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FOCK REPRESENTATION OF THE RENORMALIZED HIGHER POWERS OF WHITE NOISE AND THE VIRASORO–ZAMOLODCHIKOV–w_∞ $*$–LIE ALGEBRA

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Abstract. The identification of the $*$–Lie algebra of the renormalized higher powers of white noise (RHPWN) and the analytic continuation of the second quantized Virasoro–Zamolodchikov–w_∞ $*$–Lie algebra of conformal field theory and high-energy physics, was recently established in [3] based on results obtained in [1] and [2]. In the present paper we show how the RHPWN Fock kernels must be truncated in order to be positive definite and we obtain a Fock representation of the two algebras. We show that the truncated renormalized higher powers of white noise (TRHPWN) Fock spaces of order ≥ 2 host the continuous binomial and beta processes.

1. The RHPWN and Virasoro–Zamolodchikov–w_∞ $*$–Lie algebras

Let a_t and $a_s^\dagger$ be the standard boson white noise functionals with commutator

$$[a_t, a_s^\dagger] = \delta(t-s)\cdot 1$$

where δ is the Dirac delta function. As shown in [1] and [2], using the renormalization

$$\delta^l(t-s) = \delta(s)\,\delta(t-s), \quad l = 2, 3, ... \tag{1.1}$$

for the higher powers of the Dirac delta function and choosing test functions $f : \mathbb{R} \to \mathbb{C}$ that vanish at zero, the symbols

$$B_k^n(f) = \int_{\mathbb{R}} f(s)\, {a_s^\dagger}^n\, a_s^k\, ds \;\; ; \quad n, k \in \{0, 1, 2, ...\} \tag{1.2}$$

with involution

$$\left(B_k^n(f)\right)^* = B_n^k(\bar{f}) \tag{1.3}$$

and

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$$B_0^0(f) = \int_{\mathbb{R}} f(s)\,ds \tag{1.4}$$

satisfy the RHPWN commutation relations

$$[B_k^n(g), B_K^N(f)]_{RHPWN} := (k\,N - K\,n)\; B_{k+K-1}^{n+N-1}(gf) \tag{1.5}$$

where for $n < 0$ and/or $k < 0$ we define $B_k^n(f) := 0$. Moreover, for $n, N \geq 2$ and $k, K \in \mathbb{Z}$ the white noise operators

$$\hat{B}_k^n(f) := \int_{\mathbb{R}} f(t)\, e^{\frac{k}{2}(a_t - a_t^\dagger)} \left(\frac{a_t + a_t^\dagger}{2}\right)^{n-1} e^{\frac{k}{2}(a_t - a_t^\dagger)}\, dt$$

satisfy the commutation relations

$$[\hat{B}_k^n(g), \hat{B}_K^N(f)]_{w_\infty} := ((N-1)\,k - (n-1)\,K)\; \hat{B}_{k+K}^{n+N-2}(g\,f) \tag{1.6}$$

of the Virasoro–Zamolodchikov–w_∞ Lie algebra of conformal field theory with involution

$$\left(\hat{B}_k^n(f)\right)^* = \hat{B}_{-k}^n(\bar{f})$$

In particular, for $n = N = 2$ we obtain

$$[\hat{B}_k^2(g), \hat{B}_K^2(f)]_{w_\infty} = (k - K)\; \hat{B}_{k+K}^2(gf)$$

which are the commutation relations of the Virasoro algebra. The analytic continuation $\{\hat{B}_z^n(f)\,;\, n \geq 2, z \in \mathbb{C}\}$ of the Virasoro–Zamolodchikov–w_∞ Lie algebra, and the RHPWN Lie algebra with commutator $[\cdot,\cdot]_{RHPWN}$ have recently been identified (cf. [3]) thus bridging quantum probability with conformal field theory and high-energy physics.

Notation 1. *In what follows, for all integers n, k we will use the notation $B_k^n := B_k^n(\chi_I)$ where I is some fixed subset of $\mathbb{R}$ of finite measure $\mu := \mu(I) > 0$.*

2. The action of the RHPWN operators on the Fock vacuum vector Φ

2.1. **Definition of the RHPWN action on the Fock vacuum vector Φ.** To formulate a reasonable definition of the action of the RHPWN operators on Φ, we go to the level of white noise.

Lemma 1. *For all $t \geq s \geq 0$ and $n \in \{0, 1, 2, ...\}$*

$$(a_t^\dagger)^n\,(a_s)^n = \sum_{k=0}^{n} s_{n,k}\,(a_t^\dagger\,a_s)^k\,\delta^{n-k}(t-s)$$

where $s_{n,k}$ are the Stirling numbers of the first kind with $s_{0,0} = 1$ and $s_{0,k} = s_{n,0} = 0$ for all $n, k \geq 1$.

Proof. As shown in [4], if $[b, b^\dagger] = 1$ then

$$(b^\dagger)^k \, (b)^k = \sum_{m=0}^{k} s_{k,m} \, (b^\dagger \, b)^m \tag{2.1}$$

For fixed $t, s \in \mathbb{R}$ we define $b^\dagger$ and b through

$$\delta(t-s)^{1/2} \, b^\dagger = a_t^\dagger, \text{ and } \delta(t-s)^{1/2} \, b = a_s \tag{2.2}$$

Then $[b, b^\dagger] = 1$ and the result follows by substituting (2.2) into (2.1). □

Proposition 1. *For all integers $n \geq k \geq 0$ and for all test functions f*

$$B_k^n(f) = \int_{\mathbb{R}} f(t) \, (a_t^\dagger)^{n-k} \, (a_t^\dagger \, a_t)^k \, dt \tag{2.3}$$

Proof. For $n \geq k$ we can write

$$(a_t^\dagger)^n \, (a_s)^k = (a_t^\dagger)^{n-k} \, (a_t^\dagger)^k \, (a_s)^k$$

Multiplying both sides by $f(t) \, \delta(t-s)$ and then taking $\int_{\mathbb{R}} \int_{\mathbb{R}} ... ds \, dt$ of both sides of the resulting equation we obtain

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(t) \, (a_t^\dagger)^n \, (a_s)^k \, \delta(t-s) \, ds \, dt = \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) \, (a_t^\dagger)^{n-k} \, (a_t^\dagger)^k \, (a_s)^k \, \delta(t-s) \, ds \, dt$$

which, after applying (1.2) to its left and Lemma 1 to its right hand side, yields

$$\begin{aligned}
B_k^n(f) &= \sum_{m=0}^{k} s_{k,m} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) \, (a_t^\dagger)^{n-k} \, (a_t^\dagger \, a_s)^m \, \delta^{k-m+1}(t-s) \, ds \, dt \\
&= s_{k,k} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) \, (a_t^\dagger)^{n-k} \, (a_t^\dagger \, a_s)^k \, \delta(t-s) \, ds \, dt \\
&+ \sum_{m=0}^{k-1} s_{k,m} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) \, (a_t^\dagger)^{n-k} \, (a_t^\dagger \, a_s)^m \, \delta(s) \, \delta(t-s) \, ds \, dt \\
&= s_{k,k} \int_{\mathbb{R}} f(t) \, (a_t^\dagger)^{n-k} \, (a_t^\dagger \, a_t)^k \, dt \, + \, 0 \\
&= \int_{\mathbb{R}} f(t) \, (a_t^\dagger)^{n-k} \, (a_t^\dagger \, a_t)^k \, dt
\end{aligned}$$

where we have used the renormalization rule (1.1), $f(0) = 0$, and $s_{k,k} = 1$. □

Proposition 2. *Suppose that for all $n, k \in \{0,1,2,...\}$ and test functions f,*

$$B_k^n(f)\,\Phi := \begin{cases} 0 & \text{if } n<k \text{ or } n\cdot k<0 \\ B_0^{n-k}(f\,\sigma_k)\,\Phi & \text{if } n>k\geq 0 \\ \int_{\mathbb{R}} f(t)\,\rho_k(t)\,dt\,\Phi & \text{if } n=k \end{cases} \tag{2.4}$$

where σ_k and ρ_k are complex valued functions. Then for all $n \in \{0,1,2,...\}$

$$\sigma_n = \sigma_1^n \tag{2.5}$$

and

$$\rho_n = \frac{\sigma_1^n}{n+1} \tag{2.6}$$

Proof. By (2.4) and (1.2) for $k=0$, and by (1.4) for $n=k=0$ it follows that $\sigma_0=\rho_0=1$. For $n\geq 1$ we have

$$\begin{aligned} \langle B_0^n(f)\Phi, B_1^{n+1}(g)\Phi\rangle &= \langle B_0^n(f)\Phi, B_0^n(g\,\sigma_1)\Phi\rangle \\ &= \langle \Phi, B_n^0(\bar f)\,B_0^n(g\,\sigma_1)\,\Phi\rangle \\ &= \langle \Phi, (B_0^n(g\,\sigma_1)\,B_n^0(\bar f) + [B_n^0(\bar f), B_0^n(g\,\sigma_1)])\,\Phi\rangle \\ &= \langle \Phi, (0 + n^2\,B_{n-1}^{n-1}(\bar f\,g\,\sigma_1)])\,\Phi\rangle \\ &= n^2 \int_{\mathbb{R}} \rho_{n-1}(t)\,\sigma_1(t)\,\bar f(t)\,g(t)\,dt \end{aligned}$$

and also

$$\begin{aligned} \langle B_0^n(f)\Phi, B_1^{n+1}(g)\Phi\rangle &= \langle \Phi, B_n^0(\bar f)\,B_1^{n+1}(g)\,\Phi\rangle \\ &= \langle \Phi, (B_1^{n+1}(g)\,B_n^0(\bar f) + [B_n^0(\bar f), B_1^{n+1}(g)])\,\Phi\rangle \\ &= \langle \Phi, (0 + n\,(n+1)\,B_n^n(\bar f\,g)])\,\Phi\rangle \\ &= n\,(n+1)\int_{\mathbb{R}} \rho_n(t)\,\bar f(t)\,g(t)\,dt \end{aligned}$$

i.e., for all test functions h

$$n^2 \int_{\mathbb{R}} \rho_{n-1}(t)\,\sigma_1(t)\,h(t)\,dt = n\,(n+1)\int_{\mathbb{R}} \rho_n(t)\,h(t)\,dt$$

which implies that

$$\rho_n = \frac{n}{n+1}\,\sigma_1\,\rho_{n-1} = ... = \frac{\sigma_1^n}{n+1} \tag{2.7}$$

thus proving (2.6). Similarly,

$$\begin{aligned}
\int_{\mathbb{R}} \rho_n(t)\, f(t)\, g(t)\, dt &= \langle \Phi, B_n^n(f\, g)\, \Phi\rangle = \frac{1}{n+1}\langle \Phi, [B_n^{n-1}(f), B_1^2(g)]\, \Phi\rangle \\
&= \frac{1}{n+1}\langle \Phi, (B_n^{n-1}(f)\, B_1^2(g) - B_1^2(g)\, B_n^{n-1}(f))\, \Phi\rangle \\
&= \frac{1}{n+1}\langle \Phi, B_n^{n-1}(f)\, B_1^2(g)\, \Phi\rangle = \frac{1}{n+1}\langle B_{n-1}^n(\bar f)\, \Phi, B_1^2(g)\, \Phi\rangle \\
&= \frac{1}{n+1}\langle B_0^1(\sigma_{n-1}\, \bar f)\, \Phi, B_0^1(\sigma_1\, g)\, \Phi\rangle = \frac{1}{n+1}\langle \Phi, B_1^0(\bar\sigma_{n-1}\, f)\;\, B_0^1(\sigma_1\, g)\, \Phi\rangle \\
&= \frac{1}{n+1}\langle \Phi, [B_1^0(\bar\sigma_{n-1}\, f)\;\, B_0^1(\sigma_1\, g)]\, \Phi\rangle = \frac{1}{n+1}\langle \Phi, B_0^0(\bar\sigma_{n-1}\, f\, \sigma_1\, g)\, \Phi\rangle \\
&= \frac{1}{n+1}\int_{\mathbb{R}} \bar\sigma_{n-1}(t)\, \sigma_1(t)\, f(t)\, g(t)\, dt
\end{aligned}$$

Thus, for all test functions h

$$\int_{\mathbb{R}} \rho_n(t)\, h(t)\, dt = \frac{1}{n+1}\int_{\mathbb{R}} \bar\sigma_{n-1}(t)\, \sigma_1(t)\, h(t)\, dt$$

therefore

$$(n+1)\, \rho_n = \bar\sigma_{n-1}\, \sigma_1 \tag{2.8}$$

which combined with (2.6) implies

$$\bar\sigma_{n-1} = \sigma_1^{n-1}$$

which in turn implies that the σ_n's are real and yields (2.5).

□

In view of the interpretation of $a_t^\dagger$ and a_t as creation and annihilation densities respectively, it makes sense to assume that in the definition of the action of B_k^n on Φ it is only the difference $n-k$ that matters. Therefore we take the function σ_1 (and thus by (2.5) all the σ_n's) appearing in Proposition 2 to be identically equal to 1 and we arrive to the following definition of the action of the RHPWN operators on Φ.

Definition 1. *For $n, k \in \mathbb{Z}$ and test functions f*

$$B_k^n(f)\, \Phi := \begin{cases} 0 & \text{if } n<k \text{ or } n\cdot k<0 \\ B_0^{n-k}(f)\, \Phi & \text{if } n>k\geq 0 \\ \frac{1}{n+1}\int_{\mathbb{R}} f(t)\, dt\, \Phi & \text{if } n=k \end{cases} \tag{2.9}$$

2.2. **The n-th order RHPWN $*$–Lie algebras $\mathcal{L}_n$.**

Definition 2. *(i) $\mathcal{L}_1$ is the $*$–Lie algebra generated by B_0^1 and B_1^0 i.e., $\mathcal{L}_1$ is the linear span of $\{B_0^1, B_1^0, B_0^0\}$.*

(ii) $\mathcal{L}_2$ is the $*$–Lie algebra generated by B_0^2 and B_2^0 i.e., $\mathcal{L}_2$ is the linear span of $\{B_0^2, B_2^0, B_1^1\}$.

(iii) For $n \in \{3, 4, ...\}$, $\mathcal{L}_n$ is the $*$–Lie algebra generated by B_0^n and B_n^0 through repeated commutations and linear combinations. It consists of linear combinations of creation/annihilation operators of the form B_y^x where $x - y = k\,n$, $k \in \mathbb{Z} - \{0\}$, and of number operators B_x^x with $x \geq n - 1$.

2.3. **The Fock representation no-go theorem.** We will show that if the RHPWN action on Φ is that of Definition 1 then the Fock representation no-go theorems of [5] and [2] can be extended to the RHPWN $*$–Lie algebras $\mathcal{L}_n$ where $n \geq 3$.

Lemma 2. *For all $n \geq 3$ and with the action of the RHPWN operators on the vacuum vector Φ given by Definition 1, if a Fock space $\mathcal{F}_n$ for $\mathcal{L}_n$ exists then it contains both $B_0^n\,\Phi$ and $B_0^{2n}\,\Phi$.*

Proof. For simplicity we restrict to a single interval I of positive measure $\mu := \mu(I)$. We have

$$B_n^0\,B_0^n\,\Phi = (B_0^n\,B_n^0 + [B_n^0, B_0^n])\,\Phi = B_0^n\,B_n^0\,\Phi + n^2\,B_{n-1}^{n-1}\,\Phi = 0 + n^2\,\frac{\mu}{n}\,\Phi = n\,\mu\,\Phi$$

and

$$\begin{aligned}
B_n^0\,(B_0^n)^2\,\Phi &= B_n^0\,B_0^n\,B_0^n\,\Phi = (B_0^n\,B_n^0 + n^2\,B_{n-1}^{n-1})\,B_0^n\,\Phi \\
&= B_0^n\,n\,\mu\,\Phi + n^2\,(B_0^n\,B_{n-1}^{n-1} + [B_{n-1}^{n-1}, B_0^n])\Phi \\
&= n\,\mu\,B_0^n\,\Phi + n^2\,B_0^n\,\frac{\mu}{n}\,\Phi + n^2\,n\,(n-1)\,B_{n-2}^{2n-2}\,\Phi \\
&= 2\,n\,\mu\,B_0^n\,\Phi + n^3\,(n-1)\,B_0^n\,\Phi \\
&= (2\,n\,\mu + n^3\,(n-1))\,B_0^n\,\Phi
\end{aligned}$$

and also

$$
\begin{aligned}
B_n^0\,(B_0^n)^3\,\Phi &= (B_0^n\,B_n^0 + n^2\,B_{n-1}^{n-1})\,(B_0^n)^2\,\Phi \\
&= B_0^n\,(2\,n\,\mu + n^3\,(n-1))\,B_0^n\,\Phi + n^2\,(B_0^n\,B_{n-1}^{n-1} + n\,(n-1)\,B_{n-2}^{2n-2})\,B_0^n\,\Phi \\
&= (2\,n\,\mu + n^3\,(n-1))\,(B_0^n)^2\,\Phi + n^2\,B_0^n\,(B_0^n\,B_{n-1}^{n-1} + n\,(n-1)\,B_{n-2}^{2n-2})\,\Phi \\
&\quad + n^3\,(n-1)\,(B_0^n\,B_{n-2}^{2n-2} + n\,(n-2)\,B_{n-3}^{3n-3})\,\Phi \\
&= (2\,n\,\mu + n^3\,(n-1))\,(B_0^n)^2\,\Phi + n^2\,\frac{\mu}{n}\,(B_0^n)^2\,\Phi + n^3\,(n-1)\,(B_0^n)^2\,\Phi \\
&\quad + n^3\,(n-1)\,(B_0^n)^2\,\Phi + n^4\,(n-1)\,(n-2)\,B_0^{2n}\,\Phi \\
&= 3\,n\,(\mu + n^2\,(n-1))\,(B_0^n)^2\,\Phi + n^4\,(n-1)\,(n-2)\,B_0^{2n}\,\Phi
\end{aligned}
$$

Since $B_n^0\,(B_0^n)^3\,\Phi \in \mathcal{F}_n$ and $(B_0^n)^2\,\Phi \in \mathcal{F}_n$ it follows that $B_0^{2n}\,\Phi \in \mathcal{F}_n$.

$\square$

Theorem 1. *Let $n \geq 3$. If the action of the RHPWN operators on the vacuum vector Φ is given by Definition 1, then $\mathcal{L}_n$ does not admit a Fock representation.*

Proof. If a Fock representation of $\mathcal{L}_n$ existed then we should be able to define inner products of the form

$$\langle (a\,B_0^{2n} + b\,(B_0^n)^2)\Phi, (a\,B_0^{2n} + b\,(B_0^n)^2)\Phi \rangle$$

where $a, b \in \mathbb{R}$ and the RHPWN operators are defined on the same interval I of arbitrarily small positive measure $\mu(I)$. Using the notation $\langle x \rangle =< \Phi, x\,\Phi\rangle$ this amounts to the positive semi-definiteness of the matrix

$$A = \left[\begin{array}{cc} \langle B_{2n}^0\,B_0^{2n} \rangle & < B_{2n}^0\,(B_0^n)^2 \rangle \\ \langle B_{2n}^0\,(B_0^n)^2 \rangle & \langle (B_n^0)^2\,(B_0^n)^2 \rangle \end{array} \right]$$

Using (1.6) and Definition 1 we find that

$$\langle B_{2n}^0\,B_0^{2n} \rangle = 4\,n^2\, < B_{2n-1}^{2n-1} \rangle = 4\,n^2\,\frac{1}{2\,n}\,\mu(I) = 2\,n\,\mu(I)$$

and

$$
\begin{aligned}
\langle B_{2n}^0\,(B_0^n)^2 \rangle &= \langle B_0^{2n}\,\Phi, (B_0^n)^2\,\Phi \rangle = \langle B_n^0\,B_0^{2n}\,\Phi, B_0^n\,\Phi \rangle \\
&= 2\,n^2\,\langle B_{n-1}^{2n-1}\,\Phi, B_0^n\,\Phi \rangle = 2\,n^2\,\langle B_0^n\,\Phi, B_0^n\,\Phi \rangle \\
&= 2\,n^2\,\langle B_n^0\,B_0^n \rangle = 2\,n^2\,n^2\,\langle B_{n-1}^{n-1} \rangle \\
&= 2\,n^4\,\frac{1}{n}\,\mu(I) = 2\,n^3\,\mu(I)
\end{aligned}
$$

and also

$$\begin{aligned}
\langle (B_n^0)^2\,(B_0^n)^2\rangle &= \langle B_0^n\,\Phi, B_n^0\,(B_0^n)^2\,\Phi\rangle = \langle B_0^n\,\Phi, (B_n^0\,B_0^n)\,B_0^n\,\Phi\rangle\\
&= \langle B_0^n\,\Phi, (B_0^n\,B_n^0 + n^2\,B_{n-1}^{n-1})\,B_0^n\,\Phi\rangle\\
&= \langle B_0^n\,\Phi, B_0^n\,B_n^0\,B_0^n\,\Phi\rangle + n^2\,\langle B_0^n\,\Phi, B_{n-1}^{n-1}\,B_0^n\,\Phi\rangle\\
&= \langle B_n^0\,B_0^n\Phi, B_n^0\,B_0^n\,\Phi\rangle + n^2\,\langle B_0^n\,\Phi, (B_0^n\,B_{n-1}^{n-1} + n\,(n-1)\;B_{n-2}^{2n-2})\,\Phi\rangle\\
&= n^4\,\langle B_{n-1}^{n-1}\Phi, B_{n-1}^{n-1}\Phi\rangle + n\,\mu(I)\,\langle B_0^n\,\Phi, B_0^n\,\Phi\rangle + n^3\,(n-1)\,\langle B_0^n\,\Phi, B_{n-2}^{2n-2}\,\Phi\rangle\\
&= n^2\,\mu(I)^2 + n\,\mu(I)\,\langle B_n^0\,B_0^n\rangle + n^3\,(n-1)\,\langle B_n^0 B_{n-2}^{2n-2}\rangle\\
&= n^2\,\mu(I)^2 + n^3\,\mu(I)\,\langle B_{n-1}^{n-1}\rangle + n^4\,(n-1)\,(2\,n-2)\,\langle B_{2n-3}^{2n-3}\rangle\\
&= n^2\,\mu(I)^2 + n^2\,\mu(I)^2 + n^4\,(n-1)\,\mu(I)\\
&= 2\,n^2\,\mu(I)^2 + n^4\,(n-1)\,\mu(I)
\end{aligned}$$

Thus

$$A = \begin{bmatrix} 2\,n\,\mu(I) & 2\,n^3\,\mu(I) \\ & \\ 2\,n^3\,\mu(I) & 2\,n^2\,\mu(I)^2 + n^4\,(n-1)\,\mu(I) \end{bmatrix}.$$

A is a symmetric matrix, so it is positive semi-definite if and only if its minors are non-negative. The minor determinants of A are

$$d_1 = 2\,n\,\mu(I)$$

which is always nonnegative, and

$$d_2 = 2\,n^3\,\mu(I)^2\,(2\,\mu(I) - n^2 - n^3)$$

which is nonnegative if and only if

$$\mu(I) \geq \frac{n^2(n+1)}{2}$$

Thus the interval I cannot be arbitrarily small. □

3. The n-th order truncated RHPWN (or TRHPWN) Fock space $\mathcal{F}_n$

3.1. **Truncation of the RHPWN Fock kernels.** The generic element of the $*$-Lie algebras $\mathcal{L}_n$ of Definition 2 is B_0^n. All other elements of $\mathcal{L}_n$ are obtained by taking adjoints, commutators, and linear combinations. It thus makes sense to consider $(B_0^n(f))^k\,\Phi$ as basis vectors for the n-th particle space of the Fock space $\mathcal{F}_n$ associated with $\mathcal{L}_n$. A calculation of the "Fock kernel" $\langle (B_0^n)^k\,\Phi, (B_0^n)^k\,\Phi\rangle$ reveals that it is the terms containing $B_0^{2\,n}\,\Phi$ that prevent the kernel from being positive definite. The $B_0^{2\,n}\,\Phi$ terms appear either directly or by applying Definition 1 to terms of the form $B_y^x\,\Phi$ where $x - y = 2\,n$. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ do not contain B_0^2 and B_0^4 respectively, that problem exists for $n \geq 3$ only and the Fock spaces

$\mathcal{F}_1$ and $\mathcal{F}_2$ are actually not truncated. In what follows we will compute the Fock kernels by applying Definition 1 and by truncating "singular" terms of the form

$$\langle (B_0^n)^k\,\Phi, (B_0^n)^m\,B_y^x\,\Phi\,\rangle \tag{3.1}$$

where $n\,k = n\,m + x - y$ and $x - y = 2\,n$ i.e., $k - m = 2$. This amounts to truncating the action of the principal $\mathcal{L}_n$ number operator B_{n-1}^{n-1} on the "number vectors" $(B_0^n)^k\,\Phi$, which by commutation relations (1.5) and Definition 1 is of the form

$$B_{n-1}^{n-1}\,(B_0^n)^k\,\Phi = \left(\frac{\mu}{n} + k\,n\,(n-1)\right)\,(B_0^n)^k\,\Phi + \sum_{i\geq 1}\prod_{j\geq 1} c_{i,j}\,B_0^{\lambda_{i,j}\,n}\,\Phi$$

(where for each i not all positive integers $\lambda_{i,j}$ are equal to 1) by omitting the $\sum_{i\geq 1}\prod_{j\geq 1} c_{i,j}\,B_0^{\lambda_{i,j}\,n}\,\Phi$ part. We thus arrive to the following:

Definition 3. *For integers $n \geq 1$ and $k \geq 0$,*

$$B_{n-1}^{n-1}\,(B_0^n)^k\,\Phi := \left(\frac{\mu}{n} + k\,n\,(n-1)\right)\,(B_0^n)^k\,\Phi \tag{3.2}$$

i.e., the number vectors $(B_0^n)^k\,\Phi$ are eigenvectors of the principal $\mathcal{L}_n$ number operator B_{n-1}^{n-1} with eigenvalues $\left(\frac{\mu}{n} + k\,n\,(n-1)\right)$.

In agreement with Definition 1, for $k = 0$ Definition 3 yields $B_{n-1}^{n-1}\,\Phi := \frac{\mu}{n}\,\Phi$.

3.2. **Outline of the Fock space construction method.** We will construct the TRHPWN Fock spaces by using the following method (cf. Chapter 3 of [13]):

(i) Compute

$$\|(B_0^n)^k\,\Phi\|^2 = \langle (B_0^n)^k\,\Phi, (B_0^n)^k\,\Phi\rangle := \pi_{n,k}(\mu)$$

where $k = 0, 1, 2, \ldots$, Φ is the RHPWN vacuum vector, and $\pi_{n,k}(\mu)$ is a polynomial in μ of degree k.

(ii) Using the fact that if $k \neq m$ then $\langle (B_0^n)^k\,\Phi, (B_0^n)^m\,\Phi\rangle = 0$, for $a, b \in \mathbb{C}$ compute

$$\begin{aligned}
\langle e^{a\,B_0^n}\,\Phi, e^{b\,B_0^n}\,\Phi\rangle &= \sum_{k=0}^{\infty} \frac{(\bar{a}\,b)^k}{(k!)^2}\,\langle (B_0^n)^k\,\Phi, (B_0^n)^k\,\Phi\rangle \\
&= \sum_{k=0}^{\infty} \frac{(\bar{a}\,b)^k}{k!}\,\frac{\pi_{n,k}(\mu)}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(\bar{a}\,b)^k}{k!}\,h_{n,k}(\mu)
\end{aligned}$$

where

$$h_{n,k}(\mu) := \frac{\pi_{n,k}(\mu)}{k!} \tag{3.3}$$

(iii) Look for a function $G_n(u,\mu)$ such that

$$G_n(u,\mu) = \sum_{k=0}^{\infty} \frac{u^k}{k!}\, h_{n,k}(\mu) \tag{3.4}$$

Using the Taylor expansion of $G_n(u,\mu)$ in powers of u

$$G_n(u,\mu) = \sum_{k=0}^{\infty} \frac{u^k}{k!} \frac{\partial^k}{\partial u^k}\, G_n(u,\mu)|_{u=0} \tag{3.5}$$

by comparing (3.5) and (3.4) we see that

$$\frac{\partial^k}{\partial u^k}\, G_n(u,\mu)|_{u=0} = h_{n,k}(\mu) \tag{3.6}$$

Equation (3.6) plays a fundamental role in the search for G_n in what follows.

(iv) Reduce to single intervals and extend to step functions: For $u = \bar{a}\, b$, assuming that

$$G_n(u,\mu) = e^{\mu\, \hat{G}_n(u)} \tag{3.7}$$

which is typical for "Bernoulli moment systems" (cf. Chapter 5 of [13]), equation (3.4) becomes

$$e^{\mu\, \hat{G}_n(\bar{a}\, b)} = \sum_{k=0}^{\infty} \frac{(\bar{a}\, b)^k}{k!}\, h_{n,k}(\mu) \tag{3.8}$$

Take the product of (3.8) over all sets I, for test functions $f := \sum_i a_i\, \chi_{I_i}$ and $g := \sum_i b_i\, \chi_{I_i}$ with $I_i \cap I_j = \oslash$ for $i \neq j$, and end up with an expression like

$$e^{\int_{\mathbb{R}} \hat{G}_n(f(t)\, g(t))\, dt} = \prod \langle e^{a\, B_0^n}\, \Phi, e^{b\, B_0^n}\, \Phi\rangle \tag{3.9}$$

which we take as the definition of the inner product $\langle \psi_n(f), \psi_n(g)\rangle_n$ of the "exponential vectors"

$$\psi_n(f) := \prod_i e^{a_i\, B_0^n(\chi_{I_i})}\, \Phi \tag{3.10}$$

of the TRHPWN Fock space $\mathcal{F}_n$. Notice that $\Phi = \psi_n(0)$.

3.3. Construction of the TRHPWN Fock spaces $\mathcal{F}_n$.

Lemma 3. *Let $n \geq 1$ be fixed. Then for all integers $k \geq 0$*

$$B_n^0\,(B_0^n)^{k+1}\,\Phi := n\,(k+1)\,\left(\mu + k\,\frac{n^2\,(n-1)}{2}\right)\,(B_0^n)^k\,\Phi \tag{3.11}$$

Proof. For $k=0$ we have

$$\begin{aligned} B_n^0\,B_0^n\,\Phi &= (B_0^n\,B_n^0 + [B_n^0, B_0^n])\,\Phi = 0 + n^2\,B_{n-1}^{n-1}\,\Phi \\ &= n^2\,\frac{\mu}{n}\,\Phi = n\,\mu\,\Phi = n\,(0+1)\,\left(\mu + 0\,\frac{n^2\,(n-1)}{2}\right)\,(B_0^n)^0\,\Phi \end{aligned}$$

Assuming (3.11) to be true for k we have

$$\begin{aligned} & B_n^0\,(B_0^n)^{k+2}\,\Phi = (B_n^0\,B_0^n)\,(B_0^n)^{k+1}\,\Phi = (B_0^n\,B_n^0 + n^2\,B_{n-1}^{n-1})\,(B_0^n)^{k+1}\,\Phi \\ & = B_0^n\,B_n^0\,(B_0^n)^{k+1}\,\Phi + n^2\,B_{n-1}^{n-1}\,(B_0^n)^{k+1}\,\Phi \\ & = B_0^n\,n\,(k+1)\,\left(\mu + k\,\frac{n^2\,(n-1)}{2}\right)\,(B_0^n)^k\,\Phi + n^2\,B_{n-1}^{n-1}\,(B_0^n)^{k+1}\,\Phi \\ & = \left(n\,(k+1)\,\left(\mu + k\,\frac{n^2\,(n-1)}{2}\right) + n^2\,\left(\frac{\mu}{n} + (k+1)\,n\,(n-1)\right)\right)\,(B_0^n)^{k+1}\,\Phi \\ & = n\,(k+2)\,\left(\mu + (k+1)\,\frac{n^2\,(n-1)}{2}\right)\,(B_0^n)^{k+1}\,\Phi \end{aligned}$$

which proves (3.11) to be true for $k+1$ also, thus completing the induction. □

Proposition 3. *For all $n \geq 1$*

$$\pi_{n,k}(\mu) := \langle (B_0^n)^k\,\Phi, (B_0^n)^k\,\Phi\rangle = k!\,n^k\,\prod_{i=0}^{k-1}\left(\mu + \frac{n^2\,(n-1)}{2}\,i\right) \tag{3.12}$$

Proof. Let $n \geq 1$ be fixed. Define

$$a_k := k!\,n^k\,\prod_{i=0}^{k-1}\left(\mu + \frac{n^2\,(n-1)}{2}\,i\right)$$

Then

$$a_1 = n\,\mu$$

and for $k \geq 1$

$$a_{k+1} = n\,(k+1)\,\left(\mu + k\,\frac{n^2\,(n-1)}{2}\right)\,a_k$$

Similarly, define

$$b_k := \langle (B_0^n)^k\,\Phi, (B_0^n)^k\,\Phi\rangle$$

Then

$$b_1 = \langle B_0^n\,\Phi, B_0^n\,\Phi\rangle = \langle \Phi, B_n^0\,B_0^n\,\Phi\rangle = n^2\langle \Phi, B_{n-1}^{n-1}\,\Phi\rangle = n^2\,\frac{\mu}{n} = n\,\mu$$

and for $k \geq 1$, using Lemma 3

$$\begin{aligned} b_{k+1} &= \langle (B_0^n)^k\,\Phi, B_n^0\,(B_0^n)^{k+1}\,\Phi\rangle = n\,(k+1)\,\left(\mu + k\,\frac{n^2\,(n-1)}{2}\right)\,\langle (B_0^n)^k\,\Phi, (B_0^n)^k\,\Phi\rangle \\ &= n\,(k+1)\,\left(\mu + k\,\frac{n^2\,(n-1)}{2}\right)\,b_k \end{aligned}$$

Thus $a_k = b_k$ for all $k \geq 1$.

□

Corollary 1. *The functions $h_{n,k}$ appearing in (3.3) are given by*

$$h_{1,k} = \mu^k \tag{3.13}$$

and for $n \geq 2$

$$h_{n,k} = n^k \prod_{i=0}^{k-1} \left(\mu + \frac{n^2\,(n-1)}{2}\,i\right) \tag{3.14}$$

Proof. The proof follows from Proposition 3 and (3.3). □

Corollary 2. *The functions G_n appearing in (3.4) are given by*

$$G_1(u,\mu) = e^{u\,\mu} \tag{3.15}$$

and for $n \geq 2$

$$G_n(u,\mu) = \left(1 - \frac{n^3\,(n-1)}{2}\,u\right)^{-\frac{2}{n^2\,(n-1)}\,\mu} = e^{-\frac{2}{n^2\,(n-1)}\,\mu\,\ln\left(1 - \frac{n^3\,(n-1)}{2}\,u\right)} \tag{3.16}$$

where ln *denotes logarithm with base e.*

Proof. The proof follows from the fact that for G_n given by (3.15) and (3.16), in accordance with (3.6), we have

$$\frac{\partial^k}{\partial u^k} G_n(u,\mu)|_{u=0} = n^k \prod_{i=0}^{k-1} \left(\mu + \frac{n^2\,(n-1)}{2}\, i\right)$$

□

Corollary 3. *The functions $\hat{G}_n$ appearing in (3.5) are given by*

$$\hat{G}_1(u) = u \tag{3.17}$$

and for $n \geq 2$

$$\hat{G}_n(u) = -\frac{2}{n^2\,(n-1)}\, \ln\left(1 - \frac{n^3\,(n-1)}{2}\, u\right) \tag{3.18}$$

Proof. The proof follows directly from Corollary 2.

□

Corollary 4. *The $\mathcal{F}_n$ inner products are given by*

$$\langle \psi_1(f), \psi_1(g)\rangle_1 = e^{\int_{\mathbb{R}} \bar{f}(t)\, g(t)\, dt} \tag{3.19}$$

and for $n \geq 2$

$$\langle \psi_n(f), \psi_n(g)\rangle_n = e^{-\frac{2}{n^2\,(n-1)} \int_{\mathbb{R}} \ln\left(1 - \frac{n^3\,(n-1)}{2}\, \bar{f}(t)\, g(t)\right) dt} \tag{3.20}$$

where $|f(t)| < \frac{1}{n}\sqrt{\frac{2}{n\,(n-1)}}$ *and* $|g(t)| < \frac{1}{n}\sqrt{\frac{2}{n\,(n-1)}}$.

Proof. The proof follows from (3.9) and Corollary 2. □

The function G_1 of (3.15) and the Fock space inner product (3.19) are associated with the Heisenberg-Weyl algebra and the quantum stochastic calculus of [15]. For $n = 2$ the function G_n of (3.16) and the associated Fock space inner product (3.20) have appeared in the study of the Finite-Difference algebra and the Square of White Noise algebra in [8], [9], [11], and [12]. The functions G_n of (3.16) can also be found in Proposition 5.4.2 of Chapter 5 of [13].

Definition 4. *The n-th order TRHPWN Fock space $\mathcal{F}_n$ is the Hilbert space completion of the linear span of the exponential vectors $\psi_n(f)$ of (3.10) under the inner product $\langle\cdot,\cdot\rangle_n$ of Corollary 4. The full TRHPWN Fock space $\mathcal{F}$ is the direct sum of the $\mathcal{F}_n$'s.*

3.4. **Fock representation of the TRHPWN operators.**

Proposition 4. *For all test functions* $f := \sum_i a_i \chi_{I_i}$ *and* $g := \sum_i b_i \chi_{I_i}$ *with* $I_i \cap I_j = \oslash$ *for* $i \neq j$*, and for all* $n \geq 1$

$$B_n^0(f)\,\psi_n(g) = n \int_{\mathbb{R}} f(t)\,g(t)\,dt \;\; \psi_n(g) + \frac{n^3\,(n-1)}{2}\,\frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,\psi_n(g+\epsilon\,f\,g^2) \tag{3.21}$$

$$B_0^n(f)\,\psi_n(g) = \frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,\psi_n(g+\epsilon\,f) \tag{3.22}$$

Proof. By (3.10), the fact that $[B_n^0(\chi_{I_i}), e^{B_0^n(\chi_{I_j})}] = 0$ whenever $I_i \cap I_j = \oslash$, and by Lemma 3 we have

$$B_n^0(f)\,\psi_n(g) = \sum_{i=1}^m a_i\,B_n^0(\chi_{I_i})\,\prod_{j=1}^m e^{b_j\,B_0^n(\chi_{I_j})}\,\Phi$$

$$= \sum_{i=1}^m a_i\,\prod_{j=1}^m B_n^0(\chi_{I_i})\,e^{b_j\,B_0^n(\chi_{I_j})}\,\Phi$$

$$= \sum_{i=1}^m a_i\,\left(\prod_{\substack{j=1\\ j\neq i}}^m e^{b_j\,B_0^n(\chi_{I_j})}\right)\,B_n^0(\chi_{I_i})\,e^{b_i\,B_0^n(\chi_{I_i})}\,\Phi$$

$$= \sum_{i=1}^m a_i\,\left(\prod_{\substack{j=1\\ j\neq i}}^m e^{b_j\,B_0^n(\chi_{I_j})}\right)\,\sum_{k=0}^\infty \frac{b_i^k}{k!}\,B_n^0(\chi_{I_i})\,\left(B_0^n(\chi_{I_i})\right)^k\,\Phi$$

$$= \sum_{i=1}^{m} a_i \left(\prod_{\substack{j=1 \\ j\neq i}}^{m} e^{b_j\, B_0^n(\chi_{I_j})} \right) \sum_{k=0}^{\infty} \frac{b_i^k}{k!}\, n\, k\, \left(\mu(I_i) + (k-1)\, \frac{n^2\,(n-1)}{2} \right) \left(B_0^n(\chi_{I_i})\right)^{k-1}\, \Phi$$

$$= \sum_{i=1}^{m} a_i \left(\prod_{\substack{j=1 \\ j\neq i}}^{m} e^{b_j\, B_0^n(\chi_{I_j})} \right) \sum_{k=1}^{\infty} \frac{b_i^k}{(k-1)!}\, n\, \mu(I_i)\, \left(B_0^n(\chi_{I_i})\right)^{k-1}\, \Phi$$

$$+ \sum_{i=1}^{m} a_i \left(\prod_{\substack{j=1 \\ j\neq i}}^{m} e^{b_j\, B_0^n(\chi_{I_j})} \right) \sum_{k=2}^{\infty} \frac{b_i^k}{(k-2)!}\, \frac{n^3\,(n-1)}{2}\, \left(B_0^n(\chi_{I_i})\right)^{k-1}\, \Phi$$

$$= n\, \sum_{i=1}^{m} a_i\, b_i\, \mu(I_i) \left(\prod_{\substack{j=1 \\ j\neq i}}^{m} e^{b_j\, B_0^n(\chi_{I_j})} \right) e^{b_i\, B_0^n(\chi_{I_i})}\, \Phi$$

$$+ \frac{n^3\,(n-1)}{2} \sum_{i=1}^{m} a_i\, b_i^2\, B_0^n(\chi_{I_i}) \left(\prod_{\substack{j=1 \\ j\neq i}}^{m} e^{b_j\, B_0^n(\chi_{I_j})} \right) e^{b_i\, B_0^n(\chi_{I_i})}\, \Phi$$

$$= n\, \sum_{i=1}^{m} a_i\, b_i\, \mu(I_i) \left(\prod_{j=1}^{m} e^{b_j\, B_0^n(\chi_{I_j})} \right) \Phi$$

$$+ \frac{n^3\,(n-1)}{2} \sum_{i=1}^{m} a_i\, b_i^2\, B_0^n(\chi_{I_i})\, e^{b_i\, B_0^n(\chi_{I_i})} \left(\prod_{\substack{j=1 \\ j\neq i}}^{m} e^{b_j\, B_0^n(\chi_{I_j})} \right) \Phi$$

$$= n \int_{\mathbb{R}} f(t)\, g(t)\, dt\ \ \psi_n(g) + \frac{n^3\,(n-1)}{2} \sum_{i=1}^{m} \frac{\partial}{\partial\, \epsilon}|_{\epsilon=0}\, e^{(\epsilon\, a_i\, b_i^2 + b_i)\, B_0^n(\chi_{I_i})} \left(\prod_{\substack{j=1 \\ j\neq i}}^{m} e^{b_j\, B_0^n(\chi_{I_j})} \right) \Phi$$

$$= n \int_{\mathbb{R}} f(t)\, g(t)\, dt\ \ \psi_n(g) + \frac{n^3\,(n-1)}{2}\, \frac{\partial}{\partial\, \epsilon}|_{\epsilon=0} \left(\prod_{i=1}^{m} e^{(\epsilon\, a_i\, b_i^2 + b_i)\, B_0^n(\chi_{I_i})} \right) \Phi$$

$$= n \int_{\mathbb{R}} f(t)\, g(t)\, dt\ \ \psi_n(g) + \frac{n^3\,(n-1)}{2}\, \frac{\partial}{\partial\, \epsilon}|_{\epsilon=0}\, \psi_n(g + \epsilon\, f\, g^2)$$

To prove (3.22) we notice that for $n = 1$ (3.21) yields

$$B_1^0(f)\,\psi_1(g)=\int_{\mathbb{R}} f(t)\,g(t)\,dt\;\;\psi_1(g)$$

i.e., $B_1^0(f)=A(f)$ where $A(f)$ is the annihilation operator of Hudson-Parthasarathy calculus (cf. [15]) and so

$$B_0^1(f)\,\psi_1(g)=A^\dagger(f)\,\psi_1(g)=\frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,\psi_1(g+\epsilon\,f)$$

where $A^\dagger(f)$ is the creation operator of Hudson-Parthasarathy calculus thus proving (3.22) for $n=1$. To prove (3.22) for $n\geq 2$ we notice that by the duality condition (1.3) for all test functions f,g,ϕ

$$\begin{aligned}
&\langle B_0^n(f)\,\psi_n(\phi),\psi_n(g)\rangle_n=\langle\psi_n(\phi),B_n^0(\bar f)\,\psi_n(g)\rangle_n\\
&=n\int_{\mathbb{R}}\bar f(t)\,g(t)\,dt\;\langle\psi_n(\phi),\psi_n(g)\rangle_n+\frac{n^3\,(n-1)}{2}\,\frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,\langle\psi_n(\phi),\psi_n(g+\epsilon\,\bar f\,g^2)\rangle_n\\
&=n\int_{\mathbb{R}}\bar f(t)\,g(t)\,dt\;\langle\psi_n(\phi),\psi_n(g)\rangle_n\\
&+\frac{n^3\,(n-1)}{2}\,\frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,e^{-\frac{2}{n^2\,(n-1)}\int_{\mathbb{R}}\ln\left(1-\frac{n^3\,(n-1)}{2}\,\bar\phi(t)\,(g+\epsilon\,\bar f\,g^2)(t)\right)\,dt}\\
&=n\int_{\mathbb{R}}\bar f(t)\,g(t)\,dt\;\langle\psi_n(\phi),\psi_n(g)\rangle_n\\
&+\frac{n^3\,(n-1)}{2}\;\langle\psi_n(\phi),\psi_n(g)\rangle_n\left(-\frac{2}{n^2\,(n-1)}\int_{\mathbb{R}}\frac{-\frac{n^3\,(n-1)}{2}\,\bar\phi\,\bar f\,g^2}{1-\frac{n^3\,(n-1)}{2}\,\bar\phi\,g}(t)\,dt\right)\\
&=\left(n\int_{\mathbb{R}}\bar f(t)\,g(t)\,dt+\frac{n^4\,(n-1)}{2}\int_{\mathbb{R}}\frac{\bar\phi\,\bar f\,g^2}{1-\frac{n^3\,(n-1)}{2}\,\bar\phi\,g}(t)\,dt\right)\,\langle\psi_n(\phi),\psi_n(g)\rangle_n\\
&=n\int_{\mathbb{R}}\frac{\bar f\,g}{1-\frac{n^3\,(n-1)}{2}\,\bar\phi\,g}(t)\,dt\;\langle\psi_n(\phi),\psi_n(g)\rangle_n\\
&=\frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,e^{-\frac{2}{n^2\,(n-1)}\int_{\mathbb{R}}\ln\left(1-\frac{n^3\,(n-1)}{2}\,(\bar\phi+\epsilon\,\bar f)(t)\,g(t)\right)\,dt}\\
&=\frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,\langle\psi_n(\phi+\epsilon\,f),\psi_n(g)\rangle_n\\
&=\langle\frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,\psi_n(\phi+\epsilon\,f),\psi_n(g)\rangle_n
\end{aligned}$$

which implies (3.22).

∎

Corollary 5. *For all $n\geq 1$ and test functions f,g,h*

$$B^{n-1}_{n-1}(f\,g)\,\psi_n(h)=\frac{1}{n}\int_{\mathbb{R}} f(t)\,g(t)\;\psi_n(h)$$
$$+\frac{n\,(n-1)}{2}\,\frac{\partial^2}{\partial\epsilon\;\partial\rho}|_{\epsilon=\rho=0}\;\big(\psi_n(h+\epsilon\,g+\rho\;f\,(h+\epsilon\,g)^2)-\psi_n(h+\epsilon\,f\,h^2+\rho\,g)\big) \tag{3.23}$$

Proof.

$$B^{n-1}_{n-1}(f\,g)\,\psi_n(h)=\frac{1}{n^2}\,[B^0_n(f),B^n_0(g)]\,\psi_n(h)$$
$$=\frac{1}{n^2}\;\big(B^0_n(f)\,B^n_0(g)-B^n_0(g)\,B^0_n(f)\big)\;\psi_n(h)$$
$$=\frac{1}{n^2}\,(B^0_n(f)\,\frac{\partial}{\partial\epsilon}|_{\epsilon=0}\,\psi_n(h+\epsilon\,g)-B^n_0(g)\,(n\int_{\mathbb{R}} f(t)\,h(t)\,dt\;\;\psi_n(h)$$
$$+\frac{n^3\,(n-1)}{2}\,\frac{\partial}{\partial\epsilon}|_{\epsilon=0}\,\psi_n(h+\epsilon\,f\,h^2)))$$
$$=\frac{1}{n^2}\,\frac{\partial}{\partial\epsilon}|_{\epsilon=0}\,B^0_n(f)\,\psi_n(h+\epsilon\,g)-\frac{1}{n}\int_{\mathbb{R}} f(t)\,h(t)\,dt\;\;B^n_0(g)\,\psi_n(h)$$
$$-\frac{n\,(n-1)}{2}\,\frac{\partial}{\partial\epsilon}|_{\epsilon=0}\,B^n_0(g)\,\psi_n(h+\epsilon\,f\,h^2)$$
$$=\frac{1}{n^2}\,\frac{\partial}{\partial\epsilon}|_{\epsilon=0}\,(n\int_{\mathbb{R}} f(t)\,(h+\epsilon\,g)(t)\,dt\;\;\psi_n(h+\epsilon\,g)$$
$$+\frac{n^3\,(n-1)}{2}\,\frac{\partial}{\partial\rho}|_{\rho=0}\,\psi_n(h+\epsilon\,g+\rho\;f\,(h+\epsilon\,g)^2))$$
$$-\frac{1}{n}\int_{\mathbb{R}} f(t)\,h(t)\,dt\;\frac{\partial}{\partial\epsilon}|_{\epsilon=0}\,\psi_n(h+\epsilon\,g)-\frac{n\,(n-1)}{2}\,\frac{\partial}{\partial\epsilon}|_{\epsilon=0}\,\frac{\partial}{\partial\rho}|_{\rho=0}\,\psi_n(h+\epsilon\,f\,h^2+\rho\,g)$$
$$=\frac{1}{n}\,\left(\int_{\mathbb{R}} f(t)\,g(t)\,dt\;\;\psi_n(h)+\int_{\mathbb{R}} f(t)\,h(t)\,dt\;\;\frac{\partial}{\partial\epsilon}|_{\epsilon=0}\,\psi_n(h+\epsilon\,g)\right)$$
$$+\frac{n\,(n-1)}{2}\,\frac{\partial^2}{\partial\epsilon\,\partial\rho}|_{\epsilon=\rho=0}\,\psi_n(h+\epsilon\,g+\rho\;f\,(h+\epsilon\,g)^2)$$
$$-\frac{1}{n}\int_{\mathbb{R}} f(t)\,h(t)\,dt\;\frac{\partial}{\partial\epsilon}|_{\epsilon=0}\,\psi_n(h+\epsilon\,g)$$
$$-\frac{n\,(n-1)}{2}\,\frac{\partial^2}{\partial\epsilon\,\partial\rho}|_{\epsilon=\rho=0}\,\psi_n(h+\epsilon\,f\,h^2+\rho\,g)$$
$$=\frac{1}{n}\int_{\mathbb{R}} f(t)\,g(t)\,dt\;\;\psi_n(h)$$
$$+\frac{n\,(n-1)}{2}\,\frac{\partial^2}{\partial\epsilon\,\partial\rho}|_{\epsilon=\rho=0}\;\big(\psi_n(h+\epsilon\,g+\rho\;f\,(h+\epsilon\,g)^2)-\psi_n(h+\epsilon\,f\,h^2+\rho\,g)\big)$$

□

Using the method described in Corollary 5, i.e., using the prescription

$$B^{n+N-1}_{k+K-1}(gf) := \frac{1}{k\,N - K\,n}\,\left(B^n_k(g)\,B^N_K(f) - B^N_K(f)\,B^n_k(g)\right)$$

and suitable linear combinations, we obtain the representation of the B^x_y (and therefore of the RHPWN and Virasoro–Zamolodchikov–w_∞ commutation relations) on the appropriate Fock space $\mathcal{F}_n$.

4. Classical stochastic processes on $\mathcal{F}_n$

Definition 5. *A quantum stochastic process $x = \{x(t)\,/\,t \geq 0\}$ is a family of Hilbert space operators. Such a process is said to be classical if for all $t, s \geq 0$, $x(t) = x(t)^*$ and $[x(t), x(s)] := x(t)\,x(s) - x(s)\,x(t) = 0$.*

Proposition 5. *Let $m > 0$ and let a quantum stochastic process $x = \{x(t)/\,t \geq 0\}$ be defined by*

$$x(t) := \textstyle\sum_{n,k\in\Lambda} c_{n,k}\,B^n_k(t) \tag{4.1}$$

where $c_{n,k} \in \mathbb{C} - \{0\}$, Λ is a finite subset of $\{0, 1, 2, ...\}$ and

$$B^n_k(t) := B^n_k(\chi_{[0,t]}) \in \mathcal{F}_m$$

If for each $n, k \in \Lambda$

$$c_{n,k} = \bar{c}_{k,n} \tag{4.2}$$

then the process $x = \{x(t)\,/\,t \geq 0\}$ is classical.

Proof. By (1.3), $x(t) = x^*(t)$ for all $t \geq 0$. Moreover, by (1.5), $[x(t), x(s)] = 0$ for all $t, s \geq 0$ since each term of the form $c_{N,K}\,c_{n,k}\,[B^N_K(t), B^n_k(s)]$ is canceled out by the corresponding term of the form $c_{n,k}\,c_{N,K}\,[B^n_k(t), B^N_K(s)]$. Thus the process $x = \{x(t)\,/\,t \geq 0\}$ is classical. □

In the remaining of this section we will study the classical process $x = \{x(t)\,/\,t \geq 0\}$ whose Fock representation as a family of operators on $\mathcal{F}_n$ is

$$x(t) := B^n_0(t) + B^0_n(t)$$

By Proposition 4

$$B^n_0(t)\,\psi_n(g) = \frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,\psi_n(g + \epsilon\,\chi_{[0,t]}) \tag{4.3}$$

$$B^0_n(t)\,\psi_n(g) = n\int_0^t g(s)\,ds\;\;\psi_n(g) + \frac{n^3\,(n-1)}{2}\,\frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,\psi_n(g + \epsilon\,\chi_{[0,t]}\,g^2) \tag{4.4}$$

In particular, for $g=0$

$$B_0^n(t)\,\psi_n(0) \;=\; \frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,\psi_n(\epsilon\,\chi_{[0,t]}) \tag{4.5}$$

$$B_n^0(t)\,\psi_n(0) \;=\; 0 \tag{4.6}$$

Lemma 4 (Splitting formula)**.** *Let $s\in\mathbb{R}$. Then for $n=1$*

$$e^{s\,(B_0^1+B_1^0)}\,\Phi = e^{\frac{s^2}{2}\,\mu} e^{s\,B_0^1}\,\Phi \tag{4.7}$$

and for $n\geq 2$

$$e^{s\,(B_0^n+B_n^0)}\,\Phi = \left(\sec\left(\sqrt{\frac{n^3\,(n-1)}{2}}\,s\right)\right)^{\frac{2\,n\,\mu}{n^3\,(n-1)}}\; e^{\sqrt{\frac{2}{n^3\,(n-1)}}\,\tan\left(\sqrt{\frac{n^3\,(n-1)}{2}}\,s\right)\,B_0^n}\;\Phi \tag{4.8}$$

Proof. We will use the "differential method" of Proposition 4.1.1, Chapter 1 of [13]. So let

$$E\,\Phi := e^{s\,(B_0^n+B_n^0)}\,\Phi := e^{V(s)\,B_0^n}\,e^{W(s)}\,\Phi \tag{4.9}$$

where W,V are real-valued functions with $W(0)=V(0)=0$. Then,

$$\frac{\partial}{\partial\,s}\,E\,\Phi = (B_0^n+B_n^0)\,E\,\Phi = B_0^n\,E\,\Phi + B_n^0\,E\,\Phi \tag{4.10}$$

By Lemma 3 we have

$$B_n^0\,E\,\Phi = B_n^0\,e^{V(s)\,B_0^n}\,e^{W(s)}\,\Phi = e^{W(s)}\,B_n^0\,e^{V(s)\,B_0^n}\,\Phi$$
$$= e^{W(s)}\,\sum_{k=0}^{\infty}\frac{V(s)^k}{k!}\,B_n^0\,(B_0^n)^k\,\Phi$$
$$= e^{W(s)}\,\sum_{k=0}^{\infty}\frac{V(s)^k}{k!}\,n\,k\,\left(\mu+(k-1)\,\frac{n^2\,(n-1)}{2}\right)\,(B_0^n)^{k-1}\,\Phi$$
$$(n\,\mu\,V(s)+\frac{n^3\,(n-1)}{2}\,V(s)^2\,B_0^n)\,e^{V(s)\,B_0^n}\,e^{W(s)}\,\Phi$$
$$(n\,\mu\,V(s)+\frac{n^3\,(n-1)}{2}\,V(s)^2\,B_0^n)\,E\,\Phi$$

Thus (4.10) becomes

$$\frac{\partial}{\partial\,s}\,E\,\Phi = \left(B_0^n + n\,\mu\,V(s)+\frac{n^3\,(n-1)}{2}\,V(s)^2\,B_0^n\right)\,E\,\Phi \tag{4.11}$$

From (4.9) we also have

$$\frac{\partial}{\partial s} E\,\Phi = \left(V'(s)\, B_0^n + W'(s)\right)\, E\,\Phi \tag{4.12}$$

From (4.11) and (4.12), by equating coefficients of 1 and B_0^n, we have

$$W'(s) = n\,\mu\, V(s) \tag{4.13}$$

$$V'(s) = 1 + \frac{n^3\,(n-1)}{2}\, V(s)^2 \text{ (Riccati equation)} \tag{4.14}$$

For $n = 1$ we find $V(s) = s$ and $W(s) = \frac{s^2}{2}\,\mu$. For $n \geq 2$ by separating the variables we find

$$V(s) = \sqrt{\frac{2}{n^3\,(n-1)}}\,\tan\left(\sqrt{\frac{n^3\,(n-1)}{2}}\, s\right)$$

and so

$$W(s) = -\frac{2\,n\,\mu}{n^3\,(n-1)}\,\ln\left(\cos\left(\sqrt{\frac{n^3\,(n-1)}{2}}\, s\right)\right)$$

which implies that

$$e^{W(s)} = \left(\sec\left(\sqrt{\frac{n^3\,(n-1)}{2}}\, s\right)\right)^{\frac{2\,n\,\mu}{n^3\,(n-1)}}$$

thus completing the proof.

□

In the theory of Bernoulli systems and the Fock representation of finite-dimensional Lie algebras (cf. Chapter 5 of [13]) the Riccati equation (4.14) has the general form

$$V'(s) = 1 + 2\,\alpha\, V(s) + \beta\, V(s)^2$$

and the values of α and β determine the underlying classical probability distribution and the associated special functions. For example, for $\alpha = 1 - 2\,p$ and $\beta = -4\,p\,q$ we have the binomial process and the Krawtchouk polynomials, for $\alpha = p^{-1} - \frac{1}{2}$ and $\beta = q\,p^{-2}$ we have the negative binomial process and the Meixner polynomials, for $\alpha \neq 0$ and $\beta = 0$ we have the Poisson process and the Poisson-Charlier polynomials, for $\alpha^2 = \beta$ we have the exponential process and the Laguerre polynomials, for $\alpha = \beta = 0$ we have Brownian motion with moment generating function $e^{\frac{s^2}{2}\,t}$ and associated special functions the Hermite polynomials, and for $\alpha^2 - \beta < 0$ we have the continuous binomial and Beta processes (cf. Chapter 5 of [13] and also [14]) with moment generating function $(\sec s)^t$ and associated special functions the Meixner-Pollaczek polynomials. In the infinite-dimensional TRHPWN case the underlying classical probability distributions are given in the following.

Proposition 6 (Moment generating functions)**.** *For all $s \geq 0$*

$$\langle e^{s\,(B_0^1(t)+B_1^0(t))}\,\Phi,\Phi\rangle_1 = e^{\frac{s^2}{2}\,t} \tag{4.15}$$

i.e., $\{B_0^1(t)+B_1^0(t)\,/\,t\geq 0\}$ is Brownian motion (cf. [13], [15] *) while for $n\geq 2$*

$$\langle e^{s\,(B_0^n(t)+B_n^0(t))}\,\Phi,\Phi\rangle_n = \left(\sec\left(\sqrt{\frac{n^3\,(n-1)}{2}}\,s\right)\right)^{\frac{2\,n\,t}{n^3\,(n-1)}} \tag{4.16}$$

i.e., $\{B_0^n(t)+B_n^0(t)\,/\,t\geq 0\}$ is for each n a continuous binomial/Beta process (see Appendix)

Proof. The proof follows from Lemma 4, $\mu([0,t])=t$, and the fact that for all $n\geq 1$ we have $B_n^0(t)\,\Phi=0$.

□

5. Appendix: The continuous Binomial and Beta Processes

Let

$$b(n,k)=\binom{n}{k}\,x^k\,(1-x)^{n-k}\ \ ;\ \ n,k\in\{0,1,2,...\},\ n\geq k,\ x\in(0,1)$$

be the standard Binomial distribution. Using the Gamma function we can analytically extend from $n,k\in\{0,1,2,...\}$ to $z,w\in\mathbb{C}$ with $\Re\,z\ \geq\ \Re\,w\ >\ -1$ and we have

$$\begin{aligned}
b(z,w) &= \frac{\Gamma(z+1)}{\Gamma(z-w+1)\,\Gamma(w+1)}\,x^w\,(1-x)^{z-w}\\
&= \frac{1}{z+1}\,\frac{\Gamma(z+2)}{\Gamma(z-w+1)\,\Gamma(w+1)}\,x^w\,(1-x)^{z-w}\\
&= \frac{1}{z+1}\,\frac{\Gamma(z+2)}{\Gamma(z-w+1)\,\Gamma(w+1)}\,x^{(w+1)-1}\,(1-x)^{(n-w+1)-1}\\
&= \frac{1}{z+1}\,\beta(w+1,\ z-w+1)
\end{aligned}$$

where $\beta(w+1,\ z-w+1)$ is the analytic continuation to $\Re\,a\ >\ 0$ and $\Re\,c\ >\ 0$ of the standard Beta distribution

$$\beta(a,c)=\frac{\Gamma(a+c)}{\Gamma(a)\,\Gamma(c)}\,x^{a-1}\,(1-x)^{c-1}\ \ ;\ \ a>0\,,\,c>0$$

Proposition 7. *For each $t>0$ let X_t be a random variable with distribution given by the density*

$$p_t(x)=\frac{2^{t-1}}{2\,\pi}\,\beta(\frac{t+i\,x}{2},\frac{t-i\,x}{2})$$

Then the moment generating funcion of X_t is

$$\langle e^{s\,X_t}\rangle := \int_{-\infty}^{\infty} e^{s\,x}\, p_t(x)\, dx = (\sec s)^t \quad ; \quad \forall\, t>0,\ s\in\mathbb{R} \tag{5.1}$$

Proof. See Proposition 4.1.1, Chapter 5 of [13]. $\square$

Corollary 6. *With X_t and p_t as in Proposition 7, let*

$$Y_t := \sqrt{\frac{n^3\,(n-1)}{2}}\, X_t$$

Then the moment generating funcion of Y_t with respect to the density

$$q_t := p_{\frac{2\,n}{n^3\,(n-1)}\,t}$$

where $n\in\{1,2,...\}$, is

$$\langle e^{s\,Y_t}\rangle = \left(\sec\left(\sqrt{\frac{n^3\,(n-1)}{2}}\, s\right)\right)^{\frac{2\,n\,t}{n^3\,(n-1)}}$$

Proof. Since p_t is for each $t>0$ a probability density function we have

$$\int_{-\infty}^{\infty} p_t(x)\, dx = 1 \quad ; \quad \forall\, t>0$$

and so for $t := \frac{2\,n}{n^3\,(n-1)}\,t$

$$\int_{-\infty}^{\infty} p_{\frac{2\,n}{n^3\,(n-1)}\,t}(x)\, dx = 1 \quad ; \quad \forall\, t>0$$

i.e.,

$$\int_{-\infty}^{\infty} q_t(x)\, dx = 1 \quad ; \quad \forall\, t>0$$

so q_t is for each $t>0$ a probability density function. Moreover, letting $t := \frac{2\,n}{n^3\,(n-1)}\,t$ and $s := \sqrt{\frac{n^3\,(n-1)}{2}}\, s$ in (5.1) we obtain

$$\int_{-\infty}^{\infty} e^{s\,\sqrt{\frac{n^3\,(n-1)}{2}}\,x}\, q_t(x)\, dx = \left(\sec\left(\sqrt{\frac{n^3\,(n-1)}{2}}\, s\right)\right)^{\frac{2\,n\,t}{n^3\,(n-1)}}$$

which is precisely the moment generating function $\langle e^{s\,Y_t}\rangle$ of Y_t with respect to q_t.

$\square$

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