



A novel impact model for the rocking motion of masonry arches

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Received: 14 February 2023 / Accepted: 12 September 2023 / Published online: 9 October 2023
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Abstract The in-plane rocking motion of a masonry arch subjected to ground acceleration is investigated, focusing on the impacts at stereotomy sections, which may occur during the motion. It is assumed that the arch arrives at the impact moving along a prescribed four-hinge mechanism and that, after the impact, it continues its motion along a new four-hinge mechanism to be determined. The novel concept of impulse line, which is analogous to the thrust line computed during the smooth motion, is introduced to describe the impulsive stress state arising within the arch at the impact. That is the basis for extending the Housner impact model, initially proposed for the rocking motion of a free-standing column, to the more complicated case of a masonry arch behaving as a single-degree-of-freedom system. The mechanism after the impact is determined by minimizing the kinetic energy loss of the arch at impact, i.e. by maximizing its restitution coefficient, over the set of compatible mechanisms that fulfill a suitable formulation of the virtual work principle. The descending impulse line is proven to be equilibrated, kinematically admissible (i.e., not resisting the opening of the hinges after the impact), and statically admissible (i.e., corresponding to a compressive impulsive stress state). Numerical results are presented, discussing the restitution

coefficient of discrete and continuous circular arches with parameterized geometry, for which the four-hinge mechanism before the impact is assumed to follow from an equivalent static analysis.

Keywords Masonry · No-tension material · Arch · Limit analysis · Rocking · Restitution coefficient

1 Introduction

Arches are peculiar structural elements of historical masonry constructions, whose stability in seismic regions may be endangered by earthquakes. A common approach for a seismic safety assessment resorts to an equivalent static analysis based on the classical Heyman's assumptions [1, 2]. The minimum peak ground acceleration needed to transform the arch into a mechanism and initiate its motion is thus determined, providing a safe estimate of the arch vulnerability under a time-varying ground motion excitation.

The way towards the dynamic analysis of masonry arches subjected to ground acceleration was opened by Housner's seminal work on the rocking motion of a free-standing stone column placed upon a rigid horizontal base and hit by a horizontal acceleration impulse at its basis [3]. The motion is illustrated in Fig. 1, assuming that the column is sufficiently slender and the friction between its basis and the supporting plane is sufficiently high to avoid sliding. For an acceleration impulse from right to left, the column

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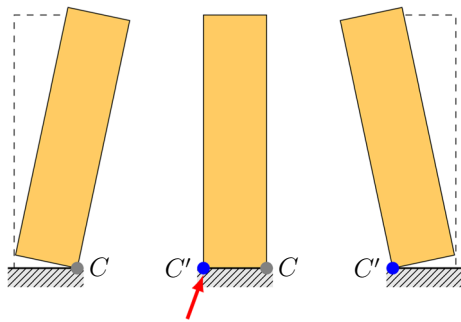


Fig. 1 Housner impact model for a free-standing stone column in rocking motion

starts to rotate about the corner C of its basis. If not overturning, the column reaches a maximum rotation angle and inverts its motion until an impact occurs with the rigid horizontal base. Then, it starts to rotate, still from right to left, about the opposite corner C' of its basis. The motion continues as a sequence of alternate rotations of the column about its bottom corners in the form of non-harmonic oscillations of reduced amplitudes and periods, progressively taking the column to rest. In [3], the equation prevailing at instants of smooth motion and the kinetic energy loss at impacts were derived. The latter computation assumed that no bouncing occurs, the impact is instantaneous, and a concentrated impulse force is transmitted through the pivotal points of the column (Fig. 1). To date, several numerical and experimental developments of the Housner model have been proposed in application to the rocking motion of single-block [4–21] or multi-block [22–25] columns.

A first dynamic analysis of a masonry arch was carried out by Oppenheim [26]. The basic assumption was that the arch moves along the four-hinge mechanism determined through an equivalent static analysis. After deriving the equation of motion for the descending single-degree-of-freedom system, direct overturning was investigated as a possible failure condition under an idealized ground motion pulse.

In [27], De Lorenzis and coauthors extended Oppenheim's approach to the rocking motion of masonry arches. That required adapting the Housner impact model, conceived for a single rigid block, to an arch behaving as a four-hinge mechanism. In particular, the two assumptions were made that (i) the hinges after the impact are at mirrored locations (about the vertical line of symmetry of the undeformed arch)

with respect to those closing during the impact and (ii) the impulse forces at the hinge sections closing during the impact are located on the opposite side of the hinges across the arch thickness. It was shown that, depending on the ground impulse, an arch might survive the first half cycle of motion but overturn during the second half cycle after an impact occurred. Moreover, the size effect that larger arches are more resistant than smaller ones was highlighted.

As noticed in [27], a possible shortcoming of that approach is that the thrust line may exit the thickness of the arch during the smooth motion, thus revealing the simplification inherent to the assumption that the arch behaves as a four-hinge mechanism. In [28], the dynamic analysis of a masonry arch regarded as a system of rigid blocks has been undertaken, assuming that hinges may open and close, and impacts may occur, at any joint between the blocks. That approach proved that a masonry arch generally moves as a four-hinge mechanism only at the beginning of the ground excitation, and multiple hinges may be opened during the smooth motion.

Alternative approaches for addressing the dynamics of multi-block masonry systems are represented by the discrete element method (DEM) [29–32], or by time-stepping schemes investigated in [33–37]. The dynamic equivalence between various rocking mechanisms and a single rocking block was investigated in [38]. It is also worth mentioning the experimental approaches proposed in [39, 40] for the seismic assessment of masonry arches through small-scale dynamic tests.

In the present work, the extension of the Housner impact model from the original case of a free-standing column to the more complicated case of a masonry arch behaving as a single-degree-of-freedom system is reconsidered. By introducing the novel concept of impulse line at impact, which is analogous to the thrust line at instants of smooth motion, the intuitive result that resultant impulses need to be transmitted through the hinges opening after the impact is demonstrated as a kinematic admissibility requirement. The minimization of kinetic energy loss of the arch at impact, i.e. the maximization of its restitution coefficient, over the set of compatible mechanisms fulfilling a suitable principle of virtual work formulation, is proposed as a variational formulation to determine the mechanism after the impact. The descending impulse line is proven to be equilibrated,

kinematically admissible, and statically admissible, i.e. corresponding to compressive resultant impulses applied within the arch thickness. A nonlinear optimization problem is thus formulated and solved in the unknown hinge locations after the impact. Numerical results are presented, discussing the restitution coefficient of discrete and continuous circular arches with parameterized geometry, which are assumed to arrive at impact along the four-hinge mechanism derived from an equivalent static analysis.

The paper is organized as follows. The concept of impulse line is introduced in Sect. 2, and the conditions for it to be kinematically admissible are discussed in Sect. 3. Section 4 addresses the impact analysis, i.e. the determination of the impulse line and of the restitution coefficient corresponding to prescribed hinge locations after the impact. The nonlinear optimization problem for deriving the optimal hinge locations is formulated in Sect. 5. Section 6 is devoted to numerical applications and conclusions are outlined in Sect. 7. Details on the kinematic analysis of a four-hinge mechanism of the arch are given in Appendix A, whereas a proof that maximizing the restitution coefficient provides an equilibrated, kinematically and statically admissible impulse line is presented in Appendix B.

2 The impulse line

A masonry arch is considered, subjected to a horizontal ground acceleration pulse, e.g. acting from right to left, (Fig. 2a). Masonry is assumed to be rigid and characterized by vanishing tensile strength, infinite compressive strength, and no-sliding behavior (i.e., the coefficient of friction of the mortar joints is assumed to be sufficiently high to prevent sliding). If the magnitude of the acceleration pulse exceeds

the incipient rocking value A_L , the arch starts moving from left to right along a four-link mechanism $\mathbf{u}$. Limit analysis theorems yield both the incipient rocking acceleration A_L and the mechanism $\mathbf{u}$ [2]. In particular, the latter is characterized by the four hinges C_1, C_{13}, C_{23}, C_2 , which determine the three rigid segments I, II, and III (Fig. 2a).

If overturning does not occur, the arch reverses its motion, and there is an instant when it, moving from right to left, returns to the undeformed configuration. At that instant, all the hinges of the mechanism $\mathbf{u}$ simultaneously close, impacts occur, and a new motion begins, still from right to left. It is assumed that the new motion develops along a new four-link mechanism $\mathbf{u}'$ to be determined. Let it be characterized by the four hinges $C'_2, C'_{23}, C'_{13}, C'_1$, which pinpoint the three rigid segments II', III', and I' (Fig. 2b).

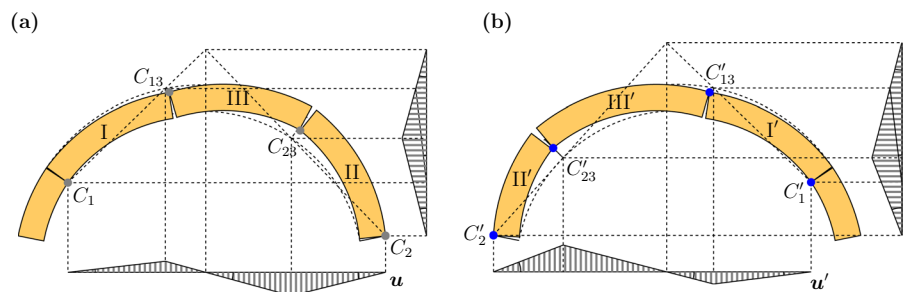
An infinitesimal voussoir of the arch is considered, located at the typical curvilinear abscissa s along the arch mid-curve and spanning the infinitesimal curvilinear length ds . Its kinematics before [resp., after] the impact is completely described by its angular velocity $\omega \mathbf{k}$ [resp., $\omega' \mathbf{k}$], with $\mathbf{k}$ as a unit vector orthogonal to the plane of the arch, and by its absolute center of rotation C [resp., C']. Angular velocity and center of rotation coincide with those relevant to the segment of the arch the voussoir belongs to before [resp., after] the impact. Due to the compatibility conditions of the mechanism $\mathbf{u}$ [resp., $\mathbf{u}'$], the angular velocities can be expressed in terms of a single free parameter α [resp., α'] by:

$$\omega = \Omega \alpha, \quad \omega' = \Omega' \alpha', \quad (1)$$

in which Ω [resp., Ω'] is a known geometrical coefficient, whose expression is detailed in Appendix A.

Equation (1) shows that, at the instant of impact, sudden and finite changes occur in the linear and

Fig. 2 Four-link mechanisms **a** before the impact $\mathbf{u}$, and **b** after the impact $\mathbf{u}'$, with highlighted hinges and subdivision into rigid segments



angular velocity of each arch voussoir, and hence in its linear and angular momentum (without significant change in the arch configuration). The principles of linear and angular impulse and momentum [41] thus imply that an impulsive stress state arises within the arch.

For a formalization, the linear momentum $d\mathbf{p}$ [resp., $d\mathbf{p}'$] and the angular momentum dL_O [resp., dL'_O] of the infinitesimal voussoir before [resp., after] the impact about an arbitrary pole O are given by (equations to be read without [resp., with] prime):

$$\begin{aligned} d\mathbf{p} &= \mathbf{k} \times (G - C)\omega\rho ds, \\ dL_O &= \omega I_G ds + (G - O) \times d\mathbf{p} \cdot \mathbf{k}, \end{aligned} \quad (2)$$

where G , ρ , and I_G respectively represent the center of gravity, the linear mass density, and the linear density of polar moment of inertia about G of the infinitesimal voussoir, and $\cdot$ and $\times$ respectively denote scalar and vector product.

Accordingly, the principles of linear and angular impulse and momentum applied to the finite portion of the arch contained between the typical curvilinear abscissas $[s_1, s_2]$ yield:

$$\begin{aligned} \int_{s_1}^{s_2} d\mathbf{p} + \mathcal{I}(s_2) - \mathcal{I}(s_1) &= \int_{s_1}^{s_2} d\mathbf{p}', \\ \int_{s_1}^{s_2} dL_O + \mathcal{M}_O(s_2) - \mathcal{M}_O(s_1) &= \int_{s_1}^{s_2} dL'_O, \end{aligned} \quad (3)$$

in which $\mathcal{I}(s)$ and $\mathcal{M}_O(s)$ respectively denote the internal linear and angular impulse about the pole O at the section of curvilinear abscissa s . In passing, it is observed that, due to the very short duration of the impact, ordinary forces (such as gravity loads) are not involved in Eq. (3), for their contribution is negligible compared to internal linear and angular impulses.

A graphical representation of the impulsive stress state arising within the arch at the impact is offered by the new concept of impulse line. Analogously to the thrust line, which depicts the equilibrium between applied loads (including inertia forces) and internal resultant forces at instants of smooth motion, the impulse line depicts the equivalence between instantaneous variations of linear and angular momenta and internal resultant impulses at the instant of impact (Eq. (3)). As the thrust line joins the centers of pressure on the sections of the arch at instants of smooth motion, the impulse line joins the centers of impulse,

i.e. the points on the sections of the arch through which the resultant impulses are transmitted. In detail, the impulse line is defined as the curve intersecting the typical section of the arch at the signed distance:

$$\lambda = \frac{\mathcal{M}_G}{\mathcal{I}_t}, \quad (4)$$

along the unit normal vector $\mathbf{n} = \mathbf{t} \times \mathbf{k}$ from the center of gravity G of the (infinitesimal voussoir containing the) section. Here, $\mathbf{t}$ is the unit tangent vector to the arch mid-curve, and:

$$\mathcal{I}_t = \mathcal{I} \cdot \mathbf{t}, \quad \mathcal{M}_G = \mathcal{M}_O + (O - G) \times \mathcal{I} \cdot \mathbf{k}, \quad (5)$$

respectively represent the tangential component of the internal linear impulse and the internal angular impulse about G .

Equations (1)–(3) imply that, for prescribed hinge locations of the new mechanism $\mathbf{u}'$, infinite impulse lines exist complying with the principles of linear and angular impulse and momentum. They correspond to the choice of the unknown parameter α' and e.g. of the internal linear and angular impulses at a specific arch section. Altogether, equilibrated impulse lines depend on four scalar parameters. Borrowing the terminology from classical thrust line analysis, among all equilibrated impulse lines, only those that are also statically and kinematically admissible are feasible. The static admissibility, which will be discussed in Sect. 5, amounts to require that (i) the internal impulse forces are compressive at any section of the arch and (ii) the impulse line is completely contained within the thickness of the arch. In turn, the kinematic admissibility is addressed in the following section.

3 Kinematic admissibility of the impulse line

The instant of impact is considered to address the kinematic admissibility of the typical impulse line.

Initially, suppose the hinges of the new mechanism $\mathbf{u}'$, i.e. $C'_2, C'_{23}, C'_{13}, C'_1$, develop at the same sections of the arch where the hinges of the mechanism $\mathbf{u}$ (respectively, C_1, C_{13}, C_{23}, C_2) are located, but on the opposite side through the arch thickness, i.e. at the points respectively denoted by $\bar{C}_1, \bar{C}_{13}, \bar{C}_{23}, \bar{C}_2$ (e.g., see Fig. 10). It is recognized that impacts occur at those points, i.e. impulsive forces are transmitted

through them (Fig. 3), thus allowing the hinges of the new mechanism u' , located at the same points, to open up. In such a case, the transition from u to u' is similar to what happens in Housner's column (Fig. 1).

Based on the analogy with Housner's column, some researchers have adopted the assumption that the resultant impulses at the sections of the arches containing the hinges of the mechanism u are transmitted through the points $\bar{C}_1, \bar{C}_{13}, \bar{C}_{23}, \bar{C}_2$ even if the hinges of the new mechanism u' develop at different sections [26, 27] (e.g., see Fig. 11a). Indeed, in that case, the internal resultant impulses flowing across the sections of the arch cause the hinges of the mechanism u to close up (Fig. 4a), and the hinges of the new mechanism u' to open up (Fig. 4c). The arch portions that enclose the new hinges suddenly begin to rotate against each other, pivoting on the points $C'_2, C'_{23}, C'_{13}, C'_1$ (Fig. 4c). For those hinges to open up, the resultant impulses transmitted through the corresponding sections (shown as red arrows in Fig. 4b) need to pass through the hinges themselves. By contrast, because the hinges of the mechanism u are closing, the resultant impulses transmitted through their sections (shown as gray arrows in Fig. 4b) do not need to pass through the points $\bar{C}_1, \bar{C}_{13}, \bar{C}_{23}, \bar{C}_2$.

Hence, it is inferred that an impulse line is kinematically admissible provided that it passes through the hinges of the new mechanism u' , analogously to the requirement for a kinematically admissible thrust line to pass through the opening hinges at an instant of smooth motion. Because that amounts to the imposition of four conditions, it follows that prescribing the new hinge locations uniquely determines the unknown parameter α' and the internal linear and angular impulses at a specific arch section. Consequently, the new mechanism u'

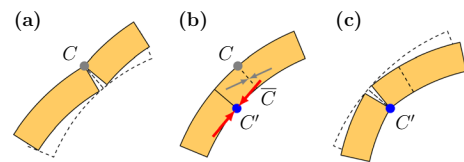


Fig. 4 The general case in which the typical hinge C' of the new mechanism u' does not develop at the point $\bar{C}$ opposite to a hinge C of the mechanism u across the arch thickness: **a** mechanism u , **b** configuration at impact, and **c** mechanism u' . At the section containing the opening hinge C' , the resultant impulse (red arrows) passes through C' . By contrast, at the section containing the closing hinge C , the resultant impulse (gray arrows) does not need to pass through $\bar{C}$

and the corresponding equilibrated and kinematically admissible impulse line are completely characterized.

4 Impact analysis

Two different strategies are proposed for deriving the unique equilibrated and kinematically admissible impulse line corresponding to prescribed hinge locations of the new mechanism u' . They are respectively based on the direct use of Eq. (3) or on a formulation of the principle of virtual work.

Preliminarily, it is convenient to introduce a subdivision of the arch into the bodies identified by considering altogether the sections containing the hinges of the mechanisms u and u' (Fig. 5). Let G_i, m_i , and $I_{O,i}$, for $i = 1, \dots, 7$, respectively denote the center of gravity, mass, and polar moment of inertia about a pole O of the i th body. By construction, such a body remains intact in both the mechanisms u and u' . In fact, the arch segments therein involved can be recovered as:

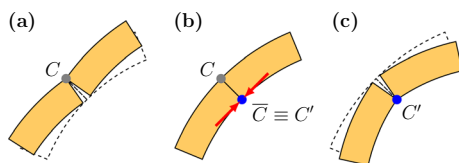


Fig. 3 The case, similar to the impact in Housner's column, in which the typical hinge C' of the new mechanism u' develops at the point $\bar{C}$ opposite to a hinge C of the mechanism u across the arch thickness: **a** mechanism u , **b** configuration at impact, and **c** mechanism u' . At the section $\bar{C}\bar{C}$, the resultant impulse (red arrows) passes through $\bar{C} \equiv C'$

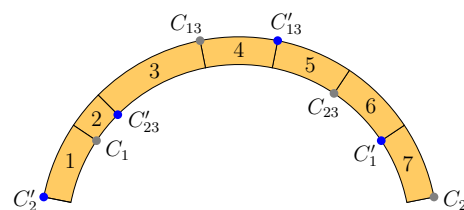


Fig. 5 Bodies identified by considering altogether the sections containing the hinges of the four-link mechanisms u and u' before and after the impact, respectively

$$\begin{aligned} \text{I} &= 2 \cup 3, & \text{II} &= 6 \cup 7, & \text{III} &= 4 \cup 5, \\ \text{I}' &= 5 \cup 6, & \text{II}' &= 1 \cup 2, & \text{III}' &= 3 \cup 4. \end{aligned} \quad (6)$$

Accordingly, the angular velocity, the absolute center of rotation, and the geometrical coefficient (as introduced in Eq. (1)) are well-defined for the i th body before [resp., after] the impact. They are respectively denoted by ω_i , C_i , Ω_i [resp., ω'_i , C'_i , Ω'_i]. The linear and angular momenta before [resp., after] the impact of the i th body, computed by integration of the counterpart infinitesimal contributions, are respectively denoted by $\mathbf{p}_i$ and $L_{O,i}$ [resp., $\mathbf{p}'_i$ and $L'_{O,i}$].

For future convenience, a straightforward computation yields the kinetic energy K [resp., K'] of the arch before [resp., after] the impact (equation to be read without [resp., with] prime):

$$K = \frac{1}{2} \Lambda \alpha^2, \quad \Lambda = \sum_{i=1}^7 I_{C_i,i} \Omega_i^2, \quad (7)$$

where $I_{C_i,i} = I_{G_i,i} + m_i \|G_i - C_i\|^2$ is the polar moment of inertia of the body about its absolute center of rotation, and $\|\cdot\|$ denotes the Euclidean norm of the enclosed argument.

4.1 The use of the principles of linear and angular impulse and momentum

Equations (3) involve the unknown parameter α' and the unknown linear and angular impulses at a specific arch section. A possible approach to determine those four scalar unknowns would be to enforce that the impulse line passes through the hinges C'_1 , C'_{13} , C'_{23} , C'_2 , amounting to four scalar equations.

An alternative approach is discussed in the following, inspired by the one proposed within a different impact model, in [27]. The unknown parameter α' , and the impulse forces $\mathcal{I}_{C'_1}$ and $\mathcal{I}_{C'_2}$ exerted by the hinges C'_1 and C'_2 , respectively, are assumed as unknowns, amounting altogether to five scalar

unknowns. Accordingly, a system of five scalar equations is formulated by applying:

- i. the principle of linear impulse and momentum to the portion of the arch between the hinges C'_1 and C'_2 (Fig. 6):

$$\sum_{i=1}^6 \mathbf{p}_i + \mathcal{I}_{C'_1} + \mathcal{I}_{C'_2} = \sum_{i=1}^6 \mathbf{p}'_i; \quad (8)$$

- ii. the principle of angular impulse and momentum about the origin O to the portion of the arch between C'_1 and C'_2 (Fig. 6):

$$\begin{aligned} \sum_{i=1}^6 L_{O,i} + \left[(C'_1 - O) \times \mathcal{I}_{C'_1} \right] \cdot \mathbf{k} \\ + \left[(C'_2 - O) \times \mathcal{I}_{C'_2} \right] \cdot \mathbf{k} = \sum_{i=1}^6 L'_{O,i}; \end{aligned} \quad (9)$$

- iii. the principle of angular impulse and momentum about the hinge C'_{13} to the portion of the arch between C'_{13} and C'_1 (Fig. 7):

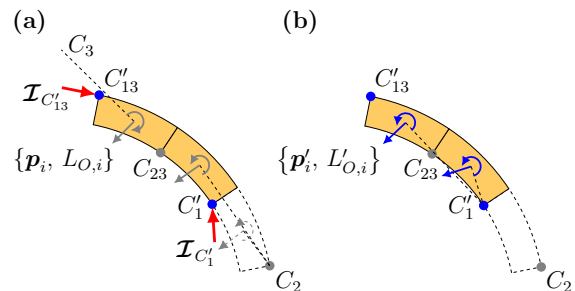
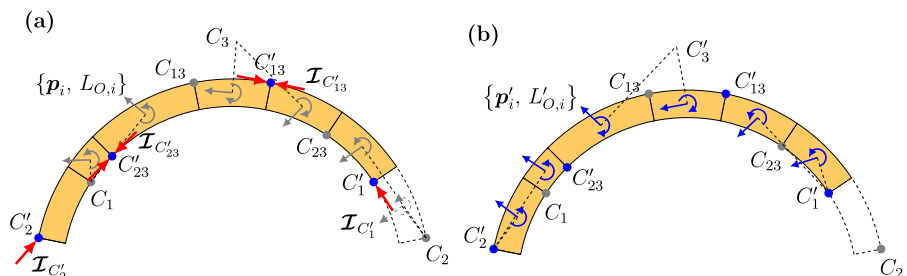


Fig. 7 Principle of angular impulse and momentum about the hinge C'_{13} applied to the portion of the arch between C'_{13} and C'_1 : **a** momenta and impulses before the impact are equivalent to **b** momenta after the impact

Fig. 6 Principle of linear impulse and momentum applied to the portion of the arch between the hinges C'_1 and C'_2 : **a** momenta and impulses before the impact are equivalent to **b** momenta after the impact



$$\sum_{i=5}^6 L_{C'_{13},i} + \left[(C'_1 - C'_{13}) \times \mathcal{I}_{C'_1} \right] \cdot \mathbf{k} = \sum_{i=5}^6 L'_{C'_{13},i}; \quad (10)$$

- iv. the principle of angular impulse and momentum about the hinge C'_{23} to the portion of the arch between C'_2 and C'_{23} (Fig. 8):

$$\sum_{i=1}^2 L_{C'_{23},i} + \left[(C'_2 - C'_{23}) \times \mathcal{I}_{C'_2} \right] \cdot \mathbf{k} = \sum_{i=1}^2 L'_{C'_{23},i}. \quad (11)$$

It is remarked that Eqs. (10) and (11) are purposely written in such a way that the impulses exerted by the internal hinges C'_{13} and C'_{23} , respectively $\mathcal{I}_{C'_{13}}$ and $\mathcal{I}_{C'_{23}}$, are not explicitly involved.

The solution of Eqs. (8)–(11) yields the impulse forces $\mathcal{I}_{C'_1}$ and $\mathcal{I}_{C'_2}$ and the desired unknown parameter α' . Starting e.g. from the section containing the hinge C'_1 , where both the linear and angular impulses are known, the equilibrated and

kinematically admissible impulse line is finally computed by Eqs. (3) and (4).

4.2 The use of the principle of virtual work

Let the arch portion constituted by the segments I', II', III' be considered, internally constrained by the hinges C'_{13} and C'_{23} and subjected to the impulse forces $\mathcal{I}_{C'_1}$, $\mathcal{I}_{C'_{13}}$, $\mathcal{I}_{C'_{23}}$, and $\mathcal{I}_{C'_2}$, respectively exerted by the hinges C'_1 , C'_{13} , C'_{23} , and C'_2 . Equations (8)–(11) imply that, on such an arch portion, (i) the linear and angular momenta before the impact together with the impulse forces $\mathcal{I}_{C'_1}$, $\mathcal{I}_{C'_{13}}$, $\mathcal{I}_{C'_{23}}$, $\mathcal{I}_{C'_2}$, and (ii) the linear and angular momenta after the impact, are statically equivalent (Fig. 6). Hence, the virtual works W_v and W'_v that (i) and (ii) respectively do for any infinitesimal virtual mechanism coincide:

$$W_v = W'_v. \quad (12)$$

By choosing the new mechanism $\mathbf{u}'$ as such virtual mechanism (Fig. 9), it is observed that the impulse forces $\mathcal{I}_{C'_1}$, $\mathcal{I}_{C'_{13}}$, $\mathcal{I}_{C'_{23}}$, $\mathcal{I}_{C'_2}$ do not work. Accordingly, it follows that:

$$W_v = \sum_{i=1}^7 \left[L_{G_i,i} \omega'_i + \mathbf{p}_i \cdot \mathbf{v}'_{G_i} \right] = \Gamma \alpha \alpha', \quad (13)$$

$$W'_v = \sum_{i=1}^7 \left[L'_{G_i,i} \omega'_i + \mathbf{p}'_i \cdot \mathbf{v}'_{G_i} \right] = \Lambda'(\alpha')^2,$$

where:

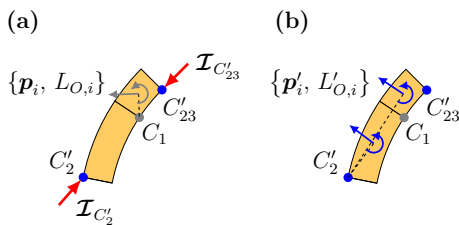


Fig. 8 Principle of angular impulse and momentum about the hinge C'_{23} applied to the portion of the arch between C'_2 and C'_{23} : **a** momenta and impulses before the impact are equivalent to **b** momenta after the impact

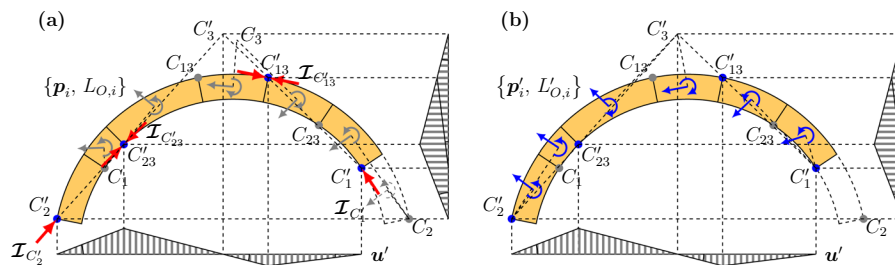


Fig. 9 Principle of virtual work for the portion of the arch between the hinges C'_1 and C'_2 assuming the mechanism $\mathbf{u}'$ after the impact as virtual mechanism: the work done by **a** momenta

and impulses before the impact is equal to the work done by **b** momenta after the impact. The impulse forces $\mathcal{I}_{C'_1}$, $\mathcal{I}_{C'_{13}}$, $\mathcal{I}_{C'_{23}}$, $\mathcal{I}_{C'_2}$ do not work

$$\Gamma = \sum_{i=1}^7 [I_{G_i,i} + m_i(G_i - C_i) \cdot (G_i - C'_i)] \Omega_i \Omega'_i, \quad (14)$$

and Λ' has been introduced in Eq. (7)₂. By substituting relationships (13) in the principle of virtual work formulation (12), a closed-form relationship is derived:

$$\alpha' = \frac{\Gamma}{\Lambda'} \alpha, \quad (15)$$

for the desired unknown parameter α' . Once it has been determined, the corresponding equilibrated and kinematically admissible impulse line is straightforwardly computed by Eqs. (3) and (4), under the condition that it passes through the hinges of the new mechanism $\mathbf{u}'$.

5 Mechanism after the impact

In the previous section, it has been shown that a unique equilibrated and kinematically admissible impulse line exists for prescribed hinge locations of the new mechanism $\mathbf{u}'$, and its computation has been discussed. However, such an impulse line may or may not be statically admissible, depending on the choice of the new hinge locations.

It is here claimed that an equilibrated and kinematically admissible impulse line is also statically admissible provided that the new hinge locations produce the minimum energy expense at the impact, i.e. maximize the restitution coefficient given by the ratio of the kinetic energy K' after the impact to the kinetic energy K before the impact [3]. The following optimization problem is thus considered:

$$\begin{aligned} \max_{\mathbf{u}'} \quad & r, \\ \text{subject to} \quad & \mathbf{u}' \text{ is compatible,} \\ & \text{Equation (15) holds.} \end{aligned} \quad (16)$$

A proof of this claim is given in Appendix B. It is noticed that relationship (7) implies the restitution coefficient r to be expressed in the optimization domain as:

$$r = \frac{K'}{K} = \frac{\Lambda'}{\Lambda} \left(\frac{\alpha'}{\alpha} \right)^2 = \frac{\Gamma^2}{\Lambda \Lambda'}. \quad (17)$$

Remark 1 By recalling Eqs. (7)₂ and (14), and by using the Cauchy-Schwarz inequality, it can be checked that the restitution coefficient descending from Eq. (17) fulfills the condition $r \leq 1$. $\square$

In closing, it is observed that the optimization problem (16)–(17) completely characterizes the impulsive stress state arising within the arch at the impact. In particular, by comparing the normal and shear components of the linear impulse at the typical arch section, a post-verification of the assumed no-sliding behavior may be performed.

6 Numerical simulations

Numerical simulations are here discussed concerning the behavior at the impact of the so-called Oppenheim's arch (Sect. 6.1) and of continuous circular arches with parameterized geometry (Sect. 6.2).

6.1 Oppenheim's arch

The Oppenheim arch is a circular arch of mid-curve radius $R = 10$ m, thickness-over-radius ratio $t/R = 0.15$, and embrace angle β . It is constituted by seven identical voussoirs, assumed to be rigid and infinitely resistant. Accordingly, hinge openings are only allowed at the eight joints between those voussoirs [26].

Initially, the case of an embrace angle $\beta = 150^\circ$ is analyzed. A limit-analysis computation shows that the horizontal acceleration of incipient rocking for the arch is $A_L = 0.444 g$, with g as the acceleration of gravity. For an acceleration acting from right to left, the arch starts moving, from left to right, along the four-link mechanism $\mathbf{u}$ characterized by the hinge locations C_1, C_{13}, C_{23}, C_2 shown as gray circles in Fig. 10. The attention is focused on the arch behavior at impact, assuming that, subjected to a certain ground acceleration, it first moves along the mechanism $\mathbf{u}$ and then reverses its motion returning to the undeformed configuration. In solution of the optimization problem (16)–(17), the maximum restitution coefficient of the arch results to be $r_{\max} = 0.532$. It is attained for a new mechanism $\mathbf{u}'$ whose hinges $C'_2, C'_{23}, C'_{13}, C'_1$ are depicted as blue circles in Fig. 10. It is noticed that the new hinges are in the mirrored locations with respect to those of the mechanism $\mathbf{u}$

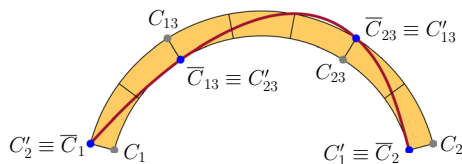


Fig. 10 Oppenheim's arch: impulse line at impact for an arch with embrace angle $\beta = 150^\circ$. The optimal new hinges $C'_2, C'_{23}, C'_{13}, C'_1$ respectively coincide with the points $\bar{C}_1, \bar{C}_{13}, \bar{C}_{23}, \bar{C}_2$ opposite to the hinges C_1, C_{13}, C_{23}, C_2 of the mechanism before the impact across the arch thickness. The attained maximum restitution coefficient is $r_{\max} = 0.532$

(respectively, C_2, C_{23}, C_{13}, C_1) and, in addition, turn out to be at the same sections, but on the opposite side through the arch thickness, of the hinges of the mechanism u (i.e., they respectively coincide with $\bar{C}_1, \bar{C}_{13}, \bar{C}_{23}, \bar{C}_2$). Consequently, the present solution reproduces the impact occurring in the Housner column and coincides with the one provided in [27]. The corresponding impulse line is plotted in Fig. 10.

Next, the case of an embrace angle $\beta = 157.5^\circ$ is considered, whose horizontal acceleration of incipient rocking is $A_L = 0.370 g$. That corresponds to the four-link mechanism u characterized by the hinges C_1, C_{13}, C_{23}, C_2 shown as grey circles in Fig. 11a, b.

Figure 11a shows the impulse line obtained under the two assumptions discussed in [27]. Accordingly, the hinges after the impact ($C'_2, C'_{23}, C'_{13}, C'_1$, shown as blue circles) are in the mirrored locations with respect to those before the impact (respectively, $C_2, C_{23}, C_{13},$

C_1 , shown as grey circles). In addition, the impulse line passes through the points $\bar{C}_1, \bar{C}_{13}, \bar{C}_{23}, \bar{C}_2$ lying on the opposite side across the arch thickness of the hinges of the mechanism u , i.e. through the points of those sections where the resultant impulses are assumed to be located. The descending impulse line is not kinematically admissible because compressive resultant impulses are predicted in the interior of the sections containing the new hinges C'_{23} and C'_{13} (green crosses), whose opening is therefore resisted.

In Fig. 11b, the impulse line computed in solution of the optimization problem (16)–(17) is depicted. It corresponds to the restitution coefficient $r_{\max} = 0.557$ and is attained for a mechanism u' after the impact whose hinge locations ($C'_2, C'_{23}, C'_{13}, C'_1$, shown as blue circles) are not mirrored with respect to those of the mechanism u . The impulse line is kinematically admissible because passing through the new hinges. Consequently, the impulsive stress state arising within the arch at the instant of impact does not resist the opening of the new hinges, i.e. it enables the activation of the new mechanism after the impact. However, it is observed that such an impulse line slightly exits the thickness of the arch at one of its joints (the one passing through C_{13}) and is thus not statically admissible. The apparent contrast with the result in Sect. 5 is resolved in light of the discrete nature of the Oppenheim arch, as opposed to the continuous format intrinsically underlying the optimization problem (16)–(17).

In fact, for achieving the maximum restitution coefficient of a continuous arch, i.e. such that hinges

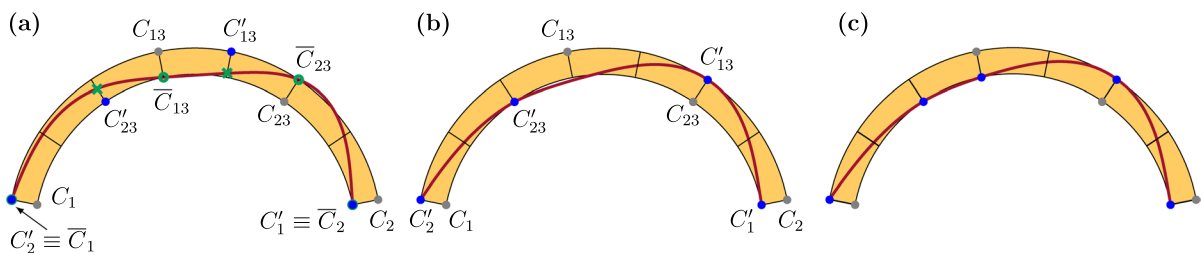


Fig. 11 Oppenheim's arch: comparison of equilibrated impulse lines at impact for an arch with embrace angle $\beta = 157.5^\circ$. **a** The impulse line passing through the points $\bar{C}_1, \bar{C}_{13}, \bar{C}_{23}, \bar{C}_2$ (green circles), respectively opposite to the hinges C_1, C_{13}, C_{23}, C_2 of the mechanism before the impact across the arch thickness, is not kinematically admissible because resisting the opening of the new hinges C'_{23} and C'_{13}

(green crosses). **b** The kinematically-admissible impulse line, i.e. passing through the new hinges $C'_2, C'_{23}, C'_{13}, C'_1$, maximizing the restitution coefficient on the restricted set of four-link mechanisms after the impact ($r = 0.557$) is not statically admissible. **c** The kinematically-admissible impulse line maximizing the restitution coefficient on the set of compatible mechanisms after the impact ($r_{\max} = 0.560$) is also statically admissible

are allowed to open at any section, it is sufficient to consider the class of four-link mechanisms after the impact that are compatible and fulfill the principle of virtual work formulation (15). By contrast, for achieving the maximum restitution coefficient of a discrete arch, i.e. such that hinges are allowed to open only at a discrete set of sections, it may be required to consider mechanisms after the impact that are still compatible and obey a principle of virtual work formulation in the fashion of Eq. (12), but that may be also characterized by more than four hinges simultaneously open.

In solution of such a generalized version of the optimization problem (16)–(17), the impulse line illustrated in Fig. 11c is obtained. The corresponding maximum restitution coefficient results to be $r_{\max} = 0.560$. It is attained for a mechanism $\mathbf{u}'$ after the impact characterized by five hinges (blue dots), which finally guarantees the kinematic and static admissibility of the impulse line. As expected, the “fifth hinge” is precisely opened at the section where the static admissibility of the impulse line computed under the assumption of four-link mechanism fails and enables a slight increase in terms of the restitution coefficient. It is also remarked that two consecutive intrados hinges are predicted to open up after the impact. Such an observation suggests that, for a continuous arch with the same geometry, the optimal hinge location would be somewhere in between the two hinges. In fact, it is the possibility to open up the new hinge at such an optimal location that makes four-hinges mechanisms after the impact enough to maximize the restitution coefficient of continuous arches. As a verification, the continuous version of the Oppenheim arch is addressed in the following section.

6.2 Continuous circular arches

Continuous circular arches are considered, i.e., constituted by many small voussoirs so that hinge openings may practically occur at any of their sections. They are characterized by their mid-curve radius R , embrace angle β , and thickness-to-radius ratio t/R . The behavior of the typical arch at impact is investigated assuming that during its motion, the arch returns to its undeformed configuration along the four-link mechanism $\mathbf{u}$ of incipient rocking (for parametric limit analyses determining the incipient

rocking horizontal acceleration of masonry structures, e.g., see [42–51]).

The particular case of a circular arch with thickness-to-radius ratio $t/R = 0.15$ and embrace angle $\beta = 157.5^\circ$ is first examined as the continuous counterpart of the Oppenheim arch discussed in the previous section. The mechanism $\mathbf{u}$ before the impact, computed as the incipient rocking mechanism, corresponds to the hinges shown as gray circles in Fig. 12. The relevant incipient rocking acceleration is $A_L = 0.353 g$. The mechanism $\mathbf{u}'$ after the impact is computed in solution of the optimization problem (16)–(17), and corresponds to the hinges shown as blue circles in Fig. 12. The maximum value of the restitution coefficient results to be $r_{\max} = 0.606$ and is attained by the impulse line shown in Fig. 12, which is statically admissible.

A parametric analysis is then carried out, investigating the maximum restitution coefficient of circular arches at varying of their geometry under the assumption that they arrive at impact along their four-link incipient rocking mechanism. Relevant results are shown in Fig. 13 where $r_{\max}$ is plotted as a function of the thickness-to-radius ratio t/R for the selected values $\beta = \{80^\circ, 100^\circ, 120^\circ, 140^\circ, 160^\circ, 180^\circ\}$ of the embrace angle. It is observed that, within the considered ranges, $r_{\max}$ decreases with t/R and increases with β . Consequently, all the obtained curves lie under an enveloping curve, representing the maximum restitution coefficient of the circular arches of minimum thickness (i.e., at incipient collapse under their self-weight) at varying of the embrace angle. As expected from Remark 1, such a curve never exceeds the limit of a unitary restitution coefficient.

In closing, it is emphasized that the present results follow from the assumption that the mechanism $\mathbf{u}$ before the impact coincides with the incipient rocking

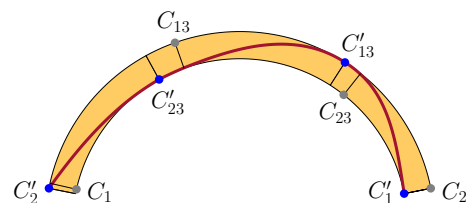


Fig. 12 Continuous circular arches: impulse line at impact for an arch with thickness-over-radius ratio $t/R = 0.15$ and embrace angle $\beta = 157.5^\circ$. The attained maximum restitution coefficient is $r_{\max} = 0.606$

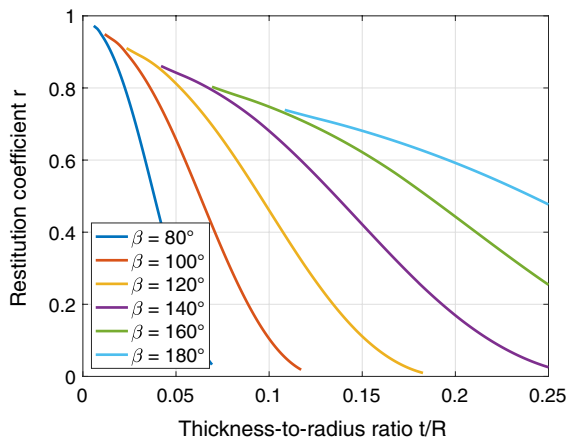


Fig. 13 Continuous circular arches: maximum restitution coefficient $r_{\max}$ versus the thickness-to-radius ratio t/R for selected values of the embrace angle β , assuming the mechanism $\mathbf{u}$ before the impact coincides with the incipient rocking mechanism

one. As discussed in [27], that may not even be the case for the first impact of a rocking motion of the arch. Nevertheless, such an assumption is of no limitation for the proposed impact model, which just takes the mechanism before the impact as an input to be determined by accurate solution strategies of the smooth motion under prescribed ground acceleration.

7 Conclusions

An extension of the Housner impact model has been proposed for addressing the impacts occurring during the rocking motion of masonry arches subjected to ground acceleration. It has been assumed that the arch arrives at the impact moving along a prescribed four-hinge mechanism and that, after the impact, it continues its motion along a new four-hinge mechanism to be determined. The proposed impact model has been founded on the introduction of the impulse line at the impact as the locus of the application points, on the sections of the arch, of the resultant impulses within the arch. That has allowed for formalizing the kinematic admissibility requirement that resultant impulses need to be transmitted through the hinges opening after the impact. The mechanism after the impact has been determined by minimizing the kinetic energy loss of the arch at impact, i.e. maximizing its restitution coefficient, over the set

of compatible mechanisms after the impact fulfilling a suitable principle of virtual work formulation. It has been proven that the descending impulse line is equilibrated, kinematically admissible, and statically admissible, i.e. corresponding to a fully compressive impulsive stress state within the arch. Numerical results have been discussed, addressing the restitution coefficient of discrete and continuous circular arches with parameterized geometry under the assumption that the four-hinge mechanism before the impact is determined by an equivalent static analysis.

Acknowledgements The financial support of MIUR, PRIN 2017 program, project “3DP_Future” (grant 2017L7X3CS_004), and of University of Rome Tor Vergata, “Ricerca Scientifica di Ateneo 2021” program, projects “OPTYMA” (CUP E83C22002290005) and “Vulnerabilità sismica di archi in muratura” (CUP E83C22002230005), is gratefully acknowledged.

Funding Open access funding provided by Università degli Studi di Roma Tor Vergata within the CRUI-CARE Agreement.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

A Kinematic analysis

The four-hinge mechanism $\mathbf{u}$ of a masonry arch before the impact is considered. It is characterized by the four hinges C_1 , C_{13} , C_{23} , C_2 , which determine the three rigid segments I, II, and III (Fig. 2a). The absolute center of rotation C_3 of the segment III lies at the intersection of the lines joining C_1 to C_{13} , and C_2 to C_{23} .

Provided that the angular velocity of the segment I is chosen as the free parameter α of the mechanism, the following expressions hold for the coefficient Ω introduced in Eq. (1)₁:

$$\begin{aligned}\Omega_I &= 1, \\ \Omega_{II} &= \frac{x_{C_{23}} - x_{C_3}}{x_{C_{23}} - x_{C_2}} \Omega_{III}, \\ \Omega_{III} &= \frac{x_{C_{13}} - x_{C_1}}{x_{C_{13}} - x_{C_3}},\end{aligned}\quad (\text{A.1})$$

respectively prevailing for an infinitesimal voussoir belonging to the segment I, II, or III of the arch. In Eq. (A.1), x_P stands for the x -coordinate, in a Cartesian reference frame, of the typical point P .

Analogous treatment holds for the mechanism $\mathbf{u}'$ after the impact and the coefficient Ω' introduced in Eq. (1)₂.

B On maximizing the restitution coefficient

In this section, it is shown that in solution of the optimization problem (16)–(17), an impulse line is obtained in the arch that is equilibrated, kinematically and statically admissible.

To such an aim, a kinematic model of the arch is considered, consistent with the classical theory of curved beams, to characterize the mechanism $\mathbf{u}'$ after the impact. The infinitesimal motion of the typical normal section of the arch is thus described by the linear velocity $\mathbf{v}' = w'\mathbf{t} + v'\mathbf{n}$ of its centroid G and by the angular velocity ω' . The following classical measures of deformation are introduced:

$$\varepsilon'_0 = \frac{dw'}{ds} - \kappa v', \quad \chi' = \frac{d\omega'}{ds}, \quad (\text{B.1})$$

where ε'_0 denotes the normal strain of the tangential fiber through G (whose geometric curvature is κ), and χ' denotes the curvature change. It follows that the normal strain of the typical tangential fiber, at distance y from G along the normal direction $\mathbf{n}$ is:

$$\varepsilon'(y) = \varepsilon'_0 + \chi'y. \quad (\text{B.2})$$

The Heyman assumptions on masonry material imply that the mechanism $\mathbf{u}'$ is compatible provided that the normal strain $\varepsilon'(y)$ is tensile for any y , and no sliding occurs. The former condition amounts to require that the normal strain, respectively ε'_+ and ε'_- , of the intrados and extrados tangential fibers, which are at a signed distance:

$$y_{\pm} = \pm t/2 + \kappa t^2/12 \quad (\text{B.3})$$

from the centroid G along the normal direction $\mathbf{n}$, is tensile:

$$\varepsilon'_{\pm} = \varepsilon'_0 + \chi'y_{\pm} \geq 0. \quad (\text{B.4})$$

On the other hand, the latter condition boils down to the requirement that the shear strain γ' between the tangential and normal fibers through G is vanishing:

$$\gamma' = \frac{dv'}{ds} + \kappa w' + \omega' = 0. \quad (\text{B.5})$$

In addition, the principle of virtual work formulation (12)–(13) is rephrased within the present notation as:

$$\int (\mathbf{p} \cdot \mathbf{v}' + L_G \omega') ds = \int (\rho \|\mathbf{v}'\|^2 + I_G \omega'^2) ds, \quad (\text{B.6})$$

where $\mathbf{p}$ and L_G denote the linear density respectively of the linear and angular momenta about the centroid G before the impact.

Since maximizing the restitution coefficient r is equivalent to maximizing the kinetic energy K' after the impact (in fact, the kinetic energy K before the impact is known), the following general form of the optimization problem (16)–(17) is arrived at:

$$\begin{aligned} \max_{\mathbf{v}', \omega'} \quad & \int \frac{1}{2} (\rho \|\mathbf{v}'\|^2 + I_G \omega'^2) ds, \\ \text{subject to} \quad & \gamma' = 0, \quad \varepsilon'_{\pm} \geq 0, \end{aligned} \quad (\text{B.7})$$

Equation (B.6) holds.

By exploiting relationships (B.1) and (B.4), and applying the Karush–Kuhn–Tucker theorem, it is inferred that $(\mathbf{v}', \omega')$ is a stationary point of the above optimization problem provided that there exist functions $\mathcal{I}_n$, $\mathcal{I}_{t+}$, $\mathcal{I}_{t-}$, to be intended as multipliers respectively of the constraints $\gamma' = 0$ and $\varepsilon'_{\pm} \geq 0$, such that:

$$\begin{aligned} p_t + \frac{d}{ds} (\mathcal{I}_{t+} + \mathcal{I}_{t-}) - \kappa \mathcal{I}_n &= \rho w', \\ p_n + \kappa (\mathcal{I}_{t+} + \mathcal{I}_{t-}) + \frac{d\mathcal{I}_n}{ds} &= \rho v', \\ L_G + \frac{d}{ds} (\mathcal{I}_{t+} y_+ + \mathcal{I}_{t-} y_-) - \mathcal{I}_n &= I_G \omega', \\ \gamma' = 0, \quad \varepsilon'_{\pm} \geq 0, \quad \mathcal{I}_{t\pm} \leq 0, \quad \mathcal{I}_{t\pm} \varepsilon'_{\pm} &= 0, \end{aligned} \quad (\text{B.8})$$

in which the decomposition $\mathbf{p} = p_t \mathbf{t} + p_n \mathbf{n}$ has been used.

For an interpretation of those conditions, the following positions are introduced:

$$\begin{aligned}\mathcal{I} &= \mathcal{I}_t \mathbf{t} + \mathcal{I}_n \mathbf{n}, \\ \mathcal{I}_t &= \mathcal{I}_{t+} + \mathcal{I}_{t-}, \\ \mathcal{M}_G &= \mathcal{I}_{t+} y_+ + \mathcal{I}_{t-} y_-, \end{aligned} \quad (\text{B.9})$$

which represent the internal linear and angular impulses about the centroid G transmitted at the typical arch section. In fact, upon noticing that:

$$\begin{aligned}\mathcal{M}_O &= \mathcal{M}_G + (G - O) \times \mathcal{I} \cdot \mathbf{k}, \\ \mathcal{L}_O &= \mathcal{L}_G + (G - O) \times \mathbf{p} \cdot \mathbf{k}, \\ \mathcal{L}'_O &= \mathcal{L}'_G + (G - O) \times \mathbf{p}' \cdot \mathbf{k}, \end{aligned} \quad (\text{B.10})$$

with $\mathbf{p}' = \rho \mathbf{v}'$ and $\mathcal{L}'_G = \mathcal{I}_G \omega'$, it is a simple matter to check that conditions (B.8)_{1–3} can be compactly written as:

$$\begin{aligned}\mathbf{p} + \frac{d\mathcal{I}}{ds} &= \mathbf{p}', \\ \mathcal{L}_O + \frac{d\mathcal{M}_O}{ds} &= \mathcal{L}'_O. \end{aligned} \quad (\text{B.11})$$

By observing that $d\mathbf{p} = \mathbf{p}ds$ and $d\mathcal{L}_O = \mathcal{L}_O ds$, those equations are recognized as the differential form of the principles of linear and angular impulse and momentum given in Eq. (3) and thus guarantee the impulse line to be equilibrated. Furthermore, because conditions (B.8)₆ can be recast in the form:

$$\mp \mathcal{M}_G \pm \mathcal{I}_t y_{\pm} \leq 0, \quad (\text{B.12})$$

it follows that the resultant impulse is compressive at any section of the arch, and the impulse line is completely contained within the arch thickness, Eq. (4), i.e. the impulse line is statically admissible. Finally, conditions (B.8)₇ enforce that the resultant impulses do zero work for the mechanism $\mathbf{u}'$ after the impact, i.e. do not resist the opening of the relevant hinges, therefore implying the impulse line to be kinematically admissible.

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