SMALL AMPLITUDE WEAK ALMOST PERIODIC SOLUTIONS FOR THE 1D NLS

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Abstract

All the almost periodic solutions for nonintegrable PDEs found in the literature are very regular (at least C^∞) and, hence, very close to quasiperiodic ones. This fact is deeply exploited in the existing proofs. Proving the existence of almost periodic solutions with finite regularity is a main open problem in KAM theory for PDEs. Here we consider the 1*-dimensional NLS with external parameters and construct almost periodic solutions which have only Sobolev regularity both in time and space. Moreover, many of our solutions are so only in a weak sense. This is the first result on existence of weak (i.e., nonclassical) solutions for nonintegrable PDEs in KAM theory.*

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1. Introduction

We present here the first result on existence of weak (i.e., nonclassical) solutions for nonintegrable PDEs in KAM theory. More precisely, we study NLS equations on the circle finding almost periodic solutions, that is, solutions which are limit (in the uniform topology in time) of time-quasiperiodic functions. We work on models with external parameters of the form

$$\mathrm{i}\mathbf{u}_t + \mathbf{u}_{xx} - V * \mathbf{u} + f\left(|\mathbf{u}|^2\right)\mathbf{u} = 0, \qquad \mathbf{u}(t,x) = \mathbf{u}(t, x+2\pi), \tag{NLS$_V$}$$

where $\mathrm{i} = \sqrt{-1}$ and $f(y)$ is real analytic in y in a neighborhood of $y = 0$ with $f(0) = 0$. Given $V = (V_j)_{j\in\mathbb{Z}} \in [-1/4, 1/4]^{\mathbb{Z}} \subset \ell^\infty(\mathbb{R})$, the Fourier multiplier $V*$ is defined as the bounded[1] linear operator $V* : \mathcal{F}(\ell^1) \to \mathcal{F}(\ell^1)$ by[2]

$$u(x) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx} \mapsto (V * u)(x) = \sum_{j\in\mathbb{Z}} V_j u_j e^{\mathrm{i}jx}, \tag{1.1}$$

where $\mathcal{F}(\ell^1)$ is the Wiener algebra of 2π-periodic functions[3] having Fourier coefficients in $\ell^1 = \ell^1(\mathbb{C}) := \{u := (u_j)_{j\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} : |u|_{\ell^1} := \sum_{j\in\mathbb{Z}} |u_j| < \infty\}$ endowed with the corresponding norm $|u(\cdot)|_{\mathcal{F}(\ell^1)} := |u|_{\ell^1}$. Note that the choice of the ball $[-1/4, 1/4]^{\mathbb{Z}}$ is rather arbitrary and any other ball is acceptable.

Besides classical solutions—namely, functions $\mathbf{u}$ admitting continuous derivatives $\mathbf{u}_t$ and $\mathbf{u}_{xx}$ solving (NLS$_V$)—we are particularly interested in weak solutions according to the following.

Definition 1.1 (Global-in-time weak solutions)

A function $\mathbf{u} : \mathbb{R}^2 \to \mathbb{C}$ which is 2π-periodic in x and such that the map $t \mapsto \mathbf{u}(t,\cdot) \in \mathcal{F}(\ell^1)$ is continuous is a weak solution of (NLS$_V$) if for any smooth compactly supported function $\chi : \mathbb{R}^2 \to \mathbb{R}$ one has

$$\int_{\mathbb{R}^2} (-\mathrm{i}\chi_t + \chi_{xx})\mathbf{u} - \left(V * \mathbf{u} - f\left(|\mathbf{u}|^2\right)\mathbf{u}\right)\chi \, dx\, dt = 0. \tag{1.2}$$

Note that according to our definition a weak solution is a continuous[4] function on $\mathbb{R}^2$.

Local well-posedness for equations as (NLS$_V$) is well known even in lower regularity (see, e.g., [9], [10]). In the context of integrable PDEs there are various results on weak almost periodic solutions, we mention [31] and [32] for the KdV and mKdV equations, [24] for the Szegö equation, and [23] for the Benjamin–Ono equation. In

[1] Clearly, the operator norm is $\|V\|_{\mathcal{L}(\mathcal{F}(\ell^1),\mathcal{F}(\ell^1))} = |V|_\infty := \sup_j |V_j|$.

[2] More precisely, $(V * \mathbf{u})(t,x) := (V * \mathbf{u}(t,\cdot))(x)$ for every $t \in \mathbb{R}$.

[3] Note that all such functions are continuous.

[4] Indeed, $|\mathbf{u}(t,x) - \mathbf{u}(t_0,x_0)| \le |\mathbf{u}(t,\cdot) - \mathbf{u}(t_0,\cdot)|_{\mathcal{F}(\ell^1)} + |\mathbf{u}(t_0,x) - \mathbf{u}(t_0,x_0)|$.

this article, we work close to the elliptic fixed point $\mathbf{u} = 0$ and consider (NLS_V) as a small (nonintegrable) perturbation of the (integrable) linear Schrödinger equation and prove the persistence of almost periodic solutions.

THEOREM 1
For almost every Fourier multiplier V, there exist infinitely many small amplitude weak almost periodic solutions of (NLS_V). Infinitely many such solutions are not classical and infinitely many are classical.

Here by *almost every* we mean a full measure set with respect to the *product probability measure* on $[-1/4, 1/4]^{\mathbb{Z}}$. The Borel sets on such a measure are with respect to the product topology on $[-1/4, 1/4]^{\mathbb{Z}}$ (see Appendix B). This theorem will be derived in Section 3 from Theorems 2 and 3, stated below.

Before describing our results in more detail, we briefly motivate why one is interested in almost periodic solutions and particularly in those of low regularity.

In the study of finite-dimensional nearly-integrable Hamiltonian systems KAM theorems play a pivotal role, casting a light that illuminates the picture quite clearly. Indeed, under some (generic) nondegeneracy assumptions most of the phase space of such systems is foliated by maximal invariant tori, whose dimension is half of the one of the whole space. In particular, the system is not ergodic, and the majority of initial data give rise to quasiperiodic solutions that densely fill some invariant torus and are, therefore, perpetually stable. Possible chaotic behavior is restricted to a set of small measure.

On the other hand, in the infinite-dimensional setting, for example, in the PDEs case, the general picture is so far rather obscure and the main questions still remain unanswered. The typical solutions of an infinite-dimensional integrable system are the *almost periodic* ones that lie on maximal infinite-dimensional invariant tori. What is their fate under perturbation? Is it still true that the majority of initial data produce perpetually stable solutions? The meaning and possible answers to these questions are deeply related to the regularity of the phase space in which one looks for solutions. These questions are completely open and only very partial answers are available. In particular, the known results are on spaces of very high regularity (contained in C^∞). On the other hand, finite regularity solutions appear naturally and are widely studied in the PDE context. For instance, the persistence of Sobolev almost periodic solutions is the first open problem on KAM for PDEs mentioned by Kuksin in [35] (see, e.g., Problem 7.1). Even though our solutions are quite special, being mainly supported on a sparse set of Fourier's modes, the present work can be seen as a first step forward in this direction.

As is usual in KAM theory, a key point in the study of the dynamics in the neighborhood of these invariant tori consists in controlling the spectral properties of appropriate linear operators and dealing with the connected problem of small divisors. The main difficulty is then to guarantee that suitable arithmetic (Diophantine) conditions on the frequencies are fulfilled all along the scheme, so that small divisors can be bounded accordingly and the almost periodic dynamics controlled. Extending Diophantine (or similar) estimates, which strongly depend on the dimension, to the infinite dimension is not straightforward. In fact, all the results on almost periodic solutions for PDEs only deal with special model cases and consider parameter-dependent equations.

Parameter versus regularity issues. Quoting Bourgain: "The role of this parameter is essential to ensure appropriate nonresonance properties of the (modulated) frequencies along the iteration" [13, p. 63].

As we shall see below, in all the existent literature, the external parameters have rather low regularity (giving a good frequency modulation), while the almost periodic solutions have an *extremely fast* decay in their Fourier coefficients, which approach zero superexponentially, exponentially, or subexponentially (Gevrey). This means that those solutions are "very close" to quasiperiodic ones. For example, Pöschel in [38] studies an NLS with a multiplicative potential in L^2 (producing an infinite set of free parameters) and smoothing nonlinearity and constructs almost periodic solutions iteratively, through successive small perturbations of finite (but at each step higher) dimensional invariant tori. This leads to a very strong compactness property: in order to overcome the dependence of the KAM estimates on the dimension, the distances of these tori have to shrink superexponentially, leading to very regular solutions. (See also [22] for a generalization of Pöschel's approach to the analytic category by using Toeplitz–Lipschitz function techniques.) In his pioneering work [13] on the quintic NLS with Fourier multipliers (providing external parameters in ℓ^∞), Bourgain proposed a different approach which does not rely on approximations by quasiperiodic functions by working directly in Fourier space, and relying on a Diophantine condition which is tailored for the infinite dimension. For most choices of the parameters, this leads to the construction of almost periodic invariant tori which support Gevrey solutions (see also [8], [17]). In any case, all the above results are valid for most choices of the (infinitely many) external parameters. An open question is whether one can achieve a similar result by modulating finitely many parameters. This would lead to applications to more natural PDEs.

Of course, the most challenging scenario is represented by a fixed PDE, where the parameters should be "extracted" from the initial data.

Almost periodic solutions for a fixed PDE. Again, quoting Bourgain: "In the absence of exterior parameters, these [nonresonance] conditions need to be realized from amplitude-frequency modulation and suitable restriction of the action-variables. This problem is harder. Indeed, a fast decay of the action-variables (enhancing convergence of the process) allows less frequency modulation and worse small divisors" [13, p. 63].

Summarizing, the possibility of constructing almost periodic solutions for a *fixed* PDE, that is, "eliminating" the external parameters through amplitude-frequency modulation, appears to be intimately related to the regularity issues. Moreover, in the context of completely integrable PDEs, the invertibility of the amplitude-frequency map is known only in spaces of very low regularity (see [30]). It then becomes fundamental to look for almost periodic solutions in lower regularity spaces if we want to bypass the introduction of external parameters. While it is possible to lower the regularity beyond the Gevrey class (see [15]), up to now all the known solutions are at least C^∞. On the other hand, finding *finite regularity* solutions appears to be a very difficult question, due to extremely small divisors.

Comparable difficulties in tackling the Sobolev case appear in Birkhoff normal form theory for PDEs leading to two different types of results. In the analytic or Gevrey case one has subexponential stability times (see [19] and [16]), whereas in the Sobolev case the stability times appear to be controlled by the Sobolev exponent (see [2], [5], [6], [20], [21]).

The counterpart of total and long-time stability results is the construction of unstable trajectories, which undergo growth of the Sobolev norms (see [11], [14], [25]–[27]).

Comparison with the quasiperiodic case. In the context of quasiperiodic solutions there is a wide literature regarding the finite regularity case (starting from the seminal paper [34]). A good strategy, which works also for fully nonlinear PDEs, is to apply a Nash–Moser scheme and prove tame estimates on the inverse of the linearized equation at an approximate solution. This method was proposed in [3] (generalizing the seminal works [12] and [18] concerning the analytic case) via multiscale analysis (see also [1] or [4] for a reducibility approach).

Note that in the existing KAM literature *finite regularity* solutions are always classical and due to *finite regularity* nonlinearities. Typically, one looks for an *invariant embedded finite-dimensional* torus in a fixed phase space of x-dependent functions. Then the regularity of the embedding is clearly related to the one of the nonlinearities, which, in the Nash–Moser schemes, is required to be high enough increasing with the dimension of the torus. This is not surprising since in n-dimensional Hamiltonian systems the minimal regularity of the Hamiltonian vector field in order to construct a maximal Diophantine torus is essentially C^n (see [29], [33], [39]). For analytic nonlinearities one obtains analytic (or, at least, C^∞) embedded tori and hence smooth-in-time solutions. By bootstrap arguments, smoothness in space follows. We finally stress that in the integrable case one can construct even periodic solutions with very low regularity (see, e.g., [23]).

In comparison, most of the almost periodic literature concentrates on the construction of the solutions rather than the invariant objects. Inspired by [13], a conceptual novelty of [8] was to look directly for an *infinite-dimensional invariant torus*.[5] The strategy developed in [8] applies also to nonmaximal tori both of infinite and finite dimension. The persistence result was achieved through an abstract normal form theorem "à la Herman" (in finite-dimensional systems; see [36], [37]), whose estimates are *uniform in the dimension* of the considered torus (see [8, Theorems 3, 7.1]).

Note that in looking for infinite-dimensional tori the topology of the phase space plays a fundamental role. In particular, the analyticity of the embedded torus does not necessarily imply the analyticity in time of the solution, at least if the frequencies are unbounded as is typical in PDEs.

Small divisors for infinite frequencies in finite regularity. As discussed above, lowering the regularity of the phase space requires dealing with very small divisors. In order to apply the general approach of [8] in a phase space with finite regularity, one needs to look at the problem from a novel perspective and introduce new ideas. The starting point is to look for *special tori which are approximately supported, in Fourier space, on a sparse subset of* $\mathbb{Z}$ called "tangential sites"(see Definition 2.2), then prove that our solutions are supported on such sites (see (3.2)) up to a close to identity change of variables. The crucial fact is that the choice of tangential sites provides an extra set of parameters which can be used, for instance, to avoid resonances or simplify small divisor estimates (see, e.g., [28], [41]). In fact, we prove that an *appropriate choice* of the tangential sites (see Definition 2.2) allows us to impose *very strong* Diophantine conditions (see Definition 2.3) so that our small divisors can be controlled similarly to the Gevrey case of [8] and [13].

Having constructed an invariant torus contained in the phase space, we show that it is the support of weak almost periodic solutions (according to Definition 1.1). Finally, we prove that many tori support almost periodic solutions which are not classical.

2. Main results

It is convenient to exploit the Hamiltonian nature of (NLS_V). As was shown by Bourgain in [13], it is most convenient to use a product space as phase space. To this purpose, we introduce the scale of Banach spaces[6]

[5] In the infinite-dimensional case, whether this is an embedding depends strongly on the chosen topology. See the discussion in Section 2.1.

[6] Obviously, one could also take the more standard weight $\langle j \rangle := \max\{1, |j|\}$ instead of $\lfloor j \rfloor$, which generates the same Banach space, namely $\mathtt{w}_p \equiv \ell^{\infty,p}$. We made such choice for merely technical reasons (see proof of the monotonicity property given in Proposition 4.3 contained in [6, Proposition 6.3]).

$$\mathtt{w}_p := \left\{u := (u_j)_{j\in\mathbb{Z}} \in \ell^1(\mathbb{C}) : |u|_p := \sup_{j\in\mathbb{Z}} |u_j|\lfloor j\rfloor^p < \infty\right\}, \quad p > 1, \tag{2.1}$$

where $\lfloor j\rfloor := \max\{2, |j|\}$. We look for solutions in the Fourier–Lebesgue space $\mathcal{F}(\mathtt{w}_p)$ of 2π-periodic functions whose Fourier coefficients belong to $\mathtt{w}_p$. In the following, we will identify functions and sequences writing, for example, ℓ^1 instead of $\mathcal{F}(\ell^1)$ or $\mathtt{w}_p$ instead of $\mathcal{F}(\mathtt{w}_p)$ and so on. Moreover, we have the following standard immersion properties.[7] For $0 \le k < p-1$, we have $C^p \subset H^p \subset \mathtt{w}_p \subset C^k$ with the estimate

$$\mathtt{c}\max\left\{\|u^{(k)}\|_{L^\infty}, \|u\|_{L^\infty}\right\} \le |u|_{\mathtt{w}_p} \le 2\max\left\{\|u^{(p)}\|_{L^2}, \|u\|_{L^2}\right\}, \tag{2.2}$$

where the first inequality holds for $u \in \mathtt{w}_p$ and the second one holds for $u \in C^p$ and where $\mathtt{c}^{-1} := \sum_j \lfloor j\rfloor^{k-p}$.

We endow $\mathtt{w}_p \subset \ell^2$ with the symplectic structure $\mathrm{i}\sum_j du_j \wedge d\bar{u}_j$ inherited from ℓ^2. We define the NLS Hamiltonian

$$H_V(u) := \sum_{j\in\mathbb{Z}} (j^2 + V_j)|u_j|^2 + P, \quad \text{with}$$
$$P := -\int_{\mathbb{T}} F\left(\left|\sum_j u_j e^{\mathrm{i}jx}\right|^2\right) dx, \qquad F(y) := \int_0^y f(s)\,ds. \tag{2.3}$$

Note that $\mathtt{w}_p$ is a Banach algebra with respect to convolution and, moreover, $V* : \mathtt{w}_p \to \mathtt{w}_p$ is a linear bounded operator with norm $|V|_\infty$. This implies that (2.3) is an analytic function on $\mathtt{w}_p \cap H^1$ (see Proposition 4.2).

By (2.2), if $p > 3/2$, then $\mathtt{w}_p \subset H^1$. Otherwise, for $1 < p < 3/2$ we might have infinite energy. Anyway, for our purposes, we only need that the Hamiltonian vector field

$$X_{H_V}^{(j)} := \mathrm{i}\partial_{\bar{u}_j} H_V = \mathrm{i}(j^2 + V_j)u_j + \mathrm{i}\partial_{\bar{u}_j} P$$

is well defined componentwise. For completeness, we remark that the flow of H_V is locally well posed on $\mathtt{w}_p$, for $p > 1$. Indeed, in Proposition 4.2 (see also (4.4)) we will show that the Hamiltonian vector field X_P is a uniformly bounded map from a suitable ball around the origin of $\mathtt{w}_p$ to $\mathtt{w}_p$; therefore, by standard variation of constants, we obtain the local well-posedness on $\mathtt{w}_p$.

The proof of Theorem 1 is based on the construction of an analytic change of variables on the phase space $\mathtt{w}_p$, which conjugates the nonlinear dynamics to a linear

[7] For $u \in \mathtt{w}_p$, we have $\|u\|_{L^\infty}, \|u^{(k)}\|_{L^\infty} \le \sum_j \langle j\rangle^k |u_j| \le \sum_j \lfloor j\rfloor^{k-p}|u|_{\mathtt{w}_p}$. Moreover, if $u \in H^p$, then by Parseval's equality, $\sum_j |j|^{2p}|u_j|^2 = \|u^{(p)}\|_{L^2}^2$ and $\sum_j |u_j|^2 = \|u\|_{L^2}^2$ so that $\sup_j |j|^p|u_j| \le \|u^{(p)}\|_{L^2}$ and $|u_0| \le \|u\|_{L^2}$, proving the second inequality in (2.2).

one, supported on an invariant flat torus. The regularity of the solution in the original variables is then deduced from the dynamics on the flat torus. Since these issues require some care, we find it instructive to first discuss the linear case, where the key difficulties become transparent.

2.1. *The linear case*

When $f = 0$ the Hamiltonian reduces to its quadratic part $\sum_{j\in\mathbb{Z}}(j^2 + V_j)|u_j|^2$, so that the linear actions $|u_j|^2$ are constants of motions and the dynamics is

$$u_j(t) = u_j(0)e^{\mathrm{i}\nu_j t}, \quad j \in \mathbb{Z}, \qquad \nu = (\nu_j)_{j\in\mathbb{Z}}, \quad \nu_j := j^2 + V_j.$$

Let us call $\mathcal{S}_* := \{j \in \mathbb{Z} \text{ s.t. } u_j(0) \neq 0\}$. If $\mathcal{S}_*$ is a finite set, then the corresponding solution $\mathbf{u}(t,x) := \sum_{j\in\mathbb{Z}} u_j(t)e^{\mathrm{i}jx}$ is quasiperiodic and analytic both in time and space. If $\mathcal{S}_*$ is infinite, then the regularity of $\mathbf{u}(t,x)$ obviously depends on the one of the initial datum. If $u(0) := (u_j(0))_{j\in\mathbb{Z}} \in \ell^1$, then $\mathbf{u}(t,x)$ is a weak solution of (NLS_V) in the sense of (1.2). Moreover, such solution is a time almost periodic function, being limit in $\mathcal{F}(\ell^1)$ of the quasiperiodic truncations $\sum_{|j|\le n} u_j(0)e^{\mathrm{i}\nu_j t+\mathrm{i}jx}$ as $n \to \infty$. Furthermore, if $(u_j(0))_{j\in\mathbb{Z}} \in \mathrm{w}_p$, then $t \mapsto \mathbf{u}(t,\cdot) \in \mathcal{F}(\mathrm{w}_p)$ and its truncations converge[8] in $\mathrm{w}_{p'}$ with $1 < p' < p$.

Note that if $p > 3$, then we have classical solutions (recall (2.2)). Otherwise, if $p \le 2$, then one can easily produce nonclassical solutions. Indeed, take any infinite subset $\mathcal{S}_*$ of $\mathbb{Z}$, and set $u_j(0) = \langle j\rangle^{-p}$ for $j \in \mathcal{S}_*$ and $u_j(0) = 0$ otherwise. Then, if $p \le 2$,

$$\limsup_{|j|\to\infty} j^2\big|u_j(t)\big| = \limsup_{|j|\to\infty} j^2\big|u_j(0)\big| > 0, \tag{2.4}$$

so $\mathbf{u}(t,\cdot) \notin H^2$. Finally, if $\mathcal{S}_* = \mathbb{Z}$ and $p \le 3/2$, then $\mathbf{u}(t,\cdot) \notin H^1$.

The support of each solution is an invariant torus in the following sense. Given $I := (I_j)_{j\in\mathbb{Z}} \in \mathrm{w}_{2p}$ with $I_j \ge 0$, we define the flat torus

$$\mathcal{T}_I := \big\{u \in \mathrm{w}_p : |u_j|^2 = I_j \ \forall j \in \mathbb{Z}\big\} \tag{2.5}$$

and the set

$$\mathcal{S}_I := \{j \in \mathbb{Z} \text{ s.t. } I_j > 0\}. \tag{2.6}$$

Then the map $\mathfrak{i} : \mathbb{T}^{\mathcal{S}_I} \to \mathcal{T}_I \subset \mathrm{w}_p, \varphi = (\varphi_j)_{j\in\mathcal{S}_I} \mapsto \mathfrak{i}(\varphi)$, with

$$\begin{cases} \mathfrak{i}_j(\varphi) := \sqrt{I_j}e^{\mathrm{i}\varphi_j} & \text{for } j \in \mathcal{S}_I, \\ \mathfrak{i}_j(\varphi) := 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

[8] Note that the convergence does not necessarily hold in w_p. See the examples below.

is an analytic immersion[9] provided that we endow $\mathbb{T}^{\mathcal{S}_I}$ with the ℓ^∞-topology (see Lemma B.2 below for details). By construction, the linear dynamics on the torus $\mathcal{T}_I$ is $\varphi \to \varphi + \nu t$.

Since the map $t \mapsto \nu t \in \mathbb{T}^{\mathcal{S}_I}$ is not even continuous (endowing $\mathbb{T}^{\mathcal{S}_I}$ with the ℓ^∞-topology), the regularity of $t \mapsto \mathrm{i}(\nu t)$ depends on the choice of the actions I_j. In the examples discussed above, it is not continuous with respect to the strong topology[10] (see also [32]). In contrast with the finite-dimensional case, even if ν has rationally independent entries, it is not straightforward to understand whether this invariant torus is densely filled[11] by the solution's orbit or not. In fact, this issue is related to the asymptotic behavior of ν. We will discuss this in Lemma B.3.

Definition 2.1 (Invariant flat torus)
Let $I \in \mathtt{w}_{2p}$, and let $\mathcal{T}_I$ be defined as in (2.5). Consider a vector field $X = (X^{(j)})_{j\in\mathbb{Z}} : \mathcal{T}_I \to \mathbb{C}^{\mathbb{Z}}$. We say that $\mathcal{T}_I$ is an invariant flat torus for X with frequency $\nu \in \mathbb{R}^{\mathbb{Z}}$ if

$$\dot u_j(t) = X^{(j)}\big(u(t)\big), \qquad u_j(t) := u_j(0)e^{\mathrm{i}\nu_j t}, \quad \forall t \in \mathbb{R},$$
$$\big|u_j(0)\big|^2 = I_j, \quad j \in \mathbb{Z}.$$

Note that the frequency ν_j is uniquely defined only for $j \in \mathcal{S}_I$.

2.2. *The nonlinear case*

In the nonlinear setting, as explained in the introduction, our invariant tori are approximately supported on $\mathcal{S}_I$ contained in an appropriately sparse set $\mathcal{S} \subset \mathbb{Z}$.

Definition 2.2 (Admissible tangential sites)
(a) An infinite subset $\mathcal{S}$ of $\mathbb{N}$, referred to as the *set of tangential sites in* $\mathbb{N}$, is said to be *admissible* if there exists a smooth strictly increasing function[12] $s : [0, +\infty) \to [0, +\infty)$ with the following properties:

- $\mathcal{S} = s(\mathbb{N})$,
- there exists $i_* \geq 21$ such that

$$s(i) \geq e^{(\log i)^{1+\eta}}, \quad \forall i \geq i_*, \text{for some given } 1 < \eta \leq 2; \tag{2.8a}$$

[9] Assuming also that $\inf_j \sqrt{I_j}\langle j\rangle^p > 0$, the map i is an embedded torus. Otherwise, i is a homeomorphism on the image only if we endow both source and target spaces with the product topology.
[10] Note that the map is continuous endowing $\mathtt{w}_p$ with the product topology, which coincides with the weak $*$ topology on bounded sets.
[11] In the product topology such solutions are always dense.
[12] The function $s(i)$ is obviously not unique, but its restriction to $\mathcal{S}$ is unique. The same holds for its inverse $i(s)$.

$$s(i+i') \geq s(i) + s(i'), \qquad s(hi) \geq hs(i), \quad \forall h \geq 1, i, i' \geq i_*; \tag{2.8b}$$

$$s(i^2) \geq s^2(i), \quad \forall i \geq i_*. \tag{2.8c}$$

We denote by $i(s)$ the inverse function of $s(i)$.

(b) An infinite subset $\mathcal{S}$ of $\mathbb{Z}$, referred to as the *set of tangential sites*, is said to be *admissible* if there exist a finite subset $\mathcal{S}_0$ of $\mathbb{Z}$ and two admissible sets of tangential sites $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{N}$ so that $\mathcal{S}$ is one of the following subsets:

$$\mathcal{S}_1, \qquad -\mathcal{S}_1, \qquad (-\mathcal{S}_1) \cup \mathcal{S}_0, \qquad \mathcal{S}_1 \cup \mathcal{S}_0 (-\mathcal{S}_1) \cup \mathcal{S}_2 \cup \mathcal{S}_0.$$

Remark 2.1

Definition 2.2 above gives a quantitative control on how "sparse" the set $\mathcal{S}$ should be. In particular, by (2.8a), the function $s(i)$ must grow faster than any polynomial.

Examples of admissible tangential sites in $\mathbb{N}$. $\mathcal{S} := \{2^i : i \in \mathbb{N}\}$; choose (see [7]) $s(x) = 2^x$. Or, given $\eta > 0$ as above, set $\mathcal{S} := \{[e^{(\log i)^{1+\eta}}] : i \in \mathbb{N}\}$ ($[\cdot]$ being the integer part), and choose $s(x) = [e^{(\log x)^{1+\eta}}]$, for any $x > 0$.

In what follows, given a sequence indexed over $\mathbb{Z}$, we systematically decompose it over $\mathcal{S}$ and $\mathcal{S}^c := \mathbb{Z} \setminus \mathcal{S}$; for example, for the potential V we write

$$V = (V_{\mathcal{S}}, V_{\mathcal{S}^c}), \qquad V_{\mathcal{S}} := (V_j)_{j \in \mathcal{S}}, \qquad V_{\mathcal{S}^c} := (V_j)_{j \in \mathcal{S}^c}.$$

Analogously, we define the set of *tangential frequencies* as

$$\mathcal{Q}_{\mathcal{S}} := \Big\{ \nu = (\nu_j)_{j \in \mathcal{S}} \in \mathbb{R}^{\mathcal{S}} : |\nu_j - j^2| < \frac{1}{2} \Big\}. \tag{2.9}$$

The cube $\mathcal{Q}_{\mathcal{S}}$ inherits the product topology[13] and the product probability measure $\mathrm{meas}_{\mathcal{Q}_{\mathcal{S}}}$ from $[-1/2, 1/2]^{\mathcal{S}}$ through the map

$$\mathcal{V}^*_{\mathcal{S}} : \mathcal{Q}_{\mathcal{S}} \to [-1/2, 1/2]^{\mathcal{S}}, \quad \text{where } \mathcal{V}^*_{\mathcal{S},j}(\nu) := \nu_j - j^2, \, j \in \mathcal{S}. \tag{2.10}$$

Moreover, the above map also endows $\mathcal{Q}_{\mathcal{S}}$ with the ℓ^∞-metric, which induces a finer topology.

Finally, we denote by $B_r(\mathrm{w}_p)$ the open ball of radius r centered at the origin of w_p, and for $\mathrm{r} > 0$ we define the *tangential actions*

$$\mathcal{I}(p, \mathrm{r}) := \big\{ I \in B_{\mathrm{r}^2}(\mathrm{w}_{2p}) : I_j = 0 \text{ for } j \in \mathcal{S}^c, I_j \geq 0 \text{ for } j \in \mathcal{S} \big\}. \tag{2.11}$$

We are now ready to state our main dynamical result regarding the existence of invariant KAM tori, parameterized by $I, V_{\mathcal{S}^c}$ and by the frequency ν.

[13] Recall that if X_i, $i \in I$ are topological metrizable spaces, then the product topology on $X := \prod_{i \in I} X_i$ is the topology of the pointwise convergence, meaning that a sequence $x^{(k)} = (x_i^{(k)})_{i \in I}$ converges if and only if $x_i^{(k)}$ converges for all $i \in I$.

THEOREM 2
Assume that $\mathcal{S} \subset \mathbb{Z}$ is an admissible set of tangential sites, that $p > 1$, and that $\mathrm{r} > 0$ *is sufficiently small. Moreover, assume that*

$$I \in \mathcal{I}(p,\mathrm{r}) \qquad \textit{and} \qquad V_{\mathcal{S}^c} = (V_j)_{j \in \mathcal{S}^c} \in [-1/4, 1/4]^{\mathcal{S}^c}.$$

Then there exists a Cantor-like set $\mathcal{C} \subset \mathcal{Q}_{\mathcal{S}}$ (recall (2.9)) of positive measure such that for any frequency $\nu \in \mathcal{C}$, there exist

$$V_{\mathcal{S}} = (V_j)_{j \in \mathcal{S}} \in [-1, 1]^{\mathcal{S}}$$

and a symplectic diffeomorphism Φ analytic on a small ball in $\mathtt{w}_p$ such that, for all $\nu \in \mathcal{C}$,

$$\mathcal{T}_I = \big\{ u \in \mathtt{w}_p : |u_j|^2 = I_j \ \forall j \in \mathbb{Z} \big\}$$

is an invariant flat torus of frequency ν (recall Definition 2.1) for the Hamiltonian vector field of $H_V \circ \Phi$, with $V = (V_{\mathcal{S}}, V_{\mathcal{S}^c})$.

Theorem 2 above contains cubes of three different sidelengths, $[-1/4, 1/4]^{\mathcal{S}^c}$, $\mathcal{Q}_{\mathcal{S}}$, and $[-1, 1]^{\mathcal{S}}$. While the choice of the lengths $1/2$, 1, and 2 is rather arbitrary, the fact that we need three different scales is unavoidable (see Remark 2.2).

By Theorem 2, equation (NLS$_V$) has invariant tori on which the dynamics is the linear translation by νt. The statement above is a typical KAM theorem regarding the existence of an invariant torus. The fact that we are looking for an infinite torus, however, introduces various new difficulties, in particular related to the regularity of the dependence on $\nu, I, V_{\mathcal{S}^c}$. This appears immediately when one wishes to prove that the Cantor set $\mathcal{C}$ of "good frequencies" is measurable (and of positive measure) with respect to the product probability measure $\mathrm{meas}_{\mathcal{Q}_{\mathcal{S}}}$ on $\mathcal{Q}_{\mathcal{S}}$. The σ-algebra of such measure, which is the natural one in this context, is given by the Borel sets of the product topology, which is coarser than the one induced by the ℓ^∞-metric. Then a crucial point is that a function $f : \mathcal{Q}_{\mathcal{S}} \to \mathbb{R}$ which is Lipschitz (with respect to the ℓ^∞-metric) might be noncontinuous with respect to the product topology and, hence, nonmeasurable. As typical in KAM schemes, $\mathcal{C}$ is defined as the intersection of sets of the form $\{|f(\nu)| > \alpha > 0\}$ (see (9.1), (9.2)) and $f : \mathcal{Q}_{\mathcal{S}} \to \mathbb{R}$ is a Lipschitz function. As explained above, this does not assure measurability.[14]

So we reformulate our theorem in a more technical way, carefully keeping track of the regularity with respect to all the parameters. To avoid working with functions defined only on a Cantor set, we suitably (see Lemma 4.1) extend all the functions so

[14]This problem does not appear if f actually depends only on a finite number of variables as in the case of maximal tori, where $f(\nu) = \nu \cdot \ell$, with $|\ell| < \infty$.

that they are defined for $\nu \in \mathcal{Q}_{\mathcal{S}}$. To summarize, the parameter dependences we need to control are:

- continuity with respect to the product topology for measure estimates both in ν and in V;
- Lipschitz dependence (w.r.t. the ℓ^∞-metric) for implicit function theorems/contractions/extensions (see Lemma B.1).

The smallness condition (i.e., how close are our solutions to zero) is tied to the size of the nonlinearity and to the regularity of the solution we are looking for. Recall that since f is analytic, for some $\mathtt{R} > 0$, we have

$$f(y) = \sum_{d=1}^{\infty} f^{(d)} y^d, \qquad |f|_{\mathtt{R}} := \sum_{d=1}^{\infty} |f^{(d)}| \mathtt{R}^d < \infty. \tag{2.12}$$

THEOREM 3
Let $p_* > 1$ and $0 < \gamma \le \min\{1/4, |f|_{\mathtt{R}}\}$. There exist $\varepsilon_* = \varepsilon_*(p_*) > 0$ and $C = C(p_*) > 1$ such that, for all $\mathrm{r} > 0$ *satisfying*

$$\varepsilon := \frac{|f|_{\mathtt{R}}}{\gamma \mathtt{R}} \mathrm{r}^2 \le \varepsilon_* \tag{2.13}$$

and for every

$$\frac{p_* + 1}{2} \le p \le p_*, \tag{2.14}$$

the following holds. There exist:

(i) a map

$$\begin{aligned} &\mathcal{V}_{\mathcal{S}} : \mathcal{Q}_{\mathcal{S}} \times [-1/4, 1/4]^{\mathcal{S}^c} \times \mathcal{I}(p, \mathrm{r}) \to [-1, 1]^{\mathcal{S}}, \\ &(\nu, V_{\mathcal{S}^c}, I) \mapsto \mathcal{V}_{\mathcal{S}}(\nu, V_{\mathcal{S}^c}, I) \end{aligned} \tag{2.15}$$

which is continuous in $\nu, V_{\mathcal{S}^c}$ with respect to the product topology and Lipschitz in all its variables with respect to the ℓ^∞-metric (in particular, $\mathcal{V}_{\mathcal{S}}$ is Lipschitz $C\varepsilon\gamma$-close with respect to ν to the map $\mathcal{V}^*_{\mathcal{S}}$);

(ii) a map

$$\begin{aligned} \Phi : &B_{3\mathrm{r}}(\mathtt{w}_p) \times \mathcal{Q}_{\mathcal{S}} \times [-1/4, 1/4]^{\mathcal{S}^c} \times \mathcal{I}(p, \mathrm{r}) \longrightarrow B_{4\mathrm{r}}(\mathtt{w}_p), \\ &(u; \nu, V_{\mathcal{S}^c}, I) \mapsto \Phi(u; \nu, V_{\mathcal{S}^c}, I) \end{aligned}$$

which is Lipschitz in all its variables and $\mathrm{r}/16$*-close to the identity with respect to u;*[15]

[15]Namely, $|\Phi(u; \nu, V_{\mathcal{S}^c}, I) - u|_p \le \mathrm{r}/16$ on $B_{3\mathrm{r}}(\mathtt{w}_p) \times \mathcal{Q}_{\mathcal{S}} \times [-1/4, 1/4]^{\mathcal{S}^c} \times \mathcal{I}(p, \mathrm{r})$.

(iii) a Cantor-like Borel set $\mathcal{C} = \mathcal{C}(V_{\mathcal{S}^c}, I, \gamma) \subset \mathcal{Q}_{\mathcal{S}}$, with[16]

$$\begin{aligned} &\operatorname{meas}_{\mathcal{Q}_{\mathcal{S}}}\big(\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C}(V_{\mathcal{S}^c}, I, \gamma)\big) \le C_0\gamma, \\ &\operatorname{meas}_{[-1/4,1/4]^{\mathcal{S}}}\big(\mathcal{V}_{\mathcal{S}}\big(\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C}(V_{\mathcal{S}^c}, I, \gamma), V_{\mathcal{S}^c}, I\big)\big) \le C_0\gamma, \end{aligned} \tag{2.16}$$

for a suitable absolute constant $C_0 > 0$.

Moreover, for any $I \in \mathcal{I}(p, \mathrm{r})$, $V_{\mathcal{S}^c} \in [-1/4, 1/4]^{\mathcal{S}^c}$, and $\nu \in \mathcal{C}(V_{\mathcal{S}^c}, I, \gamma)$, the map $\Phi(\cdot; \nu, V_{\mathcal{S}^c}, I)$ is an analytic symplectic change of variables.

Finally, for $\nu \in \mathcal{C}(V_{\mathcal{S}^c}, I, \gamma)$, $\mathcal{T}_I$ defined in (2.5) is a KAM torus of frequency ν for $H_V \circ \Phi$ with $V = (\mathcal{V}_{\mathcal{S}}(\nu, V_{\mathcal{S}^c}, I), V_{\mathcal{S}^c})$.

Remark 2.2

In Theorem 3 above, the map $\mathcal{V}_{\mathcal{S}}$ has oscillations of order $C\varepsilon\gamma$, so the target cube could be $[-1/2 - C\varepsilon\gamma, 1/2 + C\varepsilon\gamma]$, but in any case must have sidelength greater than 1. On the other hand, in order to achieve the second estimate in (2.16), we need the sidelength of $\mathcal{Q}_{\mathcal{S}}$ to be greater than $1/2$. This is the reason for choosing three distinct sidelengths (see comments after Theorem 2).

Note that the parameter γ comes from a Diophantine condition (see Definition 2.3) and consequently controls the measure estimates (2.16). Of course, this estimate is meaningful only for small γ.

We now reformulate our result in terms of the Fourier multiplier V (for the proof see Section 9).

COROLLARY 2.1

For $I \in \mathcal{I}(p, \mathrm{r})$, with p, r as in Theorem 3, let

$$\mathcal{G}(I, \gamma) := \big\{V \equiv (V_{\mathcal{S}}, V_{\mathcal{S}^c}) \in [-1/4, 1/4]^{\mathbb{Z}} : V_{\mathcal{S}} \in \mathcal{V}_{\mathcal{S}}\big(\mathcal{C}(V_{\mathcal{S}^c}, I, \gamma), V_{\mathcal{S}^c}, I\big)\big\}.$$

$\mathcal{G}(I, \gamma)$ is a Borel set in $[-1/4, 1/4]^{\mathbb{Z}}$ with measure greater than $1 - C_0\gamma$ (C_0 is the constant in (2.16)).

Therefore, for all $I \in \mathcal{I}(p, \mathrm{r})$ and all $V \in \mathcal{G}(I, \gamma)$, the equation ($\mathrm{NLS}_V$) has an invariant torus.[17]

Invariant tori and regularity of our almost periodic solutions. The flat torus $\mathcal{T}_I$ in the original variables is realized by the analytic immersion $\varphi \mapsto \Phi(\mathfrak{i}(\varphi); \nu, V_{\mathcal{S}^c}, I)$ with $\mathfrak{i}$ defined in (2.7). By construction, the NLS dynamics on the torus $\mathcal{T}_I$ is $\varphi \mapsto \varphi +$

[16] We denote by $\operatorname{meas}_{\mathcal{Q}_{\mathcal{S}}}$ and $\operatorname{meas}_{[-1/4,1/4]^{\mathcal{S}}}$ the product probability measures on $\mathcal{Q}_{\mathcal{S}}$ and $[-1/4, 1/4]^{\mathcal{S}}$, respectively. For brevity, we write $\operatorname{meas}_{\mathcal{Q}_{\mathcal{S}}}(A)$ for $\operatorname{meas}_{\mathcal{Q}_{\mathcal{S}}}(A \cap \mathcal{Q}_{\mathcal{S}})$ and similarly for $\operatorname{meas}_{[-1/4,1/4]^{\mathcal{S}}}$.

[17] Of course, in the linear case one has $\Phi = \mathrm{Id}$, $\mathcal{V}_{\mathcal{S}} = \mathcal{V}^*_{\mathcal{S}}$, $\mathcal{C} = \mathcal{Q}_{\mathcal{S}}$, $\mathcal{G} = [-\frac{1}{4}, \frac{1}{4}]^{\mathbb{Z}}$.

νt, so our candidate for an almost periodic solution is $\mathbf{u}(t,\cdot) := \Phi(\mathfrak{i}(\nu t); \nu, V_{\mathcal{S}^c}, I)$, as discussed in Section 3. We wish to stress that analyticity of $\Phi(\mathfrak{i}(\varphi))$ in the angles does not imply analyticity in time, since the map $t \mapsto \nu t \in \mathbb{T}^{\mathcal{S}}$ is not even continuous (endowing $\mathbb{T}^{\mathcal{S}}$ with the ℓ^∞-topology). On the other hand, since Φ is analytic, the *regularity issues are the same as in the linear case* (recall Section 2.1). In particular, for $p > 3$ our solutions are classical, while for $p \le 2$ we can construct also merely weak solutions (see the proof of Theorem 1).

The Cantor set $\mathcal{C}$. We can be rather explicit in our description of the set $\mathcal{C}$ of Theorem 3. We start by fixing the hypercube

$$\mathcal{Q} := \left\{\omega = (\omega_j)_j \in \mathbb{R}^{\mathbb{Z}} : |\omega_j - j^2| < \frac{1}{2}\right\}, \quad \mathcal{Q} = \mathcal{Q}_{\mathcal{S}} \times \mathcal{Q}_{\mathcal{S}^c}, \tag{2.17}$$

endowed with the product topology, and by introducing the following closed set.

Definition 2.3 (Diophantine condition)
Let $\gamma > 0$. We say that a vector $\omega \in \mathcal{Q}$ belongs to $\mathtt{D}_{\gamma,\mathcal{S}}$ if it satisfies[18]

$$\begin{aligned} |\omega \cdot \ell| \ge \gamma \prod_{s \in \mathcal{S}} \frac{1}{(1 + |\ell_s|^2 \langle i(s)\rangle^2)^{3/2}}, \\ \forall \ell : 0 < |\ell| < \infty, \quad \sum_{j \in \mathcal{S}^c} |\ell_j| \le 2, \quad \pi(\ell) = \mathfrak{m}(\ell) = 0, \end{aligned} \tag{2.18}$$

where[19] $i(s)$ is the inverse function of $s(i)$, $\pi(\ell) := \sum_{j\in\mathbb{Z}} j\ell_j$ is the "momentum," and $\mathfrak{m}(\ell) := \sum_{j\in\mathbb{Z}} \ell_j$ is the "mass."

THEOREM 4
Under the hypotheses of Theorem 3, there exists a Lipschitz map $\Omega : \mathcal{Q}_{\mathcal{S}} \times [-1/4, 1/4]^{\mathcal{S}^c} \times \mathcal{I}(p, \mathrm{r}) \to \mathcal{Q}_{\mathcal{S}^c}$ *which is continuous with respect to the product topology on* $\mathcal{Q}_{\mathcal{S}} \times [-1/4, 1/4]^{\mathcal{S}^c}$ *and*[20] *for every* $j \in \mathcal{S}^c$ *satisfies*

$$\left|\Omega_j(\nu, V_{\mathcal{S}^c}, I) - j^2 - V_j\right| \le C\gamma\varepsilon \tag{2.19}$$

and the Lipschitz estimates

$$\begin{aligned} \sup_{\nu' \ne \nu} \frac{|\Omega_j(\nu, V_{\mathcal{S}^c}, I) - \Omega_j(\nu', V_{\mathcal{S}^c}, I)|}{|\nu - \nu'|_\infty} \le C\varepsilon, \\ \sup_{I' \ne I} \frac{|\Omega_j(\nu, V_{\mathcal{S}^c}, I) - \Omega_j(\nu, V_{\mathcal{S}^c}, I')|}{|I - I'|_{2p}} \le C\gamma \mathrm{r}^{-2}\varepsilon. \end{aligned} \tag{2.20}$$

[18] Instead of $3/2$ one can put every exponent $\tau > 1$.
[19] As usual, for integer vector $\ell \in \mathbb{Z}^{\mathbb{Z}}$ we set $|\ell| = \sum_{j\in\mathbb{Z}} |\ell_j|$.
[20] The constant $\boldsymbol{C}$ is the one of Theorem 3.

Moreover, we can choose in Theorem 3

$$\mathcal{C}(V_{\mathcal{S}^c}, I, \gamma) := \{\nu \in \mathcal{Q}_{\mathcal{S}} : \omega(\nu, V_{\mathcal{S}^c}, I) \in \mathrm{D}_{\gamma,\mathcal{S}}\},$$
$$\text{where } \omega(\nu, V_{\mathcal{S}^c}, I) := (\nu, \Omega(\nu, V_{\mathcal{S}^c}, I)). \tag{2.21}$$

In this way, the torus $\mathcal{T}_I$ defined in (2.5) is an elliptic invariant torus in the sense that its linearized dynamics in the "normal" directions is $\dot{u}_j = \mathrm{i}\Omega_j u_j$ for $j \in \mathcal{S}^c$.

Remark 2.3

Note that (2.18) is a much stronger Diophantine condition than the one proposed in [13] (or [8]), where the denominators were of the form $1 + |\ell_j|^2 j^2$. Of course the reasons that we can impose such strong Diophantine conditions—still obtaining a positive measure set—are the structure of the set $\mathcal{S}$ and the fact that we only need to consider denominators with $\sum_{j\in\mathcal{S}^c} |\ell_j| \le 2$.

2.3. Plan of the paper, strategy, and main novelties

Deducing Theorem 1 from Theorems 3 and 4 is a self-contained argument, which we present in Section 3. The proof is developed as follows. $V_{\mathcal{S}^c}$ and $I \in \mathcal{J}(p, \mathrm{r})$ being fixed, we first construct a set $\mathcal{C}'(V_{\mathcal{S}^c}, I, \gamma) \subseteq \mathcal{Q}_{\mathcal{S}}$ of large relative measure such that for all $\nu \in \mathcal{C}'(V_{\mathcal{S}^c}, I, \gamma)$ the map $t \mapsto \mathbf{u}(t, \cdot) := \Phi(\mathrm{i}(\nu t); \nu, V_{\mathcal{S}^c}, I) \in \mathrm{w}_p$ is almost periodic in $\mathrm{w}_{p'}$ with $1 < p' < p$ since it is the uniform limit of quasiperiodic functions just as in the linear case (see Section 2.1). Such approximating functions are in fact classical solutions of the approximate equation (NLS_{V_n}) with $V_n = V_n(\nu) \to V(\nu)$. Second, we show that $\mathbf{u}$ is a weak solution in the sense of (1.2), for such $V = V(\nu)$. We next reformulate our result in terms of the external parameters V instead of the frequencies ν; namely, we prove that for every V in a large measure set $\mathcal{G}(I, \gamma) \subseteq [-1/4, 1/4]^{\mathbb{Z}}$ we can solve (1.2). A main point is to show that $\mathcal{G}(I, \gamma)$ is measurable. Next, taking the union over γ we obtain a full measure set, thus showing that for a.e. V there exists at least one solution. Finally, moving $\mathcal{S}$ and suitably choosing I we produce, for a.e. V, countably many different solutions.

In order to prove Theorems 3 and 4 (see the end of Section 6), we follow the general strategy of [8]. As recalled in Section 6, this consists of three main steps: the introduction of a suitable functional setting and degree decomposition, a formulation in terms of a counterterm theorem, solving a homological equation, and proving the fast convergence of a KAM scheme. Regarding the first issue, the main additional difficulty in the present setting is to keep track of the regularity with respect to all the parameters. To this purpose, in Section 4 we start by defining the Poisson algebra of "regular Hamiltonians" (basically a space of normally analytic functions H such that the corresponding Hamiltonian vector field X_H is a normally analytic map $B_r(\mathrm{w}_p) \to \mathrm{w}_p$) and further introduce the space of parameter-dependent regular Hamiltonians (see Definitions 4.3 and 4.4). In particular, we set our ambient space as the closure, with respect to a suitable norm (see (4.9) together with (4.12)–(4.13)), of functions depending only on a finite number of frequencies. A Lipschitz extension result is given in Lemma 4.1.

Theorems 3 and 4 will be derived in two steps, through a technique known as *elimination of parameters*, first introduced by Rüssmann and Herman in the 1980s. This strategy consists in introducing some extra parameters (i.e., the counterterms λ_j) in the perturbed Hamiltonian, in order to compensate its degeneracies (absence of twist properties, for instance) and conjugate the modified Hamiltonian to one that admits the desired invariant KAM torus. This first normal form step contains the hard analysis. Then we prove that we can *eliminate* the counterterms using the potential, by means of the classical implicit function theorem in Banach spaces.

Hence, in our case, we first put H_V in (2.3) in a suitable normal form with counterterms in the spirit of Herman through Theorem 5, which contains the main KAM difficulties in dealing with Sobolev regularity.

Roughly speaking, this normal form theorem states that, under appropriate smallness conditions, for all $\omega \in \mathtt{D}_{\gamma,\mathcal{S}}$, $V_{\mathcal{S}^c} \in [-1/4, 1/4]^{\mathcal{S}^c}$, and $I \in \mathcal{I}(p,r)$ there exist $\lambda \in \ell^\infty$ and a symplectic change of variables Ψ (see (6.3)) such that

$$\Big(\sum_{j\in\mathbb{Z}}(\omega_j+\lambda_j)|u_j|^2+P\Big)\circ\Psi=\sum_{j\in\mathbb{Z}}\omega_j|u_j|^2+O\big(|u|^2-I\big)^2,$$

where $R = O(|u|^2 - I)^2$ means that R is a regular Hamiltonian which has a zero of order at least 2 at the torus $\mathcal{T}_I$ defined in (2.5) (we formalize this definition in Section 5, introducing an appropriate degree decomposition). In order to deduce Theorems 3 and 4, we need to eliminate the counterterms (see equations in (6.9)): the main point is that we can solve with respect only to the tangential variables $V_{\mathcal{S}} \leftrightsquigarrow \nu$.

Coming back to the counterterm Theorem 5, let us explain the main new issues related to Sobolev regularity.

As is habitual, the map Ψ is constructed as the composition of a sequence of changes of coordinates whose generating function S at every step is determined by solving a homological equation

$$L_\omega S := \Big\{\sum_j \omega_j|u_j|^2, S\Big\} = F, \tag{2.22}$$

where F is a given analytic function on the phase space $\mathtt{w}_p$ (see Definition 4.4) which is at most quadratic in the normal variables ($j \notin \mathcal{S}$). At a formal level, a solution $S = L_\omega^{-1}F$ of (2.22) is readily determined. In order to prove that S is in fact analytic, one has to control the contribution given by small divisors (i.e., the eigenvalues of L_ω). This is possible by imposing the arithmetic Melnikov conditions on the frequencies

(2.18), where the constraint $\sum_{j\in\mathcal{S}^c}|\ell_j|\le 2$ comes from the fact that F is at most quadratic in the normal variables, while the zero mass and momentum conditions come from the presence of the corresponding quadratic constants of motion.

Also in this Sobolev context, due to the presence of small divisors, one bounds the solution S at the cost of some "loss of information." More explicitly, if the Hamiltonian vector field X_F maps $B_r(\mathtt{w}_p)\to\mathtt{w}_p$, then X_S maps $B_r(\mathtt{w}_{p+\delta})\to\mathtt{w}_{p+\delta}$ (with $\delta>0$); this means that S is analytic in a *smaller* domain, since $\mathtt{w}_{p+\delta}\subset\mathtt{w}_p$. Then, at each iteration, one is able to define the solutions S only on a (ball of a) smaller phase space of the Banach scale $(\mathtt{w}_p)_{p>1}$, the target space shrinking accordingly. Resembling the finite dimensional case, we call δ the *loss of regularity*.

Of course, the convergence of a KAM scheme is achieved only if one loses a *summable amount of regularity* δ_n at each step n. Nonetheless, the typical characteristic of the Sobolev case is the presence of a *lower bound* on the loss of regularity, so that a KAM scheme based only on such a bound cannot converge. In KAM schemes for quasiperiodic Sobolev solutions, a similar problem arises but it is bypassed by using tame estimates or approximation by analytic functions. In the present context it is not clear whether one can exploit similar ideas, since we are already working with analytic nonlinearities. Our point of view is to bypass this problem by taking full advantage of the fact that the tangential sites set $\mathcal{S}$ is sparse (recall Definition 2.2). In order to get an intuition of our strategy, take for simplicity $\mathcal{S}=\{s(i)=2^i, i\in\mathbb{N}\}$, and start by considering a toy model with a nonlinearity for which the set

$$\mathcal{U}_{\mathcal{S}}:=\{u\in\mathtt{w}_p : u_j=0\ \forall j\notin\mathcal{S}\}\approx\Big\{v\in\ell^\infty \text{ s.t. } \sup_{i\in\mathbb{N}}|v_i|2^{ip}<\infty\Big\} \tag{2.23}$$

is invariant for the dynamics so that we may study the equation restricted to $\mathcal{S}$. Then we are essentially in the analytic case (or Gevrey or slightly less if we take a slower growth for $s(i)$), and the Bourgain strategy in [13] (or [8], [15]) applies.[21] Of course for the NLS equation (NLS_V) the set $\mathcal{U}_{\mathcal{S}}$ is not invariant and the main difficulties arise from interaction between tangential and normal modes.

In the Diophantine estimates, to deal with the terms ℓ not supported only on the tangential sites, we use the constants of motion and the dispersive nature of the equation ($\omega_k\sim k^2$). Once one has guessed the correct Diophantine conditions (2.18), the proof of Proposition 7.1 (i.e., controlling the solution of the homological equation (2.22)) is the real core of our result. Again, the proof is simple if F is supported only on $\mathcal{S}$ or $\mathcal{S}^c$; on the other hand, dealing with the interaction between tangential and normal sites requires a careful case analysis.

The final goal is to control the norm of X_S as a map $B_r(\mathtt{w}_{p+\delta})\to\mathtt{w}_{p+\delta}$, for arbitrarily small δ. This should be compared with the corresponding estimate on X_S in

[21] Note that Bourgain's Diophantine condition reads $|\omega\cdot\ell|\ge\gamma\prod_{i\in\mathbb{N}}(1+\ell_i^2\langle i\rangle^2)^{-1}$. This coincides with (2.18) when ℓ is supported only on $\mathcal{S}$, modulo the renaming of the indexes in (2.23).

[6, Proposition 7.1, item (M)]. In the latter paper we take $\mathcal{S} = \mathbb{Z}$ and then, to control $L_\omega^{-1} F$, we cannot take any $\delta > 0$ but instead must require $\delta \geq \delta_*$, where $\delta_* > 0$ is some fixed quantity. As one might expect, the less *sparse* $\mathcal{S}$ is, the worst bounds one gets. The quantitative condition in Definition 2.2 is needed in order to ensure convergence of the iterative KAM scheme. We suitably choose the values of the parameter at each iterative step $n \in \mathbb{N}$; in particular, the loss of regularity δ_n must be summable, for example, $\delta_n \sim n^{-c}$, for some $c > 1$. Then the divergence due to small divisors, which is of order $\exp(\exp(n^{c/\eta}))$ by (7.3) and with η defined in (2.8a), must be compensated by the superexponential convergence $\exp(-\exp(Cn))$ given by the KAM quadratic scheme. This forces $c < \eta$ and $\eta > 1$.

The superlinearity assumptions (2.8b) and (2.8c) are essential for our estimate on the homological equation to work. The asymptotic growth in (2.8a) is only needed in the KAM step. A slower growth would give rise to a too large estimate on the solution of the homological equation which would not be compensated anymore by the quadratic convergence of the KAM scheme.

3. Proof of Theorem 1

The solutions of Theorem 1 are constructed as the limit of sequences of smooth quasiperiodic functions. Fix $p_* > 1$ and an admissible set of tangential sites $\mathcal{S}$. Fix $\gamma > 0$, and take r such that (2.13) holds. For any potential $V_{\mathcal{S}^c} \in [-1/4, 1/4]^{\mathcal{S}^c}$ and $I \in \mathcal{I}(p_*, \mathrm{r})$, we apply Theorem 3 and obtain, for all frequencies $\nu \in \mathcal{C}(V_{\mathcal{S}^c}, I, \gamma)$, an invariant torus.

Now define $I^{(n)} = (I_j^{(n)})$ by setting $I_j^{(n)} = I_j$ if $|j| \leq n$ and $I_j^{(n)} = 0$ otherwise. We apply Theorem 3 with $I \rightsquigarrow I^{(n)}$, $\gamma \rightsquigarrow \gamma/2$ and set $\mathcal{V}^{(n)}(\cdot) := (\mathcal{V}_{\mathcal{S}}(\cdot, V_{\mathcal{S}^c}, I^{(n)}), V_{\mathcal{S}^c})$ a $\Phi_n(\cdot;\cdot) := \Phi(\cdot;\cdot, V_{\mathcal{S}^c}, I^{(n)})$. We have obtained a sequence of NLS equations with potentials $\mathcal{V}^{(n)}(\nu)$ for $\nu \in [-1/4, 1/4]^{\mathcal{S}}$; each equation admits a finite-dimensional invariant torus with frequency ν, for all $\nu \in \mathcal{C}(V_{\mathcal{S}^c}, I^{(n)}, \gamma/2)$. It is not hard to see that each torus supports a smooth quasiperiodic solution.

The idea is to show that (at least up to a subsequence) the limit over n is the desired almost periodic solution.

First step (construction of the Cantor set). In order to apply Theorem 3 for each n (with the same frequency), we take the "good frequencies" in the (countable) intersection of the Cantor-like sets where all the tori are defined. Correspondingly, we define the set of "good potentials." This is the content of the following result, which is proved in Section 9.

LEMMA 3.1
There exists a subsequence $n_k \to \infty$ (independent of $V_{\mathcal{S}^c}$) such that the following holds. Defining the Borel sets

$$\mathcal{C}'(V_{\mathcal{S}^c}, I, \gamma) := \mathcal{C}(V_{\mathcal{S}^c}, I, \gamma) \bigcap_{k\in\mathbb{N}} \mathcal{C}(V_{\mathcal{S}^c}, I^{(n_k)}, \gamma/2),$$
$$\mathcal{G}_{\mathcal{S}}(V_{\mathcal{S}^c}, I, \gamma) := [-1/4, 1/4]^{\mathcal{S}} \bigcap \mathcal{V}_{\mathcal{S}}\big(\mathcal{C}'(V_{\mathcal{S}^c}, I, \gamma), V_{\mathcal{S}^c}, I\big),$$
$$\mathcal{G}(I, \gamma) := \big\{V = (V_{\mathcal{S}}, V_{\mathcal{S}^c}) \in [1/4, 1/4]^{\mathbb{Z}} \text{ s.t. } V_{\mathcal{S}} \in \mathcal{G}_{\mathcal{S}}(V_{\mathcal{S}^c}, I, \gamma)\big\},$$

we have the estimates

$$\begin{aligned} \operatorname{meas}_{[-1/4,1/4]^{\mathcal{S}}}\big(\mathcal{G}_{\mathcal{S}}(V_{\mathcal{S}^c}, I, \gamma)\big) &\geq 1 - C_0\gamma, \\ \operatorname{meas}_{[-1/4,1/4]^{\mathbb{Z}}}\big(\mathcal{G}(I, \gamma)\big) &\geq 1 - C_0\gamma, \end{aligned} \tag{3.1}$$

where C_0 is defined in Theorem 3.

Second step (construction of the convergent subsequence). We now prove that for every $V \in \mathcal{G}(I, \gamma)$, we can solve (1.2) with $V = (V_{\mathcal{S}}, V_{\mathcal{S}^c})$. Indeed, for every $V_{\mathcal{S}} \in \mathcal{G}_{\mathcal{S}}(V_{\mathcal{S}^c}, I, \gamma)$ there exists $\nu \in \mathcal{C}'(V_{\mathcal{S}^c}, I, \gamma)$ such that $\mathcal{V}_{\mathcal{S}}(\nu, V_{\mathcal{S}^c}, I) = V_{\mathcal{S}}$. We claim that

$$\mathbf{u}(t, x) := \Phi\big(v(t, x); \nu, V_{\mathcal{S}^c}, I\big), \quad \text{where } v(t, x) := \sum_{j\in\mathcal{S}} \sqrt{I_j} e^{\mathrm{i}(jx+\nu_j t)}, \tag{3.2}$$

satisfies (1.2) and, moreover, it is the uniform limit of the C^∞-quasiperiodic functions

$$\mathbf{u}_k(t, \cdot) := \Phi_{n_k}\big(v_k(t, \cdot); \nu\big), \quad \text{where } v_k(t, x) := \sum_{j\in\mathcal{S}, |j|\leq n_k} \sqrt{I_j} e^{\mathrm{i}(jx+\nu_j t)}.$$

Note that by construction $v, \mathbf{u} \in \mathrm{w}_{p_*}$ while $t \mapsto \Phi_{n_k}(v_k(t, \cdot); \nu) =: \mathbf{u}_k(t, \cdot)$ is a classical (actually C^∞-) quasiperiodic solution of ($\mathrm{NLS}_{V^{(n_k)}}$) with $V^{(n_k)} = (\mathcal{V}^{(n_k)}(\nu), V_{\mathcal{S}^c})$. Thus each $\mathbf{u}_k$ satisfies (1.2) with $V^{(n_k)}$ in place of V.

Taking $p < p_*$ satisfying (2.14), we get

$$c_k := \sup_{|j|>n_k} \sqrt{I_j} |j|^p \to 0, \quad \text{as } k \to \infty,$$

and, for every $t \in \mathbb{R}$,

$$\big|v(t, \cdot) - v_k(t, \cdot)\big|_p = c_k, \qquad |I - I^{(n_k)}|_{2p} = c_k^2.$$

Since the map $I \to \mathcal{V}_{\mathcal{S}}(\nu, V_{\mathcal{S}^c}, I)$ is Lipschitz, then $V^{(n_k)} \to V$ in ℓ^∞. Moreover, since Φ is also Lipschitz (w.r.t. u and I), for every $t \in \mathbb{R}$,

$$\begin{aligned} \big|\mathbf{u}(t, \cdot) - \mathbf{u}_k(t, \cdot)\big|_p &= \big|\Phi\big(v(t, \cdot); \nu, V_{\mathcal{S}^c}, I\big) - \Phi\big(v_k(t, \cdot); \nu, V_{\mathcal{S}^c}, I^{(n_k)}\big)\big|_p \\ &\leq L\big(\big|v(t, \cdot) - v_k(t, \cdot)\big|_p + |I - I^{(n_k)}|_{2p}\big) \leq L(c_k + c_k^2) \end{aligned}$$

for a suitable[22] $L > 0$. Then $\mathbf{u}_k \to \mathbf{u}$ uniformly[23] in $\mathbb{R}^2$. To prove that $\mathbf{u}$ satisfies (1.2), we have to show that

$$\int_{\mathbb{R}^2} (-\mathrm{i}\chi_t + \chi_{xx})(\mathbf{u} - \mathbf{u}_k) - (V * \mathbf{u} - V^{(n_k)} * \mathbf{u}_k)\chi - \big(f\big(|\mathbf{u}|^2\big)\mathbf{u} \\ - f\big(|\mathbf{u}_k|\big)\mathbf{u}_k\big)\chi\, dx\, dt$$

tends to zero as $k \to \infty$. This follows since $\mathbf{u}_k \to \mathbf{u}$ uniformly and we observe that

$$\|V^{(n_k)} * \mathbf{u}_k - V * \mathbf{u}\|_{L^\infty(\mathbb{R}^2)} \le |V^{(n_k)} * \mathbf{u}_k - V * \mathbf{u}|_p \\ \le |V^{(n_k)} - V|_{\ell^\infty} |\mathbf{u}_k|_p + |V|_{\ell^\infty} |\mathbf{u} - \mathbf{u}_k|_p \overset{k\to\infty}{\longrightarrow} 0.$$

This proves the claim in (3.2).

Third step (A set of good potentials). We now show that for almost every $V \in [-1/4, 1/4]^{\mathbb{Z}}$ there exists at least one solution of (1.2). For every integer $h \ge 1$, take $\gamma_h = |f|_{\mathrm{R}}/h$ and r_h such that (2.13) holds as an equality, namely $\mathrm{r}_h := \sqrt{\varepsilon_* \gamma_h \mathrm{R}/|f|_{\mathrm{R}}} = \sqrt{\varepsilon_* \mathrm{R}/h}$. For every given sequence $I_h \in \mathcal{J}(p_*, \mathrm{r}_h)$, we set

$$\mathcal{G} := \bigcup_{h \ge 1} \mathcal{G}(I_h, \gamma_h). \tag{3.3}$$

By (3.1), $\mathcal{G}$ has full measure in $[-1/4, 1/4]^{\mathbb{Z}}$. This implies that for almost every $V \in [-1/4, 1/4]^{\mathbb{Z}}$ there exists an integer $h \ge 1$ such that $V \in \mathcal{G}(I_h, \gamma_h)$. Then (3.2) (with $I = I_h$) gives a solution of (1.2).

Fourth step (Abundance of solutions). In order to find infinitely many solutions for almost every V in $[-1/4, 1/4]^{\mathbb{Z}}$, we proceed as follows. First, we choose in (3.3) $\sqrt{(I_h)_j} := \frac{1}{2}\mathrm{r}_h \lfloor j \rfloor^{-p_*}$. All the above construction depends on the choice of the set $\mathcal{S}$ of admissible tangential sites; in particular, this holds for the set $\mathcal{G}$ above. Let us now consider two distinct admissible sets $\mathcal{S}$ and $\mathcal{S}'$, and let us denote by $\mathcal{G}, \mathcal{G}'$ the corresponding sets of potentials. We claim that for each $V \in \mathcal{G} \cap \mathcal{G}'$ there exist (at least) two distinct almost periodic solutions. Indeed, there exist h, h' such that $V \in \mathcal{G}(I_h, \gamma_h) \cap \mathcal{G}'(I_{h'}, \gamma_{h'})$, where $\mathcal{G}(I, \gamma), \mathcal{G}'(I, \gamma)$ are the sets defined in Lemma (3.1) corresponding to $\mathcal{S}, \mathcal{S}'$. Let us call $\mathbf{u}, v, \Phi$ and $\mathbf{u}', v', \Phi'$ the functions defined in (3.2) for $\mathcal{S}$ and $\mathcal{S}'$, respectively,. Since by Theorem 3(ii) the maps Φ and Φ' are $\bar{\mathrm{r}}/16$-close to the identity, where $\bar{\mathrm{r}} := \max\{\mathrm{r}_h, \mathrm{r}_{h'}\}$, then

$$\big|\mathbf{u}(t,\cdot) - \mathbf{u}'(t,\cdot)\big|_{p_*} \ge \big|v(t,\cdot) - v'(t,\cdot)\big|_{p_*} - \bar{\mathrm{r}}/8 \ge \bar{\mathrm{r}}/2 - \bar{\mathrm{r}}/8 > 0.$$

The same holds for any translations $\mathbf{u}(t + t_0, x + x_0)$ and $\mathbf{u}'(t, x)$. Let us now consider a countable infinity of distinct admissible sets $\mathcal{S}$, and call $\mathcal{G}_*$ the *countable*

[22] We can take $L = 2$ since Φ is $C\varepsilon$-close to the identity.
[23] Given $f : \mathbb{R}^2 \to \mathbb{C}$, we have $\|f\|_{L^\infty(\mathbb{R}^2)} = \sup_{t\in\mathbb{R}} \|f(t,\cdot)\|_{L^\infty_x(\mathbb{R})} \le \sup_{t\in\mathbb{R}} |f(t,\cdot)|_p$ for every $p \ge 1$.

intersection of the corresponding sets $\mathcal{G}$. The set $\mathcal{G}_*$ has full measure in $[-1/4, 1/4]^{\mathbb{Z}}$ as well. Then for every $V \in \mathcal{G}_*$ we construct infinitely many solutions corresponding to different $\mathcal{S}$'s.

Regularity. By (3.2) and (2.2), we know that if $p_* > 3$, then v_t and v_{xx} are continuous functions on $\mathbb{R}^2$. Moreover, by analyticity of $v \to \Phi(v; \nu, I)$, $\mathbf{u}_t$ and $\mathbf{u}_{xx}$ are also continuous functions on $\mathbb{R}^2$. Therefore, u is a classical solution. On the other hand, when $p_* \le 2$ and $\sqrt{I_j} = (\mathrm{r}/2)\langle j\rangle^{-p_*}$ for all $j \in \mathcal{S}$, for every t the *function* $\mathbf{u}_{xx}(t,\cdot)$ *is not in* L^2. Otherwise its Fourier coefficients $(\mathbf{u}_{xx}(t,\cdot))_j = -j^2(\mathbf{u}(t,\cdot))_j$ would belong to ℓ_2. As in the linear case (recall (2.4)), we claim that

$$\limsup_{|j|\to\infty} j^2\big|\big(\mathbf{u}(t,\cdot)\big)_j\big| > 0.$$

Indeed, since Φ is close to the identity (in $\mathtt{w}_{p_*}$) $\Phi = \mathrm{Id} + \Phi_1$ with $\|\Phi_1\|$ uniformly $\mathrm{r}/16$-small, we have that $\mathbf{u}(t,\cdot) := \Phi(v(t,\cdot)) = v(t,\cdot) + \Phi_1(v(t,\cdot))$ and its Fourier coefficients satisfy $|(\mathbf{u}(t,\cdot))_j| \ge (\mathrm{r}/4)\langle j\rangle^{-p_*}$ for all $j \in \mathcal{S}$. Since $\mathcal{S}$ is an infinite set, this is a contradiction.

4. Functional setting

Let us introduce the spaces of Hamiltonians used in the paper.

Definition 4.1 (Multi-index notation)

In the following, with abuse of notation, we denote by $\mathbb{N}^{\mathbb{Z}}$ the set of multi-indexes $\boldsymbol{\alpha}, \boldsymbol{\beta}$, and so on such that $|\boldsymbol{\alpha}| := \sum_{j\in\mathbb{Z}} \boldsymbol{\alpha}_j$ is finite. As usual, $\boldsymbol{\alpha}! := \prod_{j\in\mathbb{Z}, \boldsymbol{\alpha}_j\neq 0} \boldsymbol{\alpha}_j!$. Moreover, $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ means that $\boldsymbol{\alpha}_j \le \boldsymbol{\beta}_j$ for every $j \in \mathbb{Z}$; then $\binom{\boldsymbol{\beta}}{\boldsymbol{\alpha}} := \frac{\boldsymbol{\beta}!}{\boldsymbol{\alpha}!(\boldsymbol{\beta}-\boldsymbol{\alpha})!}$. Finally, take $j_1 < j_2 < \cdots < j_n$ such that $\boldsymbol{\alpha}_j \neq 0$ if and only if $j = j_i$ for some $1 \le i \le n$; as usual, we set $\partial^{\boldsymbol{\alpha}} f := \partial_{u_{j_1}}^{\boldsymbol{\alpha}_{j_1}} \cdots \partial_{u_{j_n}}^{\boldsymbol{\alpha}_{j_n}} f$, and analogously for $\partial_{\bar{u}}^{\boldsymbol{\beta}} f$.

Definition 4.2 (Regular Hamiltonians)

Consider a formal power series expansion

$$H(u) = \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}} H_{\boldsymbol{\alpha},\boldsymbol{\beta}} u^{\boldsymbol{\alpha}} \bar{u}^{\boldsymbol{\beta}}, \quad u^{\boldsymbol{\alpha}} := \prod_{j\in\mathbb{Z}} u_j^{\boldsymbol{\alpha}_j}, \tag{4.1}$$

where

$$\mathcal{M} := \Big\{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathbb{N}^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}} \text{ s.t. } |\boldsymbol{\alpha}| = |\boldsymbol{\beta}| < +\infty, \sum_{j\in\mathbb{Z}} j(\boldsymbol{\alpha}_j - \boldsymbol{\beta}_j) = 0\Big\}, \tag{4.2}$$

satisfying the reality condition

$$H_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \overline{H}_{\boldsymbol{\beta},\boldsymbol{\alpha}}, \quad \forall (\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{M}. \tag{4.3}$$

We say that $H \in \mathcal{H}_{r,p}$ for $p > 1$, $r > 0$ if

$$|H|_{r,p} := \frac{1}{r}\Big(\sup_{|u|_p \le r} |X_{\underline{H}}|_p\Big) < \infty, \tag{4.4}$$

where $X_{\underline{H}}$ denotes the Hamiltonian vector field, intrinsically defined through the standard symplectic form $\varpi = \mathrm{i}\sum_j du_j \wedge d\bar{u}_j$ as

$$i_{X_{\underline{H}}}\varpi(\cdot) := \varpi(\cdot, X_{\underline{H}}) = d\underline{H}(\cdot),$$

$\underline{H}$ being the associated majorant Hamiltonian $\underline{H}(u) := \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}} |H_{\boldsymbol{\alpha},\boldsymbol{\beta}}| u^{\boldsymbol{\alpha}} \bar{u}^{\boldsymbol{\beta}}$ of $H(u)$, and $i_{X_{\underline{H}}}$ being the usual contraction of a differential form with a vector field.

We denote by $\mathcal{H}_{r,p}(\mathbb{C})$ the space of H satisfying (4.1) and (4.4) but not necessarily the reality condition (4.3).

Finally, given any $F, G \in \mathcal{H}_{r,p}(\mathbb{C})$, we define in the usual manner the Poisson brackets as $\{F, G\} := \varpi(X_F, X_G)$.

For details on the symplectic structure and Hamiltonian vector field, see, for instance, [6].

Note that by [8, Lemma 2.1],

$$|H|_{r,p} = \frac{1}{2}\sup_j \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}} |H_{\boldsymbol{\alpha},\boldsymbol{\beta}}|(\boldsymbol{\alpha}_j + \boldsymbol{\beta}_j) u_p^{\boldsymbol{\alpha}+\boldsymbol{\beta}-2e_j}, \tag{4.5}$$

where $u_p = u_p(r)$ is defined as

$$u_{p,j}(r) := r\lfloor j \rfloor^{-p}. \tag{4.6}$$

Remark 4.1
Regarding $\mathcal{M}$ in (4.2), we note that the condition $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$, that is, $\mathfrak{m}(\boldsymbol{\alpha} - \boldsymbol{\beta}) = 0$, corresponds to mass conservation; namely, the H-Poisson commutes with the *mass* $\sum_{j\in\mathbb{Z}} |u_j|^2$. Moreover, $\sum_{j\in\mathbb{Z}} j(\boldsymbol{\alpha}_j - \boldsymbol{\beta}_j) = 0$, that is, $\pi(\boldsymbol{\alpha} - \boldsymbol{\beta}) = 0$, corresponds to momentum conservation; namely, the H-Poisson commutes with the *momentum* $\sum_{j\in\mathbb{Z}} j|u_j|^2$.

Note that $|\cdot|_{r,p}$ is a seminorm on $\mathcal{H}_{r,p}$ and a norm on its subspace

$$\mathcal{H}^0_{r,p} := \big\{H \in \mathcal{H}_{r,p} \text{ with } H(0) = 0\big\}, \tag{4.7}$$

endowing $\mathcal{H}^0_{r,p}$ with a Banach space structure. Moreover, the space $\mathcal{H}_{r,p}$ enjoys the following algebra property with respect to Poisson brackets of Hamiltonians.

PROPOSITION 4.1 (Poisson structure)
For any $F, G \in \mathcal{H}_{r+\rho,p}$, with $\rho > 0$, one has $\{F, G\} \in \mathcal{H}^0_{r,p}$ with the bound

$$\big|\{F,G\}\big|_{r,p} \le 8\max\Big\{1, \frac{r}{\rho}\Big\}|F|_{r+\rho,p}|G|_{r+\rho,p}. \tag{4.8}$$

The proof is given in Appendix A. Of course, the same estimates hold in $\mathcal{H}_{r,p}(\mathbb{C})$.

The next result controls the norm of the NLS nonlinearity P defined in (2.3) and it is based on the algebra property of $\mathtt{w}_p$, $p > 1$, with respect to convolution.

PROPOSITION 4.2 ([8, Proposition 2.1])
There exist $c_1, c_2 > 1$ continuously depending on $p > 1$ such that if $c_1 r^2 \le \mathtt{R}$, then (recalling (2.12))

$$|P|_{r,p} \le c_2 \frac{|f|_{\mathtt{R}}}{\mathtt{R}} r^2.$$

4.1. Parameter families of regular Hamiltonians

Throughout the paper, our Hamiltonians will depend on two parameters, the frequency $\omega \in \mathtt{D}_{\gamma,\mathcal{S}} \subset \mathcal{Q}$ and the action $I \in \mathcal{I}(p,r)$. In order to control the regularity with respect to these parameters throughout the iterative scheme, we will introduce an appropriate weighted norm as follows.

Given $\gamma > 0$, a closed (w.r.t. the product topology) subset $\mathcal{O} \subseteq \mathcal{Q}$, and an open subset $\mathcal{I}$ of some Banach space, for $f : \mathcal{O} \times \mathcal{I} \to \mathbb{C}, (\omega, I) \mapsto f(\omega, I)$ we set[24]

$$|f|^{\gamma} = |f|^{\gamma,\mathcal{O}\times\mathcal{I}} := \sup_{\substack{\omega\in\mathcal{O}\\ I\in\mathcal{I}}} \big|f(\omega,I)\big| + \gamma \sup_{\substack{\omega\neq\omega'\in\mathcal{O}\\ I\in\mathcal{I}}} |\Delta_{\omega,\omega'} f|, \tag{4.9}$$

where as usual

$$\Delta_{\omega,\omega'} f := \frac{f(\omega,I) - f(\omega',I)}{|\omega-\omega'|_\infty}. \tag{4.10}$$

Definition 4.3
Given $\mathcal{O}$ and $\mathcal{I}$ as above, let $C_{\mathrm{Lip}}(\mathcal{O}\times\mathcal{I})$ be the Banach space of functions $f : \mathcal{O} \times \mathcal{I} \to \mathbb{C}$, which have finite norm $|f|^{\gamma,\mathcal{O}\times\mathcal{I}}$ and which are:

- continuous with respect to the product topology in $\mathcal{O}$;
- analytic in $\mathcal{I}$.

In $C_{\mathrm{Lip}}(\mathcal{O}\times\mathcal{I})$, we consider the subalgebra $\mathrm{F}(\mathcal{O}\times\mathcal{I})$ of functions which depend only on a finite number of ω_j's. The subalgebra $\mathrm{F}(\mathcal{O}\times\mathcal{I})$ can be described as follows. Given $f \in \mathrm{F}(\mathcal{O}\times\mathcal{I})$, which depends only on the variables $(\omega_{-k},\dots,\omega_k)$, there

[24] We put γ in front of the Lipschitz seminorm for dimensional reasons. See Definition 2.3.

exists a function $\hat{f} : P_k(\mathcal{O}) \times \mathcal{I} \to \mathbb{C}$, where P_k is the projection $P_k : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{2k+1}$ defined as $P_k(\omega) := (\omega_{-k}, \dots, \omega_k)$, such that $f(\omega, I) = \hat{f}(\omega_{-k}, \dots, \omega_k, I)$ for every $(\omega, I) \in \mathcal{O} \times \mathcal{I}$.

Finally, denote the closure of $\mathrm{F}(\mathcal{O} \times \mathcal{I})$ in $C_{\mathrm{Lip}}(\mathcal{O} \times \mathcal{I})$ by $\mathcal{F}(\mathcal{O} \times \mathcal{I})$.

The spaces $C_{\mathrm{Lip}}(\mathcal{O} \times \mathcal{I})$ and $\mathcal{F}(\mathcal{O} \times \mathcal{I})$ are Banach algebras (i.e., multiplicative algebras with constant equal to 1) with respect to the norm $|\cdot|^{\gamma, \mathcal{O} \times \mathcal{I}}$.

The following extension result will be proved in Appendix A.

LEMMA 4.1 (Lipschitz extension)
Given $\mathcal{O} \subset \mathcal{Q}$ and a ball B_ρ in some complex Banach space E and $f \in \mathcal{F}(\mathcal{O} \times B_\rho)$, there exists an extension $\tilde{f} : \mathcal{Q} \times B_{\rho/2} \to \mathbb{C}$ such that $|\tilde{f}|^{\gamma, \mathcal{Q} \times B_{\rho/2}} \leq 2|f|^{\gamma, \mathcal{O} \times B_\rho}$, $\tilde{f}$ is continuous with respect to the product topology in $\mathcal{Q}$, and $\tilde{f}$ is Lipschitz on $B_{\rho/2}$ with estimate

$$\big|\tilde{f}(\omega, I) - \tilde{f}(\omega, I')\big| \leq 4\rho^{-1}|f|^{\gamma, \mathcal{O} \times B_\rho}|I - I'|_E, \quad \forall \omega \in \mathcal{Q}, I, I' \in B_{\rho/2}.$$

In the following, we mainly consider $\mathcal{I} = \mathcal{I}(p, r)$ (see (2.11)); however, to use Lemma 4.1 we need to pass to the complex. Then we define

$$\mathcal{I}(\mathbb{C}) = \mathcal{I}(p, r, \mathbb{C}) := \big\{ I \in B_{r^2}(\mathrm{w}_{2p}) \text{ with } I_j = 0 \text{ for } j \in \mathcal{S}^c \big\}, \tag{4.11}$$

which we systematically identify with the open ball of radius r^2 centered at the origin of the Banach space $\{w = (w_j)_{j \in \mathcal{S}} : \sup_{j \in \mathcal{S}} |w_j| \lfloor j \rfloor^{2p} < \infty\}$.

Definition 4.4 (Real and complex Hamiltonians)
Let $0 < r_0 \leq r$, $p_0 \geq p > 1$. Let $\mathcal{H}_{r,p} = \mathcal{H}_{r,p}^{\mathcal{O} \times \mathcal{I}}$ with $\mathcal{I} = \mathcal{I}(p_0, r_0)$ be the space of parameter-dependent *real* regular Hamiltonians $H : \mathcal{O} \times \mathcal{I} \ni (\omega, I) \mapsto H(\omega, I) \in \mathcal{H}_{r,p}$ such that

$$\begin{aligned} &H_{\alpha,\beta} \in \mathcal{F}(\mathcal{O} \times \mathcal{I}), \quad \forall \alpha, \beta \in \mathcal{M} \qquad \text{and} \\ &\underline{H}_\gamma := \sum_{\alpha,\beta \in \mathcal{M}} |H_{\alpha,\beta}|^\gamma u^\alpha \bar{u}^\beta \in \mathcal{H}_{r,p}. \end{aligned} \tag{4.12}$$

We set

$$\begin{aligned} \|H\|_{r,p} &= \|H\|_{r,p}^{\mathcal{O} \times \mathcal{I}} \\ &:= |\underline{H}_\gamma|_{r,p} \overset{(4.5)}{=} \frac{1}{2} \sup_j \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{M}} |H_{\boldsymbol{\alpha},\boldsymbol{\beta}}|^\gamma (\boldsymbol{\alpha}_j + \boldsymbol{\beta}_j) u_p^{\boldsymbol{\alpha}+\boldsymbol{\beta}-2e_j}. \end{aligned} \tag{4.13}$$

Respectively for $\mathcal{I}(\mathbb{C}) = \mathcal{I}(p_0, r_0, \mathbb{C})$, we define the space $\mathcal{H}_{r,p}(\mathbb{C}) = \mathcal{H}_{r,p}^{\mathcal{O} \times \mathcal{I}(\mathbb{C})}$ of parameter-dependent *complex* Hamiltonians $H : \mathcal{O} \times \mathcal{I}(\mathbb{C}) \ni (\omega, I) \mapsto H(\omega, I) \in$

$\mathcal{H}_{r,p}(\mathbb{C})$ satisfying (4.12) with $\mathcal{I}(\mathbb{C})$ instead of $\mathcal{I}$ and verifying the reality condition $H(\omega, I) \in \mathcal{H}_{r,p}$ when $I \in \mathcal{I}$.

Finally, we denote by $\mathcal{H}^0_{r,p}$ (resp., $\mathcal{H}^0_{r,p}(\mathbb{C})$) the subspace of $\mathcal{H}_{r,p}$ (resp., of $\mathcal{H}_{r,p}(\mathbb{C})$) such that $H_{|u=0} = 0$.

The following result is proved in Appendix A.

LEMMA 4.2
$(\mathcal{H}^0_{r,p}, \|\cdot\|_{r,p})$ is a Banach–Poisson algebra in the following sense:
(1) *$(\mathcal{H}^0_{r,p}, \|\cdot\|_{r,p})$ is a Banach space.*
(2) *For any $F, G \in \mathcal{H}_{r+\rho,p}$, the following bound holds:*

$$\big\|\{F, G\}\big\|_{r,p} \le 8 \max\Big\{1, \frac{r}{\rho}\Big\} \|F\|_{r+\rho,p} \|G\|_{r+\rho,p}. \tag{4.14}$$

PROPOSITION 4.3 (Monotonicity)
The norm $\|\cdot\|_{r,p}$ is monotone decreasing in p and monotone increasing in r:

$$\|\cdot\|_{r,p+\delta} \le \|\cdot\|_{r+\rho,p} \quad \forall \rho, \delta \ge 0. \tag{4.15}$$

The fact that this norm is increasing in r follows directly from mass conservation and the fact that $H(0) = 0$. Concerning the monotonicity in p, we refer the reader to [6, Proposition 6.3], where the proof is contained, written in the case of $|\cdot|_{r,p}$. The fact that it holds also in the Lipschitz frame follows trivially.

PROPOSITION 4.4 (Hamiltonian flow)
Let $S \in \mathcal{H}_{r+\rho,p} = \mathcal{H}^{\mathcal{O}\times\mathcal{I}}_{r+\rho,p}$ with

$$\|S\|_{r+\rho,p} \le \delta := \frac{\rho}{16e(r+\rho)}. \tag{4.16}$$

Then for all $(\omega, I) \in \mathcal{O} \times \mathcal{I}$, the time 1-Hamiltonian flow of $S = S(\cdot, \omega, I)$ is well defined, analytic, and symplectic; more precisely, $\Phi^1_S : B_r(\mathtt{w}_p) \to B_{r+\rho}(\mathtt{w}_p)$ with

$$\sup_{u \in B_r(\mathtt{w}_p)} \big|\Phi^1_S(u) - u\big|_{r,p} \le (r+\rho)\|S\|_{r+\rho,p} \le \frac{\rho}{16e}. \tag{4.17}$$

For any $H \in \mathcal{H}_{r+\rho,p}$, we have that[25] *$H \circ \Phi^1_S = e^{\{S,\cdot\}} H \in \mathcal{H}_{r,p}$, $e^{\{S,\cdot\}} H - H \in \mathcal{H}^0_{r,p}$ and*

[25] $e^{\{S,\cdot\}} H := \sum_{k\in\mathbb{N}} \mathrm{ad}^k_S (H)/k!$, where $\mathrm{ad}_S = \{S, \cdot\}$.

$$\|e^{\{S,\cdot\}}H\|_{r,p} \le 2\|H\|_{r+\rho,p}, \tag{4.18}$$

$$\big\|(e^{\{S,\cdot\}}-\mathrm{Id})H\big\|_{r,p} \le \delta^{-1}\|S\|_{r+\rho,p}\|H\|_{r+\rho,p}, \tag{4.19}$$

$$\big\|\big(e^{\{S,\cdot\}}-\mathrm{Id}-\{S,\cdot\}\big)H\big\|_{r,p} \le \frac{1}{2}\delta^{-2}\big(\|S\|_{r+\rho,p}\big)^2\|H\|_{r+\rho,p}. \tag{4.20}$$

More generally, for any $h\in\mathbb{N}$ and any sequence $(c_k)_{k\in\mathbb{N}}$ with $|c_k|\le 1/k!$, we have

$$\Big\|\sum_{k\ge h} c_k\,\mathrm{ad}_S^k(H)\Big\|_{r,p} \le 2\|H\|_{r+\rho,p}\big(\|S\|_{r+\rho,p}/2\delta\big)^h, \tag{4.21}$$

where $\mathrm{ad}_S(\cdot) := \{S,\cdot\}$.

The proof is based on (4.14) and on the Lie series expansion for $e^{\{S,\cdot\}}$ (see [6, Proposition 2.1, Lemma 2.1] for details).

Remark 4.2
If we are working in $\mathcal{H}_{r,p}(\mathbb{C})$, all the estimates in Lemma 4.2, Proposition 4.3, and Proposition 4.4 still hold except for (4.17) which holds only for real I. Indeed, without the reality condition (4.3) the generating function S no longer defines a Hamiltonian flow satisfying the reality condition $\overline{u(t)}=\bar{u}(t)$.

5. Degree decompositions, projections, and normal forms

We want to prove that, in suitable variables, $\mathcal{T}_I$ introduced in (2.5) is an invariant torus on which the flow is linear. To this purpose, we introduce a suitable degree decomposition, whose main idea is to make a *power series expansion* in the variables $|v|^2-I$ and z, which control the distance to the torus $\mathcal{T}_I$, in order to highlight the terms which prevent $\mathcal{T}_I$ from being a KAM torus. In the case of finite-dimensional tori, one typically introduces action-angle variables, but it is well known that this produces a singularity at $I=0$. In the infinite-dimensional case, this should be avoided and requires some care. An effective strategy is the expansion described below, first introduced in [13] and further formalized in [8, Section 4]. For the convenience of the reader, we sketch here the main features of this decomposition. The main novelty here is to control the regularity with respect to the parameters.

Fix $\mathcal{S}$ as in (2.2). Consider a Hamiltonian $H(u)$ expanded in Taylor series at $u=0$, and tautologically rewrite H as

$$H = \sum_{\substack{m,\alpha,\beta\in\mathbb{N}^{\mathcal{S}}\\ \alpha\cap\beta=\emptyset\\ a,b\in\mathbb{N}^{\mathcal{S}^c}}} H_{m,\alpha,\beta,a,b}|v|^{2m}v^\alpha\bar{v}^\beta z^a\bar{z}^b, \tag{5.1}$$

where, by slight abuse of notation,[26] $u = (v, z)$ with $v = (v_j)_{j\in\mathcal{S}} := (u_j)_{j\in\mathcal{S}}$ and $z = (z_j)_{j\in\mathcal{S}^c} := (u_j)_{j\in\mathcal{S}^c}$. Moreover, by $\alpha\cap\beta = \emptyset$, here and in the following, we mean that $\alpha_j \neq 0$ implies $\beta_j = 0$.

Then introduce the auxiliary *action* variables $w = (w_j)_{j\in\mathcal{S}}$ substituting $|v|^{2m} v^\alpha \bar{v}^\beta z^a \bar{z}^b \rightsquigarrow w^m v^\alpha \bar{v}^\beta z^a \bar{z}^b$ in (5.1). Now we Taylor expand the Hamiltonian with respect to w and z at the point $w_j = I_j$ for $j \in \mathcal{S}$ and $z = 0$, respectively.

Definition 5.1 (Degree decomposition)
Let $I \in \mathcal{J}(p, r)$ (recall (2.11)). For every integer $d \geq -2$ and any regular Hamiltonian $H \in \mathcal{H}_{r,p}$, we define the projection $(\Pi^d H)(u) = (\Pi^d_I H)(u) = H^{(d)}(u)$ as

$$H^{(d)}(u) := \sum_{\substack{m,\alpha,\beta,\delta\in\mathbb{N}^{\mathcal{S}},a,b\in\mathbb{N}^{\mathcal{S}^c}\\ \alpha\cap\beta=\emptyset,\delta\preceq m\\ 2|\delta|+|a|+|b|=d+2}} H_{m,\alpha,\beta,a,b}\binom{m}{\delta} I^{m-\delta}\left(|v|^2 - I\right)^\delta v^\alpha \bar{v}^\beta z^a \bar{z}^b, \quad (5.2)$$

where $\delta \preceq m$ means that $\delta_s \leq m_s$ for any $s \in \mathcal{S}$, $|v|^2 = (|v_s|^2)_{s\in\mathcal{S}}$, while the multi-index notations are introduced in Definition 4.1. We also set $\Pi^{\leq d} := \sum_{d'=-2}^{d} \Pi^{d'}$ and $\Pi^{\geq d} := \mathrm{Id} - \Pi^{<d}$, analogously for $\Pi^{<d}$ and $\Pi^{>d}$. Moreover, we define, for example, $H^{(\leq d)} := \Pi^{(\leq d)} H$ and also $\mathcal{H}^d_{r,p} := \Pi^d \mathcal{H}_{r,p}$ and, for example, $\mathcal{H}^{\leq d}_{r,p} := \Pi^{\leq d}\mathcal{H}_{r,p}$.

Note that if $\mathcal{S} = \mathbb{Z}$, then the projections coincide with the ones of Section 4 of [8], while if $\mathcal{S} = \emptyset$, then $H^{(d)}$ represents the usual homogeneous degree at $z = 0$.

In this way, given $H \in \mathcal{H}_{r,p}$, then

$$H = H^{(\leq 0)} + H^{(\geq 1)} \equiv H^{(-2)} + H^{(-1)} + H^{(0)} + H^{(\geq 1)}, \quad (5.3)$$

where $H^{(-2)}$ consists of terms which are constant with respect to both z and the *auxiliary action* $w = |v|^2$, $H^{(-1)}$ is independent of the action but linear in the z_j, while $H^{(0)}$ contributes with two terms: the first one linear in the action and independent of z, the second one quadratic in z and independent of the action. Finally, $H^{(\geq 1)}$ is what is left and $X_{H^{(\geq 1)}}$ vanishes on $\mathcal{T}_I$.

The operators Π^d define continuous projections (see Section 4 of [8] and also Proposition 5.1) satisfying $\Pi^d\Pi^d = \Pi^d$ and $\Pi^{d'}\Pi^d = \Pi^d\Pi^{d'} = 0$ for every $d' \neq d$, $d' \geq -2$. Moreover, this decomposition enjoys all the crucial properties required for a KAM scheme to converge; in particular they behave well with respect to Poisson brackets, that is,

$$\forall F, G \in \mathcal{H}_{r,p} \{F, G^{\geq 1}\}^{(-2)} = 0$$

[26] Consisting in a reordering of the indexes j.

and

$$F^{(-2)} = 0 \quad \Longrightarrow \quad \{F, G^{\geq 1}\}^{(\leq -1)} = 0,$$
$$F^{(-1)} = 0 = F^{(-2)} \quad \Longrightarrow \quad \{F, G^{\geq 1}\}^{(\leq 0)} = 0.$$

Note also that if $\Pi^{<d_1} F = \Pi^{<d_2} G = 0$, then $\Pi^{<d_1+d_2}\{F, G\} = 0$. (For all the properties of the projections, see [8, Propositions 4.1 and 4.2].)

As is standard, on $\mathcal{H}_{r,p}$ we define the projections

$$\Pi^{\mathcal{K}} H := \sum_{\boldsymbol{\alpha} \in \mathbb{N}^{\mathbb{Z}}} H_{\boldsymbol{\alpha},\boldsymbol{\alpha}} |u|^{2\boldsymbol{\alpha}}, \qquad \Pi^{\mathcal{R}} H := H - \Pi^{\mathcal{K}} H, \tag{5.4}$$

which are continuous on $\mathcal{H}_{r,p}$.

Correspondingly, we define the following subspaces of $\mathcal{H}_{r,p}$:

$$\mathcal{H}_{r,p}^{\mathcal{K}} := \{H \in \mathcal{H}_{r,p} : \Pi^{\mathcal{K}} H = H\}, \qquad \mathcal{H}_{r,p}^{\mathcal{R}} := \{H \in \mathcal{H}_{r,p} : \Pi^{\mathcal{R}} H = H\}. \tag{5.5}$$

Moreover, $\mathcal{H}_{r,p}^{\leq 0,\mathcal{R}} := \mathcal{H}_{r,p}^{\leq 0} \cap \mathcal{H}_{r,p}^{\mathcal{R}}$. Note that $\mathcal{H}_{r,p}^{\mathcal{K}} \subseteq \ker L_\omega$ and if $\omega \in \mathtt{D}_{\gamma,\mathcal{S}}$, then the two spaces coincide.

Note that by (5.2) and (5.4), we have

$$d \text{ odd} \quad \Longrightarrow \quad \mathcal{H}_{r,p}^{d,\mathcal{K}} = \{0\}. \tag{5.6}$$

In [8, Lemma 4.3], we proved that the map $\lambda \to \Lambda$ defined by

$$\Lambda = \sum_{j \in \mathcal{S}} \lambda_j \big(|v_j|^2 - I_j\big) + \sum_{j \in \mathcal{S}^c} \lambda_j |z_j|^2 \tag{5.7}$$

is a linear isometry from ℓ^∞ to $\mathcal{H}_{r,p}^{0,\mathcal{K}}$ for every r, p. Note that, taking Λ as above,

$$H = \{\Lambda, G^{(d)}\} \quad \Longrightarrow \quad H = H^{(d)}. \tag{5.8}$$

The projections defined in (5.2) naturally extend to $H \in \mathcal{H}_{r,p}$ or $H \in \mathcal{H}_{r,p}(\mathbb{C})$ setting

$$(\Pi^d H)(u, \omega, I) := \big(\Pi_I^d H(\omega, I)\big)(u). \tag{5.9}$$

Similarly for (5.4).

The following result is proved in Appendix A.

PROPOSITION 5.1
For every $d \in \mathbb{N} \cup \{-2, -1\}$ and $H \in \mathcal{H}_{r',p} = \mathcal{H}_{r',p}^{\mathcal{O} \times \mathcal{I}}$, where $\mathcal{I} = \mathcal{I}(p, r)$ with $r' \geq \sqrt{2} r$, the following hold.

(i) The projection operators $\Pi^d : \mathcal{H}_{r',p} \to \mathcal{H}_{r',p}$ are continuous with bound

$$\|\Pi^d H\|_{r',p} \leq 3^{\frac{d}{2}+1}\|H\|_{r',p}. \tag{5.10}$$

(ii) Moreover, the following representation formula holds:

$$\Pi^{\geq d} H(u) = \sum_{\substack{\delta\in\mathbb{N}^{\mathcal{S}},a,b\in\mathbb{N}^{\mathcal{S}^c} \\ 2|\delta|+|a|+|b|=d+2}} \left(|v|^2 - I\right)^{\delta} z^a \bar{z}^b \check{H}_\delta(u) \quad \text{for } |u|_p < r', \tag{5.11}$$

where the $\check{H}_\delta(u)$ are analytic in $B_{r'}(\mathtt{w}_p)$ and can be written in totally convergent power series in every ball $|u|_p \leq \kappa_* r'$ with $\kappa_* < 1$.

Analogous statements hold for the complex case $H \in \mathcal{H}_{r',p}(\mathbb{C}) = \mathcal{H}_{r',p}^{\mathcal{O}\times\mathcal{J}(\mathbb{C})}$, where $\mathcal{J}(\mathbb{C}) = \mathcal{J}(p,r,\mathbb{C})$ (recall formula (4.11)).

As above, for $\mathcal{H}_{r,p}$, we define the corresponding subspaces $\mathcal{H}_{r,p}^d$, $\mathcal{H}_{r,p}^{\mathcal{K}}$, $\mathcal{H}_{r,p}^{\mathcal{R}}$, and so on of $\mathcal{H}_{r,p}$; analogously for the complex case $\mathcal{H}_{r,p}(\mathbb{C})$. In particular, we discuss the subspace $\mathcal{H}_{r,p}^{0,\mathcal{K}}$.

Definition 5.2
We denote by $\mathcal{H}^{0,\mathcal{K}} = \mathcal{H}^{0,\mathcal{K}}(\mathcal{O}\times\mathcal{J})$ the space of maps

$$\mathcal{O}\times\mathcal{J} \ni (\omega, I) \to \lambda(\omega, I) \in \ell^\infty(\mathbb{R}) \quad \text{with } \lambda_j \in \mathcal{F}(\mathcal{O}\times\mathcal{J})\ \forall j \in \mathbb{Z},$$

endowed with the norm (recall (4.9))

$$\|\lambda\|_\infty := \sup_{j\in\mathbb{Z}} |\lambda_j|^\gamma. \tag{5.12}$$

Then by (5.7), $\mathcal{H}^{0,\mathcal{K}}$ is isometrically equivalent to $\mathcal{H}_{r,p}^{0,\mathcal{K}}$ for all r, p. Similarly in the complex case $\mathcal{H}^{0,\mathcal{K}}(\mathbb{C}) = \mathcal{H}^{0,\mathcal{K}}(\mathcal{O}\times\mathcal{J}(\mathbb{C}))$.

Since are we mainly interested in the decomposition (5.3), we remark (see Proposition 5.1) that for $H \in \mathcal{H}_{r',p'}^{\mathcal{O}\times\mathcal{J}}$ with $\mathcal{J} = \mathcal{J}(p,r)$ and $r' \geq \sqrt{2}r$, $p' \leq p$, one has

$$\begin{aligned} &\|\Pi^0 H\|_{r',p'}, \|\Pi^{0,\mathcal{K}} H\|_\infty \leq 3\|H\|_{r',p'}, \\ &\|\Pi^{-1} H\|_{r',p'}, \|\Pi^{-2} H\|_{r',p'} \leq \|H\|_{r',p'}, \qquad \|\Pi^{\geq 1} H\|_{r',p'} \leq 6\|H\|_{r',p'}. \end{aligned} \tag{5.13}$$

Analogous estimates hold in the complex case $H \in \mathcal{H}_{r',p'}^{\mathcal{O}\times\mathcal{J}(\mathbb{C})}$ with $\mathcal{J}(\mathbb{C}) = \mathcal{J}(p,r,\mathbb{C})$ (see (4.11)).

6. Proofs of Theorems 3 and 4

Definition 6.1 (Normal forms)
For $\omega \in \mathcal{Q}$, let[27]

$$D : \omega \mapsto D(\omega) := \sum_{j \in \mathbb{Z}} \omega_j |u_j|^2.$$

Let $0 < r < r_0$, $1 < p_0 \le p$, and let $\mathcal{I} = \mathcal{I}(p, r)$ as in (2.11). We will say that N is an analytic family of *normal forms* if $N - D \in \mathcal{H}^{\ge 1, \mathcal{O}\times\mathcal{I}}_{r_0, p_0}$. We denote such affine subspaces by $\mathcal{N}_{r_0,p_0} = \mathcal{N}^{\mathcal{O}\times\mathcal{I}}_{r_0,p_0}$. The same definition holds in the corresponding complex spaces.

Remark 6.1
Given $N \in \mathcal{N}^{\mathcal{O}\times\mathcal{I}}_{r_0,p_0}$, for every $\omega \in \mathcal{O}$ and $I \in \mathcal{I}(p, r)$, the flat torus $\mathcal{T}_I$ is invariant for the dynamics of $N(\omega, I)$.

Let us state our counterterm KAM result.

THEOREM 5
Fix $0 < \gamma < 1$, $\mathcal{S}$ as in (2.2), $r > 0$, and $p > 1$. Set $\mathcal{O} = \mathrm{D}_{\gamma,\mathcal{S}}$ and $\mathcal{I} = \mathcal{I}(p, r)$. Consider $r_0, \rho, \delta > 0$ and $p_0 > 1$ with

$$\rho \le \frac{r_0 - r}{4}, \qquad r \le \frac{r_0}{2}, \qquad \delta \le \frac{p - p_0}{4}. \tag{6.1}$$

There exist $\bar\epsilon, 1/\bar C > 0$, decreasing functions of ρ/r_0 and δ such that the following holds. Consider $N_0 \in \mathcal{N}^{\mathcal{O}\times\mathcal{I}}_{r_0,p_0}$, $H - D \in \mathcal{H}^{\mathcal{O}\times\mathcal{I}}_{r_0,p_0}$, and assume that

$$(1+\Theta)^5 \epsilon \le \bar\epsilon, \quad \textit{where}$$
$$\epsilon := \gamma^{-1} \sup_{I \in \mathcal{I}(p,r)} \|H - N_0\|_{r_0,p_0}, \qquad \Theta := \gamma^{-1} \sup_{I \in \mathcal{I}(p,r)} \|D - N_0\|_{r_0,p_0}. \tag{6.2}$$

Then there exist a $\rho/4$-close to the identity symplectic diffeomorphism in u, Lipschitz in ω and analytic in I

$$\Psi : B_{r_0-\rho}(\mathrm{w}_p) \times \mathcal{O} \times \mathcal{I} \to B_{r_0}(\mathrm{w}_p), \quad (u; \omega, I) \mapsto \Psi(u; \omega, I) \tag{6.3}$$

and a counterterm $\Lambda \in \mathcal{H}^{0,\mathcal{K}}(\mathcal{O} \times \mathcal{I})$, where

$$\Lambda = \sum_j \lambda_j \big(|u_j|^2 - I_j\big), \qquad \|\lambda\|_\infty \le \bar C \gamma (1+\Theta)^2 \epsilon, \tag{6.4}$$

[27] Note that D is a linear map into the space of formal quadratic polynomials.

such that $\Psi(\mathcal{T}_I;\omega,I)$ is an invariant torus for the dynamics of $\Lambda(\omega,I)+H(\omega)$ for every $\omega\in \mathtt{D}_{\gamma,\mathcal{S}}$ and $I\in\mathcal{I}(p,r)$.

More precisely, there exists an analytic family of normal forms $N\in\mathcal{N}^{\mathcal{O}\times\mathcal{I}}_{r_0-\rho,p_0+\delta}$ such that

$$(\Lambda+H)\circ\Psi-N=0. \tag{6.5}$$

Finally, if N_0 and H admit a complex extension on $\mathcal{I}(p,r,\mathbb{C})$ satisfying (6.2), then also the counterterm λ extends to a holomorphic map in $\mathcal{H}^{0,\mathcal{K}}(\mathcal{O}\times\mathcal{I}(p,r,\mathbb{C}))$ satisfying (6.4).

Remark 6.2
(i) The map Ψ is $\rho/4$-close to the identity; namely,

$$\left|\Psi(u;\omega,I)-u\right|_p\le\rho/4,\quad \forall(u;\omega,I)\in B_{r_0-\rho}(\mathtt{w}_p)\times\mathcal{O}\times\mathcal{I}.$$

(ii) By (6.4) and Cauchy estimates, we get that λ is uniformly Lipschitz in $\mathcal{I}(p,r/2)$

$$\frac{|\lambda(\omega,I)-\lambda(\omega,I')|_\infty}{|I-I'|_{2p}}\le 4\bar{C}(1+\Theta)^2\gamma r^{-2}\epsilon, \tag{6.6}$$

for all $\omega\in\mathtt{D}_{\gamma,\mathcal{S}}$, $I,I'\in\mathcal{I}(p,r/2)$, $I\neq I'$.

Proofs of Theorems 3 and 4
Theorem 3 follows from Theorem 5 in a straightforward way. Fix $p_0=p-(p-1)/2$, $r_0=2r=4\mathrm{r}$, $\rho=r/8$, and $\delta=(p_*-1)/16$. Note that by these choices

$$\text{the constants } \bar{\epsilon} \text{ and } \bar{C} \text{ depend only on } p_*. \tag{6.7}$$

One first rewrites H_V in (2.3) as $D+\Lambda+P'$, where $P'=P+\sum_j\lambda_jI_j$, by setting $\lambda_j=j^2-\omega_j+V_j$. Set $H=D+P'$ and $N_0=D$ so that by definition $\Theta=0$. Since the Hamiltonian $P\in\mathcal{H}_{r_0,p_0}(\mathbb{C})$ does not depend on ω,I, trivially one has $P'\in\mathcal{H}_{r_0,p_0}(\mathbb{C})$ and $\|P'\|_{r_0,p_0}=\|P\|_{r_0,p_0}=|P|_{r_0,p_0}$.

Since by hypothesis $\gamma\le|f|_{\mathtt{R}}$, we have by (2.13) that $\mathrm{r}^2/\mathtt{R}\le\varepsilon_*$; then the hypothesis $c_1\mathrm{r}^2\le\mathtt{R}$ of Proposition 4.2 holds for all p satisfying (2.14) taking ε_* small enough. Then, by Proposition 4.2,

$$\epsilon=\gamma^{-1}\|P'\|_{r_0,p_0}=\gamma^{-1}|P|_{r_0,p_0}\le\gamma^{-1}|P|_{2r,(p_*+3)/4}\le \mathrm{c}(p_*)\varepsilon \tag{6.8}$$

for some $\mathrm{c}(p_*)>1$ by (2.13).[28] Again taking $\varepsilon_*=\varepsilon_*(p_*)$ in (2.13) small enough, condition (6.2) is satisfied and Theorem 5 gives us the desired change of variables provided that $\Lambda\in\mathcal{H}^{0,\mathcal{K}}(\mathbb{C})$ is fixed accordingly.

[28]Note that the constants c_1,c_2 in Proposition 4.2 continuously depend on p, which belongs to the compact $[(p_*+1)/2,p_*]$.

Now we denote $\omega_j = \nu_j$ if $j \in \mathcal{S}$ and $\omega_j = \Omega_j$ otherwise. We get the equations

$$\begin{cases} \Omega_j + \lambda_j(\nu, \Omega, I) = j^2 + V_j & \text{if } j \in \mathcal{S}^c, \\ \nu_j + \lambda_j(\nu, \Omega, I) = j^2 + V_j & \text{if } j \in \mathcal{S}, \end{cases} \tag{6.9}$$

for $\omega \in \mathtt{D}_{\gamma,\mathcal{S}}$ and $I \in \mathcal{I}(p, r, \mathbb{C})$. By Lemma 4.1, we Lipschitz extend the map λ to the whole $\mathcal{Q} \times \mathcal{I}(p, \mathrm{r}, \mathbb{C})$ (recall that $\mathrm{r} = r/2$) in such a way that (6.4) and (6.6) hold for $\omega, \omega' \in \mathcal{Q}$ (with $\bar{C} \rightsquigarrow 2\bar{C}$). From now on, we can work only on the real[29] case $I \in \mathcal{I}(p, \mathrm{r})$. By (6.4) and taking $\varepsilon_*(p_*)$ small enough such that $2\bar{C}\epsilon \le 2\bar{C} c(p_*)\varepsilon_*(p_*) \le 1/2$ (recall (6.8)), we use the contraction lemma (recall Lemma B.1)[30] to solve the first equation finding $\Omega : \mathcal{Q}_{\mathcal{S}} \times [-1/4, 1/4]^{\mathcal{S}^c} \times \mathcal{I}(p, \mathrm{r}) \to \mathcal{Q}_{\mathcal{S}^c}$ which is continuous in the product topology of $\mathcal{Q}_{\mathcal{S}} \times [-1/4, 1/4]^{\mathcal{S}^c}$ and satisfies

$$\begin{aligned} \left|\Omega_j(\nu, V_{\mathcal{S}^c}, I) - j^2 - V_{\mathcal{S}^c, j}\right| &\le 2\bar{C}\gamma\epsilon, \\ \sup_{\nu' \ne \nu} \frac{|\Omega_j(\nu, V_{\mathcal{S}^c}, I) - \Omega_j(\nu', V_{\mathcal{S}^c}, I)|}{|\nu - \nu'|_\infty} &\le 4\bar{C}\epsilon, \\ \sup_{V_{\mathcal{S}^c}' \ne V_{\mathcal{S}^c}} \frac{|\Omega_j(\nu, V_{\mathcal{S}^c}', I) - \Omega_j(\nu, V_{\mathcal{S}^c}, I)|}{|V_{\mathcal{S}^c}' - V_{\mathcal{S}^c}|_\infty} &\le 4\bar{C}, \\ \sup_{I' \ne I} \frac{|\Omega_j(\nu, V_{\mathcal{S}^c}, I) - \Omega_j(\nu, V_{\mathcal{S}^c}, I')|}{|I - I'|_{2p}} &\le 2\bar{C}\gamma \mathrm{r}^{-2}\epsilon, \end{aligned} \tag{6.10}$$

for every $j \in \mathcal{S}^c$ and where the first three estimates follow from (6.4) (recalling (5.12) and (4.9)) and the last one follows from (6.6). By (6.10) and (6.8), we prove (2.19) and (2.20) taking $C \ge 4\bar{C} c(p_*)$. Note that this condition depends only on p_*.

Finally, the second equation in (6.9) uniquely defines

$$\mathcal{V}_{\mathcal{S},j}(\nu, V_{\mathcal{S}^c}, I) = \nu_j + \lambda_j\big(\nu, \Omega(\nu, V_{\mathcal{S}^c}, I), I\big) - j^2 \tag{6.11}$$

for $j \in \mathcal{S}$.

By Lemma A.1(iii), we can Lipschitz extend the map $\Psi : B_{r_0-\rho}(\mathtt{w}_p) \times \mathtt{D}_{\gamma,\mathcal{S}} \times \mathcal{I}(p, r) \to B_{r_0}(\mathtt{w}_p)$ on $B_{r_0-\rho}(\mathtt{w}_p) \times \mathcal{Q} \times \mathcal{I}(p, r)$ and set $\Phi(u; \nu, V_{\mathcal{S}^c}, I) := \Psi(u; \omega(\nu, V_{\mathcal{S}^c}, I), I)$, where $\omega(\nu, V_{\mathcal{S}^c}, I)$ was defined in (2.21). The map Φ conjugates the NLS Hamiltonian H_V to normal form for all ν such that $\omega(\nu, V_{\mathcal{S}^c}, I) \in \mathtt{D}_{\gamma,\mathcal{S}}$, namely, for $\nu \in \mathcal{C}$ (recalling (2.21)). Then the proofs of Theorems 3 and 4 are concluded provided that one shows that the set $\mathcal{C}$ defined in (2.21) satisfies (2.16). This is the content of Lemma 9.1 proved in Section 9, where all the measure estimates are discussed. $\square$

[29] Obviously, the same statements hold also in the complex case.

[30] We use it with $F_j(\nu, V_{\mathcal{S}^c}, w) := -\lambda_j(\nu, q + V_{\mathcal{S}^c} + w)$, $j \in \mathcal{S}^c$, where $q = (j^2)_{j \in \mathcal{S}^c}$ and $r := \bar{C}\gamma\epsilon$.

7. Small divisors and homological equation

The proof of Theorem 5 is based on an iterative scheme that kills out the obstructing terms, namely terms belonging to $\mathcal{H}^{-2}_{r,p}$, $\mathcal{H}^{-1}_{r,p}$, and $\mathcal{H}^{0}_{r,p}$, by solving homological equations of the form

$$LF^{(d)} = G^{(d)}, \qquad G^{(d)} \in \mathcal{H}^{d}_{r,p}, \quad d = -2, -1, 0, \tag{7.1}$$

where, for all $\omega \in \mathcal{Q}$, L_ω is an operator acting on formal Hamiltonians as

$$L_\omega[\cdot] := \{D(\omega), \cdot\}, \qquad L_\omega H(\omega, I) := \mathrm{i} \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{N}^{\mathbb{Z}}} \omega \cdot (\boldsymbol{\alpha} - \boldsymbol{\beta}) H_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\omega, I) u^{\boldsymbol{\alpha}} \bar{u}^{\boldsymbol{\beta}}.$$

The convergence of the iterative KAM scheme comes from a good control of $L^{-1}G^{(d)}$ over the set $\mathtt{D}_{\gamma,\mathcal{S}}$.

For $\omega \in \mathtt{D}_{\gamma,\mathcal{S}}$, the Lie derivative operator L_ω is formally invertible on the subspace $\mathcal{H}^{\mathcal{R}}_{r,p}$ with inverse

$$\big(L_\omega^{-1} H(\omega, I)\big)_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \frac{-\mathrm{i} H_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\omega, I)}{\omega \cdot (\boldsymbol{\alpha} - \boldsymbol{\beta})}. \tag{7.2}$$

We now show that the inverse is well defined also at a nonformal level.

PROPOSITION 7.1

Let $\mathcal{O} = \mathtt{D}_{\gamma,\mathcal{S}}$ and $\mathcal{I} = \mathcal{I}(p, r)$, and set $\mathcal{H}^{\le 0,\mathcal{R}}_{r,p} = \mathcal{H}^{\le 0,\mathcal{R},\mathcal{O}\times\mathcal{I}}_{r,p}$. For every $0 < \delta < 1$, (7.2) defines a bounded linear operator $L^{-1} : \mathcal{H}^{\le 0,\mathcal{R}}_{r,p} \to \mathcal{H}^{\le 0,\mathcal{R}}_{r,p+\delta}$ with estimate[31]

$$\|L^{-1} H\|_{r,p+\delta} \le \frac{1}{\gamma} \exp\Big(\exp\Big(\frac{c}{\delta^{1/\eta}}\Big)\Big) \|H\|_{r,p}, \tag{7.3}$$

where i_*, η were introduced in Definition 2.2 and $c = c(i_*)$. The same estimate holds in the corresponding complex spaces.

Remark 7.1

In this section, by c we denote possibly different constants depending only on i_* introduced in Definition 2.2.

Proof

Let us first show that if $H_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathcal{F}(\mathcal{O} \times \mathcal{I})$, then the same holds for $(L^{-1}H)_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Indeed, since $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}| < \infty$, the expression $\omega \cdot (\boldsymbol{\alpha} - \boldsymbol{\beta})$ depends only on a finite number of frequencies ω_i and hence is continuous with respect to the product topology. As for the Lipschitz dependence, $\omega \in \mathtt{D}_{\gamma,\mathcal{S}}$ implies that $1/\omega \cdot (\boldsymbol{\alpha} - \boldsymbol{\beta})$ is C^∞ and the result follows, since the product of Lipschitz functions is Lipschitz.

[31] The constant $1 < \eta \le 2$ was introduced in (2.8a).

By (7.2) and (4.13), we get

$$\|L^{-1}H\|_{r,p+\delta} \le \frac{1}{\gamma} K \|H\|_{r,p}, \tag{7.4}$$

where

$$K = \gamma \sup_{j\in\mathbb{Z}} \sup_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}_j} \left(\frac{\lfloor j\rfloor^2}{\prod_s \lfloor s\rfloor^{\boldsymbol{\alpha}_s+\boldsymbol{\beta}_s}}\right)^{\delta} \left|\frac{1}{\omega\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta})}\right|^{\gamma}, \tag{7.5}$$

and (recall that $H \in \mathcal{H}_{r,p}^{\le 2,\mathcal{R}}$)

$$\mathcal{M}_j := \Big\{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{M} \text{ s.t. } \boldsymbol{\alpha}\ne\boldsymbol{\beta}, \sum_{s\in\mathcal{S}^c} \boldsymbol{\alpha}_s + \boldsymbol{\beta}_s \le 2, \boldsymbol{\alpha}_j + \boldsymbol{\beta}_j \ne 0\Big\}, \tag{7.6}$$

where $\mathcal{M}$ was defined in (4.2). We define

$$\mathcal{M}'_j := \Big\{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{M}_j \text{ s.t. } \Big|\sum_{s\in\mathbb{Z}} (\boldsymbol{\alpha}_s-\boldsymbol{\beta}_s)s^2\Big| < 2\sum_{s\in\mathbb{Z}} |\boldsymbol{\alpha}_s-\boldsymbol{\beta}_s|\Big\}, \tag{7.7}$$

and consider

$$K_1 = \gamma \sup_{j\in\mathbb{Z}} \sup_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}'_j} \left(\frac{\lfloor j\rfloor^2}{\prod_s \lfloor s\rfloor^{\boldsymbol{\alpha}_s+\boldsymbol{\beta}_s}}\right)^{\delta} \left|\frac{1}{\omega\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta})}\right|^{\gamma} \tag{7.8}$$

and

$$K_2 = \gamma \sup_{j\in\mathbb{Z}} \sup_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}_j\setminus\mathcal{M}'_j} \left(\frac{\lfloor j\rfloor^2}{\prod_s \lfloor s\rfloor^{\boldsymbol{\alpha}_s+\boldsymbol{\beta}_s}}\right)^{\delta} \left|\frac{1}{\omega\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta})}\right|^{\gamma}. \tag{7.9}$$

Then

$$K = \max\{K_1, K_2\}. \tag{7.10}$$

We now have to give an upper bound on K_1 and K_2. First note that, by (2.17),

$$\big|\omega\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta})\big| \ge \Big|\sum_{s\in\mathbb{Z}} s^2(\boldsymbol{\alpha}_s-\boldsymbol{\beta}_s)\Big| - \frac{1}{2}\sum_{s\in\mathbb{Z}} |\boldsymbol{\alpha}_s-\boldsymbol{\beta}_s|$$

so

$$\Big|\sum_{s\in\mathbb{Z}} (\boldsymbol{\alpha}_s-\boldsymbol{\beta}_s)s^2\Big| \ge 2\sum_{s\in\mathbb{Z}} |\boldsymbol{\alpha}_s-\boldsymbol{\beta}_s| \quad\Longrightarrow\quad \big|\omega\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta})\big| \ge |\boldsymbol{\alpha}-\boldsymbol{\beta}| \ge 1. \tag{7.11}$$

Remark 7.2

In the following, we will strongly use the fact that the involved functions depend only on a *finite* number of ω_j's. In this case, the Lipschitz seminorm is bounded by the ℓ^1-norm of the gradient.

The estimate of K_2 is trivial. Indeed, since $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{M}_j \setminus \mathcal{M}'_j$, then $|\omega\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta})| \geq |\boldsymbol{\alpha}-\boldsymbol{\beta}| \geq 1$. Therefore,

$$\Big|\frac{1}{\omega\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta}))}\Big|^{\gamma} \leq \sup_{\omega\in\mathrm{D}_{\gamma,\mathcal{S}}} \frac{1}{|\omega\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta})|} + \gamma \sup_{\omega\in\mathrm{D}_{\gamma,\mathcal{S}}} \frac{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}{(\omega\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta}))^2} \leq 2,$$

so that, in conclusion,

$$K_2 \leq 2. \tag{7.12}$$

Let us now study K_1. Since $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{M}'_j$, then (recall Definition 2.3)

$$\sup_{\omega\in\mathrm{D}_{\gamma,\mathcal{S}}} \Big|\frac{1}{\omega\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta}))}\Big| \leq \frac{2}{\gamma\,\mathrm{d}(\boldsymbol{\alpha}-\boldsymbol{\beta})}, \quad \text{with } \mathrm{d}(\ell) := \prod_{s\in\mathcal{S}} \frac{1}{(1+|\ell_s|^2\langle i(s)\rangle^2)^{3/2}};$$

as for the Lipschitz variation, we estimate it as

$$\sup_{\omega\in\mathrm{D}_{\gamma,\mathcal{S}}} \sum_j \Big|\partial_{\omega_j}\frac{1}{\omega\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta})}\Big| \leq \sup_{\omega\in\mathrm{D}_{\gamma,\mathcal{S}}} \frac{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}{(\omega\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta}))^2} \leq \frac{4}{\gamma^2(\mathrm{d}(\boldsymbol{\alpha}-\boldsymbol{\beta}))^3},$$

where the last inequality comes from $|\boldsymbol{\alpha}-\boldsymbol{\beta}| \leq (\mathrm{d}(\boldsymbol{\alpha}-\boldsymbol{\beta}))^{-1}$. Note that the sum on the left-hand side above is over a *finite* number of indexes j. In conclusion, we have proved that for $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{M}'_j$,

$$\Big|\frac{1}{\omega\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta}))}\Big|^{\gamma} \leq \frac{14}{\gamma(\mathrm{d}(\boldsymbol{\alpha}-\boldsymbol{\beta}))^3},$$

and hence, recalling the definition of η in (2.8a), we have

$$K_1 \leq 14K'_1, \quad \text{where} \tag{7.13}$$
$$K'_1 := \sup_{j\in\mathbb{Z}} \sup_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}'_j} \Big(\frac{\lfloor j\rfloor^2}{\prod_s \lfloor s\rfloor^{\boldsymbol{\alpha}_s+\boldsymbol{\beta}_s}}\Big)^{\delta} \prod_{s\in\mathcal{S}} \big(1+\langle i(s)\rangle^2|\boldsymbol{\alpha}_s-\boldsymbol{\beta}_s|^2\big)^{9/2}.$$

By (7.4), (7.10), (7.12), (7.13), and the following crucial estimate

$$\log K'_1 \leq c\exp(135\delta^{-1/\eta}), \tag{7.14}$$

Proposition 7.1 follows. □

The rest of this section is devoted to the proof of estimate (7.14). We first need some preliminaries discussed in the following subsection.

7.1. Preliminaries

We first state an elementary result that will be useful below.

LEMMA 7.1

If $\ell \in \mathbb{Z}^{\mathbb{Z}}$, with $|\ell| < \infty$, satisfies $\mathfrak{m}(\ell) = \pi(\ell) = 0$, then $|\ell|$ is even and $|\ell| \neq 2$.

Proof

As usual, we write in a unique way $\ell = \ell^+ - \ell^-$, where $\ell_s^\pm \geq 0$ and $\ell_s^+ \ell_s^- = 0$ for every $s \in \mathbb{Z}$. Since $\mathfrak{m}(\ell) = 0$, we get $|\ell^+| = |\ell^-|$; therefore $|\ell| = |\ell^+| + |\ell^-|$ is even.

Now assume by contradiction that $|\ell^+| = |\ell^-| = 1$. Then $\ell^+ = e_i, \ell^- = e_j$ for some $i \neq j$ and $\pi(\ell) = i - j$, which contradicts $\pi(\ell) = 0$. □

We now need some basic results and notation coming from [6].

Definition 7.1

Given a vector $v \in \mathbb{N}^{\mathbb{Z}}$, $0 < |v| < \infty$, we denote by $\widehat{n} = \widehat{n}(v)$ the vector $(\widehat{n}_l)_{l \in I}$ (where $I \subset \mathbb{N}$ is finite) which is the decreasing rearrangement of

$$\{\mathbb{N} \ni h > 1 \text{ repeated } v_h + v_{-h} \text{ times}\} \cup \{1 \text{ repeated } v_1 + v_{-1} + v_0 \text{ times }\}.$$

Remark 7.3

A good way of envisioning this list is as follows. Given the set of (commutative) variables $(x_j)_{j \in \mathbb{Z}}$, we consider a monomial (recall (4.1))

$$x^v = \prod_i x_i^{v_i} = x_{j_1} x_{j_2} \cdots x_{x_{|v|}}.$$

Then $\widehat{n}(v)$ is the decreasing rearrangement of the list $(\langle j_1 \rangle, \ldots, \langle j_{|v|} \rangle)$. As an example, consider $v = (v_j)_{j \in \mathbb{Z}}$

$$\text{with } v_{-6} = 1, v_{-3} = 4, v_{-1} = v_0 = 1, v_1 = v_6 = 2, \text{ and } v_j = 0 \text{ otherwise.}$$

Then $\widehat{n}(v) = (6, 6, 6, 3, 3, 3, 3, 1, 1, 1, 1)$.

Given $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{M}$ with $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}| > 1$, from now on we define

$$\widehat{n} := \widehat{n}(\boldsymbol{\alpha} + \boldsymbol{\beta}).$$

We observe that there exists a choice of $\sigma_i = \pm 1, 0$ such that by momentum conservation

$$\sum_l \sigma_l \widehat{n}_l = 0, \tag{7.15}$$

with $\sigma_l \neq 0$ if $\widehat{n}_l \neq 1$. Indeed, we recall that

$$\sum_{j\in\mathbb{Z}} j(\boldsymbol{\alpha}_j - \boldsymbol{\beta}_j) = \sum_{h\in\mathbb{N}} h\big(\boldsymbol{\alpha}_h + \boldsymbol{\beta}_{-h} - (\boldsymbol{\alpha}_{-h} + \boldsymbol{\beta}_h)\big)$$

and that each $h > 1$ appears exactly $\boldsymbol{\alpha}_h + \boldsymbol{\alpha}_{-h} + \boldsymbol{\beta}_h + \boldsymbol{\beta}_{-h}$ times in the sequence $\widehat{n}$. Hence we assign the value $\sigma = 1$, $\boldsymbol{\alpha}_h + \boldsymbol{\beta}_{-h}$ times to the $\widehat{n}_l = h$ (and $\sigma = -1$ the remaining times). The value $h = 1$ instead appears in the sequence $\widehat{n}$, $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_{-1} + \boldsymbol{\beta}_1 + \boldsymbol{\beta}_{-1} + \boldsymbol{\alpha}_0 + \boldsymbol{\beta}_0$ times. Hence we assign the value $\sigma = 1$, $\boldsymbol{\alpha}_1 + \boldsymbol{\beta}_{-1}$ times, the value $\sigma = -1$, $\boldsymbol{\alpha}_{-1} + \boldsymbol{\beta}_1$ times, and $\sigma = 0$ all the remaining times.

From (7.15), we deduce

$$\widehat{n}_1 \le \sum_{l\ge 2} \widehat{n}_l. \tag{7.16}$$

Indeed, if $\sigma_1 = \pm 1$, then the inequality follows directly from (7.15); if $\sigma_1 = 0$, then $\widehat{n}_1 = 1$ and consequently $\widehat{n}_l = 1$ $\forall l$. Since the mass is conserved, the list $\widehat{n}$ has at least two elements, and the inequality is achieved.

We finally need the following elementary result proved at the end of Appendix A.

LEMMA 7.2

Let $x_1 \ge x_2 \ge \cdots \ge x_N \ge 2$. Then

$$\frac{\sum_{1\le \ell\le N} x_\ell}{\prod_{1\le \ell \le N} \sqrt{x_\ell}} \le \sqrt{x_1} + \frac{4}{\sqrt{x_1}}.$$

7.2. *Proof of estimate (7.14)*

For $j \in \mathbb{Z}$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{M}_j$, by Lemma 7.2 we can write

$$\begin{aligned}\frac{\lfloor j \rfloor^2}{\Pi_s \lfloor s \rfloor^{\boldsymbol{\alpha}_s + \boldsymbol{\beta}_s}} &\le \frac{\lfloor \widehat{n}_1 \rfloor}{\prod_{l\ge 2} \lfloor \widehat{n}_l \rfloor} \overset{(7.16)}{\le} \frac{\sum_{l\ge 2} \lfloor \widehat{n}_l \rfloor}{\prod_{l\ge 2} \lfloor \widehat{n}_l \rfloor} \le \frac{4 + \lfloor \widehat{n}_2 \rfloor^{\frac12}}{\Pi_{l\ge 2} \lfloor \widehat{n}_l \rfloor^{\frac12}} \\ &\le \frac{3}{\Pi_{l\ge 3} \lfloor \widehat{n}_l \rfloor^{\frac12}}. \end{aligned} \tag{7.17}$$

Now, by (7.13), (7.17), and (2.18),

$$K_1' \le K_1'' := \sup_{j\in\mathbb{Z}} \sup_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}_j'} \left(\frac{3}{\Pi_{l\ge 3} \lfloor \widehat{n}_l \rfloor^{\frac12}}\right)^{\delta} \prod_{s\in\mathcal{S}} \big(1 + \langle i(s)\rangle^2 |\boldsymbol{\alpha}_s - \boldsymbol{\beta}_s|^2\big)^{9/2}. \tag{7.18}$$

Then (7.14) follows by

$$\log K_1'' \le c \exp(135\delta^{-1/\eta}). \tag{7.19}$$

It remains to prove (7.19).

In the following, we call $j_1, j_2 \in \mathcal{S}^c$ with $|j_1| \geq |j_2|$ the *possible normal sites*. Set

$$\alpha_i = \boldsymbol{\alpha}_{s(i)}, \qquad \beta_i = \boldsymbol{\beta}_{s(i)}.$$

The quantity on the right-hand side of (7.18) reads

$$\Big(\frac{3}{\Pi_{l\geq 3}\lfloor \widehat{n}_l \rfloor^{\frac{1}{2}}}\Big)^{\delta} \prod_{i\in\mathbb{N}} \big(1 + \langle i\rangle^2 |\alpha_i - \beta_i|^2\big)^{9/2}. \tag{7.20}$$

By Lemma 7.1, the constraint $\sum_{s\in\mathcal{S}^c} \boldsymbol{\alpha}_s + \boldsymbol{\beta}_s \leq 2$ implies that there exists at least one $s \in \mathcal{S}$ such that $\boldsymbol{\alpha}_s + \boldsymbol{\beta}_s \neq 0$. We denote the largest in absolute value $s \in \mathcal{S}$ with this property as s_{M} and $i_{\mathrm{M}} := i(s_{\mathrm{M}})$.

In proving (7.19), we often meet the quantity

$$A_k(\delta) := \sum_{i\leq i_{\mathrm{M}}: k_i\geq 1} -\frac{\delta}{9} k_i \log\lfloor s(i)\rfloor + \log\big(1 + \langle i\rangle^2 k_i^2\big), \tag{7.21}$$

defined for[32] $k \in \mathbb{N}^{\mathbb{Z}}$, which is estimated by the following result, whose proof is postponed to Appendix A.

LEMMA 7.3
For every $k \in \mathbb{N}^{\mathbb{Z}}$,

$$A_k(\delta) \leq c \exp(45\delta^{-1/\eta}).$$

We divide the proof of (7.19) into six cases.

Case 1: $\widehat{n}_2 > s_{\mathrm{M}}$
Here, since there are at most two normal sites we get

$$\Pi_{l\geq 3}\lfloor \widehat{n}_l \rfloor = \prod_{i\leq i_{\mathrm{M}}} \lfloor s(i)\rfloor^{\alpha_i+\beta_i}.$$

Recalling that

$$\big\langle i(s)\big\rangle |\boldsymbol{\alpha}_s - \boldsymbol{\beta}_s| = \langle i\rangle |\alpha_i - \beta_i| \leq \langle i\rangle(\alpha_i + \beta_i), \quad \text{if } s = s(i), i \in \mathbb{N},$$

we get

[32] Recall Definition 4.1. Note that the sum over i is restricted to the indexes such that $k_i \geq 1$.

$$\begin{aligned}\log K_1'' &\le \log(3^\delta)\\ &\quad + \sup_{\alpha,\beta}\Big(-\frac{\delta}{2}\sum_{i\le i_{\mathtt M}}(\alpha_i+\beta_i)\log\lfloor s(i)\rfloor + 9/2\sum_{i\le i_{\mathtt M}}\log\big(1+\langle i\rangle^2(\alpha_i+\beta_i)^2\big)\Big)\\ &= \log(3^\delta)+9/2\sup_{k\in\mathbb{N}^{\mathbb{Z}}}A_k(\delta)\le c\exp(45\delta^{-1/\eta}),\end{aligned}\tag{7.22}$$

where A_k was defined in (7.21), estimated in Lemma 7.3.

Case 2: $\widehat{n}_1 > s_{\mathtt M} = \widehat{n}_2$ and only one normal site
We have $\sum_{i\in\mathcal{S}^c}\boldsymbol{\alpha}_i+\boldsymbol{\beta}_i=1$ and the normal site must be $\widehat{n}_1$. Moreover,

$$\Pi_{l\ge 3}\lfloor\widehat{n}_l\rfloor = \lfloor s(i_{\mathtt M})\rfloor^{\alpha_{i_{\mathtt M}}+\beta_{i_{\mathtt M}}-1}\prod_{i<i_{\mathtt M}}\lfloor s(i)\rfloor^{\alpha_i+\beta_i},\tag{7.23}$$

so

$$\begin{aligned}K_1'' &\le 3^\delta\sup_{\alpha,\beta}\lfloor s(i_{\mathtt M})\rfloor^{-(\alpha_{i_{\mathtt M}}+\beta_{i_{\mathtt M}}-1)\delta/2}\big(1+\langle i_{\mathtt M}\rangle^2|\alpha_{i_{\mathtt M}}-\beta_{i_{\mathtt M}}|^2\big)^{9/2}\\ &\quad\times\prod_{i<i_{\mathtt M}}\lfloor s(i)\rfloor^{-(\alpha_i+\beta_i)\delta/2}\big(1+\langle i\rangle^2(\alpha_i+\beta_i)^2\big)^{9/2}.\end{aligned}\tag{7.24}$$

If $\alpha_{i_{\mathtt M}}=\beta_{i_{\mathtt M}}$, and hence $i_{\mathtt M}$ does not appear in the small divisors, then we proceed as in case 1.

(a) If $\alpha_{i_{\mathtt M}}+\beta_{i_{\mathtt M}}\ge 2$, then we claim that

$$\log\Big(\lfloor s(i_{\mathtt M})\rfloor^{-(\alpha_{i_{\mathtt M}}+\beta_{i_{\mathtt M}}-1)\delta/2}\big(1+\langle i_{\mathtt M}\rangle^2|\alpha_{i_{\mathtt M}}-\beta_{i_{\mathtt M}}|^2\big)^{9/2}\Big)\le\frac{c}{\delta}.\tag{7.25}$$

In order to prove our claim, we consider two cases $i_{\mathtt M}\le i_*$ and $i_{\mathtt M}>i_*$. In the first case, letting $x=\alpha_{i_{\mathtt M}}+\beta_{i_{\mathtt M}}-1\ge 1$, the left-hand side of (7.25) is bounded by

$$\frac{9}{2}\big(-\bar{\delta}x\log 2+\log\big(1+i_*^2(x+1)^2\big)\big)\quad\text{with }\bar{\delta}:=\frac{\delta}{9},$$

which is negative for $x\ge c/\bar{\delta}^2$ and, therefore, bounded by $c\log\bar{\delta}$.

Consider now the case $i_{\mathtt M}>i_*$. By (2.8a), the left-hand side of (7.25) is bounded by[33]

$$\frac{9}{2}\big(-\bar{\delta}x\log^{1+\eta}i_{\mathtt M}+\log\big(1+i_{\mathtt M}^2(x+1)^2\big)\big)\le\frac{9}{2}f(i_{\mathtt M},x),$$

with f defined in (A.6). Reasoning as above, we can estimate $f(i_{\mathtt M},x)$ by $c/\bar{\delta}$ obtaining (7.25). By (7.24) and (7.25), we have

[33]Recalling that $i_{\mathtt M}>i_*\ge 21$ and $\log 21\ge 3$.

$$\begin{aligned}\log K_1'' &\le \log(3^\delta) + \frac{c}{\delta} \\ &\quad + \sup_{\alpha,\beta}\Big(-\frac{\delta}{2}\sum_{i<i_{\mathrm{M}}}(\alpha_i+\beta_i)\log\lfloor s(i)\rfloor + \frac{9}{2}\sum_{i<i_{\mathrm{M}}}\log\big(1+\langle i\rangle^2(\alpha_i+\beta_i)^2\big)\Big) \\ &\le \log(3^\delta) + \frac{c}{\delta} + \sup_{k\in\mathbb{N}^{\mathbb{Z}}} A_k(\delta) \le c\exp(45\delta^{-1/\eta}),\end{aligned}$$

by (7.21) and Lemma 7.3.

(b) If $\alpha_{i_{\mathrm{M}}}+\beta_{i_{\mathrm{M}}}=1$ (then $|\alpha_{i_{\mathrm{M}}}-\beta_{i_{\mathrm{M}}}|=1$), here the second factor in (7.24) is equal to 1. Thus in order to bound the third factor (i.e., $(1+\langle i_{\mathrm{M}}\rangle^2)^{9/2}\le 2^{9/2}i_{\mathrm{M}}^9$), we distinguish two cases: $i_{\mathrm{M}}\le i_*$ and $i_{\mathrm{M}}>i_*$. In the first case, $2^{9/2}i_{\mathrm{M}}^9\le 2^{9/2}i_*^9$ and the estimate of K_1'' in (7.24) proceeds as in case 1 above. On the other hand, when $i_{\mathrm{M}}>i_*$ we need a different argument.

Given $u\in\mathbb{Z}^{\mathbb{Z}}$, with $|u|<\infty$, consider the set

$$\big\{j\neq 0,\ \text{repeated } |u_j| \text{ times}\big\},$$

where $D<\infty$ is its cardinality. Define the vector $m=m(u)$ as the reordering of the elements of the set above such that $|m_1|\ge|m_2|\ge\cdots\ge|m_D|\ge 1$. Given $\boldsymbol{\alpha}\neq\boldsymbol{\beta}\in\mathbb{N}^{\mathbb{Z}}$, with $|\boldsymbol{\alpha}|=|\boldsymbol{\beta}|<\infty$ we consider $m=m(\boldsymbol{\alpha}-\boldsymbol{\beta})$ and[34] $\widehat{n}=\widehat{n}(\boldsymbol{\alpha}+\boldsymbol{\beta})$.

LEMMA 7.4 ([6, Lemma C.4])
Given $\boldsymbol{\alpha}\neq\boldsymbol{\beta}\in\mathbb{N}^{\mathbb{Z}}$, with $1\le|\boldsymbol{\alpha}|=|\boldsymbol{\beta}|<\infty$ and satisfying (7.7), we have

$$|m_1|\le 31\sum_{l\ge 3}\widehat{n}_l^2. \tag{7.26}$$

Note that

$$\alpha_{i_{\mathrm{M}}}\neq\beta_{i_{\mathrm{M}}} \quad\Longrightarrow\quad s_{\mathrm{M}}\le|m_1|. \tag{7.27}$$

In the present case $|\alpha_{i_{\mathrm{M}}}-\beta_{i_{\mathrm{M}}}|=1$, by (7.27)

$$\begin{aligned}s(i_{\mathrm{M}})=s_{\mathrm{M}}\le|m_1|&\le 31\sum_{l\ge 3}\widehat{n}_l^2 \\ &=31\sum_{i<i_{\mathrm{M}}}\lfloor s(i)\rfloor^2(\alpha_i+\beta_i)\le 31\sum_{i\le i_*}\lfloor s(i)\rfloor^2(\alpha_i+\beta_i)\end{aligned}$$

[34]The relation between m and $\widehat{n}$ is the following. If we denote by D the cardinality of m and by N the one of $\widehat{n}$, respectively, we have (recalling Definition 4.1) $D+\boldsymbol{\alpha}_0+\boldsymbol{\beta}_0\le N$ and $(|m_1|,\dots,|m_D|,\underbrace{1,\dots,1}_{N-D\text{ times}})\preceq(\widehat{n}_1,\dots\widehat{n}_N)$.

$$
\begin{aligned}
&+31\sum_{i_*<i<i_{\mathtt{M}}} s^2(i)(\alpha_i+\beta_i)\\
&\le 31\sum_{i\le i_*} s_*^2(\alpha_i+\beta_i)+31\sum_{i_*<i<i_{\mathtt{M}}} s^2(i)(\alpha_i+\beta_i)
\end{aligned}
\tag{7.28}
$$

using that $s(i)$ is increasing. By (2.8a)–(2.8c), for the inverse function $i(s)$ we have for integer $j\ge 1$ and $s,s'\ge s_*:=s(i_*)$,

$$
i(s+s')\le i(s)+i(s'),\qquad i(js)\le j\,i(s),\qquad i(s^2)\le i^2(s).
\tag{7.29}
$$

Applying the inverse function $i(s)$ to the inequalities in (7.28) and using (7.29), we obtain

$$
\begin{aligned}
i_{\mathtt{M}}&\le 31 i_*^2\sum_{i\le i_*}^{*}(\alpha_i+\beta_i)+31\sum_{i_*<i<i_{\mathtt{M}}}^{*} i^2(\alpha_i+\beta_i)\le c\sum_{i<i_{\mathtt{M}}}^{*} i^2(\alpha_i+\beta_i)\\
&\le c\prod_{i<i_{\mathtt{M}}}\big(1+\langle i\rangle^2(\alpha_i+\beta_i)^2\big),
\end{aligned}
\tag{7.30}
$$

where by $\sum^*$ we mean that the sum is only over the indexes i such that $\alpha_i+\beta_i\ge 1$. By (7.24), we get

$$
\begin{aligned}
\log K_1''&\le c+\log(3^\delta)\\
&\quad+\sup_{\alpha,\beta}\sum_{i<i_{\mathtt{M}}}\Big(-\frac{\delta}{2}(\alpha_i+\beta_i)\log\lfloor s(i)\rfloor+\frac{27}{2}\log\big(1+\langle i\rangle^2(\alpha_i+\beta_i)^2\big)\Big)\\
&\le c+\log(3^\delta)+\frac{27}{2}\sup_{k\in\mathbb{N}^{\mathbb{Z}}} A_k(\delta/3)\le c\exp(135\delta^{-1/\eta}),
\end{aligned}
\tag{7.31}
$$

again by (7.21) and Lemma 7.3.

Case 3: $\widehat{n}_1>s_{\mathtt{M}}=\widehat{n}_2$ and two normal sites

Now

$$
\Pi_{l\ge 3}\lfloor\widehat{n}_l\rfloor=\lfloor j_2\rfloor\lfloor s(i_{\mathtt{M}})\rfloor^{\alpha_{h_{\mathtt{M}}}+\beta_{h_{\mathtt{M}}}-1}\prod_{i<i_{\mathtt{M}}}\lfloor s(i)\rfloor^{\alpha_i+\beta_i},
\tag{7.32}
$$

where j_2 is the smallest[35] normal site. We have

$$
\begin{aligned}
K_1''&\le 3^\delta\lfloor j_2\rfloor^{-\delta/2}\sup_{\alpha,\beta}\lfloor s(i_{\mathtt{M}})\rfloor^{-(\alpha_{i_{\mathtt{M}}}+\beta_{i_{\mathtt{M}}}-1)\delta/2}\big(1+\langle i_{\mathtt{M}}\rangle^2|\alpha_{i_{\mathtt{M}}}-\beta_{i_{\mathtt{M}}}|^2\big)^{9/2}\\
&\quad\times\prod_{i<i_{\mathtt{M}}}\lfloor s(i)\rfloor^{-(\alpha_i+\beta_i)\delta/2}\big(1+\langle i\rangle^2(\alpha_i+\beta_i)^2\big)^{9/2}.
\end{aligned}
\tag{7.33}
$$

[35] In absolute value.

(a) If $\alpha_{i_{\mathtt{M}}} = \beta_{i_{\mathtt{M}}}$ or if $\alpha_{i_{\mathtt{M}}} + \beta_{i_{\mathtt{M}}} \geq 2$, then we proceed as in case 2(a), since $\lfloor j_2 \rfloor^{-\frac{\delta}{2}} \leq 1$ and does not bother.

(b) Now let $\alpha_{i_{\mathtt{M}}} + \beta_{i_{\mathtt{M}}} = 1$ (so that (7.27) holds). The analogue of (7.30) is

$$i_{\mathtt{M}} \leq c\big(i\big(\lfloor j_2 \rfloor\big)\big)^2 \prod_{i < i_{\mathtt{M}}} \big(1 + \langle i \rangle^2 (\alpha_i + \beta_i)^2\big). \tag{7.34}$$

Then by (7.33), we get

$$\log K_1'' \leq c + \log(3^{\delta}) + 18 \log i\big(\lfloor j_2 \rfloor\big) - \frac{\delta}{2} \log \lfloor j_2 \rfloor + \sup_{\alpha,\beta} \sum_{i < i_{\mathtt{M}}} \Big(-\frac{\delta}{2} \log \lfloor s(i) \rfloor + \frac{27}{2} \log\big(1 + \langle i \rangle^2 (\alpha_i + \beta_i)^2\big)\Big). \tag{7.35}$$

Since

$$\sup_{x \geq 2} \Big(18 \log i(x) - \frac{\delta}{2} \log x\Big) = \sup_{y \geq i(2)} \Big(18 \log y - \frac{\delta}{2} \log s(y)\Big) \overset{(2.8a)}{\leq} c + \sup_{y \geq i_*} \Big(18 \log y - \frac{\delta}{2} \log^2 y\Big) \leq c + \frac{162}{\delta},$$

we obtain

$$\log K_1'' \leq c + \log(3^{\delta}) + \frac{162}{\delta} + \frac{27}{2} \sup_{k \in \mathbb{N}^{\mathbb{Z}}} A_k(\delta/3) \leq c \exp(135 \delta^{-1/\eta}),$$

again by (7.21) and Lemma 7.3.

Case 4: $\widehat{n}_1 = s_{\mathtt{M}}$ and the (eventual) normal sites are $< \widehat{n}_2$

Recall that we have to estimate from above the quantity in (7.20).

Let us start with the case of two normal sites j_1, j_2.

We start by considering the case $\widehat{n}_1 = \widehat{n}_2$, which implies that $\alpha_{i_{\mathtt{M}}} + \beta_{i_{\mathtt{M}}} \geq 2$. We obtain

$$\Pi_{l \geq 3} \lfloor \widehat{n}_l \rfloor = \lfloor j_1 \rfloor \lfloor j_2 \rfloor \lfloor s(i_{\mathtt{M}}) \rfloor^{\alpha_{i_{\mathtt{M}}} + \beta_{i_{\mathtt{M}}} - 2} \prod_{i < i_{\mathtt{M}}} \lfloor s(i) \rfloor^{\alpha_i + \beta_i}. \tag{7.36}$$

(a) If $\alpha_{i_{\mathtt{M}}} = \beta_{i_{\mathtt{M}}}$, or if $\alpha_{i_{\mathtt{M}}} + \beta_{i_{\mathtt{M}}} \geq 2$, then we are reduced to case 3(a).

(b) If $\alpha_{i_{\mathtt{M}}} = 2, \beta_{i_{\mathtt{M}}} = 0$ (or vice versa), then we proceed as in case 3(b) and apply Lemma C.4.[36] Since, again, (7.27) holds, the analogue of (7.34) is

[36] In the notation of [6]: $\widehat{n}_1 = |m_1|$.

$$
\begin{aligned}
i_{\mathrm{M}} &\leq c\big(i\left(\lfloor j_1\rfloor\right)\big)^2\big(i\left(\lfloor j_2\rfloor\right)\big)^2 \prod_{i<i_{\mathrm{M}}}\left(1+\langle i\rangle^2(\alpha_i+\beta_i)^2\right) \\
&\leq c\big(i\left(\lfloor j_1\rfloor\right)\big)^4 \prod_{i<i_{\mathrm{M}}}\left(1+\langle i\rangle^2(\alpha_i+\beta_i)^2\right). \qquad (7.37)
\end{aligned}
$$

Then the analogue of (7.35) is

$$
\begin{aligned}
\log K_1'' \leq c + \log(3^\delta) &+ 36\log i\left(\lfloor j_1\rfloor\right) - \frac{\delta}{2}\log\lfloor j_1\rfloor \\
&+ \sup_{\alpha,\beta}\sum_{i<i_{\mathrm{M}}}\Big(-\frac{\delta}{2}\log\lfloor s(i)\rfloor + \frac{27}{2}\log\left(1+\langle i\rangle^2(\alpha_i+\beta_i)^2\right)\Big).
\end{aligned}
$$

We conclude as in case 3(b).

We now consider the case $\widehat{n}_1 > \widehat{n}_2$.

Note that $\alpha_{i_{\mathrm{M}}} + \beta_{i_{\mathrm{M}}} = 1$ (so that (7.27) holds). Let us denote by s'_{M} the second largest tangential site[37] and set $i'_{\mathrm{M}} := i(s'_{\mathrm{M}})$. By construction, $s'_{\mathrm{M}} = \widehat{n}_2$ and $\alpha_{i'_{\mathrm{M}}} + \beta_{i'_{\mathrm{M}}} \geq 1$. Then

$$
\Pi_{l\geq 3}\lfloor\widehat{n}_l\rfloor = \lfloor j_1\rfloor\lfloor j_2\rfloor\lfloor s(i'_{\mathrm{M}})\rfloor^{\alpha_{i'_{\mathrm{M}}}+\beta_{i'_{\mathrm{M}}}-1}\prod_{i<i'_{\mathrm{M}}}\lfloor s(i)\rfloor^{\alpha_i+\beta_i}.
$$

In this case, the analogue of (7.37) is

$$
i_{\mathrm{M}} \leq c\big(i\left(\lfloor j_1\rfloor\right)\big)^4\prod_{i<i_{\mathrm{M}'}}\left(1+\langle i\rangle^2(\alpha_i+\beta_i)^2\right),
$$

which follows by (7.28) where the second line holds with $i_{\mathrm{M}'}$ instead of i_{M}. Then we proceed as in the case $\widehat{n}_1 = \widehat{n}_2$.

Note: If there is only one normal site or if there is none, then the same arguments apply verbatim with the only "advantage" being that there is only one $\lfloor j_1\rfloor$ or none in (7.36).

Case 5: $\widehat{n}_1 = s_{\mathrm{M}}$ and only one normal site $j_1 = \widehat{n}_2$
Here (7.23) holds and we proceed as in case 2.

Case 6: $\widehat{n}_1 = s_{\mathrm{M}}$, $j_1 = \widehat{n}_2$ and two normal sites
The proof follows that of case 3 verbatim.

This concludes the proof of (7.19), which, by (7.18), implies (7.14).

[37] Recall that the tangential sites we are considering are positive.

8. Iterative lemma and proof of Theorem 5

Let $r, r_0, p, p_0, \rho, \delta$ be as in (6.1), and let $1 < \eta \le 2$ be as in Definition 2.2. Let $\{\rho_n\}_{n\in\mathbb{N}}, \{\delta_n\}_{n\in\mathbb{N}}$ be the summable sequences[38]

$$\begin{aligned} \rho_n &= \frac{\rho}{6} 2^{-n}, \qquad \delta_n = c_\eta \delta n^{-\frac{1+\eta}{2}} \quad \forall n \ge 1, \\ c_\eta^{-1} &:= \frac{24}{5} \sum_{n\ge 1} n^{-\frac{1+\eta}{2}} > 12, \qquad \delta_0 = \frac{\delta}{8}. \end{aligned} \tag{8.1}$$

Let us recursively define

$$\begin{aligned} r_{n+1} &= r_n - 3\rho_n \to r_\infty := r_0 - \rho \ge 7r/4 \quad \text{(decreasing)}, \\ p_{n+1} &= p_n + 3\delta_n \to p_\infty := p_0 + \delta < p \quad \text{(increasing)}, \end{aligned} \tag{8.2}$$

recalling (6.1).

Set $\mathcal{O} = \mathtt{D}_{\gamma,\mathcal{g}}$, $\mathcal{I} = \mathcal{I}(p,r)$, and $\mathcal{I}(\mathbb{C}) = \mathcal{I}(p,r,\mathbb{C})$ (recall (2.18), (2.11), and (4.11)). Since these sets are fixed, we omit to write them explicitly in the notation of this section. For instance, we denote $\mathcal{H}_{r_n,p_n} = \mathcal{H}_{r_n,p_n}^{\mathcal{O}\times\mathcal{I}}$ and $\mathcal{H}_{r_n,p_n}(\mathbb{C}) = \mathcal{H}_{r_n,p_n}^{\mathcal{O}\times\mathcal{I}(\mathbb{C})}$.

By (5.13) and (8.2), we can use projections $\Pi^0, \Pi^{-1}, \Pi^{\ge 1}$ on these spaces for all n.

Remark 8.1

A crucial point for the convergence of the algorithm is that, thanks to (5.13), no small divisor appears in the estimate of the counterterms (see (8.22) and (8.24) below).

Let

$$H_0 := D + G_0 + \Lambda_0, \quad G_0 \in \mathcal{H}_{r_0,p_0}(\mathbb{C}), \Lambda_0 \in \mathcal{H}_{r,p}^{0,\mathcal{K}}, \tag{8.3}$$

where D is defined in (6.1) and the counterterms $\Lambda_0 = \sum_{j\in\mathbb{Z}} \lambda_{0,j}(|u_j|^2 - I_j)$ with $\lambda_0 = (\lambda_{0,j})_{j\in\mathbb{Z}}$ free parameters in ℓ^∞. We define

$$\begin{aligned} \varepsilon_0 &:= \gamma^{-1}\big(\|G_0^{(0,\mathcal{K})}\|_\infty + \|G_0^{(0,\mathcal{R})}\|_{r_0,p_0} + \|G_0^{(-2)}\|_{r_0,p_0} + \|G_0^{(-1)}\|_{r_0,p_0}\big), \\ \Theta_0 &:= \gamma^{-1}\|G_0^{\ge 2}\|_{r_0,p_0} + \varepsilon_0. \end{aligned} \tag{8.4}$$

LEMMA 8.1 (Iterative step)

There exists a constant $\mathfrak{C} > 1$ large enough[39] *such that if*

[38] Note that $\frac{1+\eta}{2} > 1$.

[39] Depending only on i_*, η defined in (2.8a).

$$\varepsilon_0 \le (1+\Theta_0)^{-5}\mathtt{K}^{-3}, \qquad \mathtt{K} := \Big(\frac{r_0}{\rho}\Big)^6 \sup_{n\ge 1} 2^{6n}\exp\big(\zeta^{n^{\xi}} - \chi^n(1-\chi/2)\big),$$
$$\textit{where } \zeta := \exp\Big(\frac{\mathfrak{C}}{\delta^{1/\eta}}\Big), \xi := \frac{1+\eta}{2\eta} < 1, \chi := 3/2, \tag{8.5}$$

with η as in (2.8a), then we can iteratively construct a sequence of generating functions $S_i = S_i^{(-2)} + S_i^{(-1)} + S_i^{(0)} \in \mathcal{H}_{r_i-\rho_i, p_{i+1}}(\mathbb{C})$ and a sequence of counterterms $\bar\Lambda_i \in \mathcal{H}^{0,\mathcal{K}}(\mathbb{C})$ such that the following holds, for $n \ge 0$.

(1) *For all $\omega \in \mathtt{D}_{\gamma,\mathcal{S}}$, $I \in \mathcal{I}(p,r)$, for all $i = 0,\dots,n-1$, and all $p' \ge p_{i+1}$, the time-1 Hamiltonian flow Φ_{S_i} generated by $S_i = S_i(\omega, I)$ satisfies*

$$\sup_{u\in \bar B_{r_{i+1}}(\mathtt{w}_{p'})} \big|\Phi_{S_i}(u) - u\big|_{p'} \le \rho 2^{-2i-7}. \tag{8.6}$$

Moreover,

$$\Psi_n := \Phi_{S_0}\circ\cdots\circ\Phi_{S_{n-1}} \tag{8.7}$$

is a well-defined analytic map $\bar B_{r_n}(\mathtt{w}_{p'}) \to \bar B_{r_0}(\mathtt{w}_{p'})$ for all $p' \ge p_n$ with the bound

$$\sup_{u\in \bar B_{r_n}(\mathtt{w}_{p'})} \big|\Psi_n(u) - \Psi_{n-1}(u)\big|_{p'} \le \rho 2^{-2n-3}. \tag{8.8}$$

(2) *We set $\mathcal{L}_0 := 0$, and for $i = 1,\dots,n$,*

$$\begin{aligned} &\mathcal{L}_i + \mathrm{Id} := e^{\{S_{i-1},\cdot\}}(\mathcal{L}_{i-1} + \mathrm{Id}),\\ &\Lambda_i := \Lambda_{i-1} - \bar\Lambda_{i-1}, \qquad H_i = e^{\{S_{i-1},\cdot\}}H_{i-1}, \end{aligned} \tag{8.9}$$

where $\Lambda_i \in \mathcal{H}^{0,\mathcal{K}}_{r_i,p_i}$ are free parameters and $\mathcal{L}_i : \mathcal{H}^{0,\mathcal{K}}(\mathbb{C}) \to \mathcal{H}_{r_i,p_i}(\mathbb{C})$ are bounded linear operators. Note that Λ_i are free parameters, while $\bar\Lambda_i$ are given functions of (ω, I). We have

$$H_i = D + G_i + (\mathrm{Id} + \mathcal{L}_i)\Lambda_i, \quad G_i, \in \mathcal{H}_{r_i,p_i}(\mathbb{C}). \tag{8.10}$$

Setting for $i = 0,\dots,n$,

$$\begin{aligned} &\varepsilon_i := \gamma^{-1}\big(\|G_i^{(0,\mathcal{K})}\|_\infty + \|G_i^{(0,\mathcal{R})}\|_{r_i,p_i} + \|G_i^{(-2)}\|_{r_i,p_i} + \|G_i^{(-1)}\|_{r_i,p_i}\big),\\ &\Theta_i := \gamma^{-1}\|G_i^{\ge 1}\|_{r_i,p_i} + \varepsilon_i, \end{aligned} \tag{8.11}$$

we have

$$\varepsilon_i \le \varepsilon_0 e^{-\chi^i + 1}, \qquad \Theta_i \le \Theta_0 \sum_{j=0}^{i} 2^{-j}, \tag{8.12}$$

$$\left\|(\mathcal{L}_i - \mathcal{L}_{i-1})h\right\|_{r_i,p_i} \leq \mathrm{K}\varepsilon_0(1+\Theta_0)^2 2^{-i}\|h\|_\infty, \tag{8.13}$$

$$\|\mathcal{L}_i h\|_{r_i,p_i} \leq \mathrm{K}(1+\Theta_0)^2\varepsilon_0 \sum_{j=1}^{i} 2^{-j}\|h\|_\infty,$$

for all $h \in \mathcal{H}^{0,\mathcal{K}}(\mathbb{C})$. Finally, the counterterms satisfy the bound

$$\|\bar{\Lambda}_{i-1}\|_\infty \leq \gamma \mathrm{K}\varepsilon_{i-1}(1+\Theta_0)^2, \quad i = 1,\dots,n. \tag{8.14}$$

Proof of Theorem 5
Starting from the Hamiltonian H satisfying (6.2), we set $G_0 = H - D$ in (8.3). The smallness conditions (8.5) are met, provided that we choose $\bar{\epsilon}$ and $\bar{C}$ appropriately.

Using (8.8), we define Ψ as the limit of the Ψ_n (which define a Cauchy sequence) and $\Lambda = \Lambda_0 = \sum_j \bar{\Lambda}_j < \infty$. Note that the series is summable by (8.14). (For more details, see [8, Section 6].) □

Proof of the iterative lemma
We start with a Hamiltonian $H_0 = D + \Lambda_0 + G_0$ with $\Lambda_0 \in \mathcal{H}^{0,\mathcal{K}}$ and D.

At the nth step, we have an expression of the form

$$H_n = D + (\mathrm{Id} + \mathcal{L}_n)\Lambda_n + G_n,$$

with $G_n \in \mathcal{H}_{r_n,p_n}$.

To proceed to the step $n+1$, we apply the change of variables $e^{\{S_n,\cdot\}}$. The generating function S_n and the counterterm $\bar{\Lambda}_n$ are fixed as the unique solutions of the homological equation

$$\Pi^{\leq 0}\big(\{S_n, D + G_n^{\geq 1}\} + (\mathrm{Id} + \mathcal{L}_n)\bar{\Lambda}_n + G_n\big) = G_n^{(-2,\mathcal{K})}, \tag{8.15}$$

recalling that $G_n^{(-1,\mathcal{K})} = 0$ by (5.6). This equation can be written componentwise as a triangular system and consequently solved. Indeed, we have

$$\{S_n^{(-2)}, D\} + \Pi^{-2,\mathcal{R}}\mathcal{L}_n\bar{\Lambda}_n + G_n^{(-2,\mathcal{R})} = 0, \tag{8.16}$$

$$\{S_n^{(-1)}, D\} + \Pi^{-1}\{S_n^{(-2)}, G_n^{\geq 1}\} + \Pi^{-1}\mathcal{L}_n\bar{\Lambda}_n + G_n^{(-1)} = 0, \tag{8.17}$$

$$\Pi^{0,\mathcal{K}}\{S_n^{(-2)} + S_n^{(-1)}, G_n^{\geq 1}\} + \bar{\Lambda}_n + \Pi^{0,\mathcal{K}}\mathcal{L}_n\bar{\Lambda}_n + G_n^{(0,\mathcal{K})} = 0, \tag{8.18}$$

$$\{S_n^{(0,\mathcal{R})}, D\} + \Pi^{0,\mathcal{R}}\{S_n^{(-2)} + S_n^{(-1)}, G_n^{\geq 1}\} + \Pi^{0,\mathcal{R}}\mathcal{L}_n\bar{\Lambda}_n + G_n^{(0,\mathcal{R})} = 0. \tag{8.19}$$

We start by solving the equations for S_n "modulo $\bar{\Lambda}_n$," and we then determine the counterterm by inversion of an appropriate linear operator resulting from inserting the equations for S_n into equation (8.18).

We hence have by Proposition 7.1,

$$
\begin{aligned}
S_n^{(-2)} &= L^{-1}(\Pi^{-2}\mathcal{L}_n\bar{\Lambda}_n + G_n^{(-2)}),\\
S_n^{(-1)} &= L^{-1}\big(\Pi^{-1}\{S_n^{(-2)}, G_n^{\geq 1}\} + \Pi^{-1}\mathcal{L}_n\bar{\Lambda}_n + G_n^{(-1)}\big),\\
S_n^{(0,\mathcal{R})} &= L^{-1}\big(\Pi^{0,\mathcal{R}}\{S_n^{(-2)} + S_n^{(-1)}, G_n^{\geq 1}\} + \Pi^{0,\mathcal{R}}\mathcal{L}_n\bar{\Lambda}_n + G_n^{(0,\mathcal{R})}\big).
\end{aligned}
\tag{8.20}
$$

Plugging them into (8.18), we thus get

$$
\begin{aligned}
&\Pi^{0,\mathcal{K}}\{L^{-1}\big(\Pi^{\leq -1}\mathcal{L}_n\bar{\Lambda}_n + \Pi^{-1}\{L^{-1}\Pi^{-2}\mathcal{L}_n\bar{\Lambda}_n, G_n^{\geq 1}\}\big), G_n^{\geq 1}\}\\
&\qquad + \bar{\Lambda}_n + \Pi^{0,\mathcal{K}}\mathcal{L}_n\bar{\Lambda}_n\\
&\qquad = -\Pi^{0,\mathcal{K}}\{L^{-1}\big(G_n^{\leq -1} + \Pi^{-1}\{L^{-1}G_n^{(-2)}, G_n^{\geq 1}\}\big), G_n^{\geq 1}\} - G_n^{(0,\mathcal{K})}.
\end{aligned}
$$

Note that the left-hand side of the equation above can be written as $(\mathrm{Id} + M_n)\bar{\Lambda}_n$, where $M_n : \mathcal{H}^{0,\mathcal{K}} \to \mathcal{H}^{0,\mathcal{K}}$ is

$$
\begin{aligned}
M_n h := {} & \Pi^{0,\mathcal{K}}\{L^{-1}\big(\Pi^{\leq -1}\mathcal{L}_n h + \Pi^{-1}\{L^{-1}\Pi^{-2}\mathcal{L}_n h, G_n^{\geq 1}\}\big), G_n^{\geq 1}\}\\
& + \Pi^{0,\mathcal{K}}\mathcal{L}_n h.
\end{aligned}
\tag{8.21}
$$

Similarly to Lemma 6.2 of [8], one has

$$
\|M_n h\|_\infty \leq \|h\|_\infty/2. \tag{8.22}
$$

In order to prove (8.22), we treat the three summands of M_n separately. We recall that by (5.13), (8.13), and by the smallness condition in (8.5),

$$
\|\Pi^{0,\mathcal{K}}\mathcal{L}_n h\|_\infty \leq 3\|\mathcal{L}_n h\|_{r_n,p_n} \leq 3\mathrm{K}(1+\Theta)^2\varepsilon_0 \sum_{j=1}^{n} 2^{-j}\|h\|_\infty < \frac{1}{6}\|h\|_\infty.
$$

In the other summands, we use the identification $\|\Pi^{0,\mathcal{K}}F\|_\infty \leq 3\|F\|_{r',p'}$ for any r', p' (see formulas (5.7) and (5.13)) such that $r' \geq r\sqrt{2}$ and $p' \leq p$.

We have by (4.15), (5.13), and Propositions 4.1 and 7.1 that

$$
\begin{aligned}
&\big\|\Pi^{0,\mathcal{K}}\{\Pi^{\leq -1}L_\omega^{-1}\mathcal{L}_n h, G_n^{\geq 1}\}\big\|_\infty\\
&\quad \leq 3\big\|\{\Pi^{\leq -1}L_\omega^{-1}\mathcal{L}_n h, G_n^{\geq 1}\}\big\|_{\sqrt{2}r,p}\\
&\quad \overset{(4.14),(8.2)}{\leq} 120\|\Pi^{\leq -1}L_\omega^{-1}\mathcal{L}_n h\|_{r_n,p}\|G_n^{\geq 1}\|_{r_n,p} \overset{(8.11)}{\leq} 240\gamma\|L_\omega^{-1}\mathcal{L}_n h\|_{r_n,p}\Theta_n\\
&\quad \overset{(6.1),(8.2),(7.3)}{\leq} 240\exp\Big(\exp\Big(\frac{c}{(3\delta)^{1/\eta}}\Big)\Big)\|\mathcal{L}_n h\|_{r_n,p_n}\Theta_n\\
&\quad \overset{(8.5),(8.13)}{\leq} \frac{1}{6}\mathrm{K}^2\varepsilon_0(1+\Theta_0)^3\|h\|_\infty \overset{(8.5)}{\leq} \frac{1}{6}\|h\|_\infty,
\end{aligned}
\tag{8.23}
$$

where, in order to control the exponential term, we used the definition of $\mathtt{K}$ given in (8.5).

Now we estimate the remaining term in (8.21). We have, again by (4.15), (5.13), (8.11), Proposition 4.1, and Lemma (7.1),

$$\begin{aligned}
&\big\|\Pi^{0,\mathcal{K}}\big\{L^{-1}\Pi^{-1}\{L^{-1}\Pi^{-2}\mathcal{L}_n h, G_n^{\geq 1}\}, G_n^{\geq 1}\big\}\big\|_\infty \\
&\quad \leq 3\big\|\big\{L^{-1}\Pi^{-1}\{L^{-1}\Pi^{-2}\mathcal{L}_n h, G_n^{\geq 1}\}, G_n^{\geq 1}\big\}\big\|_{\sqrt{2}r,p} \\
&\quad \overset{(4.14),(8.2)}{\leq} 400\big\|L^{-1}\Pi^{-1}\{L^{-1}\Pi^{-2}\mathcal{L}_n h, G_n^{\geq 1}\}\big\|_{3r/2,p}\|G_n^{\geq 1}\|_{3r/2,p} \\
&\quad \overset{(8.11)}{\leq} 400\gamma\big\|L^{-1}\Pi^{-1}\{L^{-1}\Pi^{-2}\mathcal{L}_n h, G_n^{\geq 1}\}\big\|_{3r/2,p}\Theta_n \\
&\quad \overset{(5.13),(7.3)}{\leq} 400\exp\Big(c\exp\Big(\Big(\frac{4}{3\delta}\Big)^{1/\eta}\Big)\Big)\big\|\{L^{-1}\Pi^{-2}\mathcal{L}_n h, G_n^{\geq 1}\}\big\|_{3r/2,(p+p_n)/2}\Theta_n \\
&\quad \overset{(4.14),(8.11)}{\leq} 2^{14}\gamma\exp\Big(c\exp\Big(\Big(\frac{4}{3\delta}\Big)^{1/\eta}\Big)\Big)\|L^{-1}\Pi^{-2}\mathcal{L}_n h\|_{r_n,(p+p_n)/2}\Theta_n^2 \\
&\quad \overset{(5.13),(7.3)}{\leq} 2^{14}\exp\Big(2c\exp\Big(\Big(\frac{4}{3\delta}\Big)^{1/\eta}\Big)\Big)\|\mathcal{L}_n h\|_{r_n,p_n}\Theta_n^2 \\
&\overset{(8.5),(8.13)}{\leq} \frac{1}{6}\mathtt{K}^2\varepsilon_0(1+\Theta_0)^4\|h\|_\infty \overset{(8.5)}{\leq} \frac{1}{6}\|h\|_\infty,
\end{aligned}$$

noting that by (6.1) and (8.2)

$$r_n - \frac{3}{2}r \geq r_\infty - \frac{3}{2}r \geq \frac{r}{4}, \qquad \frac{p-p_n}{2} \geq \frac{p-p_\infty}{2} \geq \frac{3}{2}\delta,$$

and estimating the exponential term as in (8.23). This concludes the proof of (8.22).

Then we have that

$$\begin{aligned}
\bar{\Lambda}_n = &-(\mathrm{Id}+M_n)^{-1}\Pi^{0,\mathcal{K}}\big\{L^{-1}\big(G_n^{\leq -1}\Pi^{-1}\{L^{-1}G_n^{(-2)}, G_n^{\geq 1}\}\big), G_n^{\geq 1}\big\} \\
&- (\mathrm{Id}+M_n)^{-1}G_n^{(0,\mathcal{K})}
\end{aligned} \tag{8.24}$$

is well defined. To prove (8.14), we split (8.24) into three terms and note that by (8.22) the operator norm of $(\mathrm{Id}+M_n)^{-1}$ is bounded by 2. Regarding the first term, we obtain

$$\begin{aligned}
&\big\|(\mathrm{Id}+M_n)^{-1}\Pi^{0,\mathcal{K}}\{L_\omega^{-1}G_n^{\leq -1}, G_n^{\geq 1}\}\big\|_\infty \\
&\quad \leq 6\big\|\{L_\omega^{-1}G_n^{\leq -1}, G_n^{\geq 1}\}\big\|_{\sqrt{2}r,p} \\
&\quad \overset{(4.14),(8.2)}{\leq} 240\|L_\omega^{-1}G_n^{\leq -1}\|_{r_n,p}\|G_n^{\geq 1}\|_{r_n,p} \overset{(8.11)}{\leq} 240\gamma\|L_\omega^{-1}G_n^{\leq -1}\|_{r_n,p}\Theta_n \\
&\quad \overset{(6.1),(8.2),(7.3)}{\leq} 240\exp\Big(c\exp\Big(\Big(\frac{2}{3\delta}\Big)^{1/\eta}\Big)\Big)\|G_n^{\leq -1}\|_{r_n,p_n}\Theta_n \\
&\quad \overset{(8.5),(8.11),(8.13)}{\leq} \frac{1}{4}\mathtt{K}\gamma\varepsilon_n(1+\Theta_0),
\end{aligned}$$

taking $\mathfrak{C}$ large enough here and below. Regarding the second term, we get

$$
\begin{aligned}
&\big\|(\mathrm{Id}+M_n)^{-1}\Pi^{0,\mathcal{K}}\big\{L^{-1}\Pi^{-1}\{L^{-1}G_n^{(-2)},G_n^{\geq 1}\},G_n^{\geq 1}\big\}\big\|_\infty \\
&\quad\leq 6\big\|\big\{L^{-1}\Pi^{-1}\{L^{-1}G_n^{(-2)},G_n^{\geq 1}\},G_n^{\geq 1}\big\}\big\|_{\sqrt{2}r,p} \\
&\quad\overset{(4.14),(8.2)}{\leq} 800\big\|L^{-1}\Pi^{-1}\{L^{-1}G_n^{(-2)},G_n^{\geq 1}\}\big\|_{3r/2,p}\|G_n^{\geq 1}\|_{3r/2,p} \\
&\quad\overset{(8.11)}{\leq} 800\gamma\big\|L^{-1}\Pi^{-1}\{L^{-1}G_n^{(-2)},G_n^{\geq 1}\}\big\|_{3r/2,p}\Theta_n \\
&\quad\overset{(5.13),(7.3)}{\leq} 800\exp\Big(\exp\Big(c\Big(\frac{2}{3\delta}\Big)^{1/\eta}\Big)\Big)\big\|\{L^{-1}G_n^{(-2)},G_n^{\geq 1}\}\big\|_{3r/2,(p+p_n)/2}\Theta_n \\
&\quad\overset{(4.14),(8.11)}{\leq} 2^{15}\gamma\exp\Big(\exp\Big(c\Big(\frac{2}{3\delta}\Big)^{1/\eta}\Big)\Big)\|L^{-1}G_n^{(-2)}\|_{r_n,(p+p_n)/2}\Theta_n^2 \\
&\quad\overset{(5.13),(7.3)}{\leq} 2^{15}\exp\Big(2\exp\Big(c\Big(\frac{2}{3\delta}\Big)^{1/\eta}\Big)\Big)\|G_n^{(-2)}\|_{r_n,p_n}\Theta_n^2 \\
&\quad\overset{(8.5),(8.11),(8.13)}{\leq} \frac{1}{4}\mathrm{K}\gamma\varepsilon_n(1+\Theta_0)^2.
\end{aligned}
$$

Finally, for the third term we get

$$
\big\|(\mathrm{Id}+M_n)^{-1}G_n^{0,\mathcal{K}}\big\|_\infty \overset{(8.11)}{\leq} 2\gamma\varepsilon_n,
$$

so (8.14) follows.

By substituting in the equations (8.20), we get the final expressions for $S_n^{(-2)}$ and $S_n^{(-1)}$ and finally $S_n^{(0,\mathcal{R})}$, which by (4.14), (7.3), (8.11), (8.13), (8.5), and (8.14) yield the estimates

$$
\begin{aligned}
\|S_n^{(-2)}\|_{r_n,p_n+\delta_n} &\leq \big(1+\mathrm{K}^2(1+\Theta_0)^4\varepsilon_0\big)\mathrm{D}_n\varepsilon_n \leq 2\mathrm{D}_n\varepsilon_n, \\
\|S_n^{(-1)}\|_{r_n-\rho_n,p_n+2\delta_n} &\leq (1+16r_0\rho_n^{-1}\mathrm{D}_n\Theta_0)2\mathrm{D}_n\varepsilon_n, \\
\|S_n^{(0)}\|_{r_n-2\rho_n,p_n+3\delta_n} &\leq (1+16r_0\rho_n^{-1}\mathrm{D}_n\Theta_0)^2 3\mathrm{D}_n\varepsilon_n,
\end{aligned}
\tag{8.25}
$$

where

$$
\mathrm{D}_n := \exp\Big(\exp\Big(\frac{c}{\delta_n^{1/\eta}}\Big)\Big) \leq \exp\Big(\frac{1}{3}\zeta^{n^\xi}\Big)
$$

(ζ and ξ were defined in (8.5)) systematically using the inductive hypothesis and the first bound in (8.5). The final bound thus reads (recall (8.1) and (8.2))

$$
\|S_n\|_{r_n-2\rho_n,p_{n+1}} \leq \frac{r_0^2}{\rho^2}4^{n+8}\exp(\zeta^{n^\xi})\varepsilon_n(1+\Theta_0)^2 \overset{(8.5),(8.12)}{\leq} \frac{\rho}{2^{2n+10}r_0}. \tag{8.26}
$$

Then we can apply Proposition 4.4 since (4.16) is satisfied by S_n with $\rho \to \rho_n$ and $r \to r_{n+1}$. Then Lemma 8.1(1) is easily proved. In particular, (8.6) follows by (4.17) and (8.26) (for complete details, see the analogous proof of [8, Lemma 6.1]).

Regarding Lemma 8.1(2), by construction we have

$$\mathcal{L}_{n+1} - \mathcal{L}_n = (e^{\{S_n,\cdot\}} - \mathrm{Id}) \circ (\mathcal{L}_n + \mathrm{Id});$$

hence by (4.19), (4.16), (8.1), and (5.7),

$$\begin{aligned} &\big\|(\mathcal{L}_{n+1} - \mathcal{L}_n)h\big\|_{r_{n+1},p_{n+1}} \\ &\quad \le \frac{r_0 2^{n+9}}{\rho} \|S_n\|_{r_n-2\rho_n,p_{n+1}} \big\|(\mathcal{L}_n + \mathrm{Id})h\big\|_{r_n-2\rho_n,p_{n+1}} \\ &\quad \overset{(8.26),\,(8.5),(8.13)}{\le} \frac{r_0^3}{\rho^3} 8^{n+9} \exp(\zeta^{n^\xi})\varepsilon_n (1+\Theta_0)^2 \|h\|_\infty, \end{aligned} \tag{8.27}$$

which by (8.5) proves (8.13).

As for the expression of G_{n+1}, by (8.10) and (8.9) we have

$$G_{n+1} = e^{\{S_n,\cdot\}} H_n - \big[D + (\mathrm{Id} + \mathcal{L}_{n+1})\Lambda_{n+1}\big].$$

Since S_n solves the homological equation (8.15), we have by (8.9) that

$$G_{n+1} = G_n^{(-2,\mathcal{K})} + G_n^{\ge 1} + \Pi^{\ge 1}\big(\mathcal{L}_{n+1}\bar{\Lambda}_n + \{S_n, G_n^{\ge 1}\}\big) + G_{n+1,*}, \tag{8.28}$$

$$\begin{aligned} G_{n+1,*} = {}& \{S_n, G_n^{\le 0}\} + \Pi^{\le 0}(\mathcal{L}_{n+1} - \mathcal{L}_n)\bar{\Lambda}_n + \big(e^{\{S_n,\cdot\}} - \mathrm{Id} - \{S_n,\cdot\}\big)G_n \\ & - \sum_{h=2}^{\infty} \frac{(\mathrm{ad}\, S_n)^{h-1}}{h!} \big(\Pi^{\le 0}(\mathrm{Id} + \mathcal{L}_n)\bar{\Lambda}_n + G_n^{\le 0} \\ & + \Pi^{\le 0}\{S_n^{(-1)} + S_n^{(-2)}, G_n^{\ge 1}\}\big), \end{aligned}$$

using that $\Pi^{\le 0}(\{S_n, D\}) = (\{S_n, D\})$ by (5.8) and that $\{S_n, G_n^{(-2,\mathcal{K})}\} = 0$. Note that $G_{n+1,*}$ is quadratic in $S_n \sim G_n^{\le 0}$.

Recalling (8.1), (8.2), (4.14), (5.13), (8.5), (8.11), (8.13), (8.14), and Proposition 4.4, which can be applied by (8.26), we have

$$\begin{aligned} &\|G_{n+1,*}\|_{r_{n+1},p_{n+1}} \\ &\quad \overset{(8.27)}{\le} \frac{2^{n+6} r_0}{\rho} \|S_n\|_{r_n-2\rho_n,p_{n+1}} \gamma \varepsilon_n \\ &\qquad + \frac{r_0^3}{\rho^3} 8^{n+9} \exp(\zeta^{n^\xi})\varepsilon_n (1+\Theta_0)^2 \|\bar{\Lambda}_n\|_\infty + \frac{2^{2n+21} r_0^2}{\rho^2} \|S_n\|^2_{r_n-2\rho_n,p_{n+1}} \gamma \Theta_0 \end{aligned}$$

$$
\begin{aligned}
&+ \frac{2^{n+10} r_0}{\rho} \| \Pi^{\leq 0}(\mathrm{Id} + \mathcal{L}_n)\bar{\Lambda}_n + G_n^{\leq 0} \\
&+ \Pi^{\leq 0}\{S_n^{(-1)} + S_n^{(-2)}, G_n^{\geq 1}\}\|_{r_n - 2\rho_n, p_{n+1}} \|S_n\|_{r_n - 2\rho_n, p_{n+1}} \\
&\overset{(8.26),\,(8.25)}{\leq} 2^{6n+55} \frac{r_0^6}{\rho^6} \exp(2\zeta^{n^\xi}) \varepsilon_n^2 (1+\Theta_0)^5 \gamma \\
&+ \frac{2^{n+24} r_0}{\rho} \Big(\|\bar{\Lambda}_n\|_\infty + \gamma \varepsilon_n + 2^n \frac{r_0^2}{\rho^2} \Theta_0^2 \varepsilon_n \exp(\zeta^{n^\xi})\gamma \Big) \|S_n\|_{r_n - 2\rho_n, p_{n+1}} \\
&\overset{(8.26),\,(8.14)}{\leq} 2^{6n+55} \frac{r_0^6}{\rho^6} \big(\exp(\zeta^{n^\xi}) + \mathrm{K}\big) \exp(\zeta^{n^\xi}) \varepsilon_n^2 (1+\Theta_0)^5 \gamma \\
&\overset{(8.5)}{\leq} 2^{57} \frac{\gamma}{\mathrm{K}^2} \big(\exp(\zeta^{n^\xi}) + 1\big) e^{-\chi^{n+1}/2} \varepsilon_n \leq 2^{60} \frac{\gamma}{\mathrm{K}} e^{-\chi^{n+1}} \varepsilon_0 \leq \gamma e^{-\chi^{n+1}} \varepsilon_0,
\end{aligned}
$$

taking $\mathfrak{C}$ large enough. Recalling (8.11) and (8.28), this implies[40] the first estimates in (8.12).

By the above estimate and recalling (8.11) and (8.28), we get

$$
\begin{aligned}
\Theta_{n+1} &\leq \varepsilon_{n+1} + \Theta_n + e^{-\chi^{n+1}} \varepsilon_0 + \gamma^{-1} \big\| \Pi^{\geq 1}\big(\mathcal{L}_{n+1}\bar{\Lambda}_n + \{S_n, G_n^{\geq 1}\}\big) \big\|_{r_{n+1}, p_{n+1}} \\
&\leq \Theta_n + 4 e^{-\chi^{n+1}} \varepsilon_0 + 2^4 \varepsilon_n + 2^{n+7} r_0 \rho^{-1} \|S_n\|_{r_n - 2\rho_n, p_{n+1}} \Theta_0 \\
&\leq \Theta_n + \Theta_0 2^{-n-1},
\end{aligned}
$$

again by (5.13), (8.5), (8.13), (8.14), (4.14), and (8.26). This finally implies the second estimates in (8.12).

The analyticity of Ψ_n and Λ_n follows by Proposition 7.1, Proposition 5.1(ii), and recalling Remark 4.2. □

9. Measure estimates

LEMMA 9.1 (Measure estimates)
The set $\mathcal{C}$ defined in (2.21) satisfies (2.16), namely

$$
\begin{aligned}
&\mathrm{meas}_{\mathcal{Q}_{\mathcal{S}}} \big(\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C}(V_{\mathcal{S}^c}, I, \gamma)\big) \leq C_0 \gamma, \\
&\mathrm{meas}_{[-1/4,1/4]^{\mathcal{S}}} \big(\mathcal{V}_{\mathcal{S}}\big(\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C}(V_{\mathcal{S}^c}, I, \gamma), V_{\mathcal{S}^c}, I\big)\big) \leq C_0 \gamma.
\end{aligned}
$$

Proof
We start by proving the first estimate in (2.16).

Take $\gamma \leq 1/2$. For $\ell \neq 0$ with $|\ell| < \infty$, $\sum_{s \in \mathcal{S}^c} |\ell_s| \leq 2$, $\pi(\ell) = 0$, and $\mathfrak{m}(\ell) = 0$, set for simplicity $\mathcal{R}_\ell = \mathcal{R}_\ell(V_{\mathcal{S}^c}, I, \gamma)$, and define the resonant set

[40] Recall that the seminorm of the constant term $G_n^{(-2,\mathcal{K})}$ is zero.

$$\mathcal{R}_\ell := \Big\{ \nu \in \mathcal{Q}_{\mathcal{S}} : \big|\omega(\nu, V_{\mathcal{S}^c}, I) \cdot \ell\big| \le \gamma \prod_{s \in \mathcal{S}} \frac{1}{(1 + |\ell_s|^2 \langle i(s)\rangle^2)^{3/2}} =: \gamma \mathtt{d}(\ell) \Big\}. \tag{9.1}$$

Recalling the definition of $\omega(\nu, V_{\mathcal{S}^c}, I)$ in (2.21), we note that by the continuity with respect to the product topology of the functions $\nu \mapsto \Omega_j(\nu, V_{\mathcal{S}^c}, I)$ with $j \in \mathcal{S}^c, I \in \mathcal{I}(p, \mathrm{r})$ (and since $|\ell| < \infty$) we have the crucial fact that the sets $\mathcal{R}_\ell$ are closed (w.r.t. the product topology) and, therefore, *measurable with respect to the product probability measure on* $\mathcal{Q}_{\mathcal{S}}$. Moreover, since $\mathcal{Q}_{\mathcal{S}}$ is compact, all the $\mathcal{R}_\ell$ are compact too.

In the following, we will often omit to write the immaterial dependence on $V_{\mathcal{S}^c}$ and I.

Set $\mathtt{q}(\ell) := \sum_{s \in \mathbb{N}} s^2 \ell_s$. If $|\mathtt{q}(\ell)| \ge |\ell|$; then

$$|\omega \cdot \ell| \ge \Big|\sum_{s \in \mathbb{N}} s^2 \ell_s\Big| - \frac{1}{2}|\ell| \ge \frac{1}{2}|\ell| \ge \frac{1}{2}$$

and $\mathcal{R}_\ell = \emptyset$. Recall the definition of the set $\mathcal{C}$ given in (2.21) and (2.18). Denoting by meas the product probability measure on $\mathcal{Q}_{\mathcal{S}}$, we have

$$\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C} = \bigcup_{\ell \in A} \mathcal{R}_\ell, \quad \text{which implies } \operatorname{meas}(\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C}) \le \sum_{\ell \in A} \operatorname{meas}(\mathcal{R}_\ell), \tag{9.2}$$

where

$$A := \Big\{ \ell \in \mathbb{Z}^{\mathbb{Z}} : 0 < |\ell| < \infty, \sum_{s \in \mathcal{S}^c} |\ell_s| \le 2, \mathtt{m}(\ell) = \pi(\ell) = 0, \big|\mathtt{q}(\ell)\big| < |\ell| \Big\}. \tag{9.3}$$

Fix $\ell \in A$. We note that

$$\exists \bar{s} = \bar{s}(\ell) \in \mathcal{S} \quad \text{(depending only on } \ell) \text{ such that } \ell_{\bar{s}} \ne 0; \tag{9.4}$$

otherwise $0 < |\ell| = \sum_{s \in \mathcal{S}^c} |\ell_s| \le 2$, which contradicts Lemma 7.1. Since $\omega(\nu) = (\nu, \Omega(\nu))$, we get

$$|t|^{-1} \big| \big(\omega(\nu + t e_{\bar{s}}) \cdot \ell\big) - \big(\omega(\nu) \cdot \ell\big) \big| \ge |\ell_{\bar{s}}| - 2 \sup_{s \in \mathcal{S}^c} |\Omega_s|^{\mathrm{lip}} \overset{(2.20)}{\ge} 1 - 2C\varepsilon \ge 1/2,$$

taking $\varepsilon_*(p_*)$ small enough in (2.13). Set $\hat{\nu} = (\nu_s)_{s \ne \bar{s}}$. Then for every $\hat{\nu}$ there exist $a_\ell(\hat{\nu}) = a_\ell(\hat{\nu}, V_{\mathcal{S}^c}, I, \gamma) < b_\ell(\hat{\nu}) = b_\ell(\hat{\nu}, V_{\mathcal{S}^c}, I, \gamma)$ satisfying

$$\big\{ \nu_{\bar{s}} \text{ s.t. } (\nu_{\bar{s}}, \hat{\nu}) \in \mathcal{R}_\ell \big\} \subseteq \big(a_\ell(\hat{\nu}), b_\ell(\hat{\nu})\big), \quad \text{with } b_\ell(\hat{\nu}) - a_\ell(\hat{\nu}) \le 4\gamma \mathtt{d}(\ell), \tag{9.5}$$

which implies

$$\operatorname{meas}\big\{ \nu_{\bar{s}} \text{ s.t. } (\nu_{\bar{s}}, \hat{\nu}) \in \mathcal{R}_\ell \big\} \le 4\gamma \mathtt{d}(\ell).$$

Since $\mathcal{R}_\ell$ is measurable, by Fubini's theorem we have that

$$\text{meas}(\mathcal{R}_\ell) \le 4\gamma \mathtt{d}(\ell) = 4\gamma \prod_{s\in\mathcal{S}} \frac{1}{(1+|\ell_s|^2\langle i(s)\rangle^2)^{3/2}} = 4\gamma \prod_{i\in\mathbb{N}} \frac{1}{(1+|\ell_{s(i)}|^2\langle i\rangle^2)^{3/2}}.$$

Therefore,

$$\text{meas}(\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C}) \le 4\gamma \sum_{\ell\in A}\prod_{i\in\mathbb{N}} \frac{1}{(1+|\ell_{s(i)}|^2\langle i\rangle^2)^{3/2}}. \tag{9.6}$$

We claim that

$$\sum_{\ell\in A}\mathtt{d}(\ell) = \sum_{\ell\in A}\prod_{i\in\mathbb{N}} \frac{1}{(1+|\ell_{s(i)}|^2\langle i\rangle^2)^{3/2}} \le \frac{C_0}{17}, \tag{9.7}$$

taking C_0 large enough.

Given $k\in\mathbb{Z}^{\mathbb{N}}$ with $|k|<\infty$, we define $\ell^k\in\mathbb{Z}^{\mathbb{Z}}$ supported on $\mathcal{S}$ setting

$$\ell^k_s := k_{i(s)}, \quad \text{for } s\in\mathcal{S}, \qquad \ell_s = 0, \quad \text{for } s\notin\mathcal{S}.$$

Now for each $\ell\in A$ there exist unique $k\in\mathbb{Z}^{\mathbb{N}}$ with $|k|<\infty$ and $s_1,s_2\in\mathcal{S}^c$ and $\sigma_1,\sigma_2=\pm1,0$ such that

$$\ell = \ell^k + \sigma_1\mathbf{e}_{s_1} + \sigma_2\mathbf{e}_{s_2}. \tag{9.8}$$

On the other hand, given $k\in\mathbb{Z}^{\mathbb{N}}$ with $|k|<\infty$, there exist at most $36(|k|+2)$ vectors $\ell\in A$ satisfying (9.8). Indeed, we prove that, given $\sigma_1,\sigma_2=\pm1,0$, there exist at most[41] $4(|k|+2)$ couples $(s_1,s_2)\in\mathcal{S}^c\times\mathcal{S}^c$ such that ℓ in (9.8) belongs to A. Indeed, they have to satisfy

$$\begin{cases} \sigma_1 s_1 + \sigma_2 s_2 = -\pi(\ell^k) \\ \sigma_1 s_1^2 + \sigma_2 s_2^2 = -\mathtt{q}(\ell^k) + h, \quad \text{for some } |h|<|\ell|\le|k|+2. \end{cases}$$

Then by (9.6), we get

$$\sum_{\ell\in A}\prod_{i\in\mathbb{N}} \frac{1}{(1+|\ell_{s(i)}|^2\langle i\rangle^2)^{3/2}} \le 36\gamma \sum_{k\in\mathbb{Z}^{\mathbb{N}}:0<|k|<\infty} \big(|k|+2\big)\prod_{i\in\mathbb{N}} \frac{1}{(1+|k_i|^2\langle i\rangle^2)^{3/2}}.$$

Since

$$|k|+2 \le 2\prod_{i\in\mathbb{N}}\big(1+|k_i|^2\langle i\rangle^2\big)^{1/2},$$

[41] Note that there are nine possible choices of $\sigma_1,\sigma_2=\pm1,0$.

we get

$$\sum_{\ell\in A}\prod_{i\in\mathbb{N}}\frac{1}{(1+|\ell_{s(i)}|^2\langle i\rangle^2)^{3/2}}\le 72\gamma\sum_{k\in\mathbb{Z}^{\mathbb{N}}:0<|k|<\infty}\prod_{i\in\mathbb{N}}\frac{1}{1+|k_i|^2\langle i\rangle^2},$$

where the last sum converges (see [13] or [6, Lemma 4.1]). This concludes the proof of (9.7) taking C_0 large enough and, by (9.6), the proof of the first estimate in (2.16).

Let us now prove the second estimate in (2.16). For brevity, we denote by

$$\mathrm{meas}'(E):=\mathrm{meas}_{[-1/4,1/4]^{\mathcal{S}}}\big(E\cap[-1/4,1/4]^{\mathcal{S}}\big) \tag{9.9}$$

the product probability measure on $[-1/4,1/4]^{\mathcal{S}}$. Since $\mathcal{V}_{\mathcal{S}}$ (defined in (2.15)) is Lipschitz $C\varepsilon\gamma$-close to the map $\mathcal{V}^*_{\mathcal{S}}$ (defined in (2.10)), we have that

$$\mathcal{V}_{\mathcal{S}}(\mathcal{Q}_{\mathcal{S}},V_{\mathcal{S}^c},I')\supset[-1/4,1/4]^{\mathcal{S}}$$

for every $V_{\mathcal{S}^c}\in[-1/4,1/4]^{\mathcal{S}^c}$ and[42] $I'\in\mathcal{I}(p,\mathrm{r})$. The function $\mathcal{V}_{\mathcal{S}}(\nu,V_{\mathcal{S}^c},I)$ is invertible (w.r.t. ν). In particular, there exists a function

$$\begin{aligned} g:[-3/8,3/8]^{\mathcal{S}}\times[-1/4,1/4]^{\mathcal{S}^c}\times\mathcal{I}(p,\mathrm{r})&\longrightarrow[-4\bar{C}\gamma\epsilon,4\bar{C}\gamma\epsilon]^{\mathcal{S}},\\ (V_{\mathcal{S}},V_{\mathcal{S}^c},I)&\mapsto g(V_{\mathcal{S}},V_{\mathcal{S}^c},I),\end{aligned} \tag{9.10}$$

which is continuous with respect to the product topology in $[-3/8,3/8]^{\mathcal{S}}\times[-1/4,1/4]^{\mathcal{S}^c}$ and satisfies

$$\begin{gathered}\mathcal{V}_{\mathcal{S}}\big(q+V_{\mathcal{S}}+g(V_{\mathcal{S}},V_{\mathcal{S}^c},I),V_{\mathcal{S}^c},I\big)=V_{\mathcal{S}},\\ \text{where } q=(q_j)_{j\in\mathcal{S}},q_j:=j^2,\end{gathered} \tag{9.11}$$

with Lipschitz estimates

$$\big|g(V'_{\mathcal{S}},V_{\mathcal{S}^c},I)-g(V_{\mathcal{S}},V_{\mathcal{S}^c},I)\big|_{\ell^\infty_{\mathcal{S}}}\le 6\bar{C}\epsilon|V'_{\mathcal{S}}-V_{\mathcal{S}}|_{\ell^\infty_{\mathcal{S}}}. \tag{9.12}$$

Indeed, recalling (6.11), g satisfies the fixed point equation (e.g., by a slightly modified version of Lemma B.1)

$$g=-\lambda\big(q+V_{\mathcal{S}}+g,\Omega(q+V_{\mathcal{S}}+g,V_{\mathcal{S}^c},I),I\big);$$

then (9.10) and (9.12) follow from (6.4)) (recalling (5.12) and (4.9)).

For the remainder of the proof, we drop the dependence on $V_{\mathcal{S}^c}$ and γ. Fix $\ell\in A$. First, we note that for all $I,I'\in\mathcal{I}(p,\mathrm{r})$ the set $\mathcal{V}_{\mathcal{S}}(\mathcal{R}_\ell(I);I')$ is measurable since $\mathcal{R}_\ell(I)$ is closed and $\mathcal{V}_{\mathcal{S}}$ is continuously invertible. Recalling (9.1), we have

[42] In this proof we actually take $I'=I$. We have introduced I' for estimate (9.14) that will be used in the proof of Lemma 3.1.

$$
\begin{aligned}
&\mathcal{V}_{\mathcal{S}}\big(\mathcal{R}_\ell(I), I'\big) \cap [-1/4, 1/4]^{\mathcal{S}} \\
&\quad = \big\{V_{\mathcal{S}} \in [-1/4, 1/4]^{\mathcal{S}} : \exists \nu \in \mathcal{R}_\ell(I), V_{\mathcal{S}} = \mathcal{V}_{\mathcal{S}}(\nu, I')\big\} \\
&\quad \overset{(9.11)}{=} \big\{V_{\mathcal{S}} \in [-1/4, 1/4]^{\mathcal{S}} \text{ s.t. } \big|\omega\big(q + V_{\mathcal{S}} + g(V_{\mathcal{S}}, I'), I\big) \cdot \ell\big| \le \gamma \mathtt{d}(\ell)\big\} =: \mathtt{E}.
\end{aligned}
$$

As usual, we set $\ell =: (\ell_{\mathcal{S}}, \ell_{\mathcal{S}^c})$. Let $\bar{\mathcal{S}} := \{j \in \mathcal{S} : \ell_j \neq 0\}$. We split $\ell_{\mathcal{S}} =: (\ell_{\bar{\mathcal{S}}}, 0)$.

Decomposing $V_{\mathcal{S}} = (V_{\bar{\mathcal{S}}}, V_{\mathcal{S}\setminus\bar{\mathcal{S}}})$, we set

$$
\mathtt{E}(V_{\mathcal{S}\setminus\bar{\mathcal{S}}}) := \big\{V_{\bar{\mathcal{S}}} \in [-1/4, 1/4]^{\bar{\mathcal{S}}} : (V_{\bar{\mathcal{S}}}, V_{\mathcal{S}\setminus\bar{\mathcal{S}}}) \in \mathtt{E}\big\}.
$$

We claim that for every $V_{\mathcal{S}\setminus\bar{\mathcal{S}}} \in [-1/4, 1/4]^{\mathcal{S}\setminus\bar{\mathcal{S}}}$,

$$
\operatorname{meas}_{[-1/4,1/4]^{\bar{\mathcal{S}}}}\big(\mathtt{E}(V_{\mathcal{S}\setminus\bar{\mathcal{S}}})\big) \le 16\gamma \mathtt{d}(\ell). \tag{9.13}
$$

Then by Fubini's theorem we get

$$
\operatorname{meas}_{[-1/4,1/4]^{\mathcal{S}}}(\mathtt{E}) = \operatorname{meas}'\big(\mathcal{V}_{\mathcal{S}}\big(\mathcal{R}_\ell(I), I'\big)\big) \le 16\gamma \mathtt{d}(\ell). \tag{9.14}
$$

Then

$$
\operatorname{meas}_{[-1/4,1/4]^{\mathcal{S}}}\big(\mathcal{V}_{\mathcal{S}}\big(\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C}(I), I'\big)\big) \le \sum_{\ell \in A} 16\gamma \mathtt{d}(\ell),
$$

and the second estimate in (2.16) follows by (9.7) and taking $I' = I$.

Let us finally prove the claim in (9.13). Fix $V_{\mathcal{S}\setminus\bar{\mathcal{S}}} \in [-1/4, 1/4]^{\mathcal{S}\setminus\bar{\mathcal{S}}}$. We set for brevity[43]

$$
\begin{aligned}
f(V_{\bar{\mathcal{S}}}) &:= \omega\big(q + V_{\mathcal{S}} + g(V_{\mathcal{S}}, I'), I\big) \cdot \ell \\
&\overset{(2.21)}{=} V_{\bar{\mathcal{S}}} \cdot \ell_{\bar{\mathcal{S}}} + \big(q + g(V_{\mathcal{S}}, I')\big) \cdot \ell_{\mathcal{S}} + \Omega\big(q + V_{\mathcal{S}} + g(V_{\mathcal{S}}, I'), I\big) \cdot \ell_{\mathcal{S}^c}.
\end{aligned}
$$

The aim now is to prove that the increment of f *in a suitable direction* is bounded from below. Since $\ell \in A$ by (9.4), we get $|\ell_{\mathcal{S}}| \ge 1$ and $|\ell_{\mathcal{S}^c}| \le 2$. Define $\sigma_{\bar{\mathcal{S}}}$ by $\sigma_{\bar{\mathcal{S}},j} = \operatorname{sign} \ell_j$ for $j \in \bar{\mathcal{S}}$; then $\sigma_{\bar{\mathcal{S}}} \cdot \ell_{\bar{\mathcal{S}}} = |\ell_{\bar{\mathcal{S}}}| = |\ell_{\mathcal{S}}| \ge 1$ and $|\sigma_{\bar{\mathcal{S}}}|_\infty = 1$. Then we get

$$
|t|^{-1}\big|f(V_{\bar{\mathcal{S}}} + t\sigma_{\bar{\mathcal{S}}}) - f(V_{\bar{\mathcal{S}}})\big| \ge |\ell_{\mathcal{S}}| - 16\bar{C}\epsilon|\ell_{\mathcal{S}}| \ge |\ell_{\mathcal{S}}|/2,
$$

by (9.12) and (6.10) and taking $\varepsilon_*(p_*)$ in (2.13) small enough such that $16\bar{C}\epsilon \le 16\bar{C}c(p_*)\varepsilon \le 1/2$ (recall (6.8), (6.7)). Then we get that

$$
\operatorname{meas}\big\{t \in \mathbb{R} : V_{\bar{\mathcal{S}}} + t\sigma_{\bar{\mathcal{S}}} \in \mathtt{E}(V_{\mathcal{S}\setminus\bar{\mathcal{S}}})\big\} \le 4\gamma \mathtt{d}(\ell)/|\ell_{\mathcal{S}}|.
$$

We need the following (finite-dimensional) result proved in the appendix.

[43] Recall (2.21) and (9.11).

LEMMA 9.2
Let $E \subset [-1/4, 1/4]^n$ *be a measurable set. Let* $\xi = (\hat{\xi}, \pm 1)$ *with* $\hat{\xi} \in \mathbb{R}^{n-1}$. *Assume that for every* $x \in [-1/4, 1/4]^n$ *we have*[44] $\mathrm{meas}_{\mathbb{R}}\{t \in \mathbb{R} : x + t\xi \in E\} \le \delta$. *Then* $\mathrm{meas}_{\mathbb{R}^n}(E) \le 2^{1-n}\delta|\xi|_2^2$, *where* $|\cdot|_2$ *denotes the Euclidean norm.*

Then (9.13) follows by the above lemma with[45]

$$E = \mathbb{E}(V_{\mathcal{S}\setminus\bar{\mathcal{S}}}), \qquad n = \sharp\bar{\mathcal{S}}, \qquad \xi = \sigma_{\bar{\mathcal{S}}}, \qquad x = V_{\bar{\mathcal{S}}}, \qquad \delta = 4\gamma \mathtt{d}(\ell)/|\ell_{\mathcal{S}}|$$

and noting that $|\sigma_{\mathcal{S}}|_2^2 = n \le |\ell_{\mathcal{S}}|$ and $\mathrm{meas}_{[-1/4,1/4]^n}(A) = 2^n \mathrm{meas}_{\mathbb{R}^n}(A)$ for every $A \subseteq [-1/4, 1/4]^n$. □

Proof of Corollary 2.1
It is equivalent to prove that

$$\begin{aligned}\mathcal{B}(I,\gamma) := \big\{V \equiv (V_{\mathcal{S}}, V_{\mathcal{S}^c}) \in [-1/4, 1/4]^{\mathbb{Z}} :\\ V_{\mathcal{S}} \in \mathcal{V}_{\mathcal{S}}\big(\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C}(V_{\mathcal{S}^c}, I, \gamma), V_{\mathcal{S}^c}, I\big)\big\}\end{aligned} \tag{9.15}$$

is a Borel set in $[-1/4, 1/4]^{\mathbb{Z}}$ with measure bounded by $C_0\gamma$ (C_0 is the constant in (2.16)).

The function $h : [-1/4, 1/4]^{\mathbb{Z}} \to \mathcal{Q}_{\mathcal{S}} \times [-1/4, 1/4]^{\mathcal{S}^c}$ defined by[46]

$$h(V) := \big(q + V_{\mathcal{S}} + g(V_{\mathcal{S}}, V_{\mathcal{S}^c}, I), V_{\mathcal{S}^c}\big)$$

is continuous with respect to the product topology. Then, for every $\ell \in A$ (recall (9.3)), the set

$$\begin{aligned}\mathcal{B}_\ell(I,\gamma) &:= \big\{V \equiv (V_{\mathcal{S}}, V_{\mathcal{S}^c}) \in [-1/4, 1/4]^{\mathbb{Z}} : \big|\omega\big(h(V)\big)\cdot \ell\big| \le \gamma \mathtt{d}(\ell)\big\}\\ &\overset{(9.1),(9.11)}{=} \big\{V \equiv (V_{\mathcal{S}}, V_{\mathcal{S}^c}) \in [-1/4, 1/4]^{\mathbb{Z}} : V_{\mathcal{S}} \in \mathcal{V}_{\mathcal{S}}\big(\mathcal{R}_\ell(V_{\mathcal{S}^c}, I), V_{\mathcal{S}^c}, I\big)\big\}\end{aligned}$$

is closed and, therefore, measurable. Then by Fubini's theorem and (9.14) (with $I = I'$) we get $\mathrm{meas}_{[-1/4,1/4]^{\mathbb{Z}}}(\mathcal{B}_\ell) \le 16\gamma \mathtt{d}(\ell)$. Since $\mathcal{B}$ defined in (9.15) can be written as $\mathcal{B} = \bigcup_{\ell\in A} \mathcal{B}_\ell$ (recall (9.2)), by (9.7) we get $\mathrm{meas}_{[-1/4,1/4]^{\mathbb{Z}}}(\mathcal{B}) \le C_0\gamma$. □

Proof of Lemma 3.1
The first estimate in (3.1) is equivalent to

$$\mathrm{meas}_{[-1/4,1/4]^{\mathcal{S}}}\big(\mathcal{V}_{\mathcal{S}}\big(\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C}'(V_{\mathcal{S}^c}, I, \gamma), V_{\mathcal{S}^c}, I\big)\big) \le C_0\gamma, \tag{9.16}$$

where C_0 was defined in Theorem 3.

[44] $\mathrm{meas}_{\mathbb{R}}$ (resp., $\mathrm{meas}_{\mathbb{R}^n}$) being the standard measure in $\mathbb{R}$ (resp., $\mathbb{R}^n$).
[45] Here we consider the case when $\bar{\ell}_{\mathcal{S}} = -1$; the $+1$ case is analogous.
[46] g was defined in (9.10). By (9.11), h is the inverse of the function $(v, V_{\mathcal{S}^c}) \mapsto (\mathcal{V}_{\mathcal{S}}(v, V_{\mathcal{S}^c}), V_{\mathcal{S}^c})$.

In the following, we drop the dependence on $V_{\mathcal{S}^c}$. We first note that recalling (9.1), (9.3) and setting $L := C\mathtt{r}^{-2}\epsilon$, we get

$$|I - I'|_{2p_*} \leq \mathtt{d}(\ell)/4L \quad \Longrightarrow \quad \mathcal{R}_\ell(I', \gamma/2) \subset \mathcal{R}_\ell(I, \gamma), \quad \forall \ell \in A. \tag{9.17}$$

Indeed, since $\ell \in A$,

$$\begin{aligned} \big|\omega(\nu, I) \cdot \ell\big| &\overset{(2.20)}{\leq} \big|\omega(\nu, I') \cdot \ell\big| + 2\gamma L |I - I'|_{2p_*} \\ &\leq \gamma \mathtt{d}(\ell)/2 + 2\gamma L|I - I'|_{2p_*} \leq \gamma \mathtt{d}(\ell). \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathcal{Q}_{\mathcal{S}} \setminus \big(\mathcal{C}(I, \gamma) \cap \mathcal{C}(I', \gamma/2)\big) \\ &\quad = \big(\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C}(I, \gamma)\big) \bigcup \big(\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C}(I', \gamma/2)\big) \\ &\quad \overset{(9.2),(9.17)}{=} \bigcup_{\ell \in A} \mathcal{R}_\ell(I, \gamma) \bigcup_{\substack{\ell \in A, \\ \mathtt{d}(\ell) < 4L|I - I'|_{2p_*}}} \mathcal{R}_\ell(I', \gamma/2). \end{aligned}$$

Then

$$\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C}'(I, \gamma) := \bigcup_{\ell \in A} \mathcal{R}_\ell(I, \gamma) \bigcup_{k \in \mathbb{N}} \bigcup_{\substack{\ell \in A, \\ \mathtt{d}(\ell) < 4L|I - I^{(n_k)}|_{2p_*}}} \mathcal{R}_\ell(I^{(n_k)}, \gamma/2)$$

and (recalling the notation in (9.9))

$$\begin{aligned} &\mathrm{meas}'\big(\mathcal{V}_{\mathcal{S}}\big(\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C}'(I, \gamma); I\big)\big) \\ &\quad \leq \sum_{\ell \in A} \mathrm{meas}'\big(\mathcal{V}_{\mathcal{S}}\big(\mathcal{R}_\ell(I, \gamma); I\big)\big) \\ &\qquad + \sum_{k \in \mathbb{N}} \sum_{\substack{\ell \in A, \\ \mathtt{d}(\ell) < 4L|I - I^{(n_k)}|_{2p_*}}} \mathrm{meas}'\big(\mathcal{V}_{\mathcal{S}}\big(\mathcal{R}_\ell(I^{(n_k)}, \gamma); I\big)\big) \\ &\quad \overset{(9.14)}{\leq} 16\gamma \sum_{\ell \in A} \mathtt{d}(\ell) + 16\gamma \sum_{k \in \mathbb{N}} \sum_{\substack{\ell \in A, \\ \mathtt{d}(\ell) < 4L|I - I^{(n_k)}|_{2p_*}}} \mathtt{d}(\ell) \leq 17\gamma \sum_{\ell \in A} \mathtt{d}(\ell) \overset{(9.7)}{\leq} C\gamma, \end{aligned}$$

taking the subsequence $(n_k)_{k \in \mathbb{N}}$ growing fast enough.[47] This proves (9.16). The second estimate in (3.1) follows from the first one and Fubini's theorem provided that $\mathcal{G}(I, \gamma)$ is a Borel set. This holds true since

[47] Since the positive term series in (9.7) converges.

$$\begin{aligned}\mathcal{G}(I,\gamma) = \bigcap_k \big\{V = (V_{\mathcal{S}}, V_{\mathcal{S}^c}) \in [1/4, 1/4]^{\mathbb{Z}} : V_{\mathcal{S}} \in \mathcal{V}_{\mathcal{S}}\big(\mathcal{C}(I^{(n_k)}, \gamma/2), V_{\mathcal{S}^c}, I^{(n_k)}\big)\big\} \\ \cap \big\{V = (V_{\mathcal{S}}, V_{\mathcal{S}^c}) \in [1/4, 1/4]^{\mathbb{Z}} \text{ s.t. } V_{\mathcal{S}} \in \mathcal{V}_{\mathcal{S}}\big(\mathcal{C}(I,\gamma), V_{\mathcal{S}^c}, I\big)\big\} \\ = [1/4, 1/4]^{\mathbb{Z}} \setminus \Big(\bigcup_k \mathcal{B}(I^{(n_k)}, \gamma/2) \cup \mathcal{B}(I,\gamma)\Big).\end{aligned}$$

The Borel sets $\mathcal{B}(I^{(n_k)}, \gamma/2), \mathcal{B}(I,\gamma)$ are defined in (9.15). □

Appendix A. Technicalities

Proof of Lemma 4.1
Let us consider first the case $f \in \mathrm{F}(\mathcal{O} \times B_\rho)$, namely when f depends only on a finite number of ω_j's. By definition, this means that for some $k \in \mathbb{N}$ there exists a function $\hat{f} : P_k(\mathcal{O}) \times B_\rho \to \mathbb{C}$, where P_k is the projection $P_k : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{2k+1}$ defined as $P_k(\omega) := (\omega_{-k}, \dots, \omega_k)$, such that $f(\omega, I) = \hat{f}(\omega_{-k}, \dots, \omega_k, I)$ for every $(\omega, I) \in \mathcal{O} \times B_\rho$. By Cauchy estimates,

$$\big|\hat{f}(\omega, I) - \hat{f}(\omega, I')\big| \le 2\rho^{-1}|f|^{\gamma,\mathcal{O}\times B_\rho}|I - I'|_E, \quad \forall \omega \in \mathcal{O}, I, I' \in B_{\rho/2}.$$

We need the following.

LEMMA A.1 (Lipschitz extension)
Let X be a metric space endowed with the metric $d(\cdot,\cdot)$, and let $\emptyset \ne U \subseteq X$.

(i) Let $f : U \to \mathbb{R}$ be an L-Lipschitz function, namely $|f(u) - f(v)| \le L\, d(u,v)$ for every $u, v \in U$. Then $\bar{f}(x) := \inf_{u\in U} f(u) + L\, d(x,u)$, $x \in X$, is an L-Lipschitz extension of f.

(ii) Moreover, if $\sup_U |f| =: M < \infty$, then $\tilde{f} := \max\{-M, \min\{\bar{f}, M\}\}$ is an L-Lipschitz extension of f satisfying $\sup_X |\tilde{f}| = M$.

(iii) Let $g : U \to \ell^\infty(\mathbb{R})$ be an L-Lipschitz function, namely, $|g(u) - g(v)|_{\ell^\infty} \le L\, d(u,v)$ for every $u, v \in U$. Then there exists an L-Lipschitz extension $\bar{g} : X \to \ell^\infty(\mathbb{R})$ of g. An analogous statement holds for every ℓ^∞-weighted space, in particular for w_p.

Before proving Lemma A.1, we conclude the proof of Lemma 4.1. Then Lemma A.1 with $X = P_k(\mathcal{Q}) \times B_{\rho/2}$, distance $d((\omega, I), (\omega', I')) = |\omega - \omega'|_\infty + 2\gamma\rho^{-1}|I - I'|_E$, $U = P_k(\mathcal{O}) \times B_{\rho/2}$, $f = \hat{f}$, $L = \gamma^{-1}|f|^{\gamma,\mathcal{O}\times B_\rho}$ implies Lemma 4.1 in the case $f \in \mathrm{F}(\mathcal{O} \times B_\rho)$. Note in particular that since $P_k(\mathcal{Q})$ is finite-dimensional, the product topology on it coincides with one induced by the norm $|\cdot|_\infty$. Then the extended function $\tilde{f} \in \mathrm{F}(\mathcal{Q} \times B_{\rho/2})$ and its continuity in ω follows by its Lipschitz-continuity in ω. Moreover, in this case, $|\tilde{f}|^{\gamma,\mathcal{Q}\times B_{\rho/2}} = |f|^{\gamma,\mathcal{O}\times B_\rho}$ and

$$\left|\tilde{f}(\omega, I) - \tilde{f}(\omega, I')\right| \le 2\rho^{-1}|f|^{\gamma,\mathcal{O}\times B_\rho}|I - I'|_E, \quad \forall \omega \in \mathcal{Q}, I, I' \in B_{\rho/2}.$$

Consider now the case of a general $f \in \mathcal{F}(\mathcal{O} \times B_\rho)$. By definition, there exists a sequence $f_n \in \mathrm{F}(\mathcal{O} \times B_\rho)$ such that $|f_n - f|^{\gamma,\mathcal{O}\times B_\rho} \le 4^{-n-1}|f|^{\gamma,\mathcal{O}\times B_\rho}$ and $g_0 := f_0$, $g_n := f_n - f_{n-1}$ for $n \ge 1$. Then $f = \sum_{n\ge 0} g_n$. Moreover, $|g_0|^{\gamma,\mathcal{O}\times B_\rho} \le 5/4|f|^{\gamma,\mathcal{O}\times B_\rho}$, $|g_n|^{\gamma,\mathcal{O}\times B_\rho} = |f_n - f|^{\gamma,\mathcal{O}\times B_\rho} + |f - f_{n-1}|^{\gamma,\mathcal{O}\times B_\rho} \le 5 \cdot 4^{-n-1} \times |f|^{\gamma,\mathcal{O}\times B_\rho}$ and $g_n \in \mathrm{F}(\mathcal{O} \times B_\rho)$. Then there exist extensions $\tilde{g}_n \in \mathrm{F}(\mathcal{Q} \times B_{\rho/2})$, $n \ge 0$ such that $\tilde{g}_n = g_n$ on $\mathcal{O} \times B_{\rho/2}$, $|\tilde{g}_n|^{\gamma,\mathcal{Q}\times B_{\rho/2}} \le 5 \cdot 4^{-n-1}|f|^{\gamma,\mathcal{O}\times B_\rho}$, $\tilde{g}_n$ is continuous with respect to the product topology in $\mathcal{O}$, and finally $\tilde{g}_n$ is Lipschitz on $B_{\rho/2}$ with estimate

$$\left|\tilde{g}_n(\omega, I) - \tilde{g}_n(\omega, I')\right| \le 10\rho^{-1}4^{-n-1}|f|^{\gamma,\mathcal{O}\times B_\rho}|I - I'|_E, \quad \forall \omega \in \mathcal{Q}, I, I' \in B_{\rho/2}.$$

Finally, set $\tilde{f} := \sum_{n\ge 0} \tilde{g}_n$. □

Proof of Lemma A.1

(i) It is McShane's theorem. Incidentally, we note that $\bar{f}$ is the greatest possible extension while the smaller one is $\underline{f}(x) := \sup_{u\in U} f(u) - L\, d(x, u)$.

(ii) The fact that $\tilde{f}$ is L-Lipschitz follows applying the following result twice:

Given an L-Lipschitz $g : X \to \mathbb{R}$ and $c \in \mathbb{R}$, the functions $\bar{g} := \max\{g, c\}$ and $\underline{g} := \min\{g, c\}$ are L-Lipschitz. We prove it only for $\bar{g}$, the other case being analogous. We first note that $\bar{g} = (g + c + |g - c|)/2$. Then

$$\begin{aligned}\left|\bar{g}(x) - \bar{g}(y)\right| &\le \frac{1}{2}\big(|g(x) - g(y)| + \big||g(x) - c| - |g(y) - c|\big|\big)\\ &\le \left|g(x) - g(y)\right| \le L\, d(x, y).\end{aligned}$$

(iii) By McShane's theorem in (i), every component $g_j : A \to \mathbb{R}$ of g can be extended to $\tilde{g}_j : X \to \mathbb{R}$ with the same Lipschitz constant L. Then, setting $\tilde{g} := (\tilde{g}_j)_{j\in\mathbb{Z}}$, we have that $\tilde{g} \in \ell^\infty$; in fact, for every fixed $x \in X$, $x_0 \in A$, and $j \in \mathbb{Z}$,

$$\left|\tilde{g}_j(x)\right| \le \left|\tilde{g}_j(x) - \tilde{g}_j(x_0)\right| + \left|g_j(x_0)\right| \le L\, d(x, x_0) + \left\|g(x_0)\right\|_{\ell^\infty}.$$

Finally, for every $x_1, x_2 \in X$ and $j \in \mathbb{Z}$, we have that $|g_j(x_1) - g_j(x_2)| \le L\, d(x, x_0)$ and, finally, $|g(x_1) - g(x_2)|_{\ell^\infty} \le L\, d(x, x_0)$. The extension to $\ell^\infty(\mathbb{C})$ and to every ℓ^∞-weighted space, in particular for $\mathtt{w}_p$, is straightforward. □

Proof of Proposition 4.1.

Writing $F = \sum F_{\boldsymbol{\alpha}',\boldsymbol{\beta}'} u^{\boldsymbol{\alpha}'} \bar{u}^{\boldsymbol{\beta}'}$ and $G = \sum G_{\boldsymbol{\alpha}'',\boldsymbol{\beta}''} u^{\boldsymbol{\alpha}''} \bar{u}^{\boldsymbol{\beta}''}$, we have

$$\begin{aligned}\{F,G\} &= \mathrm{i}\sum_{\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\alpha}'',\boldsymbol{\beta}''} F_{\boldsymbol{\alpha}',\boldsymbol{\beta}'} G_{\boldsymbol{\alpha}'',\boldsymbol{\beta}''} \sum_j (\boldsymbol{\alpha}'_j \boldsymbol{\beta}''_j - \boldsymbol{\beta}'_j \boldsymbol{\alpha}''_j) u^{\boldsymbol{\alpha}'+\boldsymbol{\alpha}''-e_j} \bar{u}^{\boldsymbol{\beta}'+\boldsymbol{\beta}''-e_j} \\ &=: H = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} H_{\boldsymbol{\alpha},\boldsymbol{\beta}} u^{\boldsymbol{\alpha}} \bar{u}^{\boldsymbol{\beta}},\end{aligned}$$

where

$$H_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \mathrm{i}\sum_j \sum_{\substack{\boldsymbol{\alpha}'+\boldsymbol{\alpha}''-e_j=\boldsymbol{\alpha} \\ \boldsymbol{\beta}'+\boldsymbol{\beta}''-e_j=\boldsymbol{\beta}}} F_{\boldsymbol{\alpha}',\boldsymbol{\beta}'} G_{\boldsymbol{\alpha}'',\boldsymbol{\beta}''} (\boldsymbol{\alpha}'_j \boldsymbol{\beta}''_j - \boldsymbol{\beta}'_j \boldsymbol{\alpha}''_j). \tag{A.1}$$

Note that $\{F,G\}(0)=0$. Indeed, $H_{\mathbf{0},\mathbf{0}}=0$ since $\boldsymbol{\alpha}'+\boldsymbol{\alpha}''=\boldsymbol{\beta}'+\boldsymbol{\beta}''=e_j$ (namely, $\boldsymbol{\alpha}'_j+\boldsymbol{\alpha}''_j=\boldsymbol{\beta}'_j+\boldsymbol{\beta}''_j=1$ and $\boldsymbol{\alpha}'_k+\boldsymbol{\alpha}''_k=\boldsymbol{\beta}'_k+\boldsymbol{\beta}''_k=0$ for $k\neq j$) implies that $\boldsymbol{\alpha}'_j\boldsymbol{\beta}''_j-\boldsymbol{\beta}'_j\boldsymbol{\alpha}''_j=0$ by mass conservation $|\boldsymbol{\alpha}|=|\boldsymbol{\beta}|$.

Recalling (4.5), we have

$$\begin{aligned}\big|\{F,G\}\big|_{r,p} \le \sup_\ell \sum_j \sum_{\substack{\boldsymbol{\alpha}',\boldsymbol{\beta}', \\ \boldsymbol{\alpha}'',\boldsymbol{\beta}''}} &|F_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}||G_{\boldsymbol{\alpha}'',\boldsymbol{\beta}''}|(\boldsymbol{\alpha}'_j\boldsymbol{\beta}''_j+\boldsymbol{\alpha}''_j\boldsymbol{\beta}'_j) \\ &\times(\boldsymbol{\beta}'_\ell+\boldsymbol{\beta}''_\ell) u_p^{\boldsymbol{\alpha}'+\boldsymbol{\beta}'-2e_\ell+\boldsymbol{\alpha}''+\boldsymbol{\beta}''-2e_j},\end{aligned} \tag{A.2}$$

where $u_p=u_p(r)$ was defined in (4.6). We split the right-hand side of (A.2) into four terms according to the splitting

$$(\boldsymbol{\alpha}'_j\boldsymbol{\beta}''_j+\boldsymbol{\alpha}''_j\boldsymbol{\beta}'_j)(\boldsymbol{\beta}'_\ell+\boldsymbol{\beta}''_\ell) = \boldsymbol{\alpha}'_j\boldsymbol{\beta}''_j\boldsymbol{\beta}'_\ell + \boldsymbol{\alpha}'_j\boldsymbol{\beta}''_j\boldsymbol{\beta}''_\ell + \boldsymbol{\alpha}''_j\boldsymbol{\beta}'_j\boldsymbol{\beta}'_\ell + \boldsymbol{\alpha}''_j\boldsymbol{\beta}'_j\boldsymbol{\beta}''_\ell;$$

we will consider only the term

$$A := \sup_\ell \sum_j \sum_{\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\alpha}'',\boldsymbol{\beta}''} |F_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}G_{\boldsymbol{\alpha}'',\boldsymbol{\beta}''}|\boldsymbol{\alpha}'_j\boldsymbol{\beta}''_j\boldsymbol{\beta}'_\ell \times u_p^{\boldsymbol{\alpha}'+\boldsymbol{\beta}'-2e_\ell+\boldsymbol{\alpha}''+\boldsymbol{\beta}''-2e_j},$$

the others being analogous. Noting that for all $j\in\mathbb{Z}$,

$$\sum_{\boldsymbol{\alpha}'',\boldsymbol{\beta}''} |G_{\boldsymbol{\alpha}'',\boldsymbol{\beta}''}|\boldsymbol{\beta}''_j u_p^{\boldsymbol{\alpha}''+\boldsymbol{\beta}''-2e_j} \le |G|_{r,p},$$

we have

$$\begin{aligned} A &\le |G|_{r,p} \sup_\ell \sum_{\boldsymbol{\alpha}',\boldsymbol{\beta}'} |F_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}| \sum_j \boldsymbol{\alpha}'_j \boldsymbol{\beta}'_\ell u_p^{\boldsymbol{\alpha}'+\boldsymbol{\beta}'-2e_\ell} \\ &= |G|_{r,p} \sup_\ell \sum_{\boldsymbol{\alpha}',\boldsymbol{\beta}'} |F_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}||\boldsymbol{\alpha}'|\boldsymbol{\beta}'_\ell u_p^{\boldsymbol{\alpha}'+\boldsymbol{\beta}'-2e_\ell} \\ &\le |G|_{r,p} \sup_\ell \sum_{\boldsymbol{\alpha}',\boldsymbol{\beta}'} |F_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}|\boldsymbol{\beta}'_\ell \tilde{u}_p^{\boldsymbol{\alpha}'+\boldsymbol{\beta}'-2e_\ell} |\boldsymbol{\alpha}'+\boldsymbol{\beta}'|\Big(\frac{r}{r+\rho}\Big)^{|\boldsymbol{\alpha}'+\boldsymbol{\beta}'|-2},\end{aligned}$$

where $\tilde{u}_p$ is short for $u_p(r+\rho)$ (recall (4.6)). Since[48]

$$\sup_{|\boldsymbol{\alpha}'|=|\boldsymbol{\beta}'|} |\boldsymbol{\alpha}'+\boldsymbol{\beta}'|\Big(\frac{r}{r+\rho}\Big)^{|\boldsymbol{\alpha}'+\boldsymbol{\beta}'|-2} \le \sup_{x\ge 2} x\Big(\frac{r}{r+\rho}\Big)^{x-2} \le 2\max\Big\{1,\frac{r}{\rho}\Big\},$$

we get

$$A \le 2\max\Big\{1,\frac{r}{\rho}\Big\}|G|_{r,p}\sup_{\ell}\sum_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}|F_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}|\boldsymbol{\beta}'_\ell \tilde{u}_p^{\boldsymbol{\alpha}'+\boldsymbol{\beta}'-2e_\ell},$$

and, recalling the definition in (4.5), this concludes the proof of (4.8).

Remark A.1
Note that Proposition 4.1 and its proof hold for any ℓ^∞-weighted norm, namely with norm $\sup_{j\in\mathbb{Z}} w_j|u_j|$, with $w_j\to\infty$.

Proof of Lemma 4.2
Let us first prove the Banach structure. Consider a Cauchy sequence of Hamiltonians $H^{(n)}\in\mathcal{H}_{r,p}^{\mathcal{O},0}$. For all $\omega\in\mathcal{O}$, $I\in\mathcal{J}$, $H^{(n)}(\omega,I)$ is a Cauchy sequence in $\mathcal{H}^0_{r,p}$. Then we define

$$H(\omega,I)=\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{M}} H_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\omega,I)u^{\boldsymbol{\alpha}}\bar{u}^{\boldsymbol{\beta}}\in\mathcal{H}^0_{r,p}$$

such that for all $\omega\in\mathcal{O}$, $I\in\mathcal{J}$, one has pointwise convergence $H^{(n)}(\omega,I)\to H(\omega,I)\in\mathcal{H}_{r,p}$. Moreover, for all $\boldsymbol{\alpha},\boldsymbol{\beta}$, the sequence $H^{(n)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is a Cauchy sequence with respect to the norm $|\cdot|^\gamma$. Since $\mathcal{F}(\mathcal{O}\times\mathcal{J})$ is Banach, for all $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{M}$, $H^{(n)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\to H_{\boldsymbol{\alpha},\boldsymbol{\beta}}\in\mathcal{F}(\mathcal{O}\times\mathcal{J})$.

By hypothesis, for all $\varepsilon>0$ there exists N such that for all $n,m>N$ one has

$$\frac12\sup_j\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}}|H^{(n)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}-H^{(m)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}|^\gamma(\boldsymbol{\alpha}_j+\boldsymbol{\beta}_j)u_p^{\boldsymbol{\alpha}+\boldsymbol{\beta}-2e_j}=\|H^{(n)}-H^{(m)}\|_{r,p}<\varepsilon,$$

so taking the liminf on m we get for all j,

$$\frac12\liminf_m\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}}|H^{(n)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}-H^{(m)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}|^\gamma(\boldsymbol{\alpha}_j+\boldsymbol{\beta}_j)u_p^{\boldsymbol{\alpha}+\boldsymbol{\beta}-2e_j}<\varepsilon.$$

[48] Indeed, setting $y:=\rho/r$, we have that $\sup_{x\ge2}x(\frac{r}{r+\rho})^{x-2}=\sup_{x\ge2}x(1+y)^{2-x}=2$ if $y\ge\sqrt{e}-1$. On the other hand, when $0<y<\sqrt{e}-1$ we have

$$\sup_{x\ge2}x(1+y)^{2-x}=\frac{(1+y)^2}{e\ln(1+y)}\le\frac1y\sup_{0<y<\sqrt{e}-1}\frac{(1+y)^2y}{e\ln(1+y)}=\frac2y.$$

.

Then for all $\boldsymbol{\alpha}, \boldsymbol{\beta}$, one has

$$|H^{(n)}_{\boldsymbol{\alpha},\boldsymbol{\beta}} - H_{\boldsymbol{\alpha},\boldsymbol{\beta}}|^{\gamma} \le \liminf_{m} |H^{(n)}_{\boldsymbol{\alpha},\boldsymbol{\beta}} - H^{(m)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}|^{\gamma}, \tag{A.3}$$

so

$$\begin{aligned}
&\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}} |H^{(n)}_{\boldsymbol{\alpha},\boldsymbol{\beta}} - H_{\boldsymbol{\alpha},\boldsymbol{\beta}}|^{\gamma} (\boldsymbol{\alpha}_j + \boldsymbol{\beta}_j) u_p^{\boldsymbol{\alpha}+\boldsymbol{\beta}-2e_j} \\
&\quad \le \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}} \liminf_{m} |H^{(n)}_{\boldsymbol{\alpha},\boldsymbol{\beta}} - H^{(m)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}|^{\gamma} (\boldsymbol{\alpha}_j + \boldsymbol{\beta}_j) u_p^{\boldsymbol{\alpha}+\boldsymbol{\beta}-2e_j} \\
&\quad \le \liminf_{m} \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}} |H^{(n)}_{\boldsymbol{\alpha},\boldsymbol{\beta}} - H^{(m)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}|^{\gamma} (\boldsymbol{\alpha}_j + \boldsymbol{\beta}_j) u_p^{\boldsymbol{\alpha}+\boldsymbol{\beta}-2e_j} < \varepsilon
\end{aligned}$$

by Fatou's lemma. Taking the supremum over j, we have proved that $H^{(n)} \to H$ in the $\mathcal{H}^{\mathcal{O},0}_{r,p}$-norm.

Concerning the Poisson algebra property, it suffices to use the fact that $|\cdot|^{\gamma}$ has the algebra property with respect to standard multiplication and from (A.1) we deduce

$$|H_{\boldsymbol{\alpha},\boldsymbol{\beta}}|^{\gamma} \le \sum_{j} \sum_{\substack{\boldsymbol{\alpha}'+\boldsymbol{\alpha}''-e_j=\boldsymbol{\alpha} \\ \boldsymbol{\beta}'+\boldsymbol{\beta}''-e_j=\boldsymbol{\beta}}} |F_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}|^{\gamma} |G_{\boldsymbol{\alpha}'',\boldsymbol{\beta}''}|^{\gamma} (\boldsymbol{\alpha}'_j \boldsymbol{\beta}''_j + \boldsymbol{\beta}'_j \boldsymbol{\alpha}''_j). \tag{A.4}$$

Then the proof follows verbatim the one of Proposition 4.1. □

Proof of Proposition 5.1

Items (i) and (ii) directly follow by Propositions 4.1 and 4.2 of [8], respectively. Here we only discuss the analyticity with respect to I. By formula (4.30) and recalling the notation in (5.1), for every $|u|_p \le r'$, we get the representation formula

$$\begin{aligned}
H^{(d)}(u,\omega,I) &= \sum_{\substack{\alpha,\beta,\zeta,a,b \\ 2|\zeta|+|a|+|b|\le d+2}} \sum_{\substack{\delta\succeq\zeta \\ 2|\delta|+|a|+|b|=d+2}} \sum_{m\succeq\delta} A \\
A &= \binom{m}{\delta}\binom{\delta}{\zeta}(-1)^{|\delta-\zeta|} I^{m-\zeta} H_{m,\alpha,\beta,a,b}(\omega,I)|v|^{2\zeta} v^{\alpha}\bar{v}^{\beta} z^{a}\bar{z}^{b},
\end{aligned} \tag{A.5}$$

where $m,\alpha,\beta,\zeta,\delta \in \mathbb{N}^{\mathcal{S}}, a,b \in \mathbb{N}^{\mathcal{S}^c}, \alpha\cap\beta = \emptyset$.

Set for brevity $X := X_{\Pi^d H}$ the Hamiltonian vector field of $\Pi^d H$ (recall (5.9)). Then define Y componentwise for $j \in \mathbb{Z}$ as

$$Y_j := \frac{1}{2} \sum_{\substack{\alpha,\beta,\zeta,a,b \\ 2|\zeta|+|a|+|b|\le d+2}} \sum_{\substack{\delta\succeq\zeta \\ 2|\delta|+|a|+|b|=d+2}} \sum_{m\succeq\delta} \binom{m}{\delta}\binom{\delta}{\zeta} I_p^{m-\zeta} |H_{m,\alpha,\beta,a,b}|^{\gamma} \xi_j u_p^{\xi-2e_j},$$

with $\xi := 2\zeta + \alpha + \beta + a + b$ and again $m, \alpha, \beta, \zeta, \delta \in \mathbb{N}^{\mathcal{S}}, a, b \in \mathbb{N}^{\mathcal{S}^c}, \alpha \cap \beta = \emptyset$ and where $u_p = u_p(r')$ was defined in (4.6) and $I_p := u_p^2/2$. By the formula after (4.30) in [8], we have $Y \in \mathrm{w}_p$ with $|Y|_p \le 3^{\frac{d}{2}+1}|H|_{r,p}$ and $|X_j(u,\omega,I)| \le Y_j$ for every $j \in \mathbb{Z}$, $u \in B_{r'}(\mathrm{w}_p)$, $\omega \in \mathcal{O}$, $I \in \mathcal{I}(p,r)$ (resp., $I \in \mathcal{I}(p,r,\mathbb{C})$ in the complex case). Therefore, $H^{(d)}$ is analytic in $\mathcal{I}(p,r)$ (resp., $I \in \mathcal{I}(p,r,\mathbb{C})$ in the complex case) since it can be written in a totally (a fortiori uniformly) convergent series (see, e.g., [40, Theorem 2, Appendix A]). □

Proof of Lemma 7.2
This is by induction over N.[49] The result is obvious for $N = 1$. Let us assume it for N and show it for $N+1$. We have

$$\frac{\sum_{1\le\ell\le N+1} x_\ell}{\prod_{1\le\ell\le N+1}\sqrt{x_\ell}} \le \frac{(\sum_{1\le\ell\le N} x_\ell) + x_{N+1}}{(\prod_{1\le\ell\le N}\sqrt{x_\ell})\sqrt{x_{N+1}}} \le \Big(\sqrt{x_1} + \frac{4}{\sqrt{x_1}}\Big)\frac{1}{\sqrt{x_{N+1}}} + \frac{\sqrt{x_{N+1}}}{\sqrt{2}}$$

since $x_1 \ge x_2 \ge \cdots \ge x_N \ge 2$ implies $\prod_{1\le\ell\le N}\sqrt{x_\ell} \ge \sqrt{2}$. It remains to prove that

$$\Big(\sqrt{x_1} + \frac{4}{\sqrt{x_1}}\Big)\frac{1}{\sqrt{x_{N+1}}} + \frac{\sqrt{x_{N+1}}}{\sqrt{2}} \le \sqrt{x_1} + \frac{4}{\sqrt{x_1}}.$$

Denoting $t := \sqrt{x_1}$ and $s := \sqrt{x_{N+1}}$, the above inequality is equivalent to

$$f(t,s) := 2t^2 s - \sqrt{2}ts^2 + 8s - 2t^2 - 8 \ge 0$$

for $\sqrt{2} \le s \le t$. Since f is a concave function of s, we have that

$$f(t,s) \ge \min\{f(t,\sqrt{2}), f(t,t)\}.$$

It is immediate to see that

$$\min_{t\ge\sqrt{2}} f(t,\sqrt{2}) = 7\sqrt{2} - 9 > 0, \qquad \min_{t\ge\sqrt{2}} f(t,t) = 12\sqrt{2} - 16 > 0,$$

showing that $f(t,s) > 0$ for $\sqrt{2} \le s \le t$ and concluding the proof. □

Proof of Lemma 7.3
Let $\bar{\delta} := \frac{\delta}{9}$. We split the sum in (7.21) into two terms $A_k = A_k^* + A_k^>$, with

[49] Note that [6, Lemma C.4] stated the following erroneous result: *Let* $0 < a < 1$ *and* $x_1 \ge x_2 \ge \cdots \ge x_N \ge 2$. *Then* $\frac{\sum_{1\le\ell\le N} x_\ell}{\prod_{1\le\ell\le N} x_\ell^a} \le x_1^{1-a} + \frac{2}{a x_1^a}$. Anyway in [6], we only had used the above result with $a = 1$, which is exactly the content of Lemma 7.2.

$$A_k^* := \sum_{i<i_*:k_i\geq 1} -\bar{\delta} k_i \log\lfloor s(i)\rfloor + \log\big(1+\langle i\rangle^2 k_i^2\big),$$
$$A_k^> := \sum_{i_*\leq i\leq i_{\mathrm{M}}:k_i\geq 1} -\bar{\delta} k_i \log\lfloor s(i)\rfloor + \log\big(1+\langle i\rangle^2 k_i^2\big).$$

Regarding the first term, we get (recalling that $s(i)\geq i$)

$$\begin{aligned} A_k^* &\leq \sum_{i<i_*:k_i\geq 1} -\bar{\delta} k_i \log\lfloor i\rfloor + 1 + 2\log\langle i\rangle + 2\log k_i \\ &\leq 8 i_* \log i_* - \sum_{i<i_*:k_i\geq 1} \big((\bar{\delta}\log 2)k_i - 2\log k_i\big) \leq 8 i_*\Big(\log i_* + \log\frac{1}{\bar{\delta}}\Big), \end{aligned}$$

using that $\max_{x\geq 1} -(\bar{\delta}\log 2)x + 2\log x \leq 2\log 1/\bar{\delta}$.

Consider now the second term. By (2.8a), we get

$$A_k^> \leq \sum_{i_*\leq i\leq i_{\mathrm{M}},k_i\geq 1} -\bar{\delta} k_i \log^{1+\eta}|i| + \log\big(1+\langle i\rangle^2 k_i^2\big) \leq A_{k,1}^> + A_{k,2}^>,$$

where

$$A_{k,1}^> := \sum_{i_*\leq i\leq i_{\mathrm{M}},k_i\geq 28/\bar{\delta}} 2f(i,k_i), \qquad A_{k,2}^> := \sum_{i_*\leq i\leq i_{\mathrm{M}},1\leq k_i<28/\bar{\delta}} 2f(i,k_i),$$

and[50]

$$f(i,k) := -\bar{\delta} k \log^{1+\eta} i + 3\log i + 2\log(\bar{\delta}k) + 2\log(1/\bar{\delta}). \tag{A.6}$$

When $k_i\geq 28/\bar{\delta}$, we have[51]

$$\begin{aligned} f(i,k_i) &\leq -\frac{1}{2}\bar{\delta} k_i \log^{1+\eta} i + 2\log(\bar{\delta}k_i) + 2\log(1/\bar{\delta}) \\ &\leq -\frac{1}{4}\bar{\delta} k_i \log^{1+\eta} i + 2\log(1/\bar{\delta}) \leq -7\log^{1+\eta} i + 2\log(1/\bar{\delta}), \end{aligned}$$

which is negative for $i\geq 1/\bar{\delta}$. Then we get $A_{k,1}^> \leq \frac{4}{\bar{\delta}}\log(1/\bar{\delta})$.

We finally consider the term $A_{k,2}^>$, namely when $k_i<28/\bar{\delta}$. In this case,

$$f(i,k_i) \leq (3-\bar{\delta}\log^{\eta} i)\log i + 7 + 2\log(1/\bar{\delta}).$$

We claim that the last quantity is negative for $i\geq \exp(4\bar{\delta}^{-1/\eta})$. Indeed, since $\eta\leq 2$,

[50] Recall that $i_*\geq 3$.
[51] Using that $-\frac{1}{4}x + 2\log x \leq 0$ for $x\geq 28$ and $\log i \geq \log i_* \geq 1$.

$$f(i,k_i) \le -\frac{4}{\sqrt{\bar{\delta}}} + 7 + 2\log(1/\bar{\delta}) \le 0$$

for $\bar{\delta} \le 1/2$. On the other hand, for $i < \exp(4\bar{\delta}^{-1/\eta})$, we have

$$f(i,k_i) \le \frac{12}{\bar{\delta}} + 7 + 2\log(1/\bar{\delta}) \le \frac{20}{\bar{\delta}}.$$

So we finally get

$$A^{>}_{k,2} \le \frac{40}{\bar{\delta}} \exp(4\bar{\delta}^{-1/\eta}) \le 40\exp(5\bar{\delta}^{-1/\eta}). \qquad \square$$

Proof of Lemma 9.2
To fix ideas, we consider the case $\xi_n = -1$. Set $x =: (\hat{x}, x_n)$. Let us introduce the portion of hyperplane (which is a graph over $\hat{x}$)

$$P := \{(\hat{x}, \hat{\xi} \cdot \hat{x}) : \hat{x} \in [-1/4, 1/4]^{n-1}\},$$

orthogonal to ξ. Note that for every $y \in E$ there exist unique $t \in \mathbb{R}$ and $\hat{x} \in [-1/4, 1/4]^{n-1}$ such that $y = (\hat{x}, \hat{\xi} \cdot \hat{x}) + t\xi$. Then by Fubini's theorem we have that

$$\begin{aligned} \operatorname{meas}(E) &= |\xi|_2 \int_P \operatorname{meas}\{t \in \mathbb{R} : (\hat{x}, \hat{\xi} \cdot \hat{x}) + t\xi \in \mathbb{E}\}\, d\sigma \\ &\le |\xi|_2 \delta \int_P d\sigma = |\xi|_2 \delta \int_{[-1/4,1/4]^{n-1}} \sqrt{1 + |\hat{\xi}|_2^2}\, d\hat{x} = 2^{1-n}\delta|\xi|_2^2. \qquad \square \end{aligned}$$

Appendix B. Topology, measure, and continuous functions on infinite product spaces

Product topology
Fix $\varrho > 0$. Let us consider the set $[-\varrho, \varrho]^{\mathbb{Z}}$ endowed with the *product topology*, namely the coarsest topology (i.e., the topology with the fewest open sets) for which all the projections $\pi_j : [-\varrho, \varrho]^{\mathbb{Z}} \to [-\varrho, \varrho]$, with $\pi_j(\omega) := \omega_j$, $j \in \mathbb{Z}$, are continuous.

We call *cylinder* a subset $\bigotimes_{n\in\mathbb{N}} A_n$ of $[-\varrho, \varrho]^{\mathbb{Z}}$ such that $A_n \subseteq [-\varrho, \varrho]$ with $A_n \ne [-\varrho, \varrho]$ only for finitely many $n \in \mathbb{N}$. Then a basis of the product topology is given by the open cylinders, namely cylinders $\bigotimes_{n\in\mathbb{N}} A_n$, where A_n are open.

By Tychonoff's theorem, $[-\varrho, \varrho]^{\mathbb{Z}}$ with the product topology is a compact Hausdorff space.

Product measures
The *product σ-algebra (of the Borel sets)* of $[-\varrho, \varrho]^{\mathbb{Z}}$ is generated by the set of cylinders $\bigotimes_{n\in\mathbb{N}} A_n$, where A_n are Borel sets (w.r.t. the standard topology on $[-\varrho, \varrho]$).

The *probability product measure* μ on the product σ-algebra of $[-\varrho,\varrho]^{\mathbb{Z}}$ is defined by

$$\mu\Big(\bigotimes_{n\in\mathbb{N}} A_n\Big) := \prod_{n\in\mathbb{N}} \frac{1}{2\varrho}|A_n|,$$

where $|A_n|$ denotes the usual Lebesgue measure of the Borel set A_n.

Through the bijective map

$$\mathcal{V}^* : \mathcal{Q} \to [-1/2, 1/2]^{\mathbb{Z}}, \quad \text{where } \mathcal{V}^*_j(\nu) := \nu_j - j^2, j \in \mathbb{Z},$$

we induce the product topology and the probability product measure on the set $\mathcal{Q}$ (analogously for $\mathcal{Q}_{\mathcal{S}}$ and $\mathcal{Q}_{\mathcal{S}^c}$).

Product measures

Given a compact Hausdorff space X and a Banach space E, we denote by $C(X,E)$ the Banach space of continuous functions $f : X \to E$ endowed with the uniform norm

$$|f|_{C(X,E)} := \sup_{x\in X}\big|f(x)\big|_E = \max_{x\in X}\big|f(x)\big|_E.$$

LEMMA B.1 (Lipschitz fixed point)

Let $\mathcal{C}$ be the closed subset of the Banach space $C(\mathcal{Q}_{\mathcal{S}} \times [-1/4, 1/4]^{\mathcal{S}^c}, \ell^\infty_{\mathcal{S}^c})$ (with the product topology on $\mathcal{Q}_{\mathcal{S}} \times [-1/4, 1/4]^{\mathcal{S}^c}$) defined as

$$\mathcal{C} := \big\{w \in C\big(\mathcal{Q}_{\mathcal{S}} \times [-1/4, 1/4]^{\mathcal{S}^c}, \ell^\infty_{\mathcal{S}^c}\big) \text{ s.t. } |w|_{C(\mathcal{Q}_{\mathcal{S}}\times[-1/4,1/4]^{\mathcal{S}^c},\ell^\infty_{\mathcal{S}^c})} \le r\big\},$$

for some $0 < r < 1/4$. Let $F \in C(\mathcal{Q}_{\mathcal{S}} \times [-1/4, 1/4]^{\mathcal{S}^c} \times [-r,r]^{\mathcal{S}^c}, \ell^\infty_{\mathcal{S}^c})$ with

$$\big|F(\nu, V_{\mathcal{S}^c}, w)\big|_{\ell^\infty_{\mathcal{S}^c}} \le r, \qquad \big|F(\nu, V_{\mathcal{S}^c}, w) - F(\nu, V_{\mathcal{S}^c}, w')\big|_{\ell^\infty_{\mathcal{S}^c}} \le 1/2|w - w'|_{\ell^\infty_{\mathcal{S}^c}},$$

for all $\nu \in \mathcal{Q}_{\mathcal{S}}, w, w' \in \mathcal{Q}_{\mathcal{S}^c}$. Then there exists a unique $w \in \mathcal{C}$ such that

$$w(\nu, V_{\mathcal{S}^c}) = F\big(\nu, V_{\mathcal{S}^c}, w(\nu, V_{\mathcal{S}^c})\big).$$

Moreover, if F is L-Lipschitz for some $L > 0$ with respect to $\nu \in \mathcal{Q}_{\mathcal{S}}$ (endowed with the ℓ^∞-metric), then w is $2L$-Lipschitz in ν. Analogously, if F is L'-Lipschitz with respect to some other parameter I in some Banach space, then w is $2L'$-Lipschitz in I.

Proof

First note that if $w \in \mathcal{C}$, then $F(\nu, V_{\mathcal{S}^c}, w(\nu, V_{\mathcal{S}^c})) \in \mathcal{C}$, since the product topology in $[-r,r]^{\mathcal{S}^c}$ is weaker than the one induced by the ℓ^∞-norm. Set $\Phi : \mathcal{C} \to \mathcal{C}$ by

$$\big(\Phi(w)\big)(\nu, V_{\mathcal{S}^c}) := F\big(\nu, V_{\mathcal{S}^c}, w(\nu, V_{\mathcal{S}^c})\big).$$

Let us check that Φ is a contraction on $\mathcal{C}$. Indeed,

$$\begin{aligned}
&\big|\Phi(w) - \Phi(w')\big|_{\mathcal{C}} \\
&\quad = \sup_{\nu \in \mathcal{Q}_{\mathcal{S}}, V_{\mathcal{S}^c} \in [-1/4, 1/4]^{\mathcal{S}^c}} \big|F\big(\nu, V_{\mathcal{S}^c}, w(\nu, V_{\mathcal{S}^c})\big) - F\big(\nu, V_{\mathcal{S}^c}, w'(\nu, V_{\mathcal{S}^c})\big)\big|_{\ell^\infty_{\mathcal{S}^c}} \\
&\quad \leq \frac{1}{2} \sup_{\nu \in \mathcal{Q}_{\mathcal{S}}, V_{\mathcal{S}^c} \in [-1/4, 1/4]^{\mathcal{S}^c}} \big|w(\nu, V_{\mathcal{S}^c}) - w'(\nu, V_{\mathcal{S}^c})\big|_{\ell^\infty_{\mathcal{S}^c}} = \frac{1}{2}|w - w'|_{\mathcal{C}}.
\end{aligned}$$

The existence of the fixed point follows from the contraction mapping theorem on Banach spaces.

Assume now that F is L-Lipschitz for some $L > 0$ with respect to $\nu \in \mathcal{Q}_{\mathcal{S}}$ (endowed with the ℓ^∞-metric d_∞). Then

$$\begin{aligned}
&\big|w(\nu, V_{\mathcal{S}^c}) - w(\nu', V_{\mathcal{S}^c})\big|_{\ell^\infty_{\mathcal{S}^c}} \\
&\quad \leq \big|F\big(\nu, V_{\mathcal{S}^c}, w(\nu, V_{\mathcal{S}^c})\big) - F\big(\nu', V_{\mathcal{S}^c}, w(\nu, V_{\mathcal{S}^c})\big)\big|_{\ell^\infty_{\mathcal{S}^c}} \\
&\qquad + \big|F\big(\nu', V_{\mathcal{S}^c}, w(\nu, V_{\mathcal{S}^c})\big) - F\big(\nu', V_{\mathcal{S}^c}, w(\nu', V_{\mathcal{S}^c})\big)\big|_{\ell^\infty_{\mathcal{S}^c}} \\
&\quad \leq L\, d_\infty(\nu, \nu') + \frac{1}{2}\big|w(\nu, V_{\mathcal{S}^c}) - w(\nu', V_{\mathcal{S}^c})\big|_{\ell^\infty_{\mathcal{S}^c}}
\end{aligned}$$

implying

$$\big|w(\nu, V_{\mathcal{S}^c}) - w(\nu', V_{\mathcal{S}^c})\big|_{\ell^\infty_{\mathcal{S}^c}} \leq 2L\, d_\infty(\nu, \nu'). \qquad \square$$

The next lemma regards the analyticity of the map $\mathfrak{i}$ in (2.7). Without loss of generality, we consider here only the case $\mathcal{S} = \mathbb{Z}$ since we are not assuming that $I_j \neq 0$. Let us first introduce the notation $\mathbb{C}^\infty_s := \{\varphi = (\varphi_j)_{j\in\mathbb{Z}} \text{ with } |\operatorname{Im}\varphi_j| < s, \forall\, j \in \mathbb{Z}\}$ for some $s > 0$ and the equivalence relation $\varphi \sim \varphi'$ if and only if $\varphi - \varphi' \in 2\pi\mathbb{Z}$. We set $\mathbb{T}^\infty_s := \mathbb{C}^\infty_s / \sim$.

LEMMA B.2
For $\sqrt{I} \in \bar{B}_r(\mathtt{w}_p)$, the map $\mathfrak{i}$ in (2.7) can be extended to an analytic map

$$\mathfrak{i} : \mathbb{T}^\infty_s \to \mathtt{w}_p, \quad [\varphi] = \big[(\varphi_j)_{j\in\mathbb{Z}}\big] \mapsto \big(\sqrt{I_j} e^{\mathrm{i}\varphi_j}\big)_{j\in\mathbb{Z}},$$

for any $s > 0$.

Proof
For $\varphi \in \mathbb{C}^\infty_s$, we denote by $[\varphi] \in \mathbb{T}^\infty_s$ the equivalence class of φ. $\mathbb{T}^\infty_s$ is a metric space endowed with the distance

$$d\big([\varphi],[\psi]\big) := \min_{\psi'\in[\psi]} |\varphi-\psi'|_\infty,$$

where $|\cdot|_\infty$ is the norm on $\ell^\infty=\ell^\infty(\mathbb{C})$. Moreover, it is a Banach manifold. Indeed, given every point $[\varphi]\in\mathbb{T}^\infty_s$, set $\rho:=\min\{1,s-|\operatorname{Im}\varphi|\}/2$, and consider the open ball $B_\rho([\varphi])\subset\mathbb{T}^\infty_s$ of radius ρ centered in $[\varphi]$ and the open ball U_ρ of radius ρ centered at the origin of ℓ^∞; then a local chart is $\Phi: B_\rho([\varphi])\to U_\rho$ defined so that $\Phi^{-1}(\psi):=[\varphi+\psi]$. We claim that

$$g := \mathrm{i}\circ\Phi^{-1}: U_\rho\to \mathrm{w}_p, \qquad g(\psi) := \big(\sqrt{I_j}e^{\mathrm{i}(\varphi_j+\psi_j)}\big)_{j\in\mathbb{Z}}$$

is analytic since the Fréchet derivative $Dg: U_\rho\to\mathcal{L}(\ell^\infty,\mathrm{w}_p)$ is continuous. Indeed, for every $\psi'\in\ell^\infty$,

$$Dg(\psi)[\psi'] = \big(\mathrm{i}\sqrt{I_j}e^{\mathrm{i}(\varphi_j+\psi_j)}\psi'_j\big)_{j\in\mathbb{Z}}$$

and for $\psi,\tilde\psi\in U_\rho$ the operator norm (recall that $\sqrt{I}\in\bar B_r(\mathrm{w}_p)$) satisfies

$$\begin{aligned}\big\|Dg(\psi)-Dg(\tilde\psi)\big\|_{\mathrm{op}} &= \sup_{\psi'\in\ell^\infty,|\psi'|=1}\Big|\big(\mathrm{i}\sqrt{I_j}e^{\mathrm{i}\varphi_j}(e^{\mathrm{i}\psi_j}-e^{\mathrm{i}\tilde\psi_j})\psi'_j\big)_{j\in\mathbb{Z}}\Big|_{\mathrm{w}_p}\\ &\le re^{|\operatorname{Im}\varphi|+\rho}\Big|\big((1-e^{\mathrm{i}(\tilde\psi_j-\psi_j)})\big)_{j\in\mathbb{Z}}\Big|_{\ell^\infty}\\ &\le re^{|\operatorname{Im}\varphi|+2\rho}|\tilde\psi-\psi|_{\ell^\infty}.\end{aligned}$$

□

On the density of linear flow νt on the flat torus $\mathcal{T}_I$ (recall (2.5)).

We note that if $|\nu|_{\ell^\infty}<\infty$, then the linear flow $t\mapsto\nu t$ is not dense on the torus $\mathcal{T}_I$. Indeed,[52] let $\tilde\nu\in\mathbb{R}^{\mathbb{Z}}$ defined as $\tilde\nu_j=\nu_j$ when $j\in\mathcal{S}_I$ and 0 otherwise; then $\tilde\nu\mathbb{Q}$ would be a countable dense subset but ℓ^∞-based topologies are not separable. On the other hand, if the sequence ν_j increases very fast (e.g., superexponentially), then the flow is dense.

LEMMA B.3
If for any $n\ge1$ the vector $\nu^{(n)}$ with entries ν_j, $|j|\le n$, $j\in\mathcal{S}_I$ is rationally independent and

$$\lim_{n\in\mathcal{S}_I,|n|\to\infty}|\nu_n|\sum_{j\in\mathcal{S}_I,|j|>|n|}|\nu_j|^{-1}=0,$$

then the flow is dense.

Proof
Fix $\delta>0$, $\bar\varphi$ on the torus, and call d the distance on the 1-dimensional torus

[52]Recall the definition of $\mathcal{S}_I$ in (2.6).

$d_{\mathbb{T}^1}(\varphi, \psi) = \min_{k\in\mathbb{Z}} |\varphi - \psi - 2\pi k|$ for $\varphi, \psi \in \mathbb{T}^1$. We want to show that there exists a time $T \in \mathbb{R}$ such that $d_{\mathbb{T}^1}(\bar{\varphi}_n, \nu_n T) \le \delta$ for every $n \in \mathcal{S}_I$. Let $n_0 \in \mathcal{S}_I$ be large enough such that $|\nu_n| \sum_{j\in\mathcal{S}_I, |j|>|n|} |\nu_j|^{-1} \le \delta/2\pi$ for every $n \in \mathcal{S}_I, |n| \ge |n_0|$. Since $\nu^{(n_0)}$ is rationally independent, there exists $T_0 > 0$ such that $d_{\mathbb{T}^1}(\bar{\varphi}_j, \nu_j T_0) \le \delta/2$ for every $j \in \mathcal{S}_I, |j| \le |n_0|$. For every $j \in \mathcal{S}_I, |j| > |n_0|$, there exists $t_j \in \mathbb{R}$ with $|t_j| \le \pi/|\nu_j|$ such that $d_{\mathbb{T}^1}(\bar{\varphi}_j, \nu_j(T_0 + t_j)) = 0$. Set $T := T_0 + T'$, where $T' := \sum_{j\in\mathcal{S}_I, |j|>|n_0|} t_j$. Note that $|T'| \le \delta/2|\nu_{n_0}|$. Then for $n \in \mathcal{S}_I, |n| \le |n_0|$ we have $d_{\mathbb{T}^1}(\bar{\varphi}_n, \nu_n T) \le \delta/2 + |\nu_n T'| \le \delta$. Finally, for $n \in \mathcal{S}_I, |n| > |n_0|$, we have $d_{\mathbb{T}^1}(\bar{\varphi}_n, \nu_n T) \le |\nu_n| \sum_{j\in\mathcal{S}_I, |j|>|n_0|, j\ne n_0} |t_j| \le \delta/2$, concluding the proof. $\square$

Finally, since in our application to NLS $\nu_j \sim j^2$, if $\mathcal{S}_I$ is sparse enough, then the condition of Lemma B.3 is satisfied and the flow is dense.

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