Quantum Covariance, Quantum Fisher Information, and the Uncertainty Relations

Paolo Gibilisco, Fumio Hiai, and Dénes Petz

***Abstract*—In this paper, the relation between quantum covariances and quantum Fisher informations is studied. This study is applied to generalize a recently proved uncertainty relation based on quantum Fisher information. The proof given here considerably simplifies the previously proposed proofs and leads to more general inequalities.**

***Index Terms*—Generalized variance, operator monotone functions, quantum covariance, quantum Fisher information, uncertainty principle.**

I. Introduction

FISHER information has been an important concept in mathematical statistics and it is an ingredient of the Cramér–Rao inequality. It was extended to a quantum mechanical formalism in the 1960s by Helstrom [9] and later by Yuen and Lax [28]; see [10] for the rigorous version.

The state of a finite quantum system is described by a density matrix D, which is positive semidefinite with $\mathrm{Tr} D = 1$. If D depends on a real parameter $-t < \theta < t$, then the true value of θ can be estimated by a self-adjoint matrix A, called observable, such that

$$\mathrm{Tr} D_\theta A = \theta.$$

This means that expectation value of the measurement of A is the true value of the parameter (unbiased measurement). When the measurement is performed (several times on different copies of the quantum system), the average outcome is a good estimate for the parameter θ.

It is convenient to choose the value $\theta = 0$. Then, the Cramér–Rao inequality has the form

$$\mathrm{Tr} D_0 A^2 \geq \frac{1}{\text{Fisher information}}$$

where the Fisher information quantity is determined by the parametrized family D_θ and it does not depend on the observable A; see [10] and [23].

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P. Gibilisco is with the Dipartimento SEFEMEQ, Università di Roma "Tor Vergata," 00133 Rome, Italy (e-mail: gibilisco@volterra.uniroma2.it).

F. Hiai is with Graduate School of Information Sciences, Tohoku University Aoba-ku, Sendai 980-8579, Japan (e-mail: hiai@math.is.tohoku.ac.jp).

D. Petz is with Alfréd Rényi Institute of Mathematics, H-1364 Budapest, Hungary (e-mail: petz@math.bme.hu).

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The Fisher information depends on the tangent of the curve D_θ. There are many curves through the fixed D_0 and the Fisher information is defined on the tangent space. The latter is the space of traceless self-adjoint matrices in case of the affine parametrization of the state space. The Fisher information is a quadratic form depending on the foot point D_0. If it should generate a Riemannian metric, then it should depend on D_0 smoothly [1].

II. From Coarse Graining to Fisher Information and Covariance

Heuristically, coarse graining implies loss of information, therefore Fisher information should be monotone under coarse graining. This was proved in [3] in probability theory and a similar approach was proposed in [18] for the quantum case. The approach was completed in [21], where a class of quantum Fisher information quantities was introduced; see also [22].

Assume that D_θ is a smooth curve of density matrices with tangent $A := \dot{D}_0$ at D_0. The quantum Fisher information $F_D(A)$ is an information quantity associated with the pair (D_0, A) and it appeared in the Cramér–Rao inequality above. Let now α be a coarse graining, that is, $\alpha : M_n \to M_k$ is a completely positive trace-preserving mapping. Then, $\alpha(D_\theta)$ is another curve in M_k. Due to the linearity of α, the tangent at $\alpha(D_0)$ is $\alpha(A)$. As it is usual in statistics, information cannot be gained by coarse graining, therefore we expect that the Fisher information at the density matrix D_0 in the direction A must be larger than the Fisher information at $\alpha(D_0)$ in the direction $\alpha(A)$. This is the monotonicity property of the Fisher information under coarse graining

$$F_D(A) \geq F_{\alpha(D)}(\alpha(A)). \tag{1}$$

Another requirement is that $F_D(A)$ should be quadratic in A, in other words, there exists a (nondegenerate) real positive bilinear form $\gamma_D(A, B)$ on the self-adjoint matrices such that

$$F_D(A) = \gamma_D(A, A). \tag{2}$$

The requirements (1) and (2) are strong enough to obtain a reasonable but still wide class of possible quantum Fisher informations.

The bilinear form $\gamma_D(A, B)$ can be canonically extended to the positive sesqui-linear form (denoted by the same γ_D) on the complex matrices, and we may assume that

$$\gamma_D(A, B) = \mathrm{Tr} A^* \mathbb{J}_D^{-1}(B)$$

for an operator $\mathbb{J}_D$ acting on matrices. (This formula expresses the inner product γ_D by means of the Hilbert–Schmidt inner product and the positive linear operator $\mathbb{J}_D$.) Note that this notation transforms (1) into the relation

$$\alpha^* \mathbb{J}_{\alpha(D)}^{-1} \alpha \leq \mathbb{J}_D^{-1}. \tag{3}$$

This is equivalent to $\|\mathbb{J}_{\alpha(D)}^{-1/2} \alpha \mathbb{J}_D^{1/2}\| \leq 1$ or to the inequality $\|\mathbb{J}_D^{1/2} \alpha^* \mathbb{J}_{\alpha(D)}^{-1/2}\| \leq 1$. The latter condition can be written as

$$\mathbb{J}_{\alpha(D)}^{-1/2} \alpha \mathbb{J}_D \alpha^* \mathbb{J}_{\alpha(D)}^{-1/2} \leq I.$$

So we conclude that (3) is equivalent to the following:

$$\alpha \mathbb{J}_D \alpha^* \leq \mathbb{J}_{\alpha(D)}. \tag{4}$$

Under the above assumptions, there exists a unique operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}$ such that $f(t) = t f(t^{-1})$ and

$$\mathbb{J}_D = f(\mathbf{L}_D \mathbf{R}_D^{-1}) \mathbf{R}_D \tag{5}$$

where the linear transformations $\mathbf{L}_D$ and $\mathbf{R}_D$ acting on matrices are the left and right multiplications, that is

$$\mathbf{L}_D(X) = DX \quad \text{and} \quad \mathbf{R}_D(X) = XD.$$

To be adjusted to the classical case, we always assume that $f(1) = 1$ (see [21] and [24]). It seems to be convenient to call a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ **standard** if f is operator monotone, $f(1) = 1$, and $f(t) = t f(t^{-1})$ (a standard function is essential in the context of operator means [12], [21]). If $D = \mathrm{Diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ (with $\lambda_i > 0$), then

$$\gamma_D(A, B) = \sum_{ij} \frac{1}{M_f(\lambda_i, \lambda_j)} \bar{A}_{ij} B_{ij} \tag{6}$$

where M_f is the mean induced by the function f

$$M_f(a, b) := b f(a/b).$$

When A and B are self-adjoint, the right-hand side of (6) is real as required since $M_f(a, b) = M_f(b, a)$.

Similarly to Fisher information, the covariance is a bilinear form as well. In probability theory, it is well understood but the noncommutative extension is not obvious. The monotonicity under coarse graining should hold

$$\mathrm{qCov}_D(\alpha^*(A), \alpha^*(A)) \leq \mathrm{qCov}_{\alpha(D)}(A, A) \tag{7}$$

where α^* is the adjoint with respect to the Hilbert–Schmidt inner product (α^* is a unital completely positive mapping). If the covariance is expressed by the Hilbert–Schmidt inner product as

$$\mathrm{qCov}_D(A, B) = \mathrm{Tr} A^* \mathbb{K}_D(B)$$

by means of some positive operators $\mathbb{K}_D$, then the monotonicity (7) has the form

$$\alpha \mathbb{K}_D \alpha^* \leq \mathbb{K}_{\alpha(D)}.$$

This is actually the same relation as (4). Therefore, condition (7) implies that the operator $\mathbb{K}_D$ must have the form (5) and we have

$$\mathrm{qCov}_D(A, B) = \mathrm{Tr} A^* \mathbb{J}_D(B)$$

if $\mathrm{Tr} DA = \mathrm{Tr} DB = 0$, where $\mathbb{J}_D$ is defined by (5). The general formula is obtained by replacing A by $A - (\mathrm{Tr} DA)I$ and B by $B - (\mathrm{Tr} DB)I$, so we have

$$\mathrm{qCov}_D(A, B) = \mathrm{Tr} A^* \mathbb{J}_D(B) - \overline{\mathrm{Tr} DA}\, \mathrm{Tr} DB. \tag{8}$$

This is considered to be the general definition. The one-to-one correspondence between Fisher information quantities and (generalized) covariances was discussed in [22]. The analog of formula (6) is

$$\mathrm{qCov}_D(A, B) = \sum_{ij} M_f(\lambda_i, \lambda_j) \bar{A}_{ij} B_{ij} - \left(\sum_i \lambda_i \bar{A}_{ii}\right)\left(\sum_i \lambda_i B_{ii}\right).$$

If we want to emphasize the dependence of the Fisher information and the covariance on the function f, we write γ_D^f and qCov_D^f. The usual symmetrized covariance corresponds to the function $f(t) = (t+1)/2$

$$\mathrm{qCov}_D^f(A, B) = \mathrm{Cov}_D(A, B) := \frac{1}{2}\mathrm{Tr}(D(A^*B + BA^*)) - (\mathrm{Tr} DA^*)(\mathrm{Tr} DB)$$

Of course, if D, A and B commute, then $\mathrm{qCov}_D^f(A, B) = \mathrm{Cov}_D(A, B)$ for any standard function f. Note that both qCov_D^f and γ_D^f are particular quasi-entropies [19], [20].

III. Relation to the Commutator

Let D be a density matrix and A be self-adjoint. The commutator $\mathrm{i}[D, A]$ appears in the discussion about Fisher information. One reason is that the tangent space $T_D := \{B = B^* : \mathrm{Tr} DB = 0\}$ has a natural orthogonal decomposition

$$\{B = B^* : [D, B] = 0\} \oplus \{\mathrm{i}[D, A] : A = A^*\}.$$

For self-adjoint operators $A_1, \dots, A_N$, Robertson's uncertainty principle is the inequality

$$\mathrm{Det}\left[\mathrm{Cov}_D(A_i, A_j)\right]_{i,j=1}^N \geq \mathrm{Det}\left[-\frac{\mathrm{i}}{2}\mathrm{Tr} D[A_i, A_j]\right]_{i,j=1}^N$$

see [25]. The left-hand side is known in classical probability as the generalized variance of the random vector $(A_1, \dots, A_N)$. A different kind of uncertainty principle has been recently conjectured in [5] and proved in [6] and [2]

$$\mathrm{Det}\left[\mathrm{Cov}_D(A_i, A_j)\right]_{i,j=1}^N \geq \mathrm{Det}\left[\frac{f(0)}{2}\gamma_D^f(\mathrm{i}[D, A_i], \mathrm{i}[D, A_j])\right]_{i,j=1}^N. \tag{9}$$

The inequality (9) was started with the case $N = 1$ for some special functions f. The cases $f(x) = (1+x)/2$ and $f(x) = (\sqrt{x}+1)^2/4$ were proved by Luo [13], [14]. The general case was proved by Hansen [8] and shortly after by Gibilisco, Imparato, and Isola with a different technique [7].

In the case $N = 2$, the inequality was proved by Luo, Q. Zhang, and Z. Zhang [16], [17], [15], by Kosaki [11], and by Yanagi, Furuichi, and Kuriyama [27] for some special functions f. The general case is due to Gibilisco, Imparato, and Isola [7], [4]. Gibilisco and Isola emphasized the geometric aspects of the inequality (9) and conjectured it for general quantum Fisher information [4].

Gibilisco, Imparato, and Isola proved the inequality for every $N \in \mathbb{N}$ and for every appropriate function f in [6]. Andai obtained a slightly different form by another method [2].

In (9), we have a nontrivial inequality in the case $f(0) > 0$. The inequality can be called **dynamical uncertainty principle**, since the right-hand side is the volume of a parallelepiped determined by the tangent vectors of the trajectories of the time-dependent observables $A_i(t) := D^{\mathrm{i}t}A_iD^{-\mathrm{i}t}$. Another remarkable property is that inequality (9) gives a nontrivial bound also in the odd case $N = 2m+1$ and this seems to be the first result of this type in the literature.

The right-hand side of (9) is Fisher information of commutators. If

$$\tilde{f}(x) := \frac{1}{2}\left((x+1) - (x-1)^2\frac{f(0)}{f(x)}\right) \tag{10}$$

then

$$\frac{f(0)}{2}\gamma_D^f(\mathrm{i}[D,A],\mathrm{i}[D,B]) = \mathrm{Cov}_D(A,B) - \mathrm{qCov}_D^{\tilde{f}}(A,B) \tag{11}$$

for $A, B \in T_D$. Identity (11) is easy to check but it is not obvious that for a standard f the function $\tilde{f}$ is operator monotone. It is indeed true that $\tilde{f}$ is a standard function as well; see [7, Prop. 5.2 and 6.3]. Note that the left-hand side of (11) was called (metric adjusted) skew information in [8].

IV. Inequalities

In this section, we give a simple new proof for the dynamical uncertainty principle (9). The new proof actually gives a slightly more general inequality.

Theorem 1: Assume that $f, g : \mathbb{R}^+ \to \mathbb{R}$ are standard functions such that

$$g(x) \ge c\frac{(x-1)^2}{f(x)} \tag{12}$$

for some $c > 0$. Then

$$\mathrm{qCov}_D^g(A,A) \ge c\gamma_D^f([D,A],[D,A]).$$

Proof: We may assume that $D = \mathrm{Diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\mathrm{Tr}DA = 0$. Then, the left-hand side is

$$\sum_{ij} M_g(\lambda_i,\lambda_j)|A_{ij}|^2$$

while the right-hand side is

$$c\sum_{ij}\frac{(\lambda_i-\lambda_j)^2}{M_f(\lambda_i,\lambda_j)}|A_{ij}|^2.$$

The proof is complete. □

For any standard function f and its transform $\tilde{f}$ given by (10), $\tilde{f} \ge 0$ is exactly

$$\frac{1+x}{2} - \frac{f(0)(x-1)^2}{2f(x)} \ge 0.$$

Therefore, for $g(x) = (1+x)/2$, the assumption (12) holds for any f if $c = f(0)/2$. Actually, this is the point where the operator monotonicity of f is used; in Theorem 1, only inequality (12) was essential.

The next lemma is standard but the proof is given for completeness.

Lemma 2: Let $\mathcal{K}$ be a finite-dimensional real Hilbert space with inner product $\langle\langle \cdot,\cdot \rangle\rangle$. Let $\langle \cdot,\cdot \rangle$ be a real (not necessarily strictly) positive bilinear form on $\mathcal{K}$. If

$$\langle f,f \rangle \le \langle\langle f,f \rangle\rangle$$

for every vector $f \in \mathcal{K}$, then

$$\mathrm{Det}\left([\langle f_i,f_j\rangle]_{i,j=1}^m\right) \le \mathrm{Det}\left([\langle\langle f_i,f_j\rangle\rangle]_{i,j=1}^m\right) \tag{13}$$

holds for every $f_1, f_2, \dots, f_m \in \mathcal{K}$. Moreover, if $\langle\langle \cdot,\cdot \rangle\rangle - \langle \cdot,\cdot \rangle$ is strictly positive, then inequality (13) is strict whenever $f_1, \dots, f_m$ are linearly independent.

Proof: Consider the Gram matrices $G := [\langle\langle f_i,f_j\rangle\rangle]_{i,j=1}^m$ and $H := [\langle f_i,f_j\rangle]_{i,j=1}^m$, which are symmetric and positive semidefinite. For every $a_1, \dots, a_m \in \mathbb{R}$, we get

$$\sum_{i,j=1}^m (\langle\langle f_i,f_j\rangle\rangle - \langle f_i,f_i\rangle)a_ia_j = \left\langle\left\langle \sum_{i=1}^m a_if_i, \sum_{i=1}^m a_if_i \right\rangle\right\rangle - \left\langle \sum_{i=1}^m a_if_i, \sum_{i=1}^m a_if_i \right\rangle \ge 0$$

by assumption. This says that $G - H$ is positive semidefinite, hence it is clear that $\mathrm{Det}(G) \ge \mathrm{Det}(H)$.

Moreover, assume that $\langle\langle \cdot,\cdot \rangle\rangle - \langle \cdot,\cdot \rangle$ is strictly positive and $f_1, \dots, f_m$ are linearly independent. Then, $G - H$ is positive definite, and hence, $\mathrm{Det}(G) > \mathrm{Det}(H)$. □

The previous general result is used now to have a determinant inequality, an extension of the dynamical uncertainty relation.

Theorem 3: Assume that $f, g : \mathbb{R}^+ \to \mathbb{R}$ are standard functions such that

$$g(x) \ge c\frac{(x-1)^2}{f(x)}$$

for some $c > 0$. Then, for self-adjoint matrices $A_1, A_2, \dots, A_m$, the determinant inequality

$$\mathrm{Det}\left([\mathrm{qCov}_D^g(A_i,A_j)]_{i,j=1}^m\right) \ge \mathrm{Det}\left(\left[c\gamma_D^f([D,A_i],[D,A_j])\right]_{i,j=1}^m\right) \tag{14}$$

holds.

Moreover, equality holds in (14) if and only if $A_i - (\mathrm{Tr}DA_i)I, 1 \le i \le m$, are linearly dependent, and both sides of (14) are zero in this case.

Proof: Let $\mathcal{K}$ be the real vector space $T_D = \{B = B^* : \mathrm{Tr}DB = 0\}$. We have $\mathrm{qCov}_D^g(A,A) = 0$ if and only if $A = \lambda I$, therefore

$$\langle\langle A, B\rangle\rangle := \mathrm{qCov}_D^g(A,B)$$

is an inner product on $\mathcal{K}$. From formulas (6) and (9) and from the hypothesis, we have

$$\begin{aligned} c\gamma_D^f([D,A],[D,A]) &= \sum_{ij} c\frac{(\lambda_i - \lambda_j)^2}{M_f(\lambda_i,\lambda_j)}|A_{ij}|^2 \\ &\le \sum_{ij} M_g(\lambda_i,\lambda_j)|A_{ij}| \\ &= \mathrm{qCov}_D^g(A,A) = \langle\langle A, A\rangle\rangle. \end{aligned}$$

If

$$\langle A, B\rangle := c\gamma_D^f([D,A],[D,B])$$

then $\langle A, A\rangle \le \langle\langle A, A\rangle\rangle$ holds and (13) gives the statement when $\mathrm{Tr}DA_1 = \mathrm{Tr}DA_2 = \cdots = \mathrm{Tr}DA_m = 0$. The general case follows by writing $A_i - (\mathrm{Tr}DA_i)I$ in place of $A_i, 1 \le i \le m$.

To prove the statement on equality case, we show that $g(x) > c(x-1)^2/f(x)$ or $f(x)g(x) > c(x-1)^2$ for all $x > 0$. Since $f(x)g(x)$ is increasing while $c(x-1)^2$ is decreasing for $0 < x \le 1$, it is clear that $f(x)g(x) > c(x-1)^2$ for $0 < x \le 1$. Since $f(x)$ and $g(x)$ are (operator) concave, it follows that $f(x)g(x)/x^2 = (f(x)/x)(g(x)/x)$ is decreasing for $x > 0$. But $c(x-1)^2/x^2$ is increasing for $x \ge 1$, so that we have $f(x)g(x) > c(x-1)^2$ for $x \ge 1$ as well. The inequality shown above implies that

$$M_g(\lambda_i,\lambda_j) > c\frac{(\lambda_i - \lambda_j)^2}{M_f(\lambda_i,\lambda_j)}$$

for all $1 \le i, j \le m$. Hence, $\langle\langle\cdot,\cdot\rangle\rangle - \langle\cdot,\cdot\rangle$ is strictly positive on $\mathcal{K}$, and the latter statement follows from Lemma 2. □

Recall that (9) is obtained by the choice $g(x) = (1+x)/2$ and $c = f(0)/2$. Assume we put $c = f(0)/2$. Then, (14) holds for a standard f if

$$g(x) \ge \frac{f(0)(x-1)^2}{2f(x)}.$$

In particular, $g(0) \ge 1/2$. The only standard g satisfying this inequality is $g(t) = (t+1)/2$. This corresponds to the case where the left-hand side is the usual covariance.

Motivated by [15] and [26], Kosaki studied [11] the case when $f(x)$ equals

$$h_\beta(x) = \frac{\beta(1-\beta)(x-1)^2}{(x^\beta - 1)(x^{1-\beta} - 1)}.$$

In this case, $g(x) = h_\beta(x)$ is possible for every $0 < \beta < 1$ if the constant c is chosen properly. More generally, inequality (14) holds for any standard f and g when the constant c is appropriate. It follows from Lemma 4 that $c = f(0)g(0)$ is good; see (15).

Lemma 4: For every standard function f

$$f(x) \ge f(0)|x-1|.$$

Proof: The inequality is not trivial only if $f(0) > 0$ and $x > 1$, so assume these conditions. Let $q(x_0)$ be the constant such that the tangent line to the graph of f at the point $x_0 > 1$ has the equation

$$y = f'(x_0)x + q(x_0).$$

Since f is (operator) concave, one has $q(x_0) \ge f(0)$. Using again (operator) concavity and symmetry, one has

$$\begin{aligned} f'(x_0) &\ge \lim_{x\to+\infty} f'(x) = \lim_{x\to+\infty} \frac{f(x)}{x} \\ &= \lim_{x\to+\infty} f(x^{-1}) = f(0) > 0. \end{aligned}$$

This implies

$$\begin{aligned} f(x_0) &= f'(x_0)\cdot x_0 + q(x_0) \ge f(0)\cdot x_0 + f(0) \\ &\ge f(0)\cdot x_0 - f(0) = f(0)\cdot(x_0 - 1) \end{aligned}$$

and the proof is complete. □

The lemma gives the inequality

$$f(x)g(x) \ge f(0)g(0)(x-1)^2 \tag{15}$$

for standard functions. If $f(0) > 0$ and $g(0) > 0$, then Theorem 3 applies.

Similarly to the proof of Theorem 3, one can prove that the right-hand side of (14) is a monotone function of the variable f.

Theorem 5: Assume that $f, g : \mathbb{R}^+ \to \mathbb{R}$ are standard functions. If

$$\frac{c}{f(t)} \ge \frac{d}{g(t)} \tag{16}$$

for some positive constants c, d and $A_1, A_2, \ldots, A_m$ are self-adjoint matrices, then

$$\begin{aligned} &\mathrm{Det}\left(\left[c\gamma_D^f([D,A_i],[D,A_j])\right]_{i,j=1}^m\right) \\ &\qquad \le \mathrm{Det}\left([d\gamma_D^g([D,A_i],[D,A_j])]_{i,j=1}^m\right) \end{aligned} \tag{17}$$

holds.

V. Discussion and Conclusion

Covariance and Fisher information are uniquely defined and standard concepts in mathematical statistics. In the quantum mechanical setting, the situation is very different. When the monotonicity under coarse graining is the essential requirement, then an operator monotone function appears as a parameter (if the density matrix of the quantum state commutes with the observables, then the operator monotone function does not play any role).

The uncertainty principle of quantum theory is well known. The standard uncertainty relation contains the commutator(s) of observables in a lower bound for the variance. Luo and Zhang proposed an uncertainty relation that includes the commutator of the observables and the density matrix. The generalization of the proposal has been studied by many people and the proofs have been rather complicated. In this paper, a generalized version is presented, and the variance includes an operator monotone function. In spite of the fact that the variance and the lower bound are parametrized by different functions, the proof is much simpler than the previous versions, moreover the condition of equality is also described.

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Paolo Gibilisco was born in Rome, Italy, in 1961. He graduated as a mathematician from the University of Rome "La Sapienza," Rome, Italy, in 1986 and received the Ph.D. degree in mathematics from the University of Rome "Tor Vergata," Rome, Italy, in 1992.

Currently, he is an Assistant Professor at the School of Economics, University of Rome "Tor Vergata." He published 30 papers and has been referee for several international journals. His research interests include the algebraic and geometric aspects of Fisher information, gauge theory, and foundations of probability.

Fumio Hiai was born in Toyama Prefecture of Japan in 1948. He received the M.S. degree in mathematics from Graduate School, Tokyo Institute of Technology, Tokyo, Japan, in 1975 and the Ph.D. degree from Tokyo Institute of Technology, in 1979, for a thesis entitled "Theory of conditional expectations and information channels."

He was an Associate Professor at the Institute of Applied Electricity, Hokkaido University, during 1985–1990, and a Professor at the Department of Mathematics, Ibaraki University, during 1990–1998. Since 1998, he has been the Professor at the Graduate School of Information Science, Tohoku University, Sendai, Japan. He published more than 80 papers and several books. His research interests are functional analysis, operator algebras, and noncommutative probability.

Denes Petz was born in Budapest, Hungary, in 1953. He graduated as a mathematician from the Science University, Budapest, Hungary.

Currently, he is the Professor at the Technical University and at the Science University, Budapest. He has been Researcher at the Alfréd Rényi Institute of Mathematics, Budapest, Hungary. His research interests are mathematical physics, quantum information theory, noncommutative probability, and matrix analysis. He published 120 papers and his more recent book is *Quantum information theory and quantum statistics* (New York: Springer-Verlag, 2007).

Mr. Petz was a Humboldt fellow in Tübingen, Visiting Professor in Leuven, Heidelberg, and Rome, and a Canon Fellow in Tokyo. He is the editor of two Hungarian journals and of the *Journal of Mathematical Physics*.