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On the use of the root locus of polynomials with complex coefficients for estimating the basin of attraction for the continuous-time Newton and Householder methods ☆

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ABSTRACT

The main goal of this paper is to show how Lyapunov theory and the root locus technique, widely known as powerful tools for control design, can be used to provide a characterization of the convergence properties of the celebrated continuous-time Newton and Householder methods. In particular, a Lyapunov analysis is carried out to characterize the local and global properties of these methods and it is shown how techniques to qualitatively draw the root locus of a polynomial with complex coefficients can be used to determine the boundaries of the basin of attraction of each root of the polynomial to be zeroed, which is an equilibrium of a suitably designed dynamical system. Finally, by leveraging on the properties of the root locus, an algorithm is proposed to approximate all the complex roots of a given polynomial, tackling the key issue of choosing complex initial guesses even when only real roots are sought.

1. Introduction

Nowadays, polynomial models are ubiquitous in many applications such as: robotics (Cox et al., 2006), coding theory (Van Lint, 2012), synthesis of controllers (Menini et al., 2017, 2018b), optimization (Kirk, 2012), mathematical biology (Szallasi et al., 2006), computer vision (Martinelli, 2012), game theory (Possieri & Sassano, 2016), and machine learning (Abu-Mostafa et al., 2012). Therefore, the development of tools able to determine solutions to systems of polynomial equations is of paramount importance in several control applications.

Several methods have been proposed in the literature to determine/approximate the roots of a complex polynomial: in Jenkins and Traub (1970), a three-stage process for calculating the zeros of a polynomial with real coefficients has been proposed; in Pinkert (1976), the classical theorems of Sturm and Routh are used to construct algorithms that isolate the unique roots of an univariate polynomial into disjoint squares in the complex plane; in Huang (2004), multilayer perceptron networks are used to find complex roots of polynomials; in Pan and Zheng (2011), matrix methods, Newton's linearization, repeated squaring, homotopy continuation techniques, and some heuristics are incorporated to accelerate the search for the roots; in Pan (2023), a black box polynomial root-finder, *i.e.*, involving no coefficient of the polynomial to be zeroed, has been proposed.

Some of the most used methods to determine a solution to polynomial systems are the iterative Newton, Halley, and Householder methods (Galántai, 2000). In Henrici (1993), it has been demonstrated that the discrete-time Newton method can be successfully applied to determine the complex zeros of a system of polynomial equations. Nonetheless, the boundaries of the basins of attraction in $\mathbb{C}$ of each root of the polynomials to be zeroed are generally fractals when the approximation method is implemented in discrete-time (Drexler et al., 1996; Hubbard et al., 2001). In particular, Newton's method may exhibit chaotic behavior, with some initial guesses tending either to infinity or to cycles of finite length (Strang, 1991).

Starting with the work of Ortega and Rheinboldt (2000), a large research effort has been spent to determine the properties of the continuous-time implementation of iterative algorithms originally designed in discrete-time (Allgower & Georg, 2012; Attouch et al., 2023; Romero et al., 2022; Su et al., 2014; Zghier, 1981). As an example, in Hirsch and Smale (1979) and Tanabe (1985), the local and global convergence properties of the continuous-time Newton method have been analyzed.

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1.1. Historical context

According to Deuflhard (2012), Kollerstrom (1992) and Ypma (1995), Newton's method originated from the work of the French algebraist François Viète. Viète's work (Viète, 1600) aimed at determining a root of the monic polynomial equation $f(z) = 0$. Viète's method yields individual digits of a solution z_e of $f(z) = 0$: letting $a_0, a_1, a_2, \ldots,$ be the successive significant digits of z_e so that $z_e = a_0\, 10^k + a_1\, 10^{k-1} + a_2\, 10^{k-2} + \cdots$, and letting z_i be the ith estimate of z_e, $z_i = a_0\, 10^k + a_1\, 10^{k-1} + a_2\, 10^{k-2} + \cdots + a_i\, 10^{k-i}$, the digit a_i can be computed (Ypma, 1995) as the integer part of

$$\frac{1}{2}\frac{f(x_i+10^{k-i-1})+f(x_i)}{f(x_i+10^{k-i-1})-f(x_i)-10^{(k-i-1)n}}, \tag{1}$$

where n is the degree of f. Such a method has been improved by Isaac Newton (Newton, 1669) by linearizing the successively arising polynomials: letting $g_0(z) = f(z)$, $g_0 = \sum_{i=0}^{n} \alpha_i z^i$, and $e_0 = z_e - z_0$, being z_0 an initial guess, one has that $0 = g_0(z_e) = \sum_{i=0}^{n} \alpha_i (z_0 + e_0)^i = \sum_{i=0}^{n} \alpha_i (\sum_{j=0}^{i} \binom{i}{j} z_0^j e_0^{i-j}) = g_1(e_0)$; by neglecting higher-order terms, one obtains $0 \simeq (\sum_{i=0}^{n} \alpha_i z_0^i) + e_0(\sum_{i=0}^{n} \alpha_i\, i\, z_0^{i-1})$, from which one obtains $e_0 \simeq -(\sum_{i=0}^{n} \alpha_i\, i\, z_0^{i-1})^{-1} (\sum_{i=0}^{n} \alpha_i z_0^i)$. The process can be repeated by replacing z_0 with $z_1 = z_0 + e_0$. Joseph Raphson (Raphson, 1690) managed to avoid the computation of the intermediate polynomials by making the method fully iterative:

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}. \tag{2}$$

The continuous-time analogue of (2) has been firstly considered in Gavurin (1958),

$$\dot{z}(t) = -\frac{f(z(t))}{f'(z(t))},\ z(0) = z_0. \tag{3}$$

By observing that maximal solutions $\omega(t, z_0)$ to (3) defined in the interval $[0, M)$ satisfy $f(\omega(t, z_0)) = \mathrm{e}^{-t} f(z_0)$ for all $t \in [0, M)$, one has that if $M = +\infty$, then $\lim_{t\to+\infty} f(\omega(t, z_0)) = 0$, *i.e.*, complete solutions to system (3) converge to a solution of $f(z) = 0$.

Since these early developments, a large research effort has been spent to characterize the properties of this algorithm and to accelerate its convergence. In Tanabe (1985), the local and global properties of the multi-variate version of (3) have been analyzed. In Menini et al. (2018d), it has been shown how to modify the dynamics (3) to achieve finite-time convergence, whereas in Menini et al. (2021b) it has been shown how to couple the multi-variate version of (3) with the interval arithmetic to obtain certified estimates of the solution to $f(z) = 0$. A first attempt to estimate the basin of attraction of Newton method has been made in Menini et al. (2018c).

On the other hand, focusing on the discrete-time iteration (2), a significant research effort has been expended to improve its convergence properties. Halley (1964) provided a method having super linear convergence of order 2 (whereas the methods (1) and (2) have linear and super linear convergence of order 2, respectively). Householder (1970) showed how to design iteration functions having arbitrary order of super linear convergence. Such a problem has been also analyzed in Kalantari (2008), where the *Basic Family*, *i.e.*, an infinite set of iteration functions possessing super linear convergence of assigned order together with several other useful properties from the mathematical, algorithmic and visualization points of view, has been introduced.

1.2. Contribution of the paper

The main contribution of this paper is to show how methods from control theory can be successfully used to analyze the convergence properties of the continuous-time Newton and Householder methods. In particular, a Lyapunov analysis is carried out to characterize the stability properties of these algorithms; see Sections 3 and 4. Differently from the analysis in Gavurin (1958) and Tanabe (1985), the results presented in Sections 3 and 4 leverage the LaSalle invariance principle and Lyapunov theory rather than merely computing first integrals, thus showing how control techniques can provide effective tools for the analysis of iterative methods. Furthermore, by exploiting the root locus of polynomials having complex coefficients, it is shown how to determine qualitatively and quantitatively the basin of attraction of each root of the polynomial to be zeroed, considered as an equilibrium of the corresponding dynamical system. The root locus of polynomials having complex coefficients, when a real parameter varies from $-\infty$ to $+\infty$, has been already considered in Dòria-Cerezo and Bodson (2013) to design a generalized proportional–integral controller for a simplified model of a three-phases rectifier. The rules derived in Dòria-Cerezo and Bodson (2013), recalled and extended in Section 5, are natural extensions of the traditional root locus techniques (see Evans, 1950). Several examples of application of the proposed techniques are reported in Section 6, wherein qualitative shapes of the basin of attraction of each root of $f(z) = 0$ when using the Newton's method are reported. These basins are examined in contrast with the Julia plots (Hubbard et al., 2001) of the corresponding discrete-time system to compare the performance of these two classes of root-finding algorithms. Finally, by leveraging on the properties of the root locus of a polynomial with complex coefficients, an algorithm is given in Section 7 to determine the set of all the complex roots of a polynomial, the key issue being that of choosing complex initial guesses even when only real roots are sought.

Differently from Menini et al. (2018d, 2021b) and Tanabe (1985), the focus here is on the continuous-time Newton and Householder methods to solve scalar polynomial equations. Although the convergence of the Newton method to the roots of $f(z) = 0$ has been already proved in Gavurin (1958) as highlighted in Section 1.1, in Section 3 it is shown how such an analysis can be carried out using control theoretic tools such as the analysis of the linearized system in the neighborhood of singular and equilibrium points and the LaSalle invariance principle. This paves the way to the characterization of the basin of attraction of each root of $f(z) = 0$ that is carried out in Section 5 by using the root locus of complex polynomials. It is worth pointing out that such a characterization is not an inner approximate as in Menini et al. (2018c), but allows one to precisely determine the basin of attraction of each root.

In Kalantari (2008), it has been shown that the Basic Family is equivalent to the Gerlach (1994) family of functions. In Section 4, it is shown that they are further equivalent to the Householder family. First, this allows to characterize the convergence properties of the continuous-time version of these methods. Secondly, it allows also to precisely determine the basin of attraction of each root of $f(z) = 0$ using the same root locus arguments employed for the Newton method.

2. Review of iterative methods for solving polynomial equations

Notation: Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integer, real, and complex numbers, respectively. The symbol $\imath$ denotes the imaginary unit. Given a complex number $z \in \mathbb{C}$, the symbols $\Re(z)$ and $\Im(z)$ denote its real and imaginary parts, respectively, whereas $\bar{z}$, $|z|$, and $\arg(z)$ denote its complex conjugate, its absolute value, and its argument, respectively. The symbols $\wedge$ and $\vee$ denote the logical AND and OR operator, respectively. Letting $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, $\mathbb{K}[x]$ is the set of all the polynomials in x with coefficients in $\mathbb{K}$. Given a point $z \in \mathbb{C}$ and a set $\mathcal{S} \subset \mathbb{C}$, the *distance between* z *and* $\mathcal{S}$ is $\mathrm{dist}(z, \mathcal{S}) = \min_{s\in\mathcal{S}} |z - s|$. Given a function $f : \mathcal{X} \to \mathcal{Y}$, the symbol dom($f(\cdot)$) denotes its domain $\mathcal{X}$. Given an univariate function γ, let $\gamma'(z) = \frac{\partial \gamma(z)}{\partial z}$ and $\gamma^{(i)} = \frac{\partial^i \gamma(z)}{\partial z^i}$.

Consider the problem of determining the roots of a polynomial $f \in \mathbb{R}[z]$. An iterative procedure to determine a solution of $f(z) = 0$ has the form of the discrete-time system

$$z_{k+1} = \gamma(z_k), \quad k \in \mathbb{Z}, k \geqslant 0, \tag{4}$$

where $\gamma : \mathbb{R} \to \mathbb{R}$ and $z_0 \in \mathbb{C}$ is the initial estimate; notice that z_0 must necessarily be taken into $\mathbb{C}$ in order to approximate the non-real roots of f. Letting z_e be any *root* of f, *i.e.* $f(z_e) = 0$, the function γ needs to satisfy $\gamma(z_e) = z_e$ in order to ensure that system (4) constitutes an iterative procedure to solve $f(z) = 0$. In the following, it is assumed that γ is differentiable in a neighborhood of z_e, with continuous derivatives, the required number of times. By classical Taylor expansion arguments (Traub, 1982, Sec. 2.21), it holds that $|z_{k+1} - z_e| \le |\gamma'(\zeta_k)||z_k - z_e|$, where ζ_k lies in the segment determined by z_k and z_e. Therefore, if $|\gamma'(z_e)| < 1$, then there exists a neighborhood $\mathcal{U} \subset \mathbb{C}$ of z_e such that

$$|z_k - z_e| \le L^k |z_0 - z_e|, \quad k \in \mathbb{Z}, k \geqslant 0, \tag{5}$$

with $L < 1$, for all $z_0 \in \mathcal{U}$. If the condition given in Eq. (5) holds and $\gamma'(z_e) \neq 0$, then the iterative method (4) (briefly, the function γ) has *linear convergence*. On the other hand, if $\gamma^{(i)}(z_e) = 0$, for $i = 0, \ldots, m-1$, then it holds that $|z_{k+1} - z_e| \le |\gamma^{(m)}(\zeta_k)||z_k - z_e|^m$, where ζ_k lies in the segment determined by z_k and z_e. Hence, by Traub (1982, Sec. 2.2.3), there exists a neighborhood $\mathcal{U}_m \subset \mathbb{C}$ of z_e such that, for all $z_0 \in \mathcal{U}$, one has

$$\lim_{k \to +\infty} \frac{z_{k+1} - z_e}{(z_k - z_e)^m} = \frac{\gamma^{(m)}(z_e)}{m!}. \tag{6}$$

If the condition given in Eq. (6) holds and $\gamma^{(m)}(z_e) \neq 0$, then the iterative procedure (4) (briefly, function γ) has *super linear convergence of order m*.

Note that the order m may depend on the multiplicity of z_e as a root of the polynomial f, *i.e.*, the iterative method (4) might have super linear convergence of order m for simple zeros, but different order for multiple zeros. Supposing that f has just simple zeros, by Traub (1982, Thm. 2.4), if γ_1 has order m_1 and γ_2 has order m_2, then $\gamma_1 \circ \gamma_2$ has order $m_1 m_2$. Furthermore, if $m_1 = m_2$, then there exists a function ϖ, which is well-defined at z_e and such that $\varpi(z_e) \neq 0$, for which

$$\gamma_2(z) = \gamma_1(z) + \varpi(z) \left(\frac{f(z)}{f'(z)} \right)^m, \tag{7}$$

with $m = m_1$ if $\gamma_1^{(m)}(z_e) \neq \gamma_2^{(m)}(z_e)$, or $m > m_1$ if $\gamma_1^{(m)}(z_e) = \gamma_2^{(m)}(z_e)$. In fact, for a given γ_1 such that $\gamma_1(z_e) = z_e$, having super linear convergence of order m, by Traub (1982, Thm. 2.7), the function γ_2 given in Eq. (7) has super linear convergence order m provided that $\varpi(z_e) \neq -\frac{\gamma_1^{(m)}(z_e)}{m!}$. Therefore, the availability of an iterative method of order m allows one to determine the whole family of iterative methods of order m via (7).

The most celebrated iterative method with super linear convergence of order 2 is Newton's method, whose iterations read as the following discrete-time system

$$z_{k+1} = z_k + F(z_k), \quad k \in \mathbb{Z}, k \geqslant 0, \tag{8}$$

where $z_0 \in \mathbb{C}$ is the initial estimate and $F(z) = -\frac{f(z)}{f'(z)}$. This algorithm is the first in the class of Householder's methods (Householder, 1970), which are defined as

$$z_{k+1} = z_k + F_m(z_k), \quad k \in \mathbb{Z}, k \geqslant 0, \tag{9}$$

where $z_0 \in \mathbb{C}$ is the initial estimate, $m \in \mathbb{Z}$, $m \ge 1$, and F_m is given by

$$F_m(z) = m(\tfrac{1}{f(z)})^{(m-1)}((\tfrac{1}{f(z)})^{(m)})^{-1}. \tag{10}$$

The method (9) has super linear convergence of order $m+1$ in the neighborhood of each simple root z_e of f.

3. The continuous-time Newton method

Let $f \in \mathbb{R}[z]$. The continuous-time Newton method is given by the following continuous-time system

$$\dot{z}(t) = F(z(t)), \quad t \in \mathbb{R}, t \geqslant 0, \tag{11a}$$

$$z(0) = z_0, \tag{11b}$$

where $z_0 \in \mathbb{C}$ is the initial estimate, $F(z) = -\frac{f(z)}{f'(z)}$ and $z(t) \in \mathbb{C}$ is the approximation at time t of one of the n points in the following algebraic variety:

$$\mathcal{R} = \{z \in \mathbb{C} : \quad f(z) = 0\}.$$

Note that the continuous-time system (11) is not defined at each $z \in \mathcal{S}$, where

$$\mathcal{S} = \{z \in \mathbb{C} : \quad f'(z) = 0\},$$

is the set of all the *singular* points of the Newton method (11). To overcome this issue, consider the desingularized system

$$\begin{aligned} \dot{z}(t) &= f'(z(t)) f'(\bar{z}(t)) F(z(t)) \\ &= -f'(\bar{z}(t)) f(z(t)), \quad t \in \mathbb{R}, t \geqslant 0, \end{aligned} \tag{12}$$

which has been obtained from (11) by multiplying its right hand side by $f'(z)f'(\bar{z}) = |f(z)|^2$. Since $f'(z)f'(\bar{z}) \in \mathbb{R}$, $f'(z)f'(\bar{z}) \ge 0$, for all $z \in \mathbb{C}$, and $f'(z)f'(\bar{z}) = 0$ if and only if $f'(z) = 0$, the orbits of the two systems (11) and (12) are the same, although they are traveled with a different time-behavior (but with the same direction for increasing time), with the exception of the points in $\mathcal{S}$, wherein the solution of system (11) is not defined, but which are isolated equilibria (whence orbits) of system (12), due to desingularization.

The following Lemma provides a first characterization of the equilibria of the desingularized system (12).

Lemma 1. *Let f and f' be both square free. Then, the set of all the equilibria of system (12) is $\mathcal{R} \cup \mathcal{S}$. In particular, the equilibria in $\mathcal{R}$ are locally exponentially stable, whereas the equilibria in $\mathcal{S}$ are unstable, and, in particular, are saddle points.*

Proof. Since $f \in \mathbb{R}[z]$, if $z \in \mathcal{R} \cup \mathcal{S}$ then $\bar{z} \in \mathcal{R} \cup \mathcal{S}$. Thus, the set of all the equilibria of system (12) is $\mathcal{R} \cup \mathcal{S}$. Note that $\mathcal{R} \cap \mathcal{S} = \emptyset$ since, by assumption, f is square free. Thus, the equilibria in $\mathcal{R}$ and $\mathcal{S}$ can be considered disjointly to analyze their stability properties.

Define $\alpha = \Re(z) = \frac{1}{2}z + \frac{1}{2}\bar{z}$ and $\beta = \Im(z) = -\frac{\imath}{2}z + \frac{\imath}{2}\bar{z}$, so that $z = \alpha + \imath\beta$ and $\bar{z} = \alpha - \imath\beta$, with $\alpha, \beta \in \mathbb{R}$. Hence, the "complex" differential equation (12) is equivalent to the two following "real" differential equations:

$$\begin{aligned} \dot{\alpha}(t) &= \Re\big(-f'(\alpha(t) - \imath\beta(t)) f(\alpha(t) + \imath\beta(t))\big) \\ &= -\tfrac{1}{2} f'(\alpha(t) - \imath\beta(t)) f(\alpha(t) + \imath\beta(t)) \\ &\quad - \tfrac{1}{2} f'(\alpha(t) + \imath\beta(t)) f(\alpha(t) - \imath\beta(t)), \end{aligned} \tag{13a}$$

$$\begin{aligned} \dot{\beta}(t) &= \Im\big(-f'(\alpha(t) - \imath\beta(t)) f(\alpha(t) + \imath\beta(t))\big) \\ &= \tfrac{\imath}{2} f'(\alpha(t) - \imath\beta(t)) f(\alpha(t) + \imath\beta(t)) \\ &\quad - \tfrac{\imath}{2} f'(\alpha(t) + \imath\beta(t)) f(\alpha(t) - \imath\beta(t)). \end{aligned} \tag{13b}$$

The set of all the equilibria of (13) is $\mathcal{A}_{\mathcal{R}} \cup \mathcal{A}_{\mathcal{S}}$, where

$$\begin{aligned} \mathcal{A}_{\mathcal{R}} &= \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} : \alpha + \imath\beta \in \mathcal{R}\}, \\ \mathcal{A}_{\mathcal{S}} &= \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} : \alpha + \imath\beta \in \mathcal{S}\}. \end{aligned}$$

Since $\dot{z}(t) = F(z(t))$ if and only if $\dot{\bar{z}}(t) = F(\bar{z}(t))$, letting $f''(z) = \frac{\partial^2 f(z)}{\partial z^2}$, it can be easily obtained that the linearization of system (13) about an equilibrium $(\alpha_e, \beta_e) \in \mathcal{A}_{\mathcal{R}} \cup \mathcal{A}_{\mathcal{S}}$ is given by

$$\begin{aligned}\dot{\alpha}(t) = &-\Big(\Re(f(\alpha_e + \imath\beta_e)f''(\alpha_e + \imath\beta_e)\\ &+ f'(\alpha_e + \imath\beta_e)f'(\alpha_e - \imath\beta_e))\Big)\alpha(t)\\ &+ \Im(f(\alpha_e - \imath\beta_e)f''(\alpha_e + \imath\beta_e))\beta(t), \qquad (14a)\end{aligned}$$

$$\begin{aligned}\dot{\beta}(t) = &\Im(f(\alpha_e - \imath\beta_e)f''(\alpha_e + \imath\beta_e))\alpha(t)\\ &+ \Big(\Re(f(\alpha_e + \imath\beta_e)f''(\alpha_e + \imath\beta_e))\\ &- f'(\alpha_e + \imath\beta_e)f'(\alpha_e - \imath\beta_e)\Big)\beta(t). \qquad (14b)\end{aligned}$$

Since both f and f' are square free, there does not exist $(\alpha_e + \imath\beta_e) \in \mathcal{A}_r \cup \mathcal{A}_s$ such that either $f(\alpha_e + \imath\beta_e) = 0 \wedge f'(\alpha_e + \imath\beta_e) = 0$ or $f'(\alpha_e + \imath\beta_e) = 0 \wedge f''(\alpha_e + \imath\beta_e) = 0$. Thus, if $(\alpha_e, \beta_e) \in \mathcal{A}_{\mathcal{R}}$, then the dynamic matrix of system (14) has two eigenvalues equal to $-|f'(\alpha_e + \imath\beta_e)|^2 < 0$, whereas if $(\alpha_e, \beta_e) \in \mathcal{A}_{\mathcal{S}}$, then the eigenvalues of the dynamical matrix of system (14) are $|f(\alpha_e + \imath\beta_e)f''(\alpha_e - \imath\beta_e)| > 0$ and $-|f(\alpha_e + \imath\beta_e)f''(\alpha_e - \imath\beta_e)| < 0$. Therefore, the equilibria in $\mathcal{R}$ are locally exponentially stable, whereas the equilibria in $\mathcal{S}$ are unstable for (12), and, in particular, the equilibria in $\mathcal{S}$ are saddle points. □

Let $\varsigma(t, z_0)$ denote the solution of system (12) at time $t \in \mathbb{R}$ with initial condition $z_0 \in \mathbb{C}$. On the basis of Lemma 1, the following Theorem characterizes the convergence of the desingularized Newton method to the roots of f.

Theorem 1. *Let f and f' be both square free. Then, there exists a set $\mathcal{N}$ in $\mathbb{C}$ such that $\lim_{t\to+\infty} \mathrm{dist}(\varsigma(t, z_0), \mathcal{R}) = 0$ for all $z_0 \in \mathbb{C} \setminus \mathcal{N}$.*

Proof. Consider the function $V(z) = f(z)f(\bar{z})$, whose time derivative along the trajectories of system (12) is

$$\dot{V}(z) = -2f'(z)f'(\bar{z})V(z).$$

Note that V is radially unbounded in $\mathbb{C}$. In the (α, β)-coordinates, such functions can be rewritten as $V(\alpha, \beta) = |f(\alpha + \imath\beta)|^2$ and $\dot{V}(\alpha, \beta) = -2|f'(\alpha + \imath\beta)|^2 V(\alpha, \beta)$. Thus, by LaSalle invariance principle (LaSalle, 1960, Thm. 3), every bounded solution of (13) approaches $\mathcal{A}_{\mathcal{R}} \cup \mathcal{A}_{\mathcal{S}}$ as $t \to \infty$.

Let $r > 0$ be such that $\mathcal{A}_{\mathcal{R}} \cup \mathcal{A}_{\mathcal{S}} \subset r\mathcal{B}$, where $r\mathcal{B}$ denotes the ball centered at 0 with radius r. Since $V(\alpha, \beta) > 0$ for all $(\alpha, \beta) \in \mathbb{R}^2 \setminus r\mathcal{B}$, $\dot{V}(\alpha, \beta) \le 0$ for all $(\alpha, \beta) \in \mathbb{R}^2 \setminus r\mathcal{B}$, and V is radially unbounded, by LaSalle (1960, Thm. 4), each solution of the desingularized system (12) is bounded for $t \ge 0$.

Therefore, the statement follows by the following facts: (i) all the equilibria in $\mathcal{S}$ are unstable for system (12) by Lemma 1; (ii) the vector field governing the dynamics (13) is locally Lipschitz in the neighborhood of each $(\alpha, \beta) \in \mathbb{R}^2$ and its orbits do not intersect except at equilibria; (iii) the points in $\mathcal{R}$ are locally exponentially stable for system (12). □

By Lemma 1, Theorem 1 and the correspondence between the orbits of systems (11) and (12), there exists a set $\mathcal{N}$ in $\mathbb{C}$ such that solutions of system (11) initialized over such a set do not converge to $\mathcal{R}$. In order to characterize such a set, the following Proposition provides a set containing the trajectory of system (11) passing through a point $z_0 \in \mathbb{C} \setminus \mathcal{S}$.

Proposition 1. *Given $z_0 \in \mathbb{C} \setminus \mathcal{S}$, the trajectory of system (11) passing through z_0 belongs to the set*

$$T(z_0) = \{z \in \mathbb{C} : \quad \exists k \in \mathbb{R} \mid f(z) = kf(z_0)\}. \qquad (15)$$

Proof. Let $\omega(t, z_0)$ denote the maximal solution of system (11) at time $t \in \mathbb{R}$ satisfying $\omega(0, z_0) = z_0$, $z_0 \in \mathbb{C} \setminus \mathcal{S}$, whose time domain is denoted $\mathrm{dom}(\omega(\cdot, z_0))$. Note that since the right-hand side of Eq. (11) is locally Lipschitz in the neighborhood of each $z_0 \in \mathbb{C} \setminus \mathcal{S}$, by Khalil (2002, Thm. 3.1), there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \in \mathrm{dom}(\omega(\cdot, z_0))$ for all $z_0 \in \mathbb{C} \setminus \mathcal{S}$. The continuous-time system (11) can be rewritten as

$$\frac{f'(z)}{f(z)}\mathrm{d}z = -\mathrm{d}t. \qquad (16)$$

By solving (16) and considering that $F(z)$ is locally bounded about any point $z \in \mathbb{C} \setminus \mathcal{S}$, one has that $\mathrm{dom}(\omega(\cdot, z_0))$ is open for all $z_0 \in \mathbb{C} \setminus \mathcal{S}$ and $f(\omega(t, z_0)) = \mathrm{e}^{-t}f(z_0)$ for all $z_0 \in \mathbb{C} \setminus \mathcal{S}$ and all $t \in \mathrm{dom}(\omega(\cdot, z_0))$. □

Corollary 1. *Let f be square free. Define the set*

$$T^{\ge 1}(z_0) = \{z \in \mathbb{C} : \quad \exists k \ge 1 \mid f(z) = kf(z_0)\}.$$

The set $\mathcal{N}$ is a subset of $\bigcup_{z_s \in \mathcal{S}} T^{\ge 1}(z_s)$.

Proof. By the proof of Proposition 1, one has that

$$f(\omega(t, z_0)) = \mathrm{e}^{-t}f(z_0), \ \forall t \in \mathrm{dom}(\omega(\cdot, z_0)), \qquad (17)$$

for all $z_0 \in \mathbb{C} \setminus \mathcal{S}$. Letting $\hat{z}$ be any point in $\mathcal{S}$, suppose, by absurd, that $\exists z_0 \in \mathbb{C} \setminus \mathcal{S}$ such that $\mathrm{dom}(\omega(\cdot, z_0))$ is unbounded from above and $\lim_{t\to+\infty} \omega(t, z_0) = \hat{z}$. This implies that $\lim_{t\to+\infty} f(\omega(t, z_0)) = 0$, which is in contradiction with the hypothesis that f is square free, *i.e.*, $\mathcal{R} \cap \mathcal{S} = \emptyset$. Thus, solutions to system (11) converging to $\mathcal{S}$ tend to such a set in finite time, *i.e.*, if $z_0 \in \mathcal{N}$, then there exists a finite $T \in \mathbb{R}$, $T \ge 0$, such that $\lim_{t\to T} \omega(t, z_0) = \hat{z}$, for some $\hat{z} \in \mathcal{S}$. By considering that (17) holds, if $\lim_{t\to T} \omega(t, z_0) = \hat{z}$, *i.e.*, by continuity, $f(\hat{z}) = \mathrm{e}^{-T}f(z_0)$, then $\exists k \in \mathbb{R}$, $k \ge 1$, such that $f(z_0) = kf(\hat{z})$, thus proving the statement. Namely, such a constant is given by $k = \mathrm{e}^{T}$. □

A direct consequence of Corollary 1 is that the set $\mathcal{N}$ has zero measure. In fact, one has $\mathcal{N} \subset \bigcup_{z_0 \in \mathcal{S}} T^{\ge 1}(z_0) \subset \bigcup_{z_0 \in \mathcal{S}} T(z_0)$ and the set $T(z_0)$, $z_0 \in \mathcal{S}$, is the root locus of a polynomial having complex coefficients, which is a set of zero measure in $\mathbb{C}$ (Dòria-Cerezo & Bodson, 2013).

Corollary 2. *If f is square free, then the basin of attraction of each equilibrium of system (11) in $\mathcal{R}$ is unbounded.*

Proof. By the proofs of Proposition 1 and Corollary 1, one has that (17) holds for any $z_0 \in \mathbb{C} \setminus \mathcal{S}$. Thus, one has $\lim_{t\to-\infty} |f(\omega(t, z_0))| = +\infty$, for all $z_0 \in \mathbb{C} \setminus (\mathcal{S} \cup \mathcal{R})$ such that $\mathrm{dom}(\omega(\cdot, z_0))$ is unbounded from below. Therefore, since $\lim_{|z|\to+\infty} |f(z)| = +\infty$ and there does not exist a finite $\hat{z} \in \mathbb{C}$ such that $\lim_{z\to\hat{z}} |f(z)| = +\infty$, one has that $\lim_{t\to-\infty} |\omega(t, z_0)| = +\infty$, for all $z_0 \in \mathbb{C} \setminus (\mathcal{S} \cup \mathcal{R})$ such that $\mathrm{dom}(\omega(\cdot, z_0))$ is unbounded from below. Hence, since (17) holds, hence ruling out finite escape time of the backward solutions to system (11), either $\omega(t, z_0)$ converges to $\mathcal{S}$ in finite time moving backward, or its domain is unbounded from below and $\lim_{t\to-\infty} |\omega(t, z_0)| = +\infty$, for all $z_0 \in \mathbb{C} \setminus (\mathcal{S} \cup \mathcal{R})$. Following the same reasoning used to prove Corollary 1, the set of all the $z_0 \in \mathbb{C} \setminus (\mathcal{S} \cup \mathcal{R})$ such that $\omega(t, z_0)$ converges in finite time to $\mathcal{S}$ moving backward in time is contained in $\bigcup_{z_s \in \mathcal{S}} T^{\le 1}(z_s)$, where

$$T^{\le 1}(z_0) = \{z \in \mathbb{C} : \quad \exists k \in (0, 1) \mid f(z) = kf(z_0)\}.$$

Thus, given any root $z_e \in \mathcal{R}$ of f and selecting $z_0 \notin \bigcup_{z_s \in \mathcal{S}} T^{\le 1}(z_s)$ in the basin of attraction of z_e, that is an open set by the proof of Theorem 1, the corresponding solution satisfies $\lim_{t\to-\infty} |\omega(t, z_0)| = +\infty$. □

By Corollary 1 and considering that $T(z_0)$ is, by definition (15), the root locus of a polynomial having complex coefficients, the set $\mathcal{N}$ can be determined graphically by plotting the root locus of some polynomials in $\mathbb{C}[z, k]$. Further, by Corollary 2, the set $\mathcal{N}$ divides $\mathbb{C}$ into n unbounded regions, within which a root in $\mathcal{R}$ is globally attractive. Techniques to plot such a root locus are provided in Section 5.

4. The continuous-time Householder methods

The continuous-time counterpart of the iterative, discrete-time, the Householder method (9) is

$$\dot{z}(t) = F_m(z(t)), z(0) = z_0, \tag{18}$$

where $z_0 \in \mathbb{C}$ is the initial estimate. The following Proposition provides two different expressions for the function F_m alternative to (10).

Proposition 2. *Define recursively the functions $f_2, \ldots, f_m$ as*

$$f_{i+1}(z) = \frac{f_i(z)}{\sqrt[i+1]{f_i'(z)}}, \quad i = 1, \ldots, m-1, \tag{19}$$

with $f_1(z) = f(z)$. Then, one has that

$$F_m(z) = -m\frac{f_m^m(z)}{(f_m^m)'(z)} = -\frac{f_m(z)}{f_m'(z)}.$$

Proof. Letting f_m be defined recursively as in (19), it can be easily derived, using the chain rule, that

$$f_m^m(z) = (-1)^{m+1}(m-1)!((\tfrac{1}{f(z)})^{(m-1)})^{-1}, \tag{20}$$

whence

$$(f_m^m(z))' = (-1)^m(m-1)!((\tfrac{1}{f(z)})^{(m-1)})^{-2}(\tfrac{1}{f(z)})^{(m)}.$$

This implies that F_m can be equivalently expressed as $F_m(z) = -m\frac{f_m^m(z)}{(f_m^m(z))'}$, thus proving the first equality. The second one follows by the fact that $mf_m^m(z)f_m'(z) - f_m(z)(f_m^m)'(z) = 0$. □

A consequence of Proposition 2 is that the Householder method (9) is identical to the Gerlach method (Gerlach, 1994) and hence to the basic family described in Kalantari (2008).

Since $F_m(z) = -\frac{f_m(z)}{f_m'(z)}$, where f_m is defined recursively as in (19), system (18) is the continuous-time Newton method (11) applied to the algebraic function f_m (equivalently, in view of Proposition 2, to the rational function f_m^m) rather than to f. In particular, by the equality given in (20) and considering the formula given in Shieh and Verghese (1967, Eq. (1)), it can be easily derived that

$$f_m^m(z) = \frac{f^m(z)}{\nu_m(z)}, \tag{21}$$

where $\nu_m \in \mathbb{R}[z]$. Thus, the set of the roots of f_m,

$$\mathcal{R}_m = \{z \in \mathbb{C} : \quad f_m(z) = 0\}$$

actually equals $\mathcal{R}$. Define the set of the *poles* of f_m as the set of points wherein the denominator of f_m^m vanishes,

$$\mathcal{P}_m = \{z \in \mathbb{C} : \quad \nu_m(z) = 0\}.$$

By considering that $\frac{\partial}{\partial z}f_m^m(z) = \frac{m\nu_m(z)f'(z) - f(z)\nu_m'(z)}{\nu_m^2(z)f(z)^{1-m}}$, one has that F_m can be equivalently written as

$$F_m(z) = -\frac{mf(z)\nu_m(z)}{m\nu_m(z)f'(z) - f(z)\nu_m'(z)}.$$

Therefore, the set of all the equilibria of system (18) is $\mathcal{R}_m \cup \mathcal{P}_m$. Furthermore, following a reasoning wholly similar to that used in Section 3 to analyze the Newton method (11), let

$$\mathcal{S}_m = \{z \in \mathbb{C} : \quad m\nu_m(z)f'(z) - f(z)\nu_m'(z) = 0\},$$

be the set of the *singular* points of F_m. To characterize the behavior of system (18), consider the desingularized system obtained by multiplying the right-hand side of (18) for its denominator and its complex conjugate

$$\begin{aligned}\dot{z}(t) &= -mf(z(t))\nu_m(z(t)) \cdot \\ &\quad \cdot (m\nu_m(\bar{z}(t))f'(\bar{z}(t)) - f(\bar{z}(t))\nu_m'(\bar{z})(t)).\end{aligned} \tag{22}$$

Letting $\varsigma_m(t, z_0)$ denote the solution of system (22) at time $t \in \mathbb{R}$ with initial condition $z_0 \in \mathbb{C}$, the following Theorem characterizes the behavior of the desingularized system (22).

Theorem 2. *Suppose that the pairs of polynomials (f, ν_m), (f, f'), (ν_m, ν_m') are all coprime and that $|\nu_m(z)||m^2\nu_m(z)f'(z)^2 + mf(z)\nu_m(z)f''(z) - m\nu_m(z)f'(z)^2 - f(z)^2\nu_m''(z)| \neq 0$ for all $z \in \mathcal{P}_m$. Then, the set of all the equilibria of system (22) is $\mathcal{R}_m \cup \mathcal{S}_m \cup \mathcal{P}_m$. The equilibria in $\mathcal{R}_m$ are locally exponentially stable, whereas the ones in $\mathcal{S}_m \cup \mathcal{P}_m$ are unstable. Furthermore, there exists a set $\mathcal{N}_m$ in $\mathbb{C}$ such that $\lim_{t\to+\infty} \mathrm{dist}(\varsigma_m(t, z_0), \mathcal{R}) = 0$ for all $z_0 \in \mathbb{C} \setminus \mathcal{N}_m$.*

Proof. Inspecting the dynamics (22), one has that the set of all the equilibria of system (22) is $\mathcal{R}_m \cup \mathcal{S}_m \cup \mathcal{P}_m$.

Following the same construction employed in the proof of Theorem 1, let $\alpha = \frac{1}{2}z + \frac{1}{2}\bar{z}$ and $\beta = -\frac{\imath}{2}z + \frac{\imath}{2}\bar{z}$, so that $z = \alpha + \imath\beta$ and $\bar{z} = \alpha - \imath\beta$. By noticing that the dynamics of $\bar{z}$ are given by

$$\begin{aligned}\dot{\bar{z}}(t) &= -f(\bar{z}(t))\nu_m(\bar{z}(t)) \cdot \\ &\quad \cdot (m\nu_m(z(t))f'(z(t)) - f(z(t))\nu_m'(z(t))),\end{aligned}$$

the equilibria of the system analogous to (13) in the (α, β)-coordinates is $\mathcal{A}_{\mathcal{R}_m} \cup \mathcal{A}_{\mathcal{S}_m} \cup \mathcal{A}_{\mathcal{P}_m}$, where

$$\begin{aligned}\mathcal{A}_{\mathcal{R}_m} &= \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} : \alpha + \imath\beta \in \mathcal{R}_m\}, \\ \mathcal{A}_{\mathcal{S}_m} &= \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} : \alpha + \imath\beta \in \mathcal{S}_m\}, \\ \mathcal{A}_{\mathcal{P}_m} &= \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} : \alpha + \imath\beta \in \mathcal{P}_m\}.\end{aligned}$$

By analyzing the linearization in the neighborhood of points $(\alpha_e, \beta_e) \in \mathcal{A}_{\mathcal{R}_m} \cup \mathcal{A}_{\mathcal{S}_m} \cup \mathcal{A}_{\mathcal{P}_m}$ of the system in the (α, β)-coordinates, it can be observed that its dynamic matrix has

- two negative eigenvalues for $(\alpha_e, \beta_e) \in \mathcal{A}_{\mathcal{R}_m}$;
- a negative and a positive eigenvalue for $(\alpha_e, \beta_e) \in \mathcal{A}_{\mathcal{S}_m}$;
- two positive eigenvalues for $(\alpha_e, \beta_e) \in \mathcal{A}_{\mathcal{P}_m}$.

Therefore, equilibria in $\mathcal{R}_m$ are locally exponentially stable, whereas equilibria in $\mathcal{S}_m \cup \mathcal{P}_m$ are unstable.

Consider now the function $V(z) = f_m^m(z)f_m^m(\bar{z})$. One has $\dot{V}(z) = -2(m\nu_m(z)f'(z) - f(z)\nu_m'(z))(m\nu_m(\bar{z})f'(\bar{z}) - f(\bar{z})\nu_m'(\bar{z}))V(z)$. Therefore, a reasoning wholly similar to the one used in the proof of Theorem 1 can be used to conclude the statement. □

By Theorem 2, also when dealing with the continuous-time Householder method (18), there exists a set $\mathcal{N}_m$ in $\mathbb{C}$ such that the solutions of system (18) initialized in such a set do not converge to $\mathcal{R}_m$. To characterize this set, the next Proposition provides a set containing the trajectory of system (18) that passes through $z_0 \in \mathbb{C}$.

Proposition 3. *Given $z_0 \in \mathbb{C} \setminus \mathcal{S}_m$, the trajectory of system (18) passing through z_0 belongs to the set*

$$\begin{aligned}T_m(z_0) = \{&z \in \mathbb{C} : \quad \exists k \in \mathbb{R} | \\ &f^m(z)\nu_m(z_0) = k\nu_m(z)f^m(z_0)\}.\end{aligned}$$

Proof. Let $\omega_m(t, z_0)$ denote the maximal solution of system (18) at time $t \in \mathbb{R}$ with initial condition $z_0 \in \mathbb{C} \setminus \mathcal{S}_m$, whose time domain is denoted $\mathrm{dom}(\omega_m(\cdot, z_0))$. By Proposition 2, Eq. (18) can be rewritten as

$$\frac{f_m^m(z)}{(f_m^m)'(z)}\mathrm{d}z = -\frac{1}{m}\mathrm{d}t. \tag{23}$$

By solving (23), one obtains that $f_m^m(\omega_m(t, z_0)) = \exp(-\frac{t}{m})f_m^m(z_0)$ for all $z_0 \in \mathbb{C} \setminus \mathcal{S}_m$ and all $t \in \mathrm{dom}(\omega_m(\cdot, z_0))$. Thus, the statement follows by (21). □

Corollary 3. *Define the set*

$$\begin{aligned}T_m^{\geq 1}(z_0) = \{&z \in \mathbb{C} : \quad \exists k \geq 1 \mid \\ &f^m(z)\nu_m(z_0) = k\nu_m(z)f^m(z_0)\}.\end{aligned}$$

The set $\mathcal{N}_m$ is a subset of $\mathcal{P}_m \bigcup_{z_s \in \mathcal{S}_m} T_m^{\geq 1}(z_s)$.

Proof. First note that if z_0 is such that $\nu_m(z_0) = 0$, then $T_m(z_0) = \mathcal{P}_m$ since, in this case, one has that $f^m(z)\nu_m(z_0) = k\nu_m(z)f^m(z_0)$ if and only if $\nu_m(z) = 0$ independently of k. Then, the statement follows by a reasoning wholly similar to the one used to prove Corollary 3. □

By the same reasoning given in Section 3, Corollary 3 can be used to conclude that the set $\mathcal{N}_m$ has zero measure. Moreover, similarly to what happens for the classical Newton method, the set $\mathcal{N}_m$ divides $\mathbb{C}$ into n regions within which a root in $\mathcal{R}$ is globally attractive. However, when dealing with Householder's method, these domains need not be unbounded (see, *e.g.*, the subsequent Example 3). By Corollary 3 and considering that $T_m(z_0)$ is the root locus of a polynomial having complex coefficients, the region $\mathcal{N}_m$ can be determined graphically by plotting the root locus of a polynomial in $\mathbb{C}[z, k]$, being linear in k. Techniques to plot such a root locus are provided in the next Section 5.

5. Root locus of polynomials having complex coefficients

As shown in Sections 3 and 4, the root locus of polynomials having complex coefficients is an useful tool to characterize the basin of attraction of continuous-time Newton's and higher order Householder's methods. The main goal of this section is to recall some rules (see Dòria-Cerezo & Bodson, 2013) to qualitatively plot this locus.

5.1. General rules

Given a polynomial $\phi(z, k) \in \mathbb{C}[z, k]$, its root locus (Evans, 1950) is defined as the set of all $\hat{z} \in \mathbb{C}$ such that $\phi(\hat{z}, \hat{k}) = 0$, for some $\hat{k} \in \mathbb{R}$. The following Assumption is made all throughout this section.

Assumption 1. The polynomial ϕ is linear in k and its leading monomial according to the lexicographic ordering with $z \succ k$ is independent of k.

To illustrate the rules to draw the root locus of a polynomial $\phi \in \mathbb{C}[z, k]$, linear in k, let $p \in \mathbb{C}[z]$ and $q \in \mathbb{C}[z]$ be such that $\phi(z, k) = p(z)+k\,q(z)$, where $\phi(z, k)$ is the polynomial whose root locus has to be plotted as k varies over $\mathbb{R}$. Letting n and w be the degrees of p and q, respectively, with $n > w$ by Assumption 1, such polynomials can be written as

$$p(z) = c\prod_{i=1}^{n}(z - \zeta_i),\ q(z) = \ell\prod_{j=1}^{w}(z - \varrho_j), \tag{24}$$

where $\zeta_1, \dots, \zeta_n$ and $\varrho_1, \dots, \varrho_w$ are the roots of p and q, respectively, and $|c| \neq 0$, $|\ell| \neq 0$. As in the classical root locus, the points $\hat{z} \in \mathbb{C}$ belonging to the root locus of $\phi(z, k)$ that are multiple roots of $\phi(z, \hat{k})$ for some $\hat{k} \in \mathbb{R}$ are called *singular points*. Since the curves to be plotted can be considered as implicit functions of the parameter k in the equation $\phi(z, k) = 0$, the following Rule follows directly by the implicit function theorem (Krantz & Parks, 2002).

Rule 1. *The set of the singular points of the root locus are those satisfying the following algebraic relation*

$$\phi(z, k) = 0,\ \tfrac{\partial}{\partial z}\phi(z, k) = 0. \tag{25}$$

Further, note that $f(z, 0) = p(z)$. Therefore, the following Rule immediately follows.

Rule 2. *The roots $\zeta_1, \dots, \zeta_n$ of p belong to the root locus.*

Since, by assumption, $n > w$, one has that $f(z, \hat{k}) \in \mathbb{C}[z]$ for each fixed value of $\hat{k} \in \mathbb{R}$. Therefore, the following Rule immediately follows.

Rule 3. *The root locus is composed by n branches for $k \in (0, \infty)$ and n branches for $k \in (-\infty, 0)$.*

The following Rule describes the behavior of the locus when k tends to infinity.

Rule 4. *If ϱ_s is a zero of q with multiplicity μ_s, then μ_s branches of the root locus tend to ϱ_s as $k \to \infty$ and μ_s branches tend to ϱ_s as $k \to -\infty$. The remaining $2(n - w)$ branches tend to infinity as $|k| \to \infty$.*

Proof. By using the expressions given in (24), the equation $\phi(z, k) = 0$ can be rewritten as

$$c\prod_{i=1}^{n}(z - \zeta_i) = -k\ell\prod_{j=1}^{w}(z - \varrho_j). \tag{26}$$

By taking the absolute value and the argument of such an expression, one obtains that

$$|c|\prod_{i=1}^{n}|z - \zeta_i| = |k||\ell|\prod_{j=1}^{w}|z - \varrho_j|, \tag{27a}$$

$$\arg(c) + \sum_{i=1}^{n}\arg(z - \zeta_i) = \pi + \arg(k) + \arg(\ell) + \sum_{j=1}^{w}\arg(z - \varrho_j) + 2h\pi, \tag{27b}$$

where $h \in \mathbb{Z}, h \geqslant 0$. Let ϱ_s be a root of q and let μ_s be its multiplicity. Without loss of generality, suppose that $\varrho_1 = \cdots = \varrho_{\mu_s} = \varrho_s$, whereas $\varrho_j \neq \varrho_s$ for $j = \mu_s + 1, \dots, w$. Then $\sum_{j=1}^{w}\arg(z - \varrho_j) - \sum_{i=1}^{n}\arg(z - \zeta_i) = \mu_s\arg(z - \varrho_s) + \sum_{j=\mu_s+1}^{w}\arg(z - \varrho_j) - \sum_{i=1}^{n}\arg(z - \zeta_i)$. To shorten the notation, let $\psi(z) = \sum_{i=1}^{n}\arg(z-\zeta_i) - \sum_{j=\mu_s+1}^{w}\arg(z-\varrho_j)$. Let $\mathcal{D}_{\varrho_s,\delta}$ be the circle of radius δ centered at ϱ_s, and consider the condition (27b), which can be rewritten as

$$\mu_s\arg(z - \varrho_s) = \psi(z) + \arg(c) - \arg(k) - \arg(\ell) - \pi - 2h\pi. \tag{28}$$

By considering the phase condition (28) over $\mathcal{D}_{\varrho,\delta}$, it can be easily derived that as $\delta \to 0$ one has $\psi(z) \to \psi(\varrho_s)$, whereas $\arg(z - \varrho_s)$ can attain any value in $[0, 2\pi)$. Therefore, if δ is sufficiently small, the root locus has ν_s intersections with $\mathcal{D}_{\varrho,\delta}$, thus showing, by the absolute value condition (27a), that μ_s branches of the locus tend to ϱ_s as $k \to \infty$ and μ_s branches of the locus tend to ϱ_s as $k \to -\infty$. Wholly similar arguments can be used to conclude that $2(n - w)$ branches tend to infinity. □

As a byproduct of the proof of Rule 4, the formula (28) can be used to obtain the angle of arrival at a zero of q as $|k| \to \infty$. Further, the expression (27b) can be used to determine the angle of departure of the root locus from the roots of p and from the singular points. In particular, using the convention $\arg(0) = 0$, the argument (namely, the angle formed with the positive real axis) of the root locus about a root ϱ_s of q of multiplicity μ_s is

$$\tfrac{1}{\mu_s}\Big(\arg(c) - \arg(\ell) - (2h + 1)\pi + \arg(k) + \sum_{i=1}^{n}(\varrho_s - \zeta_i) - \sum_{j=1}^{w}(\varrho_s - \varrho_j)\Big), \tag{29}$$

for $h = 0, \dots, \mu_s - 1$, whereas the argument of the root locus about a root ζ_s of p multiplicity θ_s is

$$\frac{1}{\theta_s}\Big(\arg(\ell) - \arg(c) + (2h+1)\pi - \arg(k) - \sum_{i=1}^{n}(\varrho_s - \zeta_j) + \sum_{j=1}^{w}(\varrho_s - \varrho_j)\Big), \tag{30}$$

for $h = 0, \dots, \theta_s - 1$. By noticing that if $\hat{z} \in \mathbb{C}$ is a singular point of the root locus for some $\hat{k} \in \mathbb{R}$, then it is a multiple root of the polynomial $p + \hat{k}q$, letting $\hat{\zeta}_1, \dots, \hat{\zeta}_n$ be the roots of $p + \hat{k}q$, using (30), one has that the argument of the root locus about a singular point $\hat{z}_s$ of multiplicity λ_s is

$$\frac{1}{\lambda_s}\Big(\arg(\ell) - \arg(c) + (2h+1)\pi - \arg(k) - \sum_{i=1}^{n}(\hat{z}_s - \hat{\zeta}_j) + \sum_{j=1}^{w}(\hat{z}_s - \varrho_j)\Big), \tag{31}$$

for $h = 0, \dots, \lambda_s - 1$.

The following rule (which matches with Dòria-Cerezo & Bodson, 2013, Rule 3) provides further insight on the branches of the root locus going to infinity.

Rule 5. *Each of the $2(n-w)$ branches of the root locus that do not converge to a zero of q tends to infinity along an asymptote. Such $2(n-w)$ asymptotes all pass through*

$$z^\circ = \frac{1}{n-w}\Big(\sum_{i=1}^{n}\zeta_i - \sum_{j=1}^{w}\varrho_j\Big), \tag{32}$$

and form the following angles with the positive real axis:

$$\begin{cases} \frac{1}{n-w}\left(2h\pi - \arg\left(-\frac{c}{\ell}\right)\right), & \text{if } k \to +\infty, \\ \frac{1}{n-w}\left((2h+1)\pi - \arg\left(-\frac{c}{\ell}\right)\right), & \text{if } k \to -\infty, \end{cases} \tag{33}$$

with $h = 1, 2, \dots, n-w$.

Remark 1. Rules 4 and 5 are the key results enabling the design of Algorithm 1, given in the subsequent Section 7, which is capable of approximating all the complex root of a given polynomial. In fact, by such two rules, the root locus $T^{\geq 1}(z_0)$ can be approximated, for sufficiently large z, by a straight half-line. Therefore, since $\cup_{z_0 \in \mathcal{S}} T^{\geq 1}(z_0)$ contains the boundary $\mathcal{N}$ of the basin of attraction of each root of f, by choosing the initial guesses of Algorithm 1 sufficiently large and in the cone delimited by these half-lines, Algorithm 1 determines all the complex roots of the polynomial f.

5.2. Rules for the sets $T(z_0)$ and $T_m(z_0)$

The main goal of this section is to specialize the rules given in Section 5.1 for the analysis of $T(z_0)$ and $T_m(z_0)$, with $z_0 \in \mathcal{S}$ and $z_0 \in \mathcal{S}_m$, respectively.

First, consider the following four Facts, which follows directly from the definition of $T(z_0)$ and $T_m(z_0)$.

Fact 1. The root locus $T(\bar{z}_0)$ (respectively, $T_m(\bar{z}_0)$) is obtained by mirroring $T(z_0)$ (respectively, $T(\bar{z}_0)$) with respect to the real axis.

Fact 2. If $z_0 \in \mathbb{R}$, then $T(z_0)$ (respectively, $T_m(z_0)$) is symmetric with respect to the real axis. Further, given any $z_1, z_2 \in \mathbb{R}$, one has that $T(z_1) = T(z_2)$ (respectively, $T_m(z_1) = T_m(z_2)$).

The next two Facts show that the center of the asymptotes of the root loci $T(z_0)$ and $T_m(z_0)$ can be determined directly by inspecting f and f^m, ν_m, respectively.

Fact 3. The center of the asymptotes z° of $T(z_0)$ is independent of z_0. In particular, by writing

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n,$$

one has that $z^\circ = -\frac{a_1}{a_0 n}$.

Proof. The proof follows directly from Rule 5. □

Fact 4. The center of the asymptotes z° of $T_m(z_0)$ is independent of z_0. In particular, by writing

$$f^m(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n,$$
$$\nu_m(z) = b_0 z^w + b_1 z^{w-1} + \cdots + b_w,$$

one has that $z^\circ = \frac{1}{n-w}\left(\frac{b_1}{b_0} - \frac{a_1}{a_0}\right)$.

Proof. By Rule 5 and considering that, when analyzing $T_m(z_0)$, one has $p(z) = \nu_m(z_0)f^m(z)$ and $q = f^m(z_0)\nu_m(z)$, the statement follows by the fact that the roots of p and q are the same as f^m and ν_m, respectively. □

The following two Facts provide further insights on the singular points of the root loci $T(z_0)$ and $T_m(z_0)$.

Fact 5. The point $z_0 \in \mathcal{S}$ is a singular point of $T(z_0)$.

Proof. By definition, $z_0 \in \mathcal{S}$ is such that $f'(z_0) = 0$ and $f(z_0) = \hat{k}f(z_0)$, with $\hat{k} = 1$. Therefore, since $\frac{\partial}{\partial z}(f(z) - \hat{k}f(z_0)) = f'(z)$, such a point is a multiple root of the polynomial $f(z) - \hat{k}f(z_0)$, and hence it is a singular point of the root locus $T(z_0)$. □

Fact 6. The point $z_0 \in \mathcal{S}_m$ is a singular point of $T_m(z_0)$.

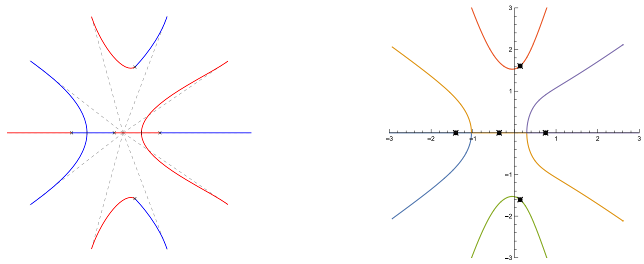
Proof. By definition, $z_0 \in \mathcal{S}$ is such that $d\nu_m(z_0)f'(z_0) - f(z_0)\nu_m'(z_0) = 0$ and $f^m(z_0)\nu_m(z_0) = k\nu_m(z_0)f^m(z_0)$, with $\hat{k} = 1$. Therefore, it is a multiple root of $f^m(z)\nu_m(z_0) - \hat{k}\nu_m(z)f^m(z_0)$ and hence it is a singular point of $T_m(z_0)$. □

The following Remarks show how the analyses carried out in this and the previous sections can be used to determine the sets $\mathcal{N}$ and $\mathcal{N}_m$, defined in Theorems 1 and 2, which delimit the basin of attraction of each root when using the continuous-time Newton (11) and Householder methods (18), respectively.

Remark 2. Since, by Fact 5, one has that $z_0 \in \mathcal{S}$ is a singular point of $T(z_0)$, the set $\mathcal{N}$ is constituted by the union, over $z_0 \in \mathcal{S}$, of the branches of each root locus $T(z_0)$ that start from the singular point z_0 (corresponding to $k = 1$) and that tend to infinity (by Rule 4) as $k \to +\infty$ along one of the asymptotes given in Rule 5.

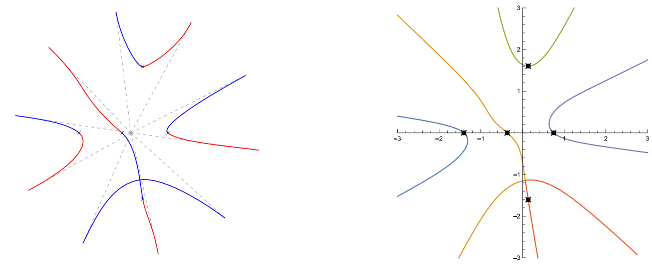
Remark 3. Since, by Fact 6, one has that $z_0 \in \mathcal{S}_m$ is a singular point of $T_m(z_0)$, the set $\mathcal{N}_m$ is constituted by the union, over $z_0 \in \mathcal{S}_m$, of the branches of each root locus $T_m(z_0)$ that start from the singular point z_0 (corresponding to $k = 1$) and that tend to either infinity or to a zero of ν_m (by Rule 4) as $k \to +\infty$.

Remark 4. By Rule 4, for sufficiently large $z \in \mathbb{C}$, the root loci $T(z_0)$ and $T_m(z_0)$ can be approximated via a set of straight half-lines that pass through the point z°, which can be determined exactly as detailed in Facts 3 and 4. Since such a point is independent of z_0, also the sets $\mathcal{N}$ and $\mathcal{N}_m$ can be approximated, for sufficiently large $z \in \mathbb{C}$, as a set of straight half-lines that pass through the point z°.



(a) Qualitative plot. (b) Actual plot.

Fig. 1. Root locus $T(\sigma_1)$.



(a) Qualitative plot. (b) Actual plot.

Fig. 2. Root locus $T(\sigma_3)$.

6. Examples

The main goal of this section is to show how the Rules and Facts presented in Section 5 can be used to determine the basin of attraction of each root of a polynomial $f(z)$ when using the continuous-time Newton and the Householder methods. All the qualitative root loci reported hereafter have been drawn by using cubic Bézier with two anchor points and two control points; the two anchor points coincide with the starting and ending points, whereas the two control points are used to impose the curvature at the starting and ending points according to the angles provided by the formulas given in Eqs. (29)–(31). The portion of the root locus corresponding to positive values of k has been drawn in blue, whereas the one corresponding to negative values of k has been drawn in red. Such qualitative plots represent what could be simply done by a hand drawing and are compared with the actual plots obtained using the `RootLocusPlot` command in the software `Mathematica`.

Example 1. Consider the polynomial

$$f(z) = -4z^5 - 3z^4 - 6z^3 - 10z^2 + 8z + 4.$$

The first derivative of f is $f'(z) = -20z^4 - 12z^3 - 18z^2 - 20z + 8$, whose approximate roots are

$$\mathcal{S} = \{-1.032, 0.297, 0.067 \pm 1.141\imath\}.$$

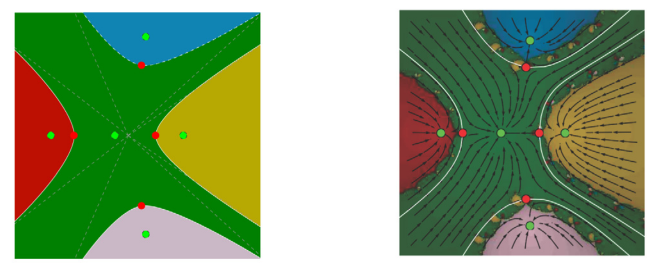
The main goal of this example is to characterize the basin of attraction of each root of $f(z)$ in $\mathbb{C}$ using the techniques outlined in Sections 3 and 5. Letting $\sigma_1 = -1.032$, the root locus $T(\sigma_1)$ can be drawn qualitatively using the rules given in Section 5. Let $p(z) = f(z)$ and $q(z) = f(\sigma_1) = 7.03215$ and note that the zeros of p are the same as the ones of f,

$$\mathcal{R} = \{-1.41, -0.366, 0.748, 0.14 \pm 1.60\imath\}$$

By Rule 1, the singular points of the root locus are $\{-1.032, 0.297\}$; by Rule 2, the set $\mathcal{R}$ is contained in the root locus, by Rule 3, it is composed by 5 branches; by Rule 4, all the 5 branches go to infinity as $|k| \to \infty$; by Rule 5, the center of the asymptotes is $z^\circ = -0.15$, and they form the angles $\{0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}\}$ with the positive real axis for $k \to +\infty$ and the angles $\{\frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}\}$ for $k \to -\infty$. These considerations allow to draw qualitatively the root locus $T(\sigma_1)$ as in Fig. 1.

Letting $\sigma_2 = 0.297$, note that $\sigma_1, \sigma_2 \in \mathbb{R}$ and $\text{sign}(f(\sigma_1)) = -\text{sign}(f(\sigma_2))$. Hence, the root locus $T(\sigma_2)$ equals $T(\sigma_1)$ with the blue and red portion swapped.

Consider now $\sigma_3 = 0.067 - 1.141\imath$. In order to draw qualitatively the root locus $T(\sigma_3)$, let $p(z) = f(z)$ and $q(z) = f(\sigma_3) = 11.8431 - 10.1405\imath$. By Rule 1, the root locus of $f(z) - kf(\sigma_3)$ has only one singular point, that is $0.0673 - 1.141\imath$; by Rule 2, the set $\mathcal{R}$ is contained in the root locus, by Rule 3, it is composed by 5 branches; by Rule 4, all the 5 branches go to infinity as $|k| \to \infty$; by Rule 5, the center of the asymptotes is $z^\circ = -0.15$, and they form the angles $\{0.486698, 1.74334, 2.99997, 4.25661, 5.51325\}$ with the positive real axis for $k \to +\infty$ and the angles $\{1.11502, 2.37165, 3.62829, 4.88493, 6.14156\}$ for $k \to -\infty$. These considerations allow to draw qualitatively the root locus $T(\sigma_3)$ as in Fig. 2.



(a) Qualitative plot. (b) Actual plot and Newton fractal.

Fig. 3. Basin of attraction of each root of f in Example 1. The points in $\mathcal{R}$ and in $\mathcal{S}$ are depicted in green and red, respectively. The stream plot of system (11) is depicted in black. The set $\mathcal{N}$ is represented in white in both plots. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

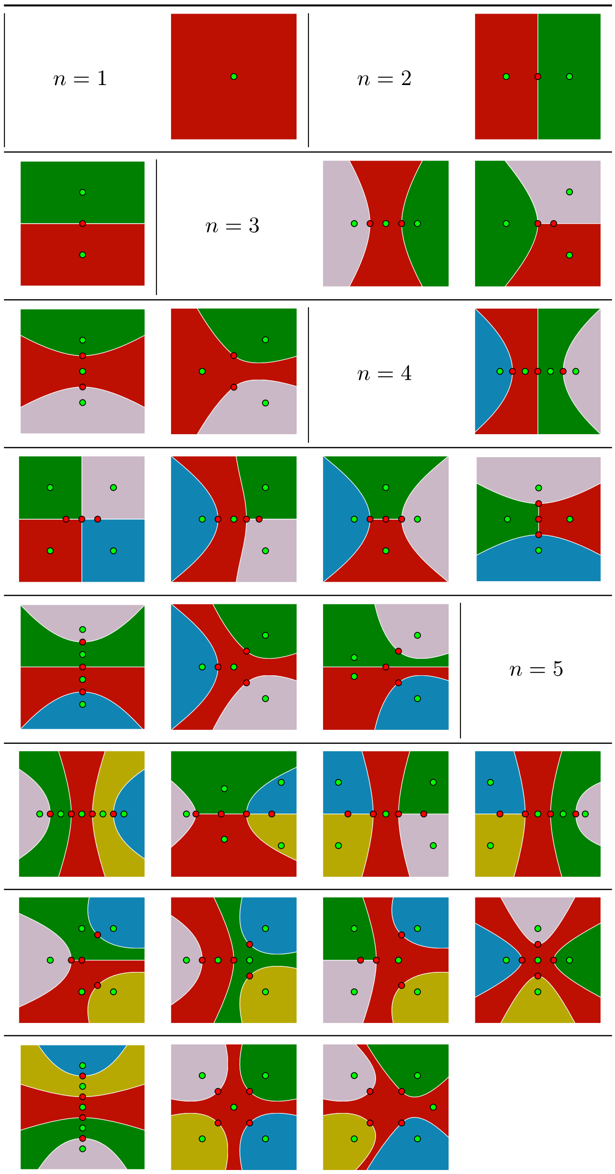
Finally, letting $\sigma_4 = 0.067 + 1.141\imath$, note that $\bar{\sigma}_4 = \sigma_3$. Thus, by Fact 1, the root locus $T(\sigma_4)$ is obtained by mirroring $T(\sigma_3)$ with respect to the real axis.

Following Remark 2, the set $\mathcal{N}$ can be obtained by joining the branches of the root loci drawn above that start from the singular point $z_0 \in \mathcal{S}$ and that tend to infinity as $k \to +\infty$. As detailed in Section 3, such a set subdivides $\mathbb{C}$ in 5 unbounded regions within each of which a root of f is attractive. Each of these region is the basin of attraction of a root in $\mathcal{R}$. Fig. 3 depicts such regions and compares them with the Newton fractal (Hubbard et al., 2001), that is the Julia set of the function $z \mapsto z + F(z)$, and with the trajectories of the continuous-time Newton method (11).

As shown by such a figure, the qualitative basin of attraction, determined by using the tools given in Section 5, is very close to the actual one. Furthermore, it can be observed that, by Remark 4, the set $\mathcal{N}$ can be approximated, for sufficiently large $z \in \mathbb{C}$, as a set of straight lines that pass through the point z°. This result is exploited in Section 7 to design an iterative method capable of determining all the roots of a polynomial.

Using a reasoning wholly similar to that employed in Example 1, one can determine the qualitative shape of the basin of attraction of each root of f when using Newton's method on the basis of the location of the points in $\mathcal{R}$ and in $\mathcal{S}$, which determine the shape of the root loci $T(z_0)$, $z_0 \in \mathcal{S}$. Table 1 reports these shapes for polynomials whose degree is lower than or equal to 5.

Table 1
Qualitative shapes of the basin of attraction of the roots of f when using Newton's method. Points in $\mathcal{R}$ are represented in green, whereas points in $\mathcal{S}$ are represented in red. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



Example 2. Consider the polynomial given in Example 1. The main goal of this example is to characterize the basin of attraction of each root of f using the continuous-time Halley method, that is the Householder method with $m = 2$. The function f_m equals

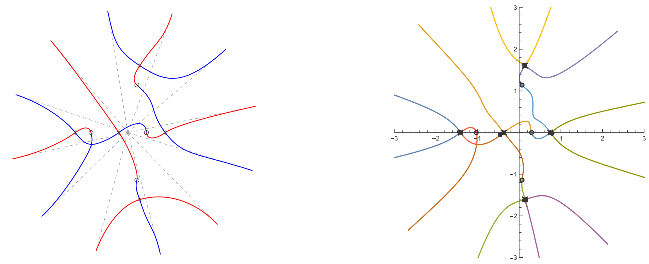
$$f_m(z) = -\frac{4z^5+3z^4+6z^3+10z^2-8z-4}{\sqrt{2}\sqrt{-10z^4-6z^3-9z^2-10z+4}},$$

and hence one has

$$F_m(z) = \frac{-(10z^4+6z^3+9z^2+10z-4)(4z^5+3z^4+6z^3+10z^2-8z-4)}{120z^8+144z^7+249z^6+315z^5+243z^4+296z^3+114z^2-84z+52}.$$

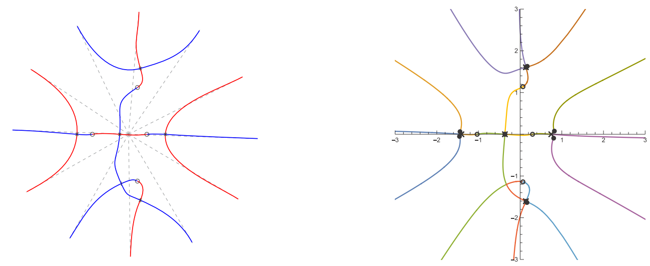
By computing the sets $\mathcal{R}_m$, $\mathcal{S}_m$, and $\mathcal{P}_m$ one obtains:

$$\begin{aligned}\mathcal{R}_m &\simeq \{-1.41, -0.366, 0.14 \pm 1.60\imath, 0.748\},\\ \mathcal{S}_m &\simeq \{-1.023 \pm 0.283\imath, -0.328 \pm 1.202\imath,\\ &\qquad 0.280 \pm 0.312\imath, 0.470 \pm 1.089\imath\},\\ \mathcal{P}_m &\simeq \{-1.032, 0.297, 0.067 \pm 1.141\imath\}.\end{aligned}$$



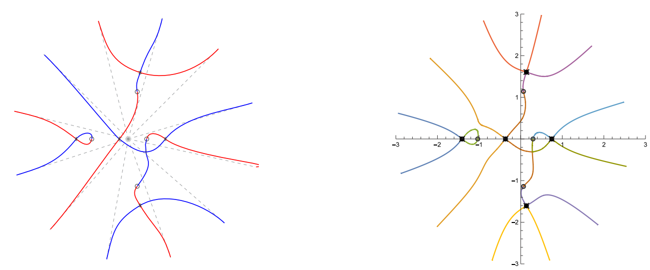
(a) Qualitative plot.   (b) Actual plot.

Fig. 4. Root locus $T_2(\sigma_1)$.



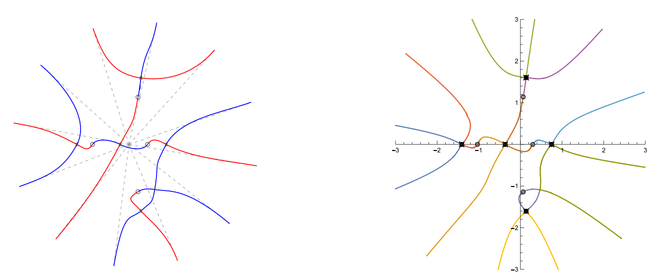
(a) Qualitative plot.   (b) Actual plot.

Fig. 5. Root locus $T_2(\sigma_3)$.



(a) Qualitative plot.   (b) Actual plot.

Fig. 6. Root locus $T_2(\sigma_5)$.

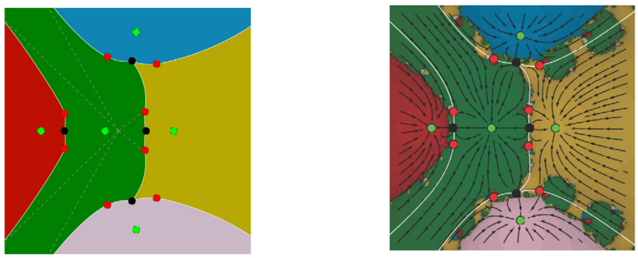


(a) Qualitative plot.   (b) Actual plot.

Fig. 7. Root locus $T_2(\sigma_7)$.

Note that $\mathcal{R}_m = \mathcal{R}$ and $\mathcal{P}_m = \mathcal{S}$, with $\mathcal{R}$ and $\mathcal{S}$ being the ones given in Example 1. Letting $\sigma_1 \simeq -1.02289 - 0.283412\imath$, $\sigma_3 \simeq -0.327797 - 1.20211\imath$, $\sigma_5 \simeq 0.280449 - 0.312178\imath$, and $\sigma_7 = 0.470235 - 1.08889\imath$, Rules 1 to 5 have been used to draw qualitatively the root loci $T(\sigma_i)$, $i = 1, 3, 5, 7$, which are depicted in Figs. 4 to 7 and compared with the actual root loci.

By noticing that the other singular points are given by the complex conjugate of σ_i, $i = 1, 3, 5, 7$, the corresponding root



(a) Qualitative plot.

(b) Actual plot and Julia plot of $z \mapsto z - \frac{f_m(z)}{f'_m(z)}$.

Fig. 8. Basin of attraction of each root of f in Example 2. The points in $\mathcal{R}_m$, in $\mathcal{S}_m$, and in $\mathcal{P}_m$ are depicted in green, red, and black, respectively. The stream plot of system (18) is depicted in black. The set $\mathcal{N}_m$ is represented in white. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

loci can be obtained by mirroring with respect to the real axis the ones depicted in Figs. 4 to 7.

Joining branches of the root loci above, following Remark 3, the set $\mathcal{N}_m$ has been obtained. As detailed in Section 4, such a set subdivides $\mathbb{C}$ in 5 regions within each of which a root of f is attractive. Fig. 8 depicts such regions and compares them with the Julia set of the function $z \mapsto z + F_m(z)$, and with the trajectories of the continuous-time Halley method that is given by (18) with $m = 2$.

Example 3. Consider the polynomial

$$f(z) = \tfrac{1}{9}z^9 + \tfrac{3}{5}z^5 - 4z.$$

The techniques outlined in Sections 3–5 have been used to determine the basin of attraction of each root of f when using either the Newton method (11) or the Householder method (18) with $m = 2, 3, 4$. Fig. 9 depicts the results of such an analysis overlapped with the Julia plots of either $z \mapsto z + F(z)$ (Fig. 9(a)) or $z \mapsto z + F_m(z)$, $m = 2, 3, 4$ (Figs. 9(b)–9(d)).

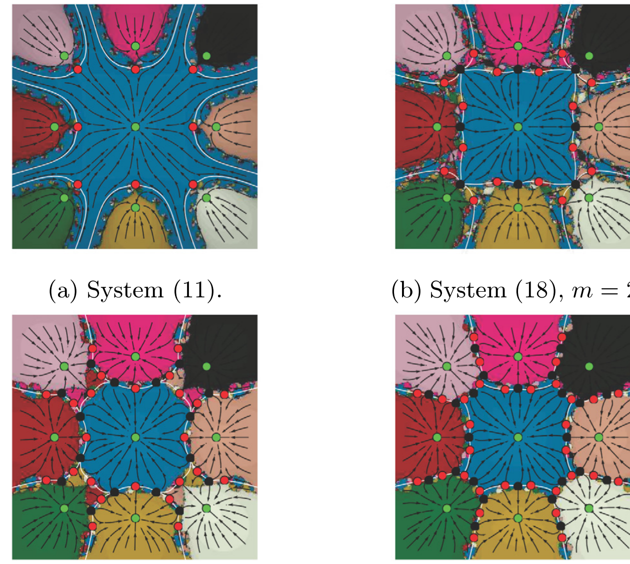
As shown by such a figure, the basin of attraction of each root of f is unbounded when using Newton's method. On the other hand, the basin of attraction of the root $z_e = 0$ is bounded when using the Householder method (18), with $m = 2, 3, 4$.

To further illustrate how to use the results of Section 5 to determine the basin of attraction of each root of f when using the Newton and the Householder methods, the following Table 2 reports such basins of attraction when these methods are applied to some noteworthy polynomials. As shown by such a figure, when using the Newton (respectively, Householder) method, the boundary of the basin of attraction of the roots of $f(z) = 0$ matches with the sets $\mathcal{N}$ (respectively, $\mathcal{N}_m$) determined using the root locus and Remarks 2 and 3.

7. Using Newton's method to determine all the roots of a polynomial

By Remark 4, the set $\mathcal{N}$ can be approximated for sufficiently large $z \in \mathbb{C}$ as a set of straight half lines passing through the point z° given in Fact 3. By Rule 4, the angle formed by these half lines with the positive real axis is $\frac{1}{n}(2h\pi + \arg(f(z_0)) - \arg(c))$ for all $z_0 \in \mathcal{S}$ and $h \in \{0, 1, \ldots, n-1\}$, where c is the leading coefficient of f. Therefore, letting

$$\begin{aligned}\Theta &= \bigcup_{z_0 \in \mathcal{S}} \bigcup_{h=0}^{n-1} \left\{ \tfrac{1}{n}(2h\pi + \arg(f(z_0)) - \arg(c)) \right\}, \\ &= \{\theta_1, \theta_2, \ldots, \theta_N\},\end{aligned}$$



(a) System (11).

(b) System (18), $m = 2$.

(c) System (18), $m = 3$.

(d) System (18), $m = 4$.

Fig. 9. Basin of attraction of each root of f in Example 3 when using either the Newton method (11) or the Householder method (18), with $m = 2, 3, 4$. The points in $\mathcal{R}$ ($\mathcal{R}_m$), in $\mathcal{S}$ ($\mathcal{S}_m$), and in $\mathcal{P}_m$ are depicted in green, red, and black, respectively. The stream plot of systems (11) and (18) is depicted in black. The set $\mathcal{N}$ ($\mathcal{N}_m$) is represented in white. The background is the Julia plot of $z \mapsto z - \frac{f_m(z)}{f'_m(z)}$ ($d = 1$ in the first panel). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

with $\theta_i \leq \theta_{i+1}$, $i = 1, \ldots, N-1$, and taking $\vartheta_i \in \mathbb{R}$ so that $\theta_i < \vartheta_i < \theta_{i+1}$, $i = 1, \ldots, N$, with $\theta_N = \theta_1 + 2\pi$, one obtains that, for each root $z_e \in \mathcal{R}$ of f, there exists (at least one) $i \in \{1, \ldots, N\}$ such that the point $z_i(0) = (z^\circ + r\cos(\vartheta_i)) + \imath r \sin(\vartheta_i)$ is in the basin of attraction of z_e, provided that $r \in \mathbb{R}$ is sufficiently large. Hence, by integrating system (11) initialized at each $z_1(0), z_2(0), \ldots, z_N(0)$, one can determine the set of all the roots of f.

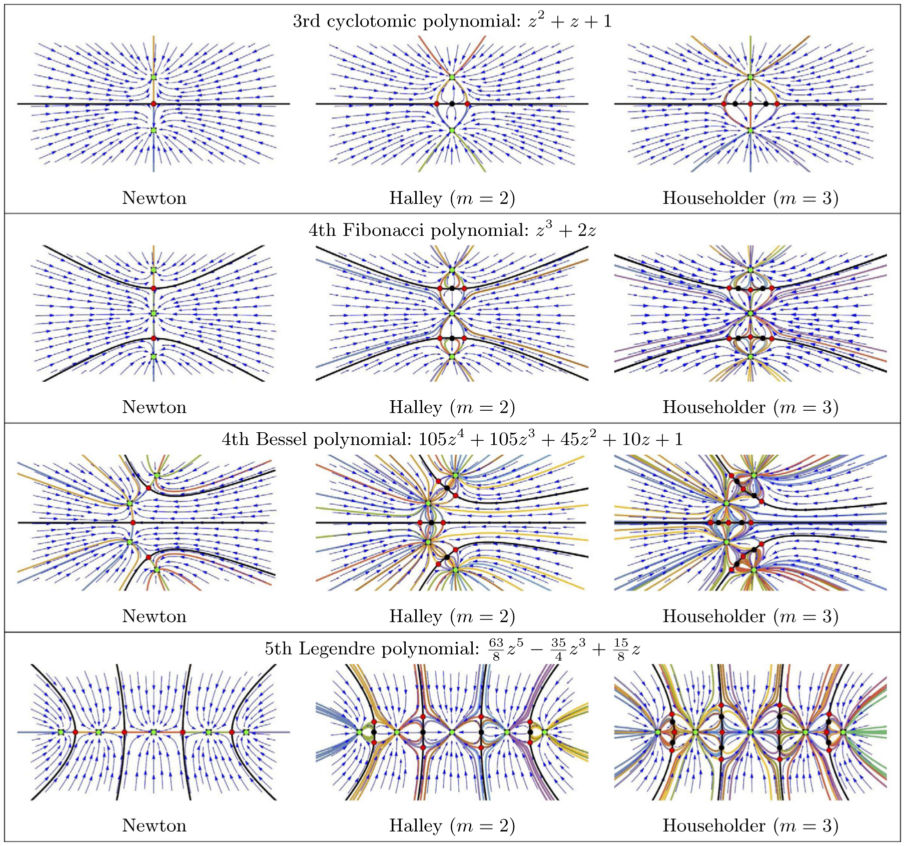
Remark 5. If $\mathcal{S} \subset \mathbb{R}$, the set Θ can be determined even without any knowledge about the points $z_0 \in \mathcal{S}$. In fact, in this case, it can be easily observed that

$$\Theta \subset \left(\bigcup_{h=0}^{n-1} \left\{ \tfrac{1}{n} 2h\pi \right\} \right) \cup \left(\bigcup_{h=0}^{n-1} \left\{ \tfrac{1}{n}(2h\pi + 1) \right\} \right).$$

It is worth pointing out that, except for the case $\mathcal{S} \subset \mathbb{R}$ considered in Remark 5, the set $\mathcal{S}$ is generally not known a priori. In order to overcome this issue, the following Algorithm 1 firstly determines the unique root of $\psi_{n-1}(z) = \frac{\partial^{n-1} f(z)}{\partial z^{n-1}}$, whose basin of attraction is the whole $\mathbb{C}$ by Table 1. Then, it iteratively finds the set of all the roots of $\psi_{n-i}(z) = \frac{\partial^{n-i} f(z)}{\partial z^{n-i}}$ by leveraging the knowledge of the roots of $\psi_{n-i+1}(z) = \psi'_{n-i}(z)$, using the reasoning given at the beginning of this section, $i = 2, \ldots, n$. Thus, noticing that $\psi_0 = f$, the roots determined at the final iteration of this algorithm are the points in $\mathcal{R}$, provided that the parameters r and T are sufficiently large, that the numerical tolerance δ is sufficiently small, and that the polynomials $\psi_0, \ldots, \psi_{n-1}$ are square-free.

Remark 6. It is worth pointing out that even if the goal is to obtain just the real roots of f, Algorithm 1 has to be initialized with complex initial guesses. In fact, as shown in Table 1, there

Table 2
Application to some noteworthy polynomials. The stream plots of systems (11) and (18) are depicted in blue, the sets $\mathcal{N}$ and $\mathcal{N}_m$ are depicted in black, whereas the colored lines represent the root loci $T(z_0)$, $z_0 \in \mathcal{S}$ and $T_m(z_0)$, $z_0 \in \mathcal{S}_m$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



Algorithm 1 Set of all the roots of f

Input: polynomial $f \in \mathbb{R}[z]$, sufficiently large $r, T \in \mathbb{R}$, sufficiently small numerical tolerance $\delta \in \mathbb{R}$
Output: roots of f
1: $n \leftarrow \deg(f, z)$
2: $\psi_{n-1}(z) = \frac{\partial^{n-1} f(z)}{\partial z^{n-1}}$
3: numerically integrate $\dot{z} = -\frac{\psi_{n-1}(z)}{\psi'_{n-1}(z)}$ with $z(0) = r$ in $[0, T]$
4: $\mathcal{S}_{n-1} \leftarrow z(T)$
5: **for** $i = 2$ **to** n **do**
6: $\quad \psi_{n-i}(z) = \frac{\partial^{n-i} f(z)}{\partial z^{n-i}}$
7: $\quad$ let c_{n-i} be the leading coefficient of ψ_{n-i}
8: $\quad \Theta_{n-i} = \bigcup_{z_0 \in \mathcal{S}_{n-i+1}} \bigcup_{h=0}^{n-i-1} \frac{1}{n-i}(2h\pi + \arg(\frac{f(z_0)}{c_{n-i}}))$
9: $\quad$ sort Θ_{n-i}, thus obtaining the set $\{\theta_1, \ldots, \theta_{N_i}, \theta_{N_i+1}\}$, with $\theta_{N_i+1} = \theta_1 + 2\pi$
10: $\quad$ take ϑ_i such that $\theta_i \leq \vartheta_i \leq \theta_{i+1}$, $i = 1, \ldots, N_i$
11: $\quad \mathcal{S}_{n-i} \leftarrow \emptyset$
12: $\quad$ compute z° as in Fact 3
13: $\quad$ **for** $j = 1$ **to** N_i **do**
14: $\quad\quad$ let $z(0) = z^\circ + r\cos(\vartheta_i) + \imath r \sin(\vartheta_i)$
15: $\quad\quad$ integrate $\dot{z} = -\frac{\psi_{n-i}(z)}{\psi'_{n-i}(z)}$ from $z(0)$ in $[0, T]$
16: $\quad\quad$ append $z(T)$ to $\mathcal{S}_{n-i}$
17: $\quad$ delete duplicates from $\mathcal{S}_{n-i}$ with tolerance δ
18: $\mathcal{R} \leftarrow \mathcal{S}_0$
19: **return** $\mathcal{R}$

exist polynomials f such that the intersection between the real axis and the set $\mathcal{N}$ is nonempty. In that case, the key feature guaranteeing that Algorithm 1 determines the set of all the roots of f is that the initial guess chosen by such an algorithm to implement the continuous-time Newton method is complex.

The following Example illustrates Algorithm 1.

Example 4. Consider the polynomial

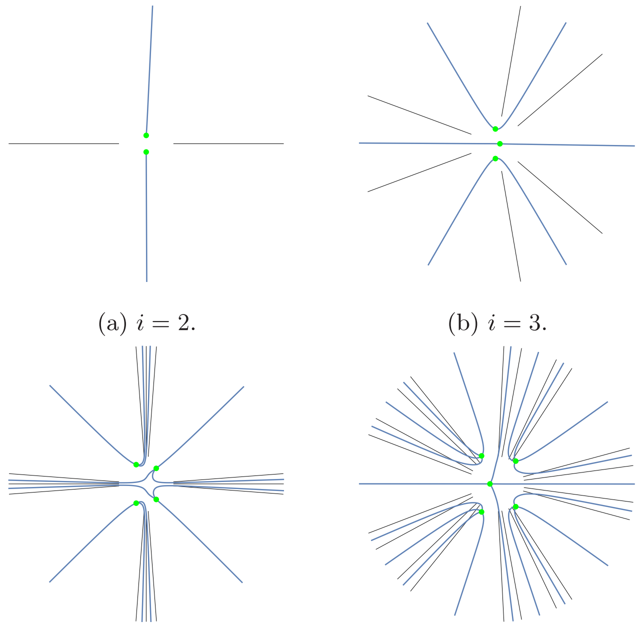
$$f(z) = -7z^5 - 8z^4 - 10z^3 - 3z^2 - 10z - 5.$$

Algorithm 1, with $r = 5$, $T = 30$, and $\delta = 10^{-6}$, has been used to determine all its roots. At the first iteration, one has $\psi_4(z) = -192 - 840z$ and integrating $\dot{z}(t) = -8/35 - z(t)$ starting from $z(0) = r$, one obtains $z(T) = -0.228571$ and hence $\mathcal{S}_{n-1} = \{-0.228571\}$. Thus, one has $\Theta_{n-2} = \{0., 3.14286\}$ and carrying out Steps 9–16, one obtains $\mathcal{S}_{n-2} = \{-0.228571 \pm 0.301018\imath\}$, see Fig. 10(a). Thus, one has $\Theta_{n-3} = \{0.7, 1.4, 2.8, 3.5, 4.88889, 5.58333\}$ and carrying out Steps 9–16, one obtains $\mathcal{S}_{n-3} = \{-0.283547 \pm 0.530004\imath, -0.118619\}$, see Fig. 10(b). Thus, one has

$$\Theta_{n-4} = \{0., 0.0769231, 1.5, 1.57143, 1.63636, 3.07143, 3.14286, 3.22222, 4.63636, 4.71429, 4.77778, 6.2\}$$

and carrying out Steps 9–16, one has $\mathcal{S}_{n-4} = \{-0.593997 \pm 0.700573\imath, 0.136855 \pm 0.565634\imath\}$, see Fig. 10(c). Finally, one has

$$\Theta_{n-5} = \{0.142857, 0.272727, 1., 1.125, 1.4, 1.52941, 2.25, 2.375, 2.64286, 2.77778, 3.5, 3.63636, 3.9, 4.03333, 4.76923, 4.88889, 5.16667, 5.28571, 6.02041, 6.14286\}$$



(a) $i = 2$. (b) $i = 3$. (c) $i = 4$. (d) $i = 5$.

Fig. 10. Iterations carried out by Algorithm 1 in Example 4. The black half-lines represent the asymptotes determined using the solution gathered at the previous step. The blue lines represent the trajectories of the continuous-time Newton method. The green points represent the solutions determined at each step. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

and carrying out Steps 9–16, one obtains $\mathcal{S}_{n-5} = \{-0.787368 \pm 1.01795\imath, -0.482553, 0.457217 \pm 0.727471\imath\}$, see Fig. 10(d).

8. Conclusions and further applications

The Newton and the Householder iterative methods are strong instruments to compute sufficiently accurate approximations of the roots of a polynomial in a single variable, especially when they are implemented as continuous-time dynamic systems. In this paper, it is shown how tools widely known in the control theory community can be used to characterize the stability properties of the Newton and the Householder method implemented in continuous-time. This analysis allows one to sketch by hand the basin of attraction of each root of the polynomial system, seen as an equilibrium of the dynamical systems associated with either the Newton or the Householder method, by using the qualitative rules for drawing the root locus of a polynomial with complex coefficients (such rules are similar to the usual ones for polynomials with real coefficients); if an accurate description of the basin of attraction is needed, such root loci can be plotted using `Matlab` or `Mathematica`. As a by-product of the analysis of the Newton method, the root locus description has shown how the basin of attraction can accurately be described by planar straight cones at points sufficiently far from the center of asymptotes, thus showing how to conveniently initialize the Newton method with complex initial guesses so that all the roots of the considered polynomial can be approximately found.

Future work will deal with the analysis of the continuous-time Newton and Householder methods in degenerate cases, *e.g.*, in the case that f and f' are not coprime, with the analysis of their basin of attraction in the non-polynomial case, and with the design of other algorithms to determine the set of all the complex roots of a polynomial.

As shown in the following two sections, the techniques analyzed in this paper can be extended to deal with systems of polynomial equations and with polynomial optimization problems. The notation used in these two sections to deal with algebraic geometry concepts is taken from Cox et al. (2006) and Menini et al. (2021a).

8.1. Extension to systems of equations

It is worth pointing out that, given polynomials $p_1, \ldots, p_s$, with real coefficients in the variables $x_1, \ldots, x_s$, denoted briefly $p_1, \ldots, p_s \in \mathbb{R}[x]$, the problem of determining the set of all the roots of the polynomial system

$$p_i(x) = 0, \quad i = 1, \ldots, s, \tag{34}$$

may be reduced to determine the roots of a single polynomial under some assumptions. In fact, if system (34) admits a finite number of solutions in $\mathbb{C}$ and each of these solutions has multiplicity 1, then there exist $q_1, \ldots, q_s, q_{s+1} \in \mathbb{R}[z]$, where z is a single-variable, such that, letting $z^1, \ldots, z^n \in \mathbb{C}$ be the set of all the distinct roots of q_{s+1}, the set of all the roots of (34) is given by:

$$x^i = [\ q_1(z^i) \quad \cdots \quad q_s(z^i)\ ]^\top, \quad i = 1, \ldots, n. \tag{35}$$

In particular, by the Shape Lemma (see, *e.g.*, Sturmfels, 2002, Prop. 2.3. and Menini et al., 2021a, Lem. 1.8.1), the method to reduce the problem of computing the solutions to (34) is to compute the Gröbner basis of the ideal $\langle p_1(x), \ldots, p_s(x), z - g(x)\rangle$ in $\mathbb{R}[z, x]$, assumed to be radical, according to the lexicographic monomial order with $x_1 \succ \ldots \succ x_s \succ z$, where g is a *separating* polynomial in x, *i.e.*, a polynomial attaining different values at the different roots of system (34). Under such positions, the reduced Gröbner basis of the ideal $\langle p_1(x), \ldots, p_s(x), z - g(x)\rangle$ is $\{x_1 - q_1(z), \ldots, x_n - q_s(z), q_{s+1}\}$. Hence, by computing such a basis, one can find all the solutions to system (34). In fact, using the Shape Lemma, the problem of determining the roots of a polynomial system can be recast into the problem of determining the roots of a single polynomial with real coefficients. Hence, using the Shape Lemma, the techniques proposed in this paper can be used to deal with systems of polynomial equations.

8.2. Application to polynomial optimization problems

Following the construction given in Menini et al. (2018a), the techniques given in this paper can be used to determine the solution to polynomial optimization problems. Namely, consider the *minimization problem*

$$\left|\begin{array}{rl} \text{minimize} & f(x), \\ \text{with} & g_i(x) \le 0, \quad i = 1, \ldots, r, \\ & h_j(x) = 0, \quad j = 1, \ldots, p, \end{array}\right. \tag{MP}$$

where $f, g_1, \ldots, g_r, h_1, \ldots, h_p \in \mathbb{R}[x]$. Let Θ be the set of all the subsets of $\{1, \ldots, r\}$ and, for each $\theta \in \Theta$, define $f_\theta = f + \sum_{j=1}^p \gamma_j h_j + \sum_{k\in\theta} \lambda_k g_k$ and $\tilde{f}_\theta = \sum_{j=1}^p \gamma_j h_j + \sum_{k\in\theta} \lambda_k g_k$. Thus, for each $\theta \in \Theta$, define the ideals

$$\mathcal{I}_{\theta,a} = \langle \tfrac{\partial f_\theta}{\partial x}, \tfrac{\partial f_\theta}{\partial \gamma}, \tfrac{\partial \tilde{f}_\theta}{\partial \lambda}, z - f(x)\rangle, \qquad \mathcal{I}_{\theta,b} = \mathcal{I}_{\theta,a} \cap \mathbb{R}[z],$$
$$\tilde{\mathcal{I}}_{\theta,a} = \langle \tfrac{\partial \tilde{f}_\theta}{\partial x}, \tfrac{\partial \tilde{f}_\theta}{\partial \gamma}, \tfrac{\partial \tilde{f}_\theta}{\partial \lambda}, z - f(x)\rangle, \qquad \tilde{\mathcal{I}}_{\theta,b} = \tilde{\mathcal{I}}_{\theta,a} \cap \mathbb{R}[z],$$

where z is an auxiliary single variable. Thus, letting q_θ and $\tilde{q}_\theta$ be generators of $\mathcal{I}_{\theta,b}$ and $\tilde{\mathcal{I}}_{\theta,b}$, $\theta \in \Theta$, and letting $q = \prod_{\theta\in\Theta} q_\theta \tilde{q}_\theta$, by Proposition 1 of Menini et al. (2018a), the optimal value of the optimization problem (MP) is a root of the polynomial q. Therefore, the techniques outlined in this paper can be used to determine the optimal value of the optimization problem (MP), and then, as explained in Menini et al. (2018a), the optimal points.

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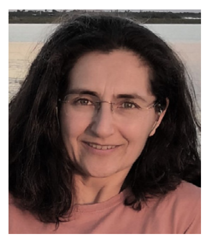

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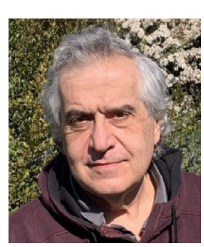

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