PROBLEM DEPARTMENT

ASHLEY AHLIN* AND HAROLD REITER†

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@email.uncc.edu`*. Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by March 1, 2007. Solutions identified as by students are given preference.*

Problems for Solution.

Solutions. We would like to make the following corrections with apologies. In the Spring 2006 issue, we failed to credit **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan with the solution to problem 1116. Also, we failed to credit **The Armstrong Problem Solvers**, Armstrong State University, Savannah, GA. with solutions to problems 1110, 1112, 1114, 1115, 1116, 1117, 1118, 1120, 1121, and 1122.

1123. *Proposed by Mike Pinter, Belmont University, Nashville, TN*

This problem is in honor of the 300^{th} birthday of Benjamin Franklin. Consider the base 9 cryptarithm $K\ I\ T\ E\ +\ K\ E\ Y\ =\ S\ H\ O\ C\ K$. Find a solution that minimizes the $S\ H\ O\ C\ K$.

Solution by **Frank Battles***, Massachusetts Maritime Academy, Buzzards Bay, MA*

Clearly $S = 1,\ H = 0$, and $K = 8$. We now aim for a minimum $SHOCK$. If $O = 2$, then $I = 3$, but $E + Y = 8$ is not possible with the remaining digits, (4, 5, 6, 7). Trying $O = 3$, then $I = 4$. For $E + Y = 8$, we must have either (i) $E = 6,\ Y = 2$, but $T + 6 = C$ is impossible with the remaining digits or (ii) $E = 2,\ Y = 6$. Now we have a solution with $T = 5,\ C = 7$. Our solution is $8452 + 826 = 10378$.

Also solved by **ABC Student Problem Solving Group**, Mountain Lakes High School, Mountain Lakes, NJ; **Paul S. Bruckman**, Sointula, BC; **Josh Caswell, undergraduate**, Eastern Kentucky University, Richmond, KY; **Thomas Dence**, Ashland University, Ashland, OH; **Billy Dobbs, graduate student**, Eastern Kentucky University, Richmond, KY; **Clayton Dodge**, Orono, ME; **Mark Evans**, Louisville, KY; **Robert Gebhardt**, Hopatcong, NJ; **Bohyun Areum Han, undergraduate**, Eastern Kentucky University, Richmond, KY; **Richard Hess**, Rancho Palos Verdes, CA; **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan; **Rex H. Wu**, Brooklyn, NY; and the **Proposer**.

1124. *Proposed by Paul S. Bruckman, Sointula, BC, Canada*

*Nashville, TN
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Given positive integers a and b, let $S(a,b) = \sum_{j=0}^{a}\{\lfloor b(1-j^2/a^2)^{1/2}\rfloor + 1\}$. Prove that $S(a,b) = S(b,a)$.

Solution by **Jaree Hudson, student and the Armstrong Problem Solvers**, *Armstrong State University, Savannah, GA*

If we set $y = b\sqrt{1-x^2/a^2}$, then for integers j with $0 \le j \le a$,

$$\left\lfloor b\sqrt{1-j^2/a^2}\right\rfloor + 1$$

counts the number of points (j,y) on the vertical line $x = j$ on or above the x-axis and on or below the ellipse $x^2/a^2 + y^2/b^2 = 1$. Thus, $S(a,b)$ counts the number of points (x,y) with nonnegative integer coordinates inside or on the ellipse $x^2/a^2 + y^2/b^2 = 1$. Similarly, if we set $x = a\sqrt{1-y^2/b^2}$, then for integers j with $0 \le j \le b$,

$$\left\lfloor a\sqrt{1-j^2/b^2}\right\rfloor + 1$$

counts the number of points (x,j) on the horizontal line $y = j$ on or to the right of the y-axis and on or to the left of the ellipse $x^2/a^2 + y^2/b^2 = 1$. Thus $S(b,a)$ also counts the number of points (x,y) with nonnegative integer coordinates inside or on the ellipse $x^2/a^2 + y^2/b^2 = 1$, and hence must be equal to $S(a,b)$.

Also solved by the **Proposer**.

1125. *Proposed by David Wells, Penn State New Kensington, Upper Burrell, PA*

For each positive integer n, let $P(n)$ be the product of the decimal digits of n, let $P_1(n) = P(n)$, and for $k \ge 2$, let $P_k(n) = P(P_{k-1}(n))$. Prove that $P_k(n) = 1$ for some k if and only if n contains no digits other than 1.

Solution by **Clayton Dodge**, *Orono, ME*

If the positive integer n contains an even digit, then $P(n)$ is an even number and hence cannot be equal to 1. Similarly, if n contains the digit 5, then $P(n)$ terminates in either 0 or 5 and cannot equal 1. Therefore, we assume that n contains only the odd digits in the set $S = \{1,3,7,9\}$ in its base ten representation. It is easy to check that the product of any two not-necessarily-distinct members of that set has a tens digit that is even and a units digit that is a member of S. Now, if n is a number whose digits are in S, then $P(n)$ will be of the form $P(n) = 1^r 3^s y^7 9^u$ for some nonnegative integers r, s, t, and u. Furthermore, since $9 = 3^2, P(n)$ reduces to the form $P(n) = 3^s 7^t$. By testing each product of 3 and 7 times every two-digit number whose tens digit is even and whose units digit is a member of S, one discovers that all such products terminate in a member of S and have an even tens digit. It follows by induction that if the digits of n are members of S, then $P(n)$ is even and cannot be equal to 1. Hence, only if n is a number all of whose digits are 1s will $P(n) = 1$, and clearly $P(n) = 1$ for such an n.

Also solved by **Armstrong Problem Solvers**, Armstrong Atlantic State University, Savannah, GA; **Paul S. Bruckman**, Sointula, BC; **Mark Evans**, Louisville, KY; **James Hall**, La Crescenta, CA; **Richard Hess**, Rancho Palos Verdes, CA; **Mike Pinter**, Belmont University, Nashville, TN; **Harry Sedinger**, St. Bonaventure University, St. Bonaventure, NY; **Mike Stein, undergraduate**, The College Of New Jersey, Ewing NJ; **Rex H. Wu**, Brooklyn, NY; and the **Proposer**.

1126. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA*

Find a rational function $f(x)$ with integer coefficients such that

$$\cos\theta = f(\sin\theta - \cos\theta)$$

is an identity or prove that no identity of this form exists.

Solution by **Paola Perfetti**, *Università degli Studi di Roma "Tor Vergata", Rome, Italy*

No identity of the given form can exist because $\sin\theta - \cos\theta = 1$ for $\theta_1 = \dfrac{\pi}{2}$ and $\theta_2 = \pi$ but $\cos\pi = -1$ while $\cos\dfrac{\pi}{2} = 0$.

Also solved by **Armstrong Problem Solvers**, Armstrong Atlantic State University, Savannah, GA; **Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie**, Angelo State University, San Angelo, TX; **Frank Battles**, Massachusetts Maritime Academy, Buzzards Bay, MA; **Brian Bradie**, Christopher Newport University, Newport News VA; **Paul S. Bruckman**, Sointula, BC; **Richard Hess**, Rancho Palos Verdes, CA; **Brendan Kelly, student**, The College Of New Jersey, Ewing NJ; **Miguel Lerma**, Northwestern University Problem Solving Group, Evanston, IL; **David E. Manes**, Oneonta, NY; **Raúl A. Simón**, LAMB, Santiago, CHILE; **Rex H. Wu**, Brooklyn, NY; and the **Proposer**.

1127. *Proposed by Arthur Holshouser, Charlotte, NC*

A bug starts from the origin on the plane and crawls one unit upwards to $(0,1)$ after one minute. During the second minute, it crawls two units to the right ending at $(2,1)$. Then during the third minute, it crawls three units upward, arriving at $(2,4)$. It makes another right turn and crawls four units during the fourth minute. From here it continues to crawl n units during minute n and then making a $90°$, either left or right. The bug continues this until after 16 minutes, it finds itself back at the origin. Its path does not intersect itself. What is the maximum possible area of the 16-gon traced out by its path?

Solution by **Armstrong Problem Solvers**, *Armstrong Atlantic State University, Savannah, GA*

The maximum possible area is 664 square units. The bug's path is determined for the first four minutes, after which its location is $(6,4)$. Since the bug moves vertically during each odd minute, the condition that it finishes at the origin means that $4 \pm 5 \pm 7 \pm 9 \pm 11 \pm 13 \pm 15 = 0$. Since $\pm 5 \pm 7 \pm 9 = \pm 3$, ± 7, ± 11, or ± 21; and $\pm 11 \pm 13 \pm 15 = \pm 9$, ± 13, ± 17, or ± 39; the only possibility is that $4 - 21 + 17 = 0$, so that $4 - 5 - 7 - 9 - 11 + 13 + 15 = 0$.

Similarly, since the bug moves horizontally during each even minute, we must have $6 \pm 6 \pm 8 \pm 10 \pm 12 \pm 14 \pm 16 = 0$. Since $\pm 6 \pm 8 \pm 10 = \pm 4$, ± 8, ± 12, or ± 24; and $\pm 12 \pm 14 \pm 16 = \pm 10$, ± 14, ± 18, or ± 42; there are four possible sums: $6 + 4 - 10 = 0$, $6 + 8 - 14 = 0$, $6 + 12 - 18 = 0$, or $6 - 24 + 18 = 0$. Since the bug crosses the x-axis at $(6,0)$ during the fifth minute, the last move must be to the right, rather than to the left, ruling out the second and third paths. This leaves only two possible sequences of horizontal moves: $6+6+8-10-12-14+16 = 0$ and $6-6-8-10-12+14+16 = 0$. The first path encloses an area of 664 square units, whereas the second encloses an area of 402 square units; thus the maximum possible area is 664 square units.

Also solved by **ABC Student Problem Solving Group**, Mountain Lakes High School, Mountain Lakes, NJ; **Paul S. Bruckman**, Sointula, BC; **Cal Poly Pomona Problem Solving Group**, Pomona, CA; **Mark Evans**, Louisville, KY; **Richard Hess**, Rancho Palos Verdes, CA; **Andy Shapiro, undergraduate**, The College of New Jersey, Bensalem, PA; **Raúl A. Simón**, LAMB, Santiago, CHILE; **Rex H. Wu**, Brooklyn, NY; and the **Proposer**.

1128. *Proposed by Brian Bradie, Christopher Newport University, Newport News, VA*

Evaluate

$$\int_0^{\pi/2} \frac{1}{1+\tan^n x}\ dx,$$

for $n = 0, 1, 2, \ldots$.

Solution by **Lee Kennard, student**, *Kenyon College, Gambier, OH*

Let I be the value of the given integral. Then, on one hand, we can rewrite I as

$$I = \int_0^{\pi/2} \frac{1}{1+\tan^n x} dx = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x} dx.$$

On the other hand, we can make the substitution $u = \frac{\pi}{2} - x$ to obtain another expression for I:

$$\begin{aligned} I &= \int_{\pi/2}^{0} \frac{1}{1+\tan^n(\frac{\pi}{2}-u)}(-du) \\ &= -\int_{\pi/2}^{0} \frac{1}{1+\cot^n(u)} du \\ &= \int_0^{\pi/2} \frac{\sin^n u}{\sin^n u + \cos^n u} du. \end{aligned}$$

Adding these two expressions for I gives us

$$2I = \int_0^{\pi/2} \left(\frac{\cos^n x}{\cos^n x + \sin^n x} + \frac{\sin^n x}{\sin^n x + \cos^n x} \right) dx = \int_0^{\pi/2} 1 dx = \frac{\pi}{2}$$

and, therefore, $I = \frac{\pi}{4}$. (Note that we have proven that, in fact, $I = \frac{\pi}{4}$ for all real numbers n.)

Also solved by **Armstrong Problem Solvers**, Armstrong Atlantic State University, Savannah, GA; **Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie**, Angelo State University, San Angelo, TX; **Frank Battles**, Massachusetts Maritime Academy, Buzzards Bay, MA; **Paul S. Bruckman**, Sointula, BC; **Printhwijit De**, University College Cork, Cork, Ireland; **Thomas Dence**, Ashland University, Ashland, OH; **Mark Evans**, Louisville, KY; **Robert Gebhardt**, Hopatcong, NJ; **Miguel Lerma**, Northwestern University Problem Solving Group, Evanston, IL; **David E. Manes**, Oneonta, NY; **Ángel Plaza** , Universidad de Las Palmas de Gran Canaria, Las Palmas G.C., Spain; **Henry Ramirez**, Fundacion Universitaria SAN MARTIN, Bogota, Colombia; **Henry Ricardo**, Medgar Evers College, Brooklyn, NY; **Harry Sedinger**, St. Bonaventure University, St. Bonaventure, NY; **Jonathan Strzelec, student** Christopher Newport University Newport News VA; and the **Proposer**. **Perfetti Paolo**, Università degli Studi di Roma "Tor Vergata", Rome, Italy points to A.K.Arora; S.K.Goel; D.M.Rodriguez: *Special Integration Techniques for Trigonometric Integrals*, The American Mathematical Monthly, vol.95, No.2.(Feb.,1988),pp.126–130, where integrals of this type are studied using elementary methods.

1129. *Proposed by Arthur Holshouser and Stas Molchanov*

Let R denote the real numbers and Q the rational numbers. A function f has a local minimum at the point x_0 if there exists an open neighborhood U of x_0 such that $f(x_0) \leq f(x)$ for all $x \in U$.

1. Find a non-constant function $f : R \to R$ such that f has a local minimum at each point.

2. Find a function $g : Q \to Q$ such that for each rational number r, there is neighborhood U of r such that $g(r) < g(x)$ for each $x \in U$.

Solution by **Paola Perfetti**, *Università degli Studi di Roma "Tor Vergata", Rome, Italy*

1. Let $f\colon R \to R$ be defined by $f(x) = 1$ if $x < 0$ and $f(x) = 0$ for $x \geq 0$. f is non–constant and every point is a minimum.

2. $g\colon Q \to Q$, $g(p/q) = -1/q$, $g(0) = -1$ and (1 is the greatest common divisor of p and q: $(p|q) = 1$). For any p/q let's define a neighborhood by $\left|\frac{p}{q} - \frac{p'}{q'}\right| < \frac{1}{q^2}$, $p'/q' \neq p/q$. $\frac{1}{q^2} > \left|\frac{pq' - p'q}{qq'}\right| \geq \frac{1}{qq'}$ whence $q' \geq q + 1$. It follows $g(p'/q') = -1/q' > -1/q$. Of Course $x = 0$ is a minimum.

This function is not new. As far as I know, it goes back to Riemann.

- By the way $f\colon R \to R$,

$$f(x) = \begin{cases} -1/q & x = p/q,\ (p|q) = 1 \\ 0 & \text{otherwise} \end{cases}$$

has a dense set of minima of 0 Lebesgue measure. The set of minima of the everywhere discontinuous Dirichlet function: $f(x) = 0$ $x \in Q$ and $f(x) = -1$ for $x \in R\backslash Q$ is of full measure.
In [2] the authors construct an example of *continuous* function $f\colon R \to R$ having a dense set of *proper* minima $(f(x_0) < f(x))$. This type of functions can be proved to be dense (residual) in the set of continuous functions $C([0,1])$ with the sup norm [3]. A considerably more difficult example of a *differentiable* function having a dense set of maxima and minima is constructed in [4], see also [5] p.141.

- As for point 1 we could have inserted "more steps": $f(x) = -k$ for $k \leq x < k+1$ whose graph is a "staircase". The steps can be made as close as we want but there does not exist a function satisfying 1 and not constant on every interval. We then prove the theorem

Theorem *There does not exist a function $f\colon R \to R$, having a minimum at each point and not constant on every open neighborhood*

The initial step of the proof is the following interesting lemma (proved as early as 1900 [1]).

Lemma *The set of the ordinates of maxima or minima is a countable set for any function $f\colon R \to R$,*

Proof of the theorem Let be $f\colon R \to R$ and $B = f(R)$. By hypotheses each point of B is a minimum and

the lemma implies the countability of B: $B = \bigcup_{k=1}^{\infty} y_k$. Let's define $A_k \doteq f^{-1}(y_k)$

so that $R = \bigcup_{k=1}^{\infty} A_k$, $(A^{(k_0)} \doteq \bigcup_{k=1, k\neq k_0}^{\infty} A_k)$.

By the Baire–category theorem applied on R which is complete, we have $\overset{\circ}{\overline{A}}_k \neq \emptyset$ for at least a k, say $k = k_0$. Hence there exists an open set, say C, such that for any $p \in C$, $A_{k_0} \cap U_p \neq \emptyset$ for any open neighborhood $U_p \ni p$.

There are two possibilities:

i) If $U_p \cap A^{(k_0)} = \emptyset$ for a pair p–U_p, then $U_p \subset A_{k_0}$ and this would imply f constant and equal to y_{k_0} in U_p hence the thesis.

ii) If $U_p \cap A^{(k_0)} \neq \emptyset$ for any $p \in C$ and for any $U_p \ni p$, then $A^{(k_0)}$ would be dense in C. In this case the density of $A^{(k_0)}$ and A_{k_0} contradicts the fact that each point must be a minimum.

• The Lemma is false if one wants countable the set of the inflection points and in fact a C^1 counterexample is easily constructed. Let be $F \subset [0,1]$ the Cantor–ternary–set and for $x \in [0,1]$, $\rho(x,F)$ is the distance between x and F. Let be $h(x) \doteq \int_0^x \rho(t,F)dt$.
1) h is differentiable and $h'(x) = \rho(x,F)$ being $\rho(x,F)$ continuous , 2) $h'(x) \geq 0$ being $\rho(x,F) \geq 0$. The derivative is zero if and only if $x \in F$ being F a closed set 3) h is injective. In fact $\int_x^{x'} \rho(t,F)dt > 0$ if $x < x'$ because F is completely disconnected, (F does not contain any interval), 4) the points of zero derivative are uncountable being F uncountable (as well known).

The Cantor–ternary–set has zero Lebesgue measure but this is inessential. We could have taken a Cantor set of positive measure.

See also

[1] A.Shoenflies: Die Entwickelung der Lenhre von dem Punktmannigflatigkeiten, Jahrebsbericht Deutschen Mathematiker–Vereinigung 8, Leipzig, 1900.

[2] E.E.Posey, J.E.Vaughan: Functions with a Proper Local Maximum in Each Interval, Amer.Math.Monthly, Vol.90, Issue 4 (Apr.,1983), 281–283.

[3] V Drobot, M.Motayne: Continuous Functions with a dense set of Proper Local Maxima, Amer.Math.Monthly, Vol.92, Issue 3 (Mar.,1985), 209–211.

[4] Y.Katznelson, K.Stromberg: Everywhere differentiable, Nowhere Monotone Functions, Amer.Math.Monthly, Vol.81, Issue 4 (Apr.,1974), 349–354.

[5] A.M.Bruckner, J.Marik, C.E.Weil: Some Aspects of Products of Derivatives, Amer.Math.Monthly, Vol.99, Issue 2 (Feb.,1992), 134–145.

Solution by **Gabriel T. Prăjitură**, *SUNY Brockport, Brockport, NY*

1. The function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is non-constant and has a local minimum at every point.

2. Let $\{x_1, x_2, x_3, \dots\}$ be an enumeration of $\mathbb{Q}$ and $g : \mathbb{Q} \to \mathbb{Q}$, $g(x_n) = n$. We define $U_1 = \mathbb{Q}$ and, for $n \geq 2$, $U_n = (x_n - r_n, x_n + r_n)$, where

$$r_n = \min\{|x_n - x_1|, |x_n - x_2|, \dots, |x_n - x_{n-1}|\}.$$

It is obvious that $g(x_n) < g(x)$ for each $x \in U_n$, $x \neq x_n$.

Also solved by **Armstrong Problem Solvers**, Armstrong Atlantic State University, Savannah, GA; **Cal Poly Pomona Problem Solving Group**, Pomona, CA; **Don Hancock**, Pepperdine University, Malibu, CA; **Mike Stein and Brendan Kelly, undergraduates**, The College Of New Jersey, Ewing NJ; and the **Proposers**.

1130. *Proposed by Marcin Kuczma, University of Warsaw, Warsaw, Poland*

Twins is the keyword for this season. Let $t(1) = 5$, $t(2) = 7$, $t(3) = 13$, $t(4) = 19, \ldots$ be the increasing sequence (finite or infinite?) of all primes such that, for each i, $t(i) - 2$ is also a prime –and let $t(t(t(2)))$ be nice and lucky and happy for you!!! Editor's note: this puzzle was sent to friends of the poser in December of a certain year as a gift. This is the fourth of several such problems we plan for this column.

Solution by **Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie**, *Angelo State University, San Angelo, TX*

Since

$$\begin{aligned} t(1) &= 5, \\ t(2) &= 7, \\ t(3) &= 13, \quad \text{and} \\ t(4) &= 19, \end{aligned}$$

then

$$t(7) = 61.$$

Then continuing in the same manner,

$$t(t(t(2))) = t(t(7)) = t(61) = 1999.$$

Also solved by **Armstrong Problem Solvers**, Armstrong Atlantic State University, Savannah, GA; **Avery Cotton, student**, Western Oregon University, Monmouth, OR; **Frank Battles**, Massachusetts Maritime Academy, Buzzards Bay, MA;

Paul S. Bruckman, Sointula, BC; **Mark Evans**, Louisville, KY; **Robert Gebhardt**, Hopatcong, NJ; **Richard Hess**, Rancho Palos Verdes, CA; **Peter Lindstrom**, Batavia, NY; **David E. Manes**, Oneonta, NY; **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan; **Mike Pinter**, Belmont University, Nashville, TN; **Rex H. Wu**, Brooklyn, NY; and the **Proposer**.

1131. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington, PA*

$ABCD$ is a convex quadrilateral in which ΔBCD is equilateral and $m\angle DAB = 30°$. Show that $(AC)^2 = (AD)^2 + (AB)^2$.

Solution by **Brian Bradie**, *Christopher Newport University, Newport News VA*

Without loss of generality, let $BC = CD = BC = 1$. Further, let $m\angle ADB = \theta$. Then $m\angle ABD = 150° - \theta$. Applying the Law of Sines twice to ΔABD, we find

$$AB = 2\sin\theta \quad \text{and} \quad AD = \cos\theta + \sqrt{3}\sin\theta.$$

Thus,

$$\begin{aligned} (AD)^2 + (AB)^2 &= \cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 3\sin^2\theta + 4\sin^2\theta \\ &= 6\sin^2\theta + 1 + 2\sqrt{3}\sin\theta\cos\theta. \end{aligned} \tag{0.1}$$

Finally, applying the Law of Cosines to ΔABC, we find

$$\begin{aligned}(AC)^2 &= 4\sin^2\theta + 1 - 4\sin\theta\cos(210^\circ - \theta)\\ &= 4\sin^2\theta + 1 + 2\sin\theta\left(\sqrt{3}\cos\theta + \sin\theta\right)\\ &= 6\sin^2\theta + 1 + 2\sqrt{3}\sin\theta\cos\theta. \qquad (0.2)\end{aligned}$$

Comparing (1) and (2), we see that $(AC)^2 = (AD)^2 + (AB)^2$.

Also solved by **Armstrong Problem Solvers**, Armstrong Atlantic State University, Savannah, GA; **Paul S. Bruckman**, Sointula, BC; **Miguel Amengual Covas**, Mallorca, Spain; **Robert Gebhardt**, Hopatcong, NJ; **Richard Hess**, Rancho Palos Verdes, CA; **Jaree Hudson, student**, Armstrong State University, Savannah, GA; **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan; **Rex H. Wu**, Brooklyn, NY; and the **Proposer**.

1132. *Proposed by Leo Schneider, John Carroll University, Cleveland, OH*

The two parallel sides of a trapezoid are of length a and b. A segment of length m parallel to these two sides divides the trapezoid into two trapezoids, each of area equal to one half of the original trapezoid. Prove that if a, b, and m are relatively prime positive integers, then neither 2 nor 3 is a prime factor of any of these integers.

Solution by **Rex H. Wu**, *Brooklyn, NY*

Let h be the height of the trapezoid with parallel sides a and b, h_1 be the height of the trapezoid with parallel sides a and m. Then the height for the trapezoid with parallel sides m and b is $h_2 = h - h_1$. Equating the areas, we have

$$\frac{1}{2}(a+m)h_1 = \frac{1}{2}(m+b)(h-h_1) = \frac{1}{4}(a+b)h$$

From the above, the first and the last parts give

$$h_1 = \frac{a+b}{2(a+m)}h$$

and the first two parts give

$$h_1 = \frac{m+b}{a+b+2m}h$$

Equating h_1 and simplify gives

$$a^2 + b^2 = 2m^2$$

The original problem is now reduced to showing the primitive solutions, i.e. $\gcd(a, b, m) = 1$, of the diophantine equation $a^2 + b^2 = 2m^2$ cannot be divisible by 2 or 3.

Let's show the primitive solutions are not divisible by 2. Suppose $a = 2\alpha$, for some integer α. Since $a^2+b^2 = 2m^2$, or a^2+b^2 is even, it follows that b^2 must be even. Therefore, $b = 2\beta$, for some integer β. Then $a^2+b^2 = 4(\alpha^2+\beta^2) = 2m^2$ which implies $m^2 = 2(\alpha^2 + \beta^2)$. So $m = 2\mu$ for some integer μ. But then 2 is a common factor of a, b and m, contradicting the assumption that $\gcd(a, b, m) = 1$. Similar argument can be applied to see b cannot be even.

Suppose $m = 2\mu$ for some integer μ. Then $a^2 + b^2 = 2(2\mu)^2 = 8\mu^2$. This implies 4 divides a^2+b^2. Observe that a and b cannot be one odd and one even since the sum

a^2+b^2 would be odd. Therefore, a and b must be both odd or both even. Suppose a and b are both odd, $a=2i+1$ and $b=2j+1$ for some integers i and j. We have $a^2+b^2=4i^2+4i+4j^2+4j+2$, not divisible by 4. Therefore, a and b must be both even. But then we have the contradiction again, that a, b and m have a common factor 2.

To show 3 cannot be a factor of a, b and m is more involved. I need to characterize the primitive solutions (a,b,m) of the equation $a^2+b^2=2m^2$. The equation $a^2+b^2=2m^2$ is closely related to the well-known equation $x^2+y^2=z^2$.

LEMMA 0.1. *$n=x^2+y^2$ has a solution if and only if n does not contain any prime $p\equiv 3 \pmod 4$ and the exponent of p is odd.* This is a well know fact. A proof can be found in many elementary number theory books.

LEMMA 0.2. *If p is a prime and $p\equiv 3 \pmod 4$, then $p^2=x^2+y^2$ has no primitive solutions in positive integers.*

Proof. All the primitive solutions to $p^2=x^2+y^2$ can be generated by $p=r^2+s^2$, $x=r^2-s^2$ and $y=2rs$, with $r\not\equiv s \pmod 2$ and $\gcd(r,s)=1$. From Lemma 1, we know there is no integer solution to $p=r^2+s^2$. □

LEMMA 0.3. *If p is a prime and $p\equiv 3 \pmod 4$, $Q=q_1q_2\cdots q_j$ where q_i is prime and $q_i\not\equiv 3 \pmod 4$, then the solution to $n=x^2+y^2$, where $n=p^{2k}Q$, is not primitive.*

Proof. p^{2k} can be viewed as $(p^k)^2$. If k is odd, by lemma 1, p^k cannot be expressed as the sum of two squares and therefore, $(p^k)^2$ cannot be the sum of two squares. If k is even, then the only way to express p^{2k} as the sum of two squares is $p^{2k}=0^2+\left(p^k\right)^2$. Suppose $Q=r^2+s^2$, then $p^{2k}Q=p^{2k}(r^2+s^2)=p^{2k}r^2+p^{2k}s^2$, i.e. $\gcd(n,x,y)=\gcd(p^{2k}Q,p^{2k}r^2,p^{2k}s^2)=p^{2k}$. □

LEMMA 0.4. *The product of the sum of two squares is again a sum of two squares.*

Proof.

$$(r^2+s^2)(t^2+u^2)=(rt+su)^2+(ru-st)^2=(rt-su)^2+(ru+st)^2.$$

□

THEOREM 0.5. *All the primitive solutions of the diophantine equation $a^2+b^2=2m^2$ are generated by $(a,b,m)=(|r^2-s^2-2rs|,|r^2-s^2+2rs|,r^2+s^2)$, where $\gcd(r,s)=1$ and $r\not\equiv s \pmod 2$.*

Proof. It is easy to verify that $(a,b,m)=(|r^2-s^2-2rs|,|r^2-s^2+2rs|,r^2+s^2)$ is a solution to $a^2+b^2=2m^2$.

Suppose there is a primitive solution to $a^2+b^2=2m^2$. By Lemma 1, we know $2m^2$ cannot contain a prime p such that $p\equiv 3 \pmod 4$ and the exponent of p is odd.

From Lemma 3, we know m cannot contain a prime p such that $p\equiv 3 \pmod 4$ and the exponent of p is even.

That means $2m^2$ can only contain primes p such that $p\not\equiv 3 \pmod 4$.

Since 2 is the only even prime, $m\equiv 1 \pmod 4$.

From Lemma 1, $m^2=x^2+y^2$, for some integers x and y. This representation of m is unique if m is prime. However, from Lemma 4, if m is composite, there is at least one representation.

The primitive solutions of $m^2=x^2+y^2$ are the Pythagorean triples, $(x,y,m)=(r^2-s^2,2rs,r^2+s^2)$, where $\gcd(r,s)=1$ and $r\not\equiv s \pmod 2$.

Going back to the original equation, $a^2+b^2=2m^2=2(x^2+y^2)=(1+1)(x^2+y^2)=(x-y)^2+(x+y)^2$. Thus, we have $2m^2=2(r^2+s^2)^2=(r^2-s^2-2rs)^2+(r^2-s^2+2rs)^2=a^2+b^2$. □

Lemma 0.6. *If (x,y,z) is a primitive solution of $x^2+y^2=z^2$, then $gcd(x,y) = gcd(x,z) = gcd(y,z) = 1$.*

Lemma 0.7. *If (x,y,z) is a primitive solution of $x^2+y^2=z^2$, then only one of x or y is a multiple of 3.*

Proofs to these two lemmas can be found in many elementary number theory textbooks.

Lemma 0.8. *If (x,y,z) is a primitive solution of $x^2+y^2=z^2$, then $x+y$ and $x-y$ cannot contain the factor 3.*

Proof. This follows from Lemmas 5 and 6 since $\gcd(x,y)=1$ and only one of x and y is a multiple of 3. □

Now, we are ready to show 3 cannot be a factor of a, b and m in the equation $a^2+b^2=2m^2$. Here I will also refer to the equation $x^2+y^2=z^2$.

a and b cannot contain a factor 3 because $a=|r^2-s^2-2rs|=|x-y|$ and $b=|r^2-s^2+2rs|=|x+y|$, both as a consequence of Lemma 7.

That m cannot be a multiple of 3 is already shown in the proof to Theorem 1. In fact, m cannot contain any prime p such that $p \equiv 3 \pmod 4$.

Also solved by **Armstrong Problem Solvers**, Armstrong Atlantic State University, Savannah, GA; **Paul S. Bruckman**, Sointula, BC; **Richard Hess**, Rancho Palos Verdes, CA; **Cal Poly Pomona Problem Solving Group**, Pomona, CA; **Harry Sedinger**, St. Bonaventure University, St. Bonaventure, NY; and the **Proposer**.

1133. *Proposed by Arthur Holshouser, Charlotte, NC; Anita Chatelain and Joe Albree, Auburn University at Montgomery*

In Pillow Problem 14, Lewis Carroll proved in his head that 3 times the sum of 3 squares is also the sum of 4 squares. (Of course, 0^2 is considered to be a square).

1. Prove pillow problem 14.
2. Prove that $\left(a^2+b^2+c^2\right)\left(x^2+y^2+z^2\right)$ is also the sum of 4 squares.
3. Prove that $\prod_{i=1}^{n}\left(x_i^2+y_i^2+z_i^2+v_i^2\right)$ is also the sum of 4 squares by first proving that $\left(a^2+b^2+c^2+d^2\right)\left(x^2+y^2+z^2+v^2\right)$ is the sum of 4 squares.

Solution by **Mike Pinter**, *Belmont University, Nashville, TN*

1. The following identity is verified by routine algebraic manipulation:

$$(a-b)^2+(a-c)^2+(b-c)^2+(a+b+c)^2 = 3(a^2+b^2+c^2).$$

2. After some trial and error, one is led to the following identity, which again is verified by algebraic manipulation:

$$(az-cx)^2+(bx-ay)^2+(cy-bz)^2+(ax+by+cz)^2 = (a^2+b^2+c^2)(x^2+y^2+z^2).$$

3. Similar to above, after some trial and error the following identity is obtained:

$$[(ay-bx)+(cv-dz)]^2+[(cx-az)+(bv-dy)]^2+[(av-dx)+(bz-cy)]^2$$

$$+(ax+by+cz+dv)^2 = (a^2+b^2+c^2+d^2)(x^2+y^2+z^2+v^2).$$

Assume inductively that $\displaystyle\prod_{i=1}^{n-1}(x_i^2+y_i^2+z_i^2+v_i^2) = (a^2+b^2+c^2+d^2)$. Thus, $\displaystyle\prod_{i=1}^{n}(x_i^2+y_i^2+z_i^2+v_i^2) = (x_n^2+y_n^2+z_n^2+v_n^2)\prod_{i=1}^{n-1}(x_i^2+y_i^2+z_i^2+v_i^2)$. From our basis for induction (shown immediately above), we know this last expression can be represented as the sum of 4 squares.

The desired result follows by induction.

Also solved by **Armstrong Problem Solvers**, Armstrong Atlantic State University, Savannah, GA; **Paul S. Bruckman**, Sointula, BC; **Miguel Amengual Covas**, Mallorca, Spain; **Thomas Dence**, Ashland University, Ashland, OH; **Charles R. Diminnie**, Angelo State University, San Angelo, **James Hall**, La Crescenta, CA; TX; **Richard Hess**, Rancho Palos Verdes, CA; **Peter Lindstrom**, Batavia, NY; **David E. Manes**, Oneonta, NY; **Mike Pinter**, Belmont University, Nashville, TN; **Raúl A. Simón**, LAMB, Santiago, CHILE; **Rex H. Wu**, Brooklyn, NY; and the **Proposer**.

1134. *Proposed by Paul S. Bruckman, Sointula, BC*

For all $x > 0$, let $\pi(x)$ denote the number of positive integers less than or equal to x that are prime. Prove the following inequality :

$$\pi(2n+4) > 1 + \frac{3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n+1)}{2 \cdot 4 \cdot 6 \cdot \ldots 2n}$$

$n = 1, 2, \ldots$.

Solution by ,

Also solved by , and the **Proposer**.

1135. *Proposed by Cecil Rousseau, University of Memphis, Memphis, TN*

Let $\phi = (1+\sqrt{5})/2$ denote the golden ratio. Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 |\sin(n\pi\phi)|}$$

converges.

Solution by **Paul S. Bruckman**, *Sointula, BC*

Solution by **the proposer.**

A much stronger result is true; ϕ can be replaced by any algebraic irrational and n^2 can be replaced by $n^{1+\epsilon}$ where $\epsilon > 0$. Let α be an algebraic irrational and let $\delta = \epsilon/2$. Then there is a positive constant $c = c(\alpha, \delta)$ such that

$$\left|\alpha - \frac{m}{n}\right| > \frac{c}{n^{2+\delta}} \tag{0.3}$$

for all integers $n > 0$ and m (Thue-Siegel-Roth Theorem). Let $\langle x \rangle$ denote the distance from x to the nearest integer. Consider the sequence of numbers

$$\langle \alpha \rangle, \langle 2\alpha \rangle, \ldots, \langle n\alpha \rangle.$$

Then

$$\langle k\alpha \rangle > \frac{c}{k^{1+\delta}} \geq \frac{c}{n^{1+\delta}} \qquad (k = 1, 2, \ldots, n),$$

and since

$$|\langle x \rangle - \langle y \rangle| = \min\{\langle x+y \rangle,\ \langle x-y \rangle\},$$

it follows that for all $1 \leq j < k \leq n$, the value of $|\langle k\alpha \rangle - \langle j\alpha \rangle|$ is one of the numbers

$$\langle \alpha \rangle, \langle 2\alpha \rangle, \ldots, \langle (2n-1)\alpha \rangle.$$

Hence

$$|\langle k\alpha\rangle - \langle j\alpha\rangle| > \frac{c}{(2n)^{1+\delta}}.$$

Hence if the sequence $\langle\alpha\rangle, \langle 2\alpha\rangle, \dots, \langle n\alpha\rangle$ is arranged in increasing order, the kth number on the list is greater than $kc/(2n)^{1+\delta}$. Thus we obtain

$$\sum_{k=1}^{n} \frac{1}{\langle k\alpha\rangle} < \frac{(2n)^{1+\delta}}{c} \sum_{k=1}^{n} \frac{1}{k} \le \frac{(2n)^{1+\delta}(\log n + 1)}{c}.$$

The convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}\langle n\alpha\rangle}$$

now follows by applying Abel's summation by parts formula (see note below) with $a_k = 1/\langle k\alpha\rangle$ and $b_k = k^{-(1+\epsilon)}$. Since

$$2\langle x\rangle \le |\sin(\pi x)| \le \pi\langle x\rangle,$$

we see that

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}|\sin(n\pi\alpha)|}$$

converges if α is any algebraic irrational and ϵ is positive. Of course, for the particular case of ϕ (or any other quadratic irrational) the strength of the Thue-Siegel-Roth Theorem is not required; one can use Liouville's Theorem or simply prove directly that

$$\left|\phi - \frac{m}{n}\right| > \frac{1}{3n^2}$$

for all positive rational numbers m/n.

Note. Abel's formula is the discrete version of integration by parts, namely

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (A_k - A_{k-1}) b_k = A_n b_n - \sum_{k=1}^{n-1} A_k (b_{k+1} - b_k),$$

where $A_k = \sum_{i=1}^{k} a_i$ for $k \ge 1$ and $A_0 = 0$.

1137. *Proposed by Peter Lindstrom, Batavia, NY*

Let n be a positive integer and T_i be the i^{th} triangular number.

Find the value of $\lim_{k\to\infty} \sum_{i=1}^{k} \frac{\sum_{j=1}^{n} \binom{n}{j} i^{n-j}}{(T_i)^n}$.

Solution by **Printhwijit De**, *University College Cork, Cork, Ireland*

$\sum_{j=1}^{n} \binom{n}{j} i^{n-j} = i^n \sum_{j=1}^{n} \binom{n}{j} i^{n-j} = i^n \left(\left(1 + \frac{1}{i}\right)^n - 1\right) = (1+i)^n - i^n.$

As $T_i = \frac{i(i+1)}{2}$ we have,

$$\frac{\sum_{j=1}^{n} \binom{n}{j} i^{n-j}}{(T_i)^n} = 2^n \cdot \frac{((1+i)^n - i^n)}{(i\,(i+1))^n} = 2^n \left(\frac{1}{i^n} - \frac{1}{(i+1)^n}\right).$$

$$\therefore \sum_{i=1}^{k} \frac{\sum\limits_{j=1}^{n} \binom{n}{j} i^{n-j}}{(T_i)^n} = 2^n \sum_{i=1}^{k} \left(\frac{1}{i^n} - \frac{1}{(i+1)^n} \right)$$
$$= 2^n \left(1 - \frac{1}{(k+1)^n} \right).$$

$$\therefore \sum_{i=1}^{k} \left(\frac{\sum\limits_{j=1}^{n} \binom{n}{j} i^{n-j}}{(T_i)^n} \right) = 2^n.$$

*S*olution by **Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie**, Angelo State University, San Angelo, TX

Using the Binomial Theorem,

$$\begin{aligned}
\lim_{k\to\infty} \sum_{i=1}^{k} \frac{\sum_{j=1}^{n} \binom{n}{j} i^{n-j}}{(T_i)^n} &= \lim_{k\to\infty} \sum_{i=1}^{k} \frac{(i+1)^n - i^n}{(T_i)^n} \\
&= \lim_{k\to\infty} \sum_{i=1}^{k} \frac{(i+1)^n - i^n}{\left(\frac{i(i+1)}{2}\right)^n} \\
&= \lim_{k\to\infty} \sum_{i=1}^{k} \left[\left(\frac{2}{i}\right)^n - \left(\frac{2}{i+1}\right)^n \right] \\
&= \lim_{k\to\infty} \left[2^n - \left(\frac{2}{k+1}\right)^n \right] \\
&= 2^n.
\end{aligned}$$

Also solved by **Armstrong Problem Solvers**, Armstrong Atlantic State University, Savannah, GA; **Frank Battles**, Massachusetts Maritime Academy, Buzzards Bay, MA; **Brian Bradie**, Christopher Newport University, Newport News VA; **Paul S. Bruckman**, Sointula, BC; **Kenny Davenport**, Dallas, PA; **Billy Dobbs, graduate student**, Eastern Kentucky University, Richmond, KY; **Nate Dorr, undergraduate**, Rose-Hulman Institute of Technology, Terre Haute, IN; **Henry Ricardo**, Medgar Evers College, Brooklyn, NY; **Rex H. Wu**, Brooklyn, NY; and the **Proposer**.