTO damping falls.

4.2. Effect of PTO system

We now investigate the influences of PTO coefficient ν_{PTO} and PTO distribution on power extraction efficiency for a plate of length $L = 20 \text{ m}$. Again, we assume fixed flexural rigidity $D = \bar{D} = 6.9 \times 10^4 \text{ kg m}^3 \text{ s}^{-2}$, mass per unit length $\mu = 44 \text{ kg/m}$, and $k_{PTO} = 0$. Fig. 5 shows the surface plot of the capture-width ratio C_W as a function of incident wave frequency and PTO coefficient for four different PTO distributions. Fig. 5(a) refers to two PTO devices located at the plate edges $x_i = \pm L$, whereas Figs. 5(b)–5(c) correspond to five and nine equally spaced PTO devices located across the plate. Fig. 5(d) presents results for the idealised case of a continuous system of PTO cables, simulating the effect of a great number of discrete devices beneath the plate. As the number of PTOs increases, the bandwidth of C_W also increases and the system becomes more efficient. This suggests that there could be an optimal distribution of PTO devices, maximising power extraction. From a theoretical perspective, such a distribution should maximise the radiated waves in a direction that opposes the incident waves (Mei et al., 2005; Linton and McIver, 2001). Several peaks of magnitude $C_W \sim 1$ can be discerned. Note that $C_W = 1$ corresponds to the maximum capture-width ratio of a WEC radiating in a single direction only (Mei et al., 2005); hence, plate elasticity and rigid motion act to further increase power extraction. Our finding that multiple-degree-of-freedom WECs are more efficient than rigid devices has also been confirmed for a circular floating elastic plate of constant properties in regular waves (Michele et al., 2022). Note also that the maxima correspond to the first eigenfrequencies of the system. This means that, as in oscillating wave surge converters, oscillating water columns and three-dimensional plates, the resonance of natural modes is beneficial in terms of power extraction efficiency (Michele et al., 2016a,b).

We now focus on the behaviour of $C_{W\alpha}$ for cases involving two PTO cables at $x_i = \pm L$ and continuous PTO distributions (see also related Figs. 5(a) and 5(d)). Figs. 6(a)–6(c) are for cases involving two PTO cables and $\nu_{PTO} = [10^3, 10^4, 10^5] \text{ kg m}^{-1} \text{ s}^{-1}$, respectively. Extraction efficiency peaks occur at similar frequencies, independent of PTO coefficient strength. These peaks correspond to bending mode resonant frequencies, with plate elasticity the dominant factor in enhancing C_W . The contributions from even and odd modes

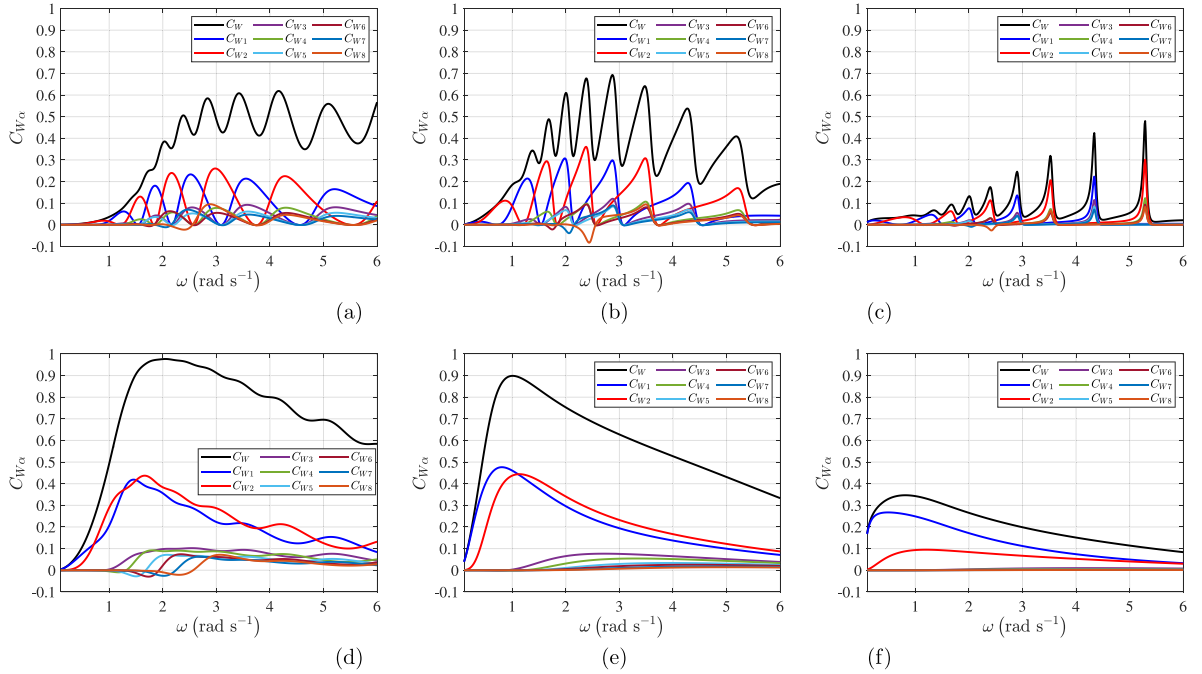


Fig. 6. Capture-width ratio component $C_{W\alpha}$ as a function of incident wave frequency ω , PTO distribution and fixed plate length $L = 20$ m for the first eight natural modes: (a) $x_i = \pm L$, $\nu_{PTO} = 10^3 \text{ kg m}^{-1} \text{ s}^{-1}$; (b) $x_i = \pm L$, $\nu_{PTO} = 10^4 \text{ kg m}^{-1} \text{ s}^{-1}$; (c) $x_i = \pm L$, $\nu_{PTO} = 10^5 \text{ kg m}^{-1} \text{ s}^{-1}$; (d) continuous PTO, $\nu_{PTO} = 10^3 \text{ kg m}^{-2} \text{ s}^{-1}$; (e) continuous PTO, $\nu_{PTO} = 10^4 \text{ kg m}^{-2} \text{ s}^{-1}$; and (f) continuous PTO, $\nu_{PTO} = 10^5 \text{ kg m}^{-2} \text{ s}^{-1}$.

switch between each other beneath each peak. This is confirmed theoretically by (82) which shows that there is no contribution from coupling between even and odd modes because the second term in (82) is identically equal to zero. Figs. 6(d)–6(f) represent the case of continuous PTO distribution and ν_{PTO} as considered before. When $\nu_{PTO} = 10^3 \text{ kg m}^{-1} \text{ s}^{-1}$ (Fig. 6(d)) the efficiency becomes very high at a frequency of about $\omega \sim 2 \text{ rad s}^{-1}$, with C_W close to unity mainly because of the combined effect of rigid and bending modes. Fig. 6(e) shows the capture-width ratio component versus wave frequency for $\nu_{PTO} = 10^4 \text{ kg m}^{-1} \text{ s}^{-1}$, in which case plate elasticity effects start to decrease (and become less relevant than C_{W1} and C_{W2}). Fig. 6(f) shows the behaviour for a very large value of PTO coefficient, with dominant effects activated by heaving modes and in a minor way by pitching at small frequencies. Effects from elastic bending modes are negligible and the overall efficiency attains smaller values than in previous cases.

Given that the PTO system plays a dominant role in the C_W behaviour, we now investigate two additional different configurations, characterised by uneven spacing. For brevity, let us consider the following squared-root distributions of seven PTO cables

$$x_i = L \left(\sqrt{2(i-1)/3} - 1 \right), \quad i = 1, \dots, 7, \quad (92)$$

$$x_i = -L \left(\sqrt{2(i-1)/3} - 1 \right), \quad i = 1, \dots, 7, \quad (93)$$

where (92) and (93) correspond to PTO devices concentrated at the front and back respectively of the plate. Figs. 7(a) and 7(b) present results for seven PTO devices located according to (92) and (93). The significantly different overall behaviour between the two cases is mainly due to amplification of different natural modes. It appears that the PTO system concentrated at the front is less efficient overall because the system maximises wave radiation in the same direction as incident waves, and maximum efficiency occurs when radiated waves propagate towards $x \rightarrow \infty$.

Fig. 8 depicts the frequency dependent behaviour of $C_{W\alpha}$ for fixed $\nu_{PTO} = [10^3, 10^4, 10^5] \text{ kg m}^{-1} \text{ s}^{-1}$ and the two square-root PTO distributions mentioned above. We first consider the distribution (92) and PTO coefficient $\nu_{PTO} = 10^3 \text{ kg m}^{-1} \text{ s}^{-1}$. The behaviour of $C_{W\alpha}$ depicted in Fig. 8(a) is very similar to that in Figs. 6(a) and 3(d), implying that the PTO distribution does not affect overall efficiency when ν_{PTO} assumes small values. For the larger PTO coefficient $\nu_{PTO} = 10^4 \text{ kg m}^{-1} \text{ s}^{-1}$ we obtain the curves represented in Fig. 8(b) where it is possible to observe that significant hydroelastic contributions occur at $\omega > 2 \text{ rad s}^{-1}$. The maximum efficiency is located at about $\omega \sim 1.5 \text{ rad s}^{-1}$, primarily due to optimised heaving and pitching motion. Fig. 8(c) shows the corresponding results for $\nu_{PTO} = 10^5 \text{ kg m}^{-1} \text{ s}^{-1}$; here the main contribution over the entire range of frequencies is provided by heaving and pitching modes. Figs. 8(d)–8(f) present results for PTO devices concentrated at the back at locations described by (93). For the

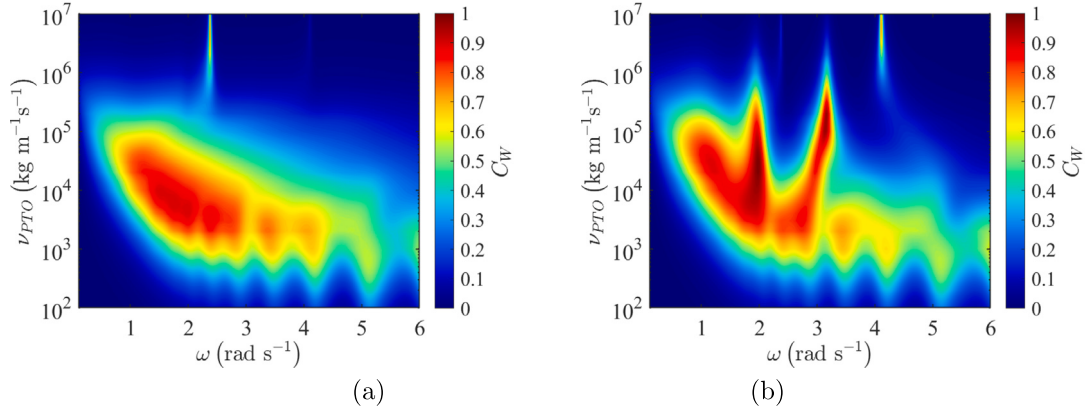


Fig. 7. Capture-width ratio C_W as a function of incident wave frequency ω and PTO coefficient ν_{PTO} for a plate of length $L = 20$ m and two squared-root PTO distributions: (a) $x_i = -L \left(\sqrt{2(i-1)/3} - 1 \right)$; and (b) $x_i = -L \left(\sqrt{2(i-1)/3} - 1 \right)$ where $i = 1, \dots, 7$.

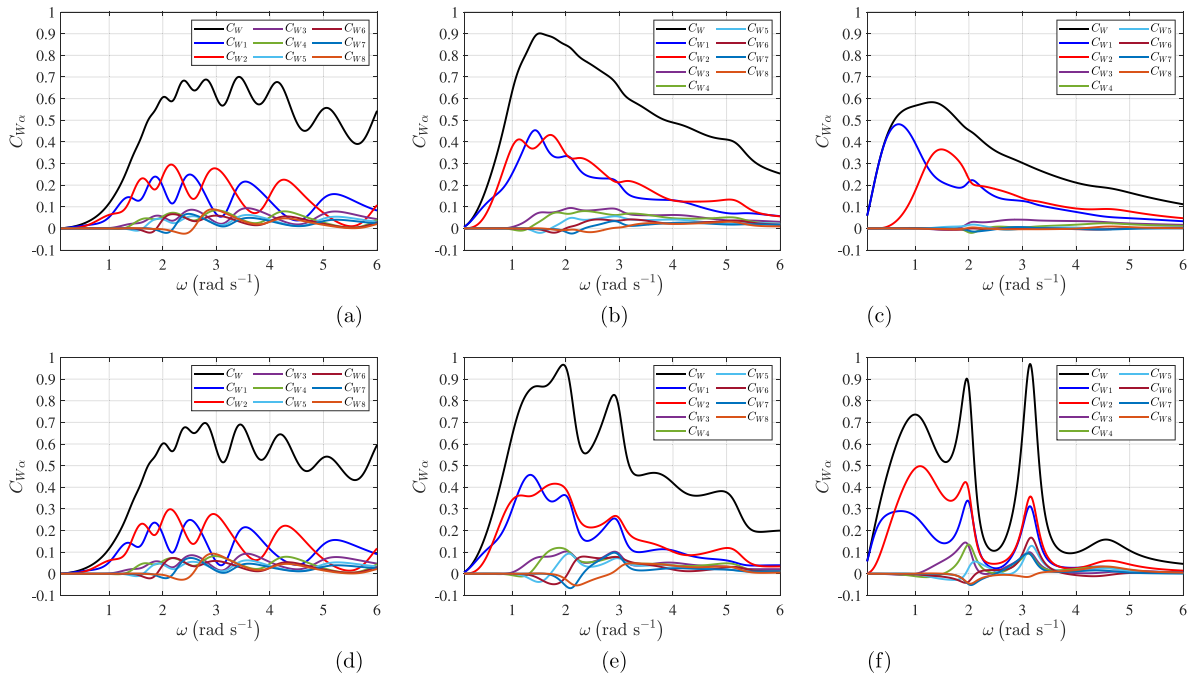


Fig. 8. Capture-width ratio component $C_{W\alpha}$ as a function of incident wave frequency ω , PTO distribution and fixed plate length $L = 20$ m for the first eight natural modes: (a) $x_i = -L \left(\sqrt{2(i-1)/3} - 1 \right)$, $\nu_{PTO} = 10^3$ kg m⁻¹ s⁻¹; (b) $x_i = -L \left(\sqrt{2(i-1)/3} - 1 \right)$, $\nu_{PTO} = 10^4$ kg m⁻¹ s⁻¹; (c) $x_i = -L \left(\sqrt{2(i-1)/3} - 1 \right)$, $\nu_{PTO} = 10^5$ kg m⁻¹ s⁻¹; (d) $x_i = -L \left(\sqrt{2(i-1)/3} - 1 \right)$, $\nu_{PTO} = 10^3$ kg m⁻¹ s⁻¹; (e) $x_i = -L \left(\sqrt{2(i-1)/3} - 1 \right)$, $\nu_{PTO} = 10^4$ kg m⁻¹ s⁻¹; and (f) $x_i = -L \left(\sqrt{2(i-1)/3} - 1 \right)$, $\nu_{PTO} = 10^5$ kg m⁻¹ s⁻¹, where $i = 1, \dots, 7$.

PTO coefficient $\nu_{PTO} = 10^3$ kg m⁻¹ s⁻¹ we obtain similar behaviour to that in Fig. 8(a) again confirming that the PTO distribution has hardly any effect. Conversely, as the PTO coefficient increases we obtain significant secondary peaks located at $\omega \sim 1.96$ rad s⁻¹ and $\omega \sim 3.15$ rad s⁻¹ (see also Fig. 7(b)). The pitching efficiency is higher at low frequency values where there are significant contributions from the first bending modes linked to large-amplitude hydroelastic oscillations of the plate front. This allows us to maximise wave radiation as $x \rightarrow +\infty$. The foregoing results further confirm the role of the PTO distribution in maximising power extraction efficiency.

Table 2

First eight eigenfrequency values (in rad/s) for different flexural rigidity distributions. Numerical values in the last row correspond to the case of uniform flexural rigidity analysed in Section 4.1.

D_m	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$
$\bar{D} \times [0.1, 10, 0.1]$	0.70	1.08	1.48	1.71	1.96	2.23	2.38	2.69
$\bar{D} \times [10, 0.1, 10]$	0.70	1.08	1.51	1.74	2.02	2.36	2.86	3.28
$\bar{D} \times [0.1, 1, 10]$	0.70	1.08	1.49	1.73	2.02	2.30	2.67	3.00
$\bar{D}$	0.70	1.08	1.48	1.72	2.02	2.29	2.65	3.05

4.3. Effect of plate flexural rigidity

In this section we investigate the effect of flexural rigidity distribution D on power extraction efficiency. As already pointed out in Section 2, in this paper we focus on stepped variation in flexural rigidity. For simplicity, we consider a plate made of $M = 3$ segments each of length $2L/3$ and constant D_m , $m = 1, 2, 3$. From (16),

$$D = \sum_{m=1}^3 D_m H[x - x_{m-1}] H[x_m - x], \quad (94)$$

where

$$x_0 = -L, \quad x_1 = -L/3, \quad x_2 = L/3, \quad x_3 = L. \quad (95)$$

The plate has length $L = 20$ m, fixed mass per unit length $\mu = 44$ kg/m and $k_{PTO} = 0$, and possesses 7 equally spaced PTO devices located at $x_i = \pm[0, L/3, 2L/3, L]$.

Fig. 9 shows the C_W map with respect to incident wave frequency and PTO coefficient for various distributions of flexural rigidity. Fig. 9(a) refers to a plate with a stiff middle segment, $D_m = \bar{D} \times [0.1, 10, 0.1]$, whereas Fig. 9(b) refers to the opposite case of a plate with stiff ends $D_m = \bar{D} \times [10, 0.1, 10]$. Fig. 9(a) shows the presence of at least four regions located within the range $1 < \omega < 4$ rad s^{-1} in which the capture-width ratio is close to unity. Conversely, Fig. 9(b) shows the presence of a single peak at $\omega \sim 1.5$ rad s^{-1} with lower efficiency elsewhere. This means that the plate with softer ends exhibits greater efficiency, and highlights a detrimental property of rigid plates with respect to power extraction. Fig. 9(c) concerns a case where the plate flexural rigidity increases towards the plate front, such that $D_m = \bar{D} \times [0.1, 1, 10]$. The efficiency is rather poor because the plate enhances wave radiation as $x \rightarrow -\infty$. Fig. 9(d) shows the map obtained for D increasing towards $x = -L$ such that $D_m = \bar{D} \times [10, 1, 0.1]$. This case achieves much greater efficiency, with the map also containing several additional peaks at high frequencies.

Similarly to Section 4.1, we evaluate the first eight eigenfrequencies (Table 2) corresponding with the above plate distributions. The main discrepancies appear for higher bending modes $k > 5$, whereas the first two modes affected by pitching and heaving motion have essentially the same eigenfrequencies because of unchanged mass per unit length μ and plate length L . Table 2 contains information for the case of a plate of uniform rigidity. Corresponding k eigenfrequencies are very close to the case $\bar{D} \times [0.1, 1, 10]$.

Figs. 9(e)–9(h) present C_W maps for continuous distributions of PTO systems with the same combinations of D_m as analysed previously. In these cases, the overall efficiency is much greater than evident in Figs. 9(a)–9(d), thus highlighting the positive effect of a great number of PTO devices. Fig. 9(h) shows the results obtained for the optimal plate configuration comprising a flexible plate with softened front. This case attains high efficiency over a rather large frequency bandwidth, and is the best configuration analysed herein from an energy extraction efficiency perspective.

We now consider the capture-width ratio $C_{W\alpha}$ versus frequency ω curves for three values of $\nu_{PTO} = [10^3, 10^4, 10^5]$ kg $m^{-1} s^{-1}$. For brevity, we analyse the optimised cases of plate flexural rigidity increasing towards the plate back $D_m = \bar{D} \times [10, 1, 0.1]$. The corresponding C_W maps are shown in Figs. 9(d) and 9(h). Fig. 10(a) refers to the case of a finite number of PTO cables and $\nu_{PTO} = 10^3$ kg $m^{-1} s^{-1}$, and shows that overall efficiency oscillates dramatically because of strong coupling between the modes. Of significance is the strong destructive effect of certain bending modes about $\omega \sim 2$ rad s^{-1} . For larger PTO coefficient values we obtain the curves shown in Figs. 10(b)–10(c). Low frequency dynamics results are again governed by heaving and pitching, whereas a secondary peak appears at $\omega \sim 3.7$ rad s^{-1} . It can be shown that near this value the total radiated waves result is maximised as is the amplitude of oscillations of the plate front. Figs. 10(d)–10(f) refer to the continuous PTO distribution which leads to very high efficiency. Here the contribution from bending modes is as large as for the heaving and pitching modes, especially for $\omega > 2$ rad s^{-1} . Moreover, the behaviour of the curves lacks spikes; in other words, the presence of a large number of PTO cables might simplify WEC design because of the broader and more regular capture-width ratio of each mode.

Given that flexural rigidity plays a similarly important role as that of the PTO distribution, Fig. 11 presents capture-width ratio maps for uniform plates with different values of D . Fig. 11(a) concerns a softened plate with $D = 10^{-2} \times \bar{D}$, and shows that as D decreases, the efficiency of the system increases. Fig. 11(b) refers to a rigid plate for which $D \rightarrow \infty$; in this case there are no contributions from the bending modes and the dynamics is governed by pitching and heaving motion only. There is a single maximum located at $\omega \sim 0.90$ rad s^{-1} with value $C_W \sim 0.8$. This frequency falls within the eigenfrequencies of the heave mode, $\omega = 0.70$ rad s^{-1} , and pitching mode, $\omega = 1.08$ rad s^{-1} . The overall efficiency is much lower than in the previous cases because of the smaller number of eigenfrequencies and possible modal optimisations. This highlights the beneficial effect of bending elastic modes on power extraction efficiency.

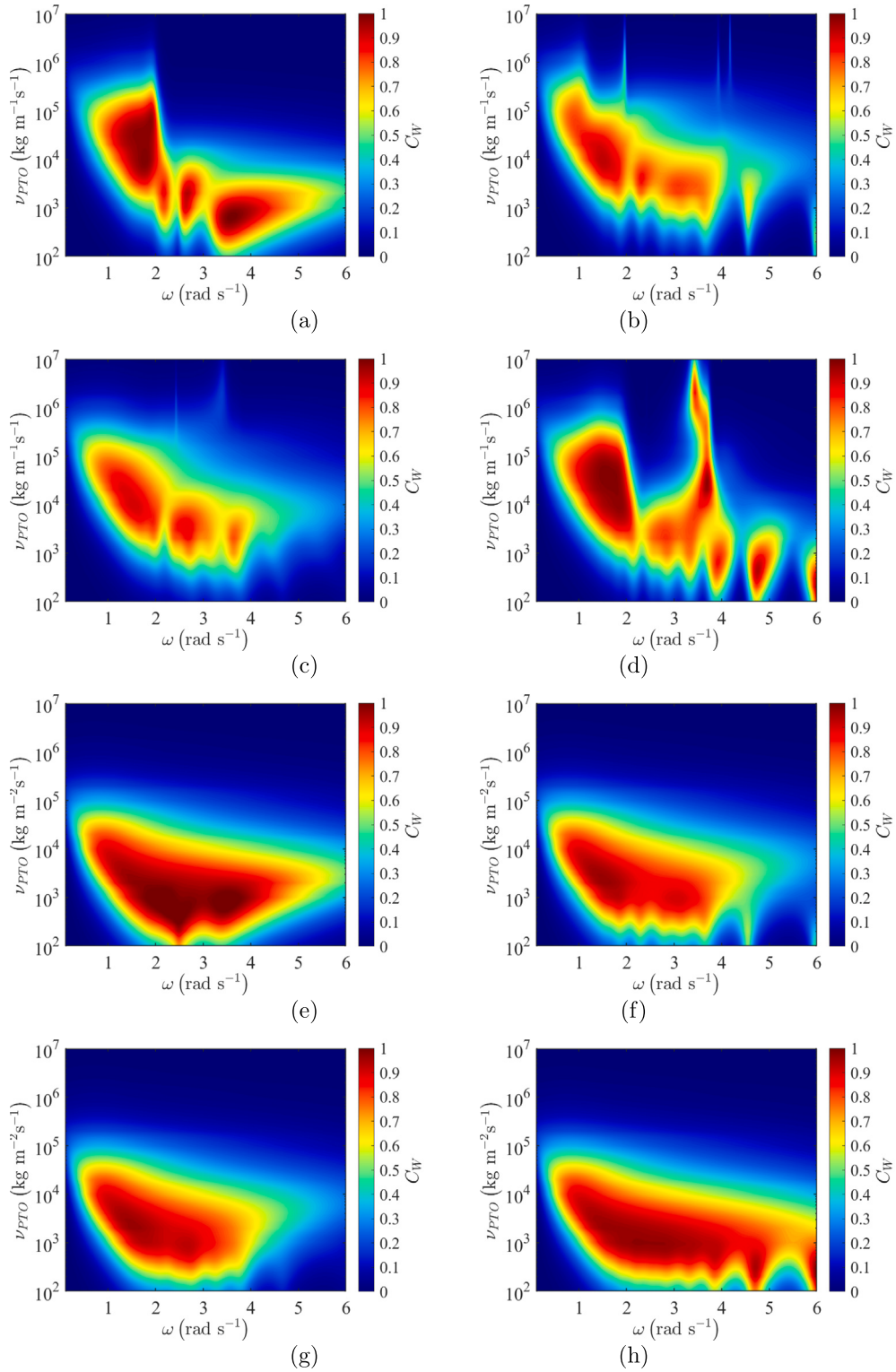


Fig. 9. Capture-width ratio C_W as a function of incident wave frequency ω and PTO coefficient ν_{PTO} for the following stepped variations in D : (a) $D_m = \bar{D} \times [0.1, 10, 0.1]$; (b) $D_m = \bar{D} \times [10, 0.1, 10]$; (c) $D_m = \bar{D} \times [0.1, 1, 10]$; (d) $D_m = \bar{D} \times [10, 1, 0.1]$; (e) $D_m = \bar{D} \times [0.1, 10, 0.1]$, continuous PTO; (f) $D_m = \bar{D} \times [10, 0.1, 10]$, continuous PTO; (g) $D_m = \bar{D} \times [0.1, 1, 10]$, continuous PTO; and (h) $D_m = \bar{D} \times [10, 1, 0.1]$, continuous PTO.

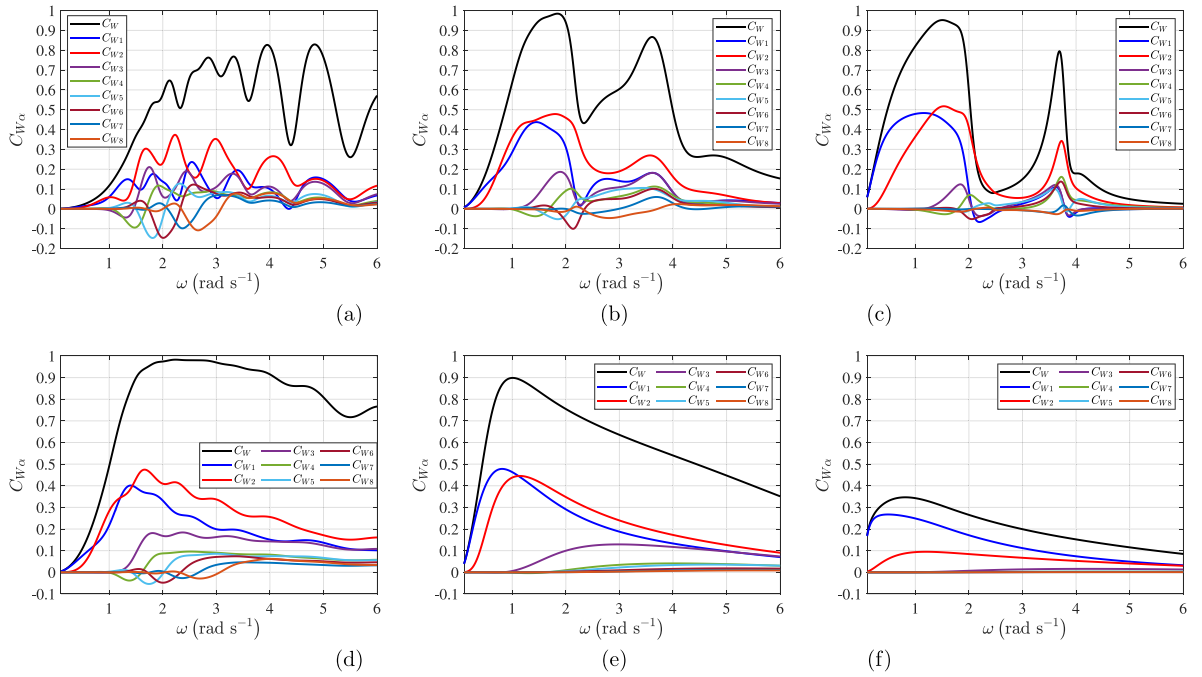


Fig. 10. Capture-width ratio component $C_{W\alpha}$ as a function of incident wave frequency ω , PTO distribution and fixed plate length $L = 20$ m and $D_m = \bar{D} \times [10, 1, 0.1]$ for the first eight natural modes: (a) $x_i = \pm[0, L/3, 2L/3, L]$, $\nu_{PTO} = 10^3$ kg m⁻¹ s⁻¹; (b) $x_i = \pm[0, L/3, 2L/3, L]$, $\nu_{PTO} = 10^4$ kg m⁻¹ s⁻¹; (c) $x_i = \pm[0, L/3, 2L/3, L]$, $\nu_{PTO} = 10^5$ kg m⁻¹ s⁻¹; (d) continuous PTO, $\nu_{PTO} = 10^3$ kg m⁻² s⁻¹; (e) continuous PTO, $\nu_{PTO} = 10^4$ kg m⁻² s⁻¹; and (f) continuous PTO, $\nu_{PTO} = 10^5$ kg m⁻² s⁻¹.

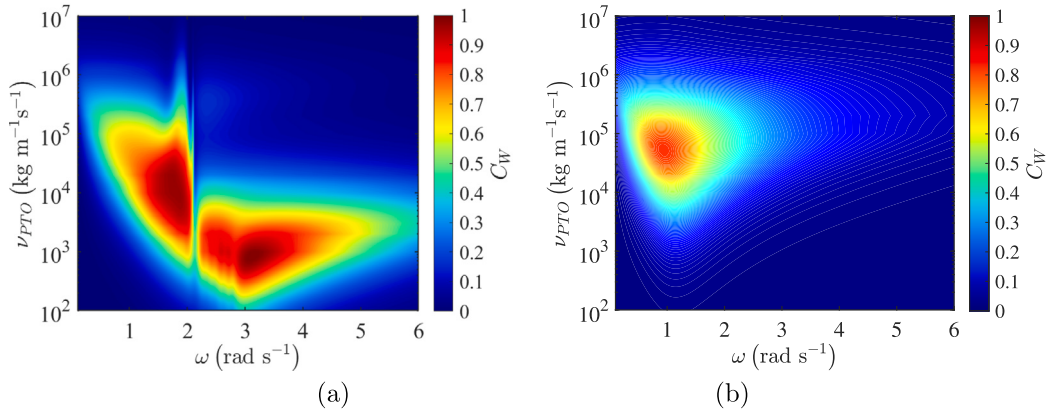


Fig. 11. Capture-width ratio C_W as a function of incident wave frequency ω and PTO coefficient ν_{PTO} for the following constant values of D : (a) $D = 0.01 \times \bar{D}$; and (b) Rigid plate, $D \rightarrow \infty$.

4.4. Offshore structures

The model presented herein can be extended to complex floating offshore platforms combined with mooring systems and, specifically, to the analysis of corresponding mooring forces. This aspect is of practical significance in offshore engineering applications where the flexural rigidity of the floating system is not constant and affects plate hydrodynamics, such as in flexible floating solar farms. Even though the mooring force is modelled through a simple elastic spring of constant stiffness k_{PTO} , results from this section may lead to new design guidelines for offshore mooring industry and renewable energy sector. To this end, let us consider a plate comprising $M = 3$ equal segments, fixed $L = 20$ m, $\nu_{PTO} = 0$, and mass per unit length $\mu = 44$ kg/m. The simplified mooring system comprises two elastic springs located at the plate ends $x_i = \pm L$.

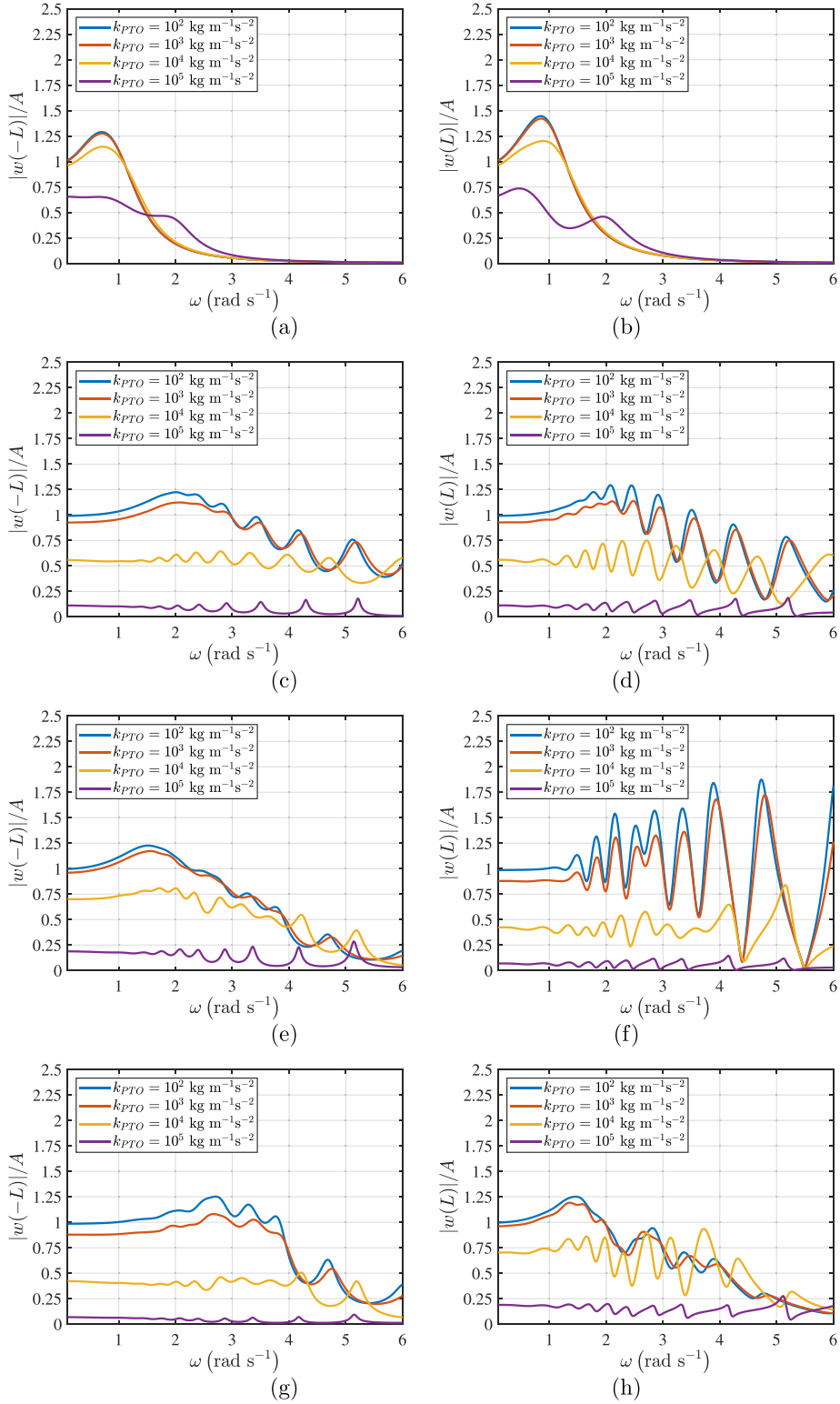


Fig. 12. Variation in mooring displacement with incident wave frequency ω for the following elastic plate distributions: (a) Rigid plate, $|w(-L)|$; (b) Rigid plate, $|w(L)|$; (c) $D = \bar{D}$, $|w(-L)|$; (d) $D = \bar{D}$, $|w(L)|$; (e) $D_m = \bar{D} \times [10, 1, 0.1]$, $|w(-L)|$; (f) $D_m = \bar{D} \times [10, 1, 0.1]$, $|w(L)|$; (g) $D_m = \bar{D} \times [0.1, 1, 10]$, $|w(-L)|$; and (h) $D_m = \bar{D} \times [0.1, 1, 10]$, $|w(L)|$.

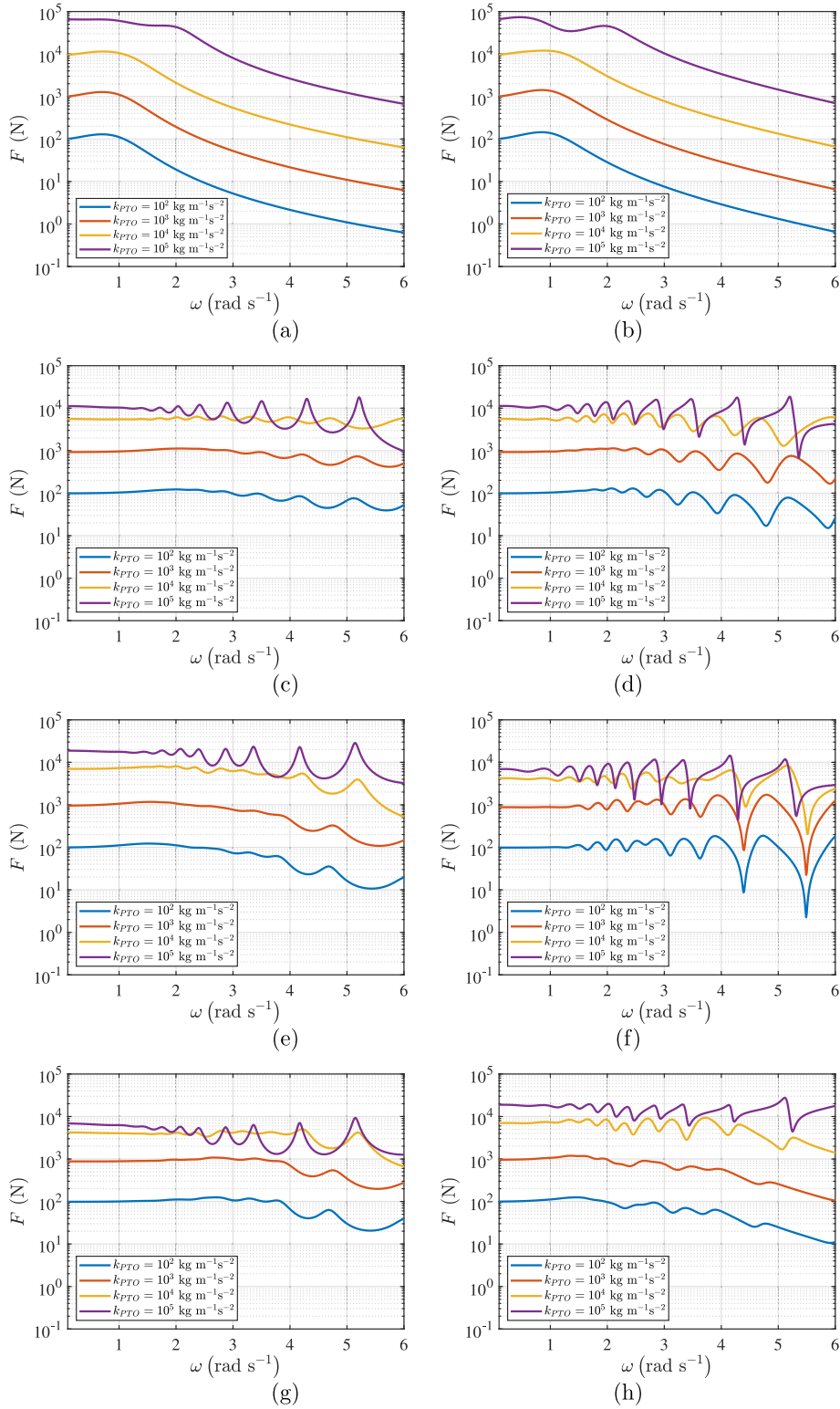


Fig. 13. Variation in mooring force with incident wave frequency ω for the following elastic plate distributions: (a) Rigid plate, $F(x_i) = -L$; (b) Rigid plate, $F(x_i) = L$; (c) $D = \bar{D}$, $F(x_i) = -L$; (d) $D = \bar{D}$, $F(x_i) = L$; (e) $D_m = \bar{D} \times [10, 1, 0.1]$, $F(x_i) = -L$; (f) $D_m = \bar{D} \times [10, 1, 0.1]$, $F(x_i) = L$; (g) $D_m = \bar{D} \times [0.1, 1, 10]$, $F(x_i) = -L$; and (h) $D_m = \bar{D} \times [0.1, 1, 10]$, $F(x_i) = L$.

Fig. 12 shows the displacement Response Amplitude Operator (RAO) of each mooring line $|w(\pm L)|/A$ with respect to incident wave frequency for various plate flexural rigidity distributions and the following values of mooring stiffness coefficient $k_{PTO} = [10^2; 10^3; 10^4; 10^5] \text{ kg m}^{-1} \text{ s}^{-2}$. Figs. 12(a)–12(b) refer to a rigid plate. In each case, the displacement RAO exhibits a maximum of two visible peaks due solely to resonance at the heave and pitch eigenfrequencies. For small values of k_{PTO} the plate displacement at $x_i = L$ is larger than at $x_i = -L$ because of the greater exciting force localised at the plate front. Figs. 12(c)–12(d) depict mooring displacements of the elastic plate displacements for constant $D = \bar{D}$. All curves display several peaks related to resonant bending modes at high frequencies. This implies that special care is needed during design to avoid susceptibility to large displacements. Figs. 12(e)–12(f) concern plates with flexural rigidity increasing towards $x_i = -L$, such that $D_m = \bar{D} \times [10, 1, 0.1]$. The overall displacement behaviour of the mooring line at $x_i = -L$ is very similar to that shown in Fig. 12(c), although significant differences occur at the front of the plate where $x_i = L$. In this case, the mooring displacement curves are characterised by large fluctuations, especially for moderate values of $k_{PTO} \leq 10^3 \text{ kg m}^{-1} \text{ s}^{-2}$. Conversely, when the stiffness coefficient is very large, the left edge experiences greater oscillations. Figs. 12(g)–12(h) show the behaviour of a plate having increasing flexural rigidity towards the front of the plate, such that $D_m = \bar{D} \times [0.1, 1, 10]$. Although the mooring displacement at $x_i = -L$ does not appear to be strongly affected by plate stiffness, the oscillations at the front of the plate are significantly damped for low values of $k_{PTO} \leq 10^3 \text{ kg m}^{-1} \text{ s}^{-2}$. In a similar manner to the previous case, the mooring line beneath the plate experiences greater displacement when the stiffness coefficient is large.

For completeness, we consider the behaviour of the mooring elastic force, which is defined as

$$F = |w(\pm L)|k_{PTO}. \quad (96)$$

Fig. 13 shows the dependence of the mooring elastic force on incident wave frequency for different stiffness coefficients and elastic distributions. In general, the mooring elastic force increases with k_{PTO} coefficient, and large fluctuations appear at resonant frequencies for elastic plates. However, for the rigid plate case shown in Figs. 13(a)–13(b), the total mooring elastic force decreases smoothly with ω . Figs. 13(c)–13(d) display the mooring force behaviour for homogeneous plates of constant flexural rigidity $D = \bar{D}$. Corresponding mooring displacements are shown in 12(c)–12(d). At small wave frequencies, the moorings experience the same force; minor differences appear at high wave frequencies. The behaviour is more complicated when the flexural rigidity is no longer constant along the length of the plate. For example, Figs. 13(e)–13(f) refer to cases where the plate stiffness increases towards $x_i = -L$ (see also Figs. 12(e)–12(f) for the corresponding plate displacements). The mooring force becomes very great at $x_i = -L$ for large k_{PTO} ; and this phenomenon is more evident for small ω where dynamic effects are negligible. The opposite occurs with decreasing k_{PTO} , with the mooring force at $x_i = L$ greater than that at $x_i = -L$ because of larger resonant displacement. Figs. 13(g)–13(h) show the results for $D_m = \bar{D} \times [0.1, 1, 10]$ where the plate is stiffer at its front. Here the $F - \omega$ curves behave in the opposite way to that shown in Figs. 13(e)–13(f) indicating that mooring lines connected to stiffer regions of the plate are in general subject to greater forces. This interesting result should be taken into account by designers when analysing wave loads on offshore flexible structures.

5. Conclusions

This paper has described a mathematical model of the hydroelastic load-response behaviour of a flexible plate of variable stiffness properties with PTO and mooring system. The theory is particularly applicable to hybrid wave energy extraction devices combined with solar panels affecting localised plate rigidity. The mathematical model is based on linearised potential-flow theory, whereby the method of dry modes is combined with matched eigenfunction expansions and Fourier series to solve the hydrodynamics of radiation and diffraction velocity potentials. The analytical model demonstrates that the effect of plate elasticity is to increase the number of resonant frequencies with respect to a rigid plate, whereas wave power extraction and bandwidth of capture-width ratio become greater. The location of each PTO device plays a significant role in system performance; by increasing the number of PTO devices and/or modifying the PTO distribution, it is possible to achieve optimisation of overall efficiency of the system. For example, we found that the optimal configuration is made of a softened plate front and continuous PTO system. This case attains high efficiency over a large frequency bandwidth, therefore in the case of hybrid systems we recommend the solar panels or stiffer elements be placed in the lee side region of the plate. This will enhance wave radiation in the direction opposite to that of the incident waves which is equivalent to optimising power extraction. Furthermore, plates of longer length yield greater efficiency at lower frequency. In such cases, the number of peaks increases due to there being more resonant frequencies. By investigating the effects of variable plate flexural rigidity on the mooring system, we also found that the elastic mooring force increases in stiffer regions of a given plate. Given its promising results, the present model highlights a need to improve the experimental and numerical investigation of flexible hybrid offshore renewable energy systems. We point out that the theoretical model analysed herein is limited to two-dimensional configurations and stepped variations of flexural rigidity. Extensions to realistic three-dimensional geometries such as circular or rectangular plates could lead to the analysis of important effects arising from incident directional spectra, the three-dimensional modal shapes of flexible structures, etc. We recommend that more complicated analysis of the optimal distribution of both PTO devices and flexural rigidity, and theoretical investigations of optimal capture-width ratio be undertaken. Such studies would lead to a better understanding of related topics of considerable practical interest in ocean engineering and ORE, such as floating offshore solar farms, ice floes and VLFS.

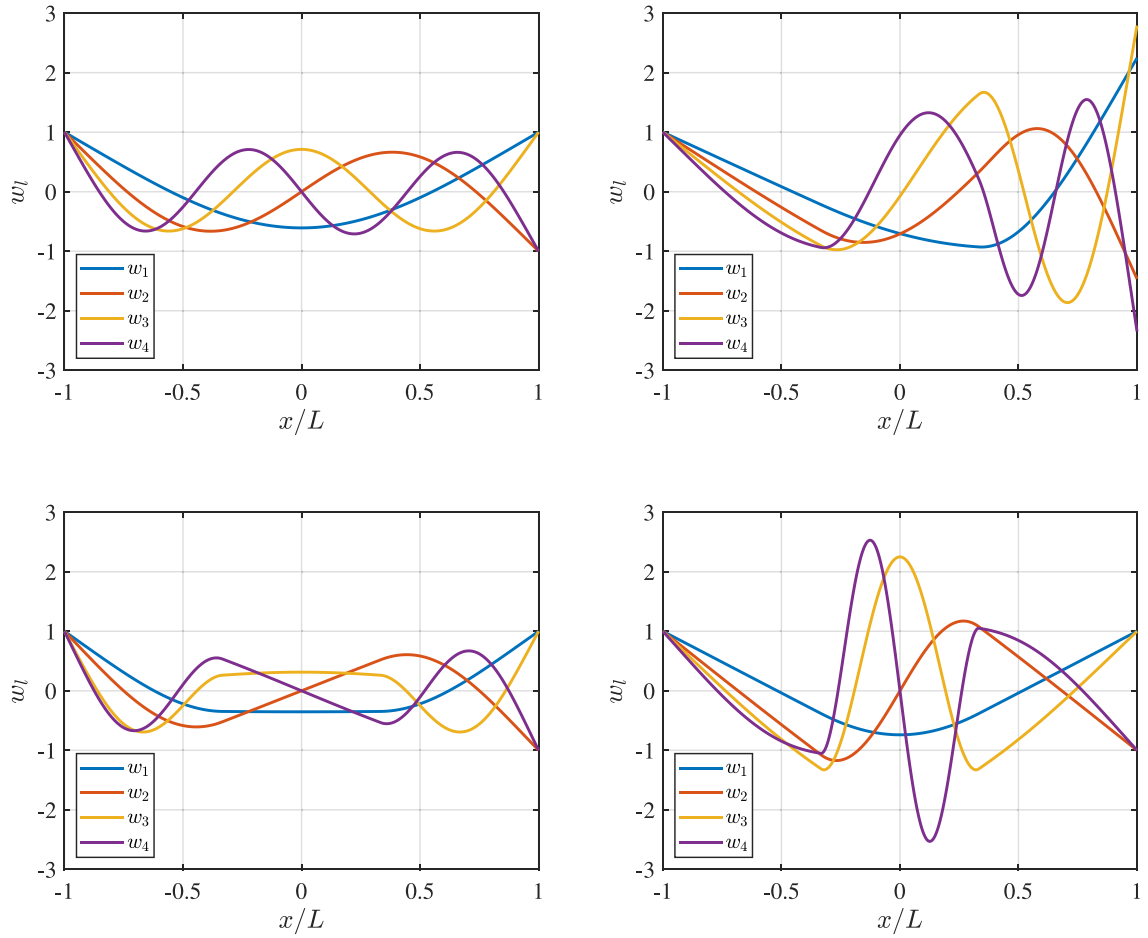


Fig. 14. Natural modal shapes of elastic plates for the following flexural rigidity distributions: (a) Uniform rigidity, $D_m = \bar{D}$; (b) $D_m = \bar{D} \times [10, 1, 0.1]$; (c) $D_m = \bar{D} \times [0.1, 10, 0.1]$; and (d) $D_m = \bar{D} \times [10, 0.1, 10]$.

CRedit authorship contribution statement

S. Michele: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. **S. Zheng:** Conceptualization, Funding acquisition, Methodology, Project administration, Visualization, Writing – original draft, Writing – review & editing. **E. Renzi:** Supervision, Visualization, Writing – original draft, Writing – review & editing. **A.G.L. Borthwick:** Supervision, Visualization, Writing – original draft, Writing – review & editing. **D.M. Greaves:** Supervision, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Simone Michele reports financial support was provided by University of Plymouth.

Data availability

Data will be made available on request.

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Appendix. Natural modes of stepped plates with free edges

This section considers the first four natural bending shapes w_l , $l = 1, 2, 3, 4$, of a stepped plate characterised by various distributions of D and free edges. To be consistent with foregoing text, we consider a plate constructed from three segments of equal length $2L/3$. Fig. 14(a) shows the well-established symmetric and antisymmetric modal forms of a homogeneous plate (35), and Fig. 14(b) depicts the first natural mode shapes for $D_m = \bar{D} \times [10, 1, 0.1]$ where the plate is stiffer in the region $x < 0$. Fig. 14(c) shows the behaviour obtained for a plate with a very rigid middle segment. Conversely, Fig. 14(d) depicts the symmetric and antisymmetric natural bending shapes obtained for a symmetric plate stiffer at its ends.

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