Inclusions and positive cones of von Neumann algebras

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Abstract

We consider cones in a Hilbert space associated to two von Neumann algebras and determine when one algebra is included in the other. If a cone is associated to a von Neumann algebra, the Jordan structure is naturally recovered from it and we can characterize projections of the given von Neumann algebra with the structure in some special situations.

1 Introduction

The natural positive cone $\mathcal{P}^\natural = \overline{\Delta^{\frac{1}{4}}\mathcal{M}_+\xi_0}$ plays a significant role in the theory of von Neumann algebras (see, for example, [1, 5]) where $\mathcal{M}$ is a von Neumann algebra, ξ_0 is a cyclic separating vector for $\mathcal{M}$ and Δ is the Tomita-Takesaki modular operator associated to ξ_0. Among them, the result of Connes [6] is of particular interest which characterized the natural positive cones with their geometric properties called selfpolarity, facial homogeneity and orientability, and showed that if two von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ share a same cone, then there is a central projection q of $\mathcal{M}$ such that $N = q\mathcal{M} \oplus q^\perp\mathcal{M}'$. Connes used the Lie algebra with an involution of the linear transformation group of $\mathcal{P}^\natural$ in his paper.

In the present paper, instead of $\mathcal{P}^\natural$, we study $\mathcal{P}^\sharp = \overline{\mathcal{M}_+\xi_0}$, which holds more informations of $\mathcal{M}$, for example, the subalgebra structure.

In the second section, we study what occurs when $\overline{\mathcal{N}_+\xi_0} \subset \mathcal{P}^\sharp$ where $\mathcal{N}$ is another von Neumann algebra. We consider first the case when ξ_0 is not cyclic for $\mathcal{N}$ and then assume the cyclicity. It turns out that in the latter case $\mathcal{N}$ is included in $\mathcal{M}$ except the part where ξ_0 is tracial.

In the third section, we characterize central projections of $\mathcal{M}$ in terms of $\mathcal{P}^\sharp$. A projection p is in $\mathcal{M} \cap \mathcal{M}'$ if and only if p and its orthogonal complement $p^\perp$ preserve $\mathcal{P}^\sharp$.

In the fourth and fifth sections, the Jordan structure on $\mathcal{P}^\sharp$ is studied. We can recover the lattice structure of projections and the operator norm from the order structure of $\mathcal{P}^\sharp$. Then we can define the square operation on $\mathcal{P}^\sharp$.

In the final section, using the Jordan structure, a characterization of projections in $\mathcal{M}$ is obtained when the modular automorphism with respect to ξ_0 acts ergodically.

The result of the second section has an easy application to the theory of half-sided modular inclusions [12, 2]. Let $\{U(t)\}$ be a one-parameter group of unitary operators with a generator H which kills ξ_0. Assume that $\mathcal{M}$ is a factor of type III_1 (or more generally a properly infinite algebra). It is easy to see that $U(t)\mathcal{M}U(t)^* \subset \mathcal{M}$ for $t \geq 0$ if and only if $U(t)$ preserves $\mathcal{P}^\sharp$ for $t \geq 0$. A similar result for $\mathcal{P}^\natural$ and $\{e^{-tH}\}$ has been obtained by Borchers with additional conditions on H [4].

Davidson has obtained conditions for $\{U(t)\}$ to generate a one-parameter semigroup of endomorphisms [7]. The relations with the modular group have been shown to be important in his study.

2 Inclusions of positive cones

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and ξ_0 be a cyclic separating vector for $\mathcal{M}$. We denote the modular group by Δ^{it}, the modular conjugation by J, modular automorphism by σ_t and the canonical involution by $S = J\Delta^{\frac{1}{2}}$. The positive cone associated to ξ_0 is denoted by $\mathcal{P}^\sharp = \overline{\mathcal{M}_+\xi_0}$.

Suppose there is another von Neumann algebra $\mathcal{N}$ such that $\overline{\mathcal{N}_+\xi_0} \subset \mathcal{P}^\sharp$. We can define a positive contractive map α from $\mathcal{N}$ into $\mathcal{M}$ as follows.

Lemma 2.1. *For $a \in \mathcal{N}_+$ there is the unique positive element $\alpha(a) \in \mathcal{M}$ satisfying $a\xi_0 = \alpha(a)\xi_0$. In addition, α is contractive on $\mathcal{M}_+$.*

Proof. By the assumption, we have $a\xi_0 \in \mathcal{P}^\sharp$. Recall that for a vector $a\xi_0$ in $\mathcal{P}^\sharp$ there is a positive linear operator $\alpha(a)$ affiliated to $\mathcal{M}$ such that $a\xi_0 = \alpha(a)\xi_0$ [11].

Since $\|a\|I - a$ is positive, we have$(\|a\|I - a)\xi_0 \in \mathcal{P}^\sharp$. This implies, for every $y \in \mathcal{M}'$,

$$\begin{aligned}
\langle \alpha(a)y\xi_0, y\xi_0\rangle &= \langle \alpha(a)\xi_0, y^*y\xi_0\rangle \\
&= \langle a\xi_0, y^*y\xi_0\rangle \\
&\leq \|a\|\langle \xi_0, y^*y\xi_0\rangle = \|a\|\|y\xi_0\|^2.
\end{aligned}$$

Hence $\alpha(a)$ is bounded and in $\mathcal{M}$. □

We can easily see that α extends to $\mathcal{N}$ by linearity. Since α is contractive on $\mathcal{N}_+$, α is bounded on $\mathcal{N}_{sa}$.

Lemma 2.2. *The map α maps every projection to a projection.*

Proof. Take a projection $e \in \mathcal{N}$. Note that, since α maps $\mathcal{N}_+$ into $\mathcal{M}_+$ and is contractive, we have $\alpha(e) \geq \alpha(e)^2$.

Recall that, by the definition of α, we have $\alpha(e)\xi_0 = e\xi_0$. We calculate as follows.

$$\begin{aligned}
\langle \alpha(e)^2\xi_0, \xi_0\rangle &= \langle \alpha(e)\xi_0, \alpha(e)\xi_0\rangle \\
&= \langle e\xi_0, e\xi_0\rangle \\
&= \langle e\xi_0, \xi_0\rangle \\
&= \langle \alpha(e)\xi_0, \xi_0\rangle.
\end{aligned}$$

This implies that $\langle \left(\alpha(e) - \alpha(e)^2\right)\xi_0, \xi_0\rangle = 0$. As we noted above, $\alpha(e) - \alpha(e)^2$ must be positive, hence the vector $\left(\alpha(e) - \alpha(e)^2\right)^{\frac{1}{2}}\xi_0$ must vanish. By the separating property of ξ_0, we see $\alpha(e) = \alpha(e)^2$. □

Recall that a linear mapping ϕ which preserves every anticommutator is called a Jordan homomorphism:

$$\phi(xy + yx) = \phi(x)\phi(y) + \phi(y)\phi(x).$$

Now we show the following lemma. The proof of it is essentially taken from [9].

Lemma 2.3. *The map α is a Jordan homomorphism.*

Proof. Let e and f be mutually orthogonal projections in $\mathcal{N}$. Then $e + f$, $\alpha(e)$, $\alpha(f)$ and $\alpha(e) + \alpha(f)$ are projections. We see the range of $\alpha(e)$ and the range of $\alpha(f)$ are mutually orthogonal because if not, then the sum $\alpha(e) + \alpha(f)$ could not be a projection. This implies that

$$\alpha(e)\alpha(f) = \alpha(f)\alpha(e) = 0.$$

In particular, α maps the positive (resp. negative) part of a self-adjoint element x to the positive (reps. negative) part of $\alpha(x)$. From this we see that α is contractive on $\mathcal{N}_{sa}$.

Next suppose we have commuting projections $e, f \in \mathcal{N}$. Remark that, since $ef \leq e$, positivity of α assures $\alpha(ef) \leq \alpha(e)$. Recalling that in this case ef and e are projections, we see the range of $\alpha(ef)$ is included in the range of $\alpha(e)$. Thus we have $\alpha(ef)\alpha(e) = \alpha(ef)$.

Now noting $e - ef$ and f are mutually orthogonal projections, we have

$$0 = \alpha(e - ef)\alpha(e) = \alpha(e)\alpha(f) - \alpha(ef).$$

Hence α preserves products of commuting projections.

Since every self-adjoint element in a von Neumann algebra is a uniform limit of linear combinations of mutually orthogonal projections, and since α is continuous in norm on $\mathcal{N}_{sa}$, α preserves products of commuting self-adjoint elements. In particular, α preserves the square of self-adjoint elements.

This implies that, firstly, α preserves Jordan products of self-adjoint elements $ab + ba = (a + b)^2 - a^2 - b^2$. This shows

$$\begin{aligned} \alpha(ab + ba) &= \alpha\left((a + b)^2\right) - \alpha(a^2) - \alpha(b^2) \\ &= \alpha(a + b)^2 - \alpha(a)^2 - \alpha(b)^2 \\ &= \alpha(a)\alpha(b) + \alpha(b)\alpha(a). \end{aligned}$$

Secondly, α preserves squares of arbitrary elements $(a + ib)^2 = a^2 + i(ab + ba) - b^2$:

$$\begin{aligned} \alpha\left((a + ib)^2)\right) &= \alpha\left(a^2 + i(ab + ba) - b^2\right) \\ &= \alpha(a^2) + i\alpha(ab + ba) - \alpha(b^2) \\ &= \alpha(a)^2 + i\left(\alpha(a)\alpha(b) + \alpha(b)\alpha(a)\right) - \alpha(b)^2 \\ &= \left(\alpha(a) + i\alpha(b)\right)^2. \end{aligned}$$

Finally, α preserves Jordan products of arbitrary elements $xy + yx = (x + y)^2 - x^2 - y^2$:

$$\begin{aligned} \alpha(xy + yx) &= \alpha\left((x + y)^2\right) - \alpha(x^2) - \alpha(y^2) \\ &= \alpha(x + y)^2 - \alpha(x)^2 - \alpha(y)^2 \\ &= \alpha(x)\alpha(y) + \alpha(y)\alpha(x). \end{aligned}$$

This completes the proof. □

Here we need the following result on Jordan homomorphisms of Jacobson and Rickart [8].

Proposition 2.4. *Suppose ϕ is a unital Jordan homomorphism from an algebra $\mathcal{A}$ into $\mathcal{B}$. Suppose further that $\mathcal{A}$ has a system of matrix units. Then there is a central idempotent g of the algebra generated by $\phi(\mathcal{A})$ such that $\phi(\cdot)g$ is homomorphic and $\phi(\cdot)(I-g)$ is antihomomorphic.*

Note that every von Neumann algebra $\mathcal{N}$ decomposes into the commutative part, the I_n parts, the II_1 part, and the properly infinite part. On the first one α causes no problem and on the remaining parts we can apply Proposition 2.4 to the case in which $\phi = \alpha$, $\mathcal{A} = \mathcal{N}$, $\mathcal{B} = \mathcal{M}$. Examining the proof, we see if ϕ is self-adjoint, then g is a central projection of $\alpha(\mathcal{N})''$ (the argument here is due to Kadison [9]).

Next, we show the normality of α.

Lemma 2.5. *The map α is a normal linear mapping from $\mathcal{N}$ into $\mathcal{M}$.*

Proof. We only have to show that for any normal functional φ on $\mathcal{M}$ the functional $\varphi \circ \alpha$ on $\mathcal{N}$ is normal. Note that, since $\mathcal{M}$ has a separating vector ξ_0, we may assume $\varphi(\cdot) = \langle \cdot \eta_1, \eta_2 \rangle$ for some $\eta_1, \eta_2 \in \mathcal{H}$.

Recall that a linear functional on a von Neumann algebra is normal if and only if it is continuous on every bounded set in the weak operator topology.

Now suppose that we have a convergent bounded net in the weak operator topology $x_i \to x$ in $\mathcal{N}$. Obviously $\{x_i\xi_0\}$ converges to $x\xi_0$ weakly. By the definition of α, we see $\{\alpha(x_i)\xi_0\}$ converges to $\alpha(x)\xi_0$ weakly. We have, for any $y_1, y_2 \in \mathcal{M}'$,

$$\begin{aligned}
\langle \alpha(x_i)y_1\xi_0, y_2\xi_0 \rangle &= \langle y_1\alpha(x_i)\xi_0, y_2\xi_0 \rangle \\
&= \langle \alpha(x_i)\xi_0, y_1^* y_2\xi_0 \rangle \\
&\to \langle \alpha(x)\xi_0, y_1^* y_2\xi_0 \rangle \\
&= \langle \alpha(x)y_1\xi_0, y_2\xi_0 \rangle.
\end{aligned}$$

First we assume $\{x_i\}$ is a net of self-adjoint elements. Then for arbitrary $\eta_1, \eta_2 \in \mathcal{H}$ the convergence $\langle \alpha(x_i)\eta_1, \eta_2 \rangle \to \langle \alpha(x)\eta_1, \eta_2 \rangle$ holds since $\{x_i\}$ is a bounded net, α is contractive on $\mathcal{N}_{sa}$, and ξ_0 is cyclic for $\mathcal{M}'$.

Then we can obtain the convergence for arbitrary bounded convergent net in WOT $\{x_i\}$ since we have the decomposition

$$x_i = \frac{x_i + x_i^*}{2} + i\frac{x_i - x_i^*}{2i}$$

and each part of the net is self-adjoint or antiself-adjoint, bounded and WOT-converging. □

We combine this lemma and the proposition of Jacobson and Rickart to get the following.

Lemma 2.6. *There is a normal homomorphism β and normal antihomomorphism γ of $\mathcal{N}$ into $\mathcal{M}$ such that $\alpha(x) = \beta(x) + \gamma(x)$ and the the range of β and γ are mutually orthogonal.*

In addition, there are central projections $e, f \in \mathcal{N}$ and a central projection $g \in \alpha(\mathcal{N})''$ such that $\alpha(e\,\cdot)g = \beta(\cdot)$ is an isomorphism of $\mathcal{N}e$ and $\alpha(f\,\cdot)g^\perp = \gamma(\cdot)$ is an antiisomorphism of $\mathcal{N}f$.

Proof. We know from Proposition 2.4 that there is a central projection $g \in \alpha(\mathcal{N})''$ such that $\beta(\cdot) = \alpha(\cdot)g$ is a homomorphism of $\mathcal{N}$ and $\gamma(\cdot) = \alpha(\cdot)g^{\perp}$ is an antihomomorphism of $\mathcal{N}f$. Then just take e as the support of β and f as the support of γ. Since α is normal, so are β and γ and the definitions of e and f are legitimate. □

Lemma 2.7. *The von Neumann algebra $\mathcal{N}f$ is finite.*

Proof. Let $\mathcal{N}h$ be the properly infinite part of $\mathcal{N}f$. We have $g^{\perp}\alpha(xy) = g^{\perp}\alpha(y)\alpha(x) = \alpha(y)g^{\perp}\alpha(x)$ for $x, y \in \mathcal{N}h$.

Again take $x, y \in \mathcal{N}h$. By the definition of α, we have

$$\begin{aligned} g^{\perp}xy\xi_0 &= g^{\perp}\alpha(xy)\xi_0 \\ &= \alpha(y)g^{\perp}\alpha(x)\xi_0 \\ \left\langle g^{\perp}xy\xi_0, \xi_0 \right\rangle &= \left\langle \alpha(y)g^{\perp}\alpha(x)\xi_0, \xi_0 \right\rangle \\ &= \left\langle g^{\perp}\alpha(x)\xi_0, \alpha(y^*)\xi_0 \right\rangle \\ &= \left\langle g^{\perp}x\xi_0, y^*\xi_0 \right\rangle \\ &= \left\langle yg^{\perp}x\xi_0, \xi_0 \right\rangle. \end{aligned}$$

Since $\mathcal{N}h$ is properly infinite, there is a sequence of isometries $\{v_n\} \subset \mathcal{N}h$ such that $v_n v_n^* \to 0$ in SOT-topology (That they are isometries means $v_n^* v_n = h$). Now

$$\begin{aligned} \langle \gamma(h)\xi_0, \xi_0 \rangle &= \left\langle g^{\perp}h\xi_0, \xi_0 \right\rangle \\ &= \left\langle g^{\perp}v_n^* v_n\xi_0, \xi_0 \right\rangle \\ &= \left\langle v_n g^{\perp}v_n^*\xi_0, \xi_0 \right\rangle \\ &\leq \langle v_n v_n^*\xi_0, \xi_0 \rangle \to 0. \end{aligned}$$

But since $\gamma(h)$ is a projection in $\alpha(\mathcal{N})'' \subset \mathcal{M}$ and since ξ_0 is separating for $\mathcal{M}$, $\gamma(h)$ must be zero. Recalling that h is a subprojection of f and that f is the support of γ, we see that $h = 0$. □

Theorem 2.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras and ξ_0 is a cyclic separating vector for $\mathcal{M}$. Suppose $\overline{\mathcal{N}_+\xi_0} \subset \mathcal{P}^\sharp$.*

Then we have two disjoint possibilities:

1. *The von Neumann algebra $\mathcal{M}$ has a subalgebra $\mathcal{M}_1$ such that $\overline{\mathcal{M}_{1+}\xi_0} = \overline{\mathcal{N}_+\xi_0}$.*
2. *For any subalgebra $\mathcal{M}_2$ of $\mathcal{M}$, its "sharpened cone" $\overline{\mathcal{M}_{2+}\xi_0}$ cannot coincide with $\overline{\mathcal{N}_+\xi_0}$ and $\mathcal{N}$ has a finite ideal $\mathcal{N}_1$ such that there is a subalgebra of $\mathcal{M}$ which is isomorphic to the direct sum of $\mathcal{N}_1$ and $\mathcal{N}_1^{\mathrm{opp}}$.*

Proof. Suppose that e and f defined above are mutually orthogonal. Then let us define $\mathcal{M}_1 = \alpha(\mathcal{N})$. Since we have $ef = 0$, it decomposes as follows.

$$\begin{aligned} \alpha(\mathcal{N}) &= \alpha\left(\mathcal{N}[e + e^{\perp}][f + f^{\perp}]\right) \\ &= \alpha\left(\mathcal{N}[ef^{\perp} + fe^{\perp} + e^{\perp}f^{\perp}]\right) \\ &= \beta\left(\mathcal{N}ef^{\perp}\right) + \gamma\left(\mathcal{N}fe^{\perp}\right), \end{aligned}$$

by noting that $\mathcal{N}e^{\perp}f^{\perp}$ is the kernel of α.

Since the range of β and γ are mutually orthogonal, and since e and f are central projections, $\alpha(\mathcal{N})$ is a direct sum of $\beta\left(\mathcal{N}ef^{\perp}\right)$ and $\gamma\left(\mathcal{N}fe^{\perp}\right)$.

Let a be a positive element of $\mathcal{N}$. Then we have

$$\begin{aligned} a\xi_0 &= \alpha(a)\xi_0 \\ &= \beta(ae)\xi_0 + \gamma(af)\xi_0 \\ &= \beta(aef^{\perp})\xi_0 + \gamma(afe^{\perp})\xi_0. \end{aligned}$$

Conversely it is easy to see that for $b \in \alpha(\mathcal{N})_+$ there is $a \in \mathcal{N}_+$ such that $\alpha(a) = b$, hence we have $a\xi_0 = b\xi_0$. This completes the proof of the claimed equality $\overline{\mathcal{M}_{1+}\xi_0} = \overline{\mathcal{N}+\xi_0}$.

Next, we assume that $ef \neq 0$. Note that $\mathcal{N}ef$ is noncommutative since by the definition of β and γ the commutative part of $\mathcal{N}$ is left to β. In particular g is a nontrivial central projection in $\alpha(\mathcal{N}ef)''$. By Lemma 2.7, $\mathcal{N}ef$ is finite. One can easily see that $\alpha(\mathcal{N}ef)''$ is a subalgebra of $\mathcal{M}$ which decomposes into the direct sum of $\beta(\mathcal{N}ef)$ and $\gamma(\mathcal{N}ef)$ where the latter is isomorphic to $(\mathcal{N}ef)^{\mathrm{opp}}$.

What remains to prove is that for any subalgebra $\mathcal{M}_2$ of $\mathcal{M}$ we cannot have the equality $\overline{(\mathcal{N}ef)_+\xi_0} = \overline{\mathcal{M}_{2+}\xi_0}$. To see this impossibility, recall that

$$\overline{\mathcal{M}_+\xi_0} = \{A\xi_0 \,|A \text{ is a closed positive operator affiliated to } \mathcal{M}\},$$

since ξ_0 is a separating vector for $\mathcal{M}$ [11]. Similarly we have

$$\overline{\mathcal{M}_{2+}\xi_0} = \{A\xi_0 \,|A \text{ is a closed positive operator affiliated to } \mathcal{M}_2\}.$$

Now suppose $a\xi_0 \in \overline{\mathcal{M}_{2+}\xi_0}$ for a positive element a of $\mathcal{N}ef$. By the above remark, we have a positive operator A affiliated to $\mathcal{M}_2$ such that $a\xi_0 = \alpha(a)\xi_0 = A\xi_0$. Then for $y \in \mathcal{M}'$ we have

$$\alpha(a)y\xi_0 = y\alpha(a)\xi_0 = yA\xi_0 = Ay\xi_0,$$

hence A is bounded and $\alpha(a) = A$. This implies $\alpha(a) \in \mathcal{M}_2$ and $\alpha(\mathcal{N}ef) \subset \mathcal{M}_2$. But by Proposition 2.4 $\alpha(\mathcal{N}ef)$ generates $\beta(\mathcal{N}ef) \oplus \gamma(\mathcal{N}ef)$. We have $\beta(\mathcal{N}ef) \oplus \gamma(\mathcal{N}ef) \subset \mathcal{M}_2$.

We will show that this leads to a contradiction. By the observation above we see that $\overline{\mathcal{M}_{2+}\xi_0}$ contains vectors of the form $ga\xi_0, g^{\perp}b\xi_0$ where $a, b \in (\mathcal{N}ef)_+$.

Suppose the contrary that $ga\xi_0 \in \overline{(\mathcal{N}ef)_+\xi_0}$. By the argument similar to the above one, there is a self-adjoint positive operator A affiliated to $\mathcal{N}ef$ such that $A\xi_0 = ga\xi_0$. Then $g^{\perp}A\xi_0 = 0$. Noting that f is the support of γ and that ξ_0 is separating for $\mathcal{M}$, we see $g^{\perp}e_A\xi_0 = \gamma(e_A)\xi_0$ cannot vanish for any nontrivial projection e_A of $\mathcal{N}ef$.

There are a spectral projection e_A of A, a positive scalar ϵ and $y \in \mathcal{M}'$ such that $A \geq \epsilon e_A$ and $\langle \gamma(e_A)y\xi_0, y\xi_0\rangle > 0$. Remark that

$$\begin{aligned} g^{\perp}(A - \epsilon e_A)\xi_0 &\in g^{\perp}\overline{(\mathcal{N}ef)_+\xi_0} \\ &\subset \overline{g^{\perp}(\mathcal{N}ef)_+\xi_0} \\ &= \overline{\gamma(\mathcal{N}ef)_+\xi_0}. \end{aligned}$$

Then we have

$$
\begin{aligned}
0 &= \langle y g^{\perp} A \xi_0, y\xi_0 \rangle \\
&= \langle g^{\perp} A \xi_0, y^* y \xi_0 \rangle \\
&= \langle g^{\perp}(A - \epsilon e_A)\xi_0, y^* y \xi_0 \rangle + \langle g^{\perp} \epsilon e_A \xi_0, y^* y \xi_0 \rangle \\
&\geq \langle g^{\perp} \epsilon e_A \xi_0, y^* y \xi_0 \rangle \\
&= \langle y \gamma(\epsilon e_A) \xi_0, y \xi_0 \rangle \\
&= \epsilon \langle \gamma(e_A) y \xi_0, y \xi_0 \rangle \\
&> 0.
\end{aligned}
$$

This contradiction completes the proof of that $\overline{(\mathcal{N}ef)_+\xi_0} \neq \overline{\mathcal{M}_{2+}\xi_0}$. □

If we further assume the cyclicity of ξ_0 for $\mathcal{N}$, we have a stronger result. For the proof of it, we need the following lemma. This can be found, for example in [3], but here we present another simple proof.

Lemma 2.9. *If $\mathcal{A} \subset \mathcal{B}$ is a proper inclusion of von Neumann algebras on a Hilbert space $\mathcal{K}$ and if ζ is a common cyclic separating vector, then $\mathcal{B}$ cannot be finite.*

Proof. Suppose the contrary, that $\mathcal{B}$ is finite. Then $\mathcal{A}$ must be finite, too. Hence there is a faithful trace τ on $\mathcal{B}$. Since ζ is separating for $\mathcal{B}$, there is a vector η such that $\tau(x) = \langle x\eta, \eta \rangle$ by the Radon-Nikodym type theorem. Since τ is faithful, η must be separating for $\mathcal{B}$.

We can see that η is cyclic for $\mathcal{B}$ as follows. Denote the orthogonal projection onto $\overline{\mathcal{B}\eta}$ by p. By separation verified above, we have $\overline{\mathcal{B}'\eta} = \mathcal{K}$. On the other hand, by assumption, $\overline{\mathcal{B}\zeta} = \overline{\mathcal{B}'\zeta} = \mathcal{K}$. By the general theory of equivalence of projections, $p \sim I$ in $\mathcal{B}$. But recalling that $\mathcal{B}$ is finite, we see that $p = I$, i.e., η is cyclic.

By the same reasoning, η is cyclic separating tracial for $\mathcal{A}$. Then the modular conjugations $J_{\mathcal{A}}$ and $J_{\mathcal{B}}$ with respect to η must coincide and we have the required equation.

$$\mathcal{A}' \supset \mathcal{B}' = J_{\mathcal{B}} \mathcal{B} J_{\mathcal{B}} = J_{\mathcal{A}} \mathcal{B} J_{\mathcal{A}} \supset J_{\mathcal{A}} \mathcal{A} J_{\mathcal{A}} = \mathcal{A}'.$$

This contradicts the assumption that the inclusion $\mathcal{A} \subset \mathcal{B}$ is proper. □

Theorem 2.10. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras and ξ_0 be a vector cyclic separating for $\mathcal{M}$ and cyclic for $\mathcal{N}$. Suppose $\overline{\mathcal{N}_+\xi_0} \subset \mathcal{P}^\sharp$.*

Then we have the following.

1. *The vector ξ_0 is also separating for $\mathcal{N}$.*

2. *There is a central projection e in $\mathcal{N}$ such that $\mathcal{N}e \subset \mathcal{M}$.*

3. *The vector $e^\perp \xi_0$ is tracial for $\mathcal{N}e^\perp$.*

4. *$J_{e^\perp} \mathcal{N} e^\perp J_{e^\perp} \subset \mathcal{M}$.*

In particular, $\mathcal{N}$ and $\mathcal{N}e \oplus J_{e^\perp}\mathcal{N}e^\perp J_{e^\perp}$ share the same positive cone $\mathcal{P}^\sharp_{\mathcal{N}}$ where $\mathcal{N}e \oplus J_{e^\perp}\mathcal{N}e^\perp J_{e^\perp} \subset \mathcal{M}$.

Proof. First we show that the induction by g realizes $\beta(\cdot) = g\alpha(\cdot)$. For arbitrary $x, y \in \mathcal{N}$ we have

$$\begin{aligned} gxy\xi_0 &= g\alpha(xy)\xi_0 \\ &= g\alpha(x)\alpha(y)\xi_0 \\ &= g\alpha(x)y\xi_0 \\ &= \alpha(x)gy\xi_0. \end{aligned}$$

Taking it into consideration that ξ_0 is cyclic for $\mathcal{N}$, we see that $gx = g\alpha(x) = \alpha(x)g$. But, since this holds for arbitrary $x \in \mathcal{N}$, in particular for self-adjoint elements. If $x = x^*$, then we have

$$gx = \alpha(x)g = (g\alpha(x))^* = (gx)^* = xg.$$

Since this equation is linear for x, we see that $g \in \mathcal{N}'$ and $gx = g\alpha(x)$.

Now recall that we have decomposed α into a normal homomorphism β and a normal antihomomorphism γ. We again denote the support of β by e and the support of γ by f.

Let $\mathcal{N}h$ be the properly infinite part. By Lemma 2.7 the intersection of h and f is trivial. Thus we have

$$ghx\xi_0 = hg\alpha(hx)\xi_0 = h\alpha(hx)\xi_0 = hx\xi_0,$$

for $x \in \mathcal{N}$. Cyclicity of ξ_0 tells us that $gh = h$. Then for $hx \in \mathcal{N}h$ we get that

$$\alpha(hx) = ghx = hx.$$

In other words, α maps identically on $\mathcal{N}h$. In particular, α is decomposed by h, that is, we have

$$h\alpha(h^\perp) = \alpha(h)\alpha(h^\perp) = 0,$$

since α maps orthogonal projections to orthogonal projections.

Note that $h\xi_0$ is cyclic for $\mathcal{N}h$ since ξ_0 is cyclic for $\mathcal{N}$. The vector $h\xi_0$ is also separating for $\mathcal{N}h$ since

$$\mathcal{N}h = \alpha(\mathcal{N}h) \subset \mathcal{M}$$

and ξ_0 is separating for $\mathcal{M}$.

For the proof of remaining part of the theorem, we may assume $\mathcal{N}$ is finite.

Recall that $g^\perp$ commutes with $\mathcal{N}$. Take $x, y \in \mathcal{N}$ and let us calculate

$$\begin{aligned} \langle xyg^\perp\xi_0, g^\perp\xi_0\rangle &= \langle g^\perp y\xi_0, g^\perp x^*\xi_0\rangle \\ &= \langle g^\perp\alpha(y)\xi_0, g^\perp\alpha(x^*)\xi_0\rangle \\ &= \langle g^\perp\alpha(x)\alpha(y)\xi_0, g^\perp\xi_0\rangle \\ &= \langle g^\perp\alpha(yx)\xi_0, g^\perp\xi_0\rangle \\ &= \langle g^\perp yx\xi_0, g^\perp\xi_0\rangle \\ &= \langle yxg^\perp\xi_0, g^\perp\xi_0\rangle \end{aligned}$$

This shows that $g^\perp\xi_0$ is a tracial vector for $\mathcal{N}g^\perp$. By assumption, ξ_0 is cyclic for $\mathcal{N}$, hence $g^\perp\xi_0$ is cyclic for $\mathcal{N}g^\perp$. In addition, it is also separating as follows. If $xg^\perp\xi_0 = 0$ for some

$x \in \mathcal{N}g^{\perp}$, then for any $y \in \mathcal{N}g^{\perp}$ we have

$$\begin{aligned}\left\| xyg^{\perp}\xi_0 \right\|^2 &= \langle y^*x^*xyg^{\perp}\xi_0, g^{\perp}\xi_0\rangle \\ &= \langle xyy^*x^*g^{\perp}\xi_0, g^{\perp}\xi_0\rangle \\ &\leq \|y\|^2\langle xx^*g^{\perp}\xi_0, g^{\perp}\xi_0\rangle \\ &= \|y\|^2\langle x^*xg^{\perp}\xi_0, g^{\perp}\xi_0\rangle \\ &= 0,\end{aligned}$$

then the cyclicity implies the separation by $g^{\perp}\xi_0$.

Now $\mathcal{N}g^{\perp}$ has the canonical conjugation $J_{g^{\perp}}$ defined as (the closure of)

$$J_{g^{\perp}} : g^{\perp}\mathcal{H} \ni x\xi_0 \longmapsto x^*\xi_0 \in g^{\perp}\mathcal{H}.$$

On $\mathcal{N}g^{\perp}$ we have the canonical antihomomorphism

$$\mathcal{N}g^{\perp} \ni x \longmapsto J_{g^{\perp}}x^*J_{g^{\perp}} \in \mathcal{N}g^{\perp}.$$

In our situation the composition of the induction by $g^{\perp}$ and this antihomomorphism coincide with the composition of α and the induction by $g^{\perp}$. In fact, for any elements $x, y, z \in \mathcal{N}g^{\perp}$ we have

$$\begin{aligned}\langle J_{g^{\perp}}(xg^{\perp})^*g^{\perp}J_{g^{\perp}}yg^{\perp}\xi, zg^{\perp}\xi_0\rangle &= \langle z^*g^{\perp}\xi_0, x^*y^*g^{\perp}\xi_0\rangle \\ &= \langle yxz^*g^{\perp}\xi_0, g^{\perp}\xi_0\rangle \\ &= \langle z^*yxg^{\perp}\xi_0, g^{\perp}\xi_0\rangle \\ &= \langle g^{\perp}yx\xi_0, zg^{\perp}\xi_0\rangle \\ &= \langle g^{\perp}\alpha(yx)\xi_0, zg^{\perp}\xi_0\rangle \\ &= \langle g^{\perp}\alpha(x)\alpha(y)\xi_0, zg^{\perp}\xi_0\rangle \\ &= \langle g^{\perp}\alpha(x)y\xi_0, zg^{\perp}\xi_0\rangle \\ &= \langle g^{\perp}\alpha(x)y\xi_0, zg^{\perp}\xi_0\rangle.\end{aligned}$$

The cyclicity of $g^{\perp}\xi_0$ shows that $g^{\perp}\alpha(x) = J_{g^{\perp}}(xg^{\perp})^*J_{g^{\perp}}$.

Summing up, we get the following formula for α:

$$\begin{aligned}\alpha(x) &= g\alpha(x) + g^{\perp}\alpha(x) \\ &= gx + J_{g^{\perp}}g^{\perp}x^*J_{g^{\perp}}.\end{aligned}$$

Note that $g\xi_0$ is cyclic separating for $\mathcal{N}g$. In fact, the cyclicity comes from the assumption of ξ_0's cyclicity and separating property can be seen by observing

$$\mathcal{N}g = g\alpha(\mathcal{N}) \subset \mathcal{M}$$

and by separating property of ξ_0 for $\mathcal{M}$.

On the other hand, we have seen that $g^{\perp}\xi_0$ is cyclic separating for $\mathcal{N}g^{\perp}$ in the way proving that $g^{\perp}\xi_0$ is a faithful tracial vector.

The direct sum of $\mathcal{N}g$ and $\mathcal{N}g^{\perp}$ has a cyclic separating vector ξ_0. These summands are finite because we are assuming that $\mathcal{N}$ is finite and they are induced part of it. Hence $\mathcal{N}g \oplus \mathcal{N}g^{\perp}$ is also finite.

Clearly $\mathcal{N}$ is a subalgebra of $\mathcal{N}g \oplus \mathcal{N}g^\perp$. So ξ_0 is separating for $\mathcal{N}$. This is the first statement of the theorem.

Now we have an inclusion of finite von Neumann algebras

$$\mathcal{N} \subset \mathcal{N}g \oplus \mathcal{N}g^\perp$$

and ξ_0 is a common cyclic separating vector. Then they must coincide by Lemma 2.9. This happens only if g is a projection of $\mathcal{N}$ from the beginning, i.e, g is a central projection of $\mathcal{N}$.

Recall that induction by g coincides with the homomorphic part of α. Now we know that g is central. Then the support e of the homomorphic part β must be exactly g.

On the other hand, the intersection $e^\perp f^\perp$ of kernels of the homomorphic part β and the antihomomorphic part γ must be trivial. To see this, take $x \in \mathcal{N}$. We have

$$\begin{aligned} e^\perp f^\perp x\xi_0 &= xe^\perp f^\perp \xi_0 \\ &= x\alpha\left(e^\perp f^\perp\right)\xi_0 \\ &= 0. \end{aligned}$$

Since ξ_0 is cyclic for $\mathcal{N}$, we get that $e^\perp f^\perp = 0$.

Since the induction by e realizes the homomorphic part β of α, for the antihomomorphic part γ it holds

$$\gamma(e) = e^\perp \alpha(e) = \alpha(e) - e\alpha(e) = 0.$$

This implies e must be orthogonal to f, which is the support of γ. As their intersection vanishes, we get $f = I - e$.

Recalling $g = e$, we saw that $e^\perp \xi_0$ is a cyclic separating tracial vector for $\mathcal{N}e^\perp$ and the canonical antiisomorphism with respect to $e^\perp \xi_0$ coincides with $e^\perp \alpha$. Then the proof of all the statements in the theorem is done. □

3 Recovery of central projections

In the following sections we turn to the study of single von Neumann algebra. Again let $\mathcal{M}$ be a von Neumann algebra and ξ_0 be a cyclic separating vector for $\mathcal{M}$. By Connes' result, $\mathcal{P}^\natural$ determines $\mathcal{M}$ up to center.

Here we show that the center is easily recovered from $\mathcal{P}^\sharp$. Let p be a projection $\mathcal{B}(\mathcal{H})$ such that $p\mathcal{P}^\sharp \subset \mathcal{P}$ and $p^\perp \mathcal{P}^\sharp \subset \mathcal{P}^\sharp$.

In this situation, we can define a mapping from $\mathcal{M}$ into $\mathcal{M}$ using p.

Lemma 3.1. *For every $a \in \mathcal{M}_+$ there is $\alpha(a) \in \mathcal{M}_+$ such that $pa\xi_0 = \alpha(a)\xi_0$.*

Proof. As in the proof of Lemma 2.1, we have a positive operator $\alpha(a)$ affiliated to $\mathcal{M}$ such that $pa\xi_0 = \alpha(a)\xi_0$ since $pa\xi_0$ is a vector of the positive cone $\mathcal{P}^\sharp$. This is again bounded for a different reason. In fact, for $y \in \mathcal{M}'$ we have

$$\begin{aligned}\langle \alpha(a) y\xi_0, y\xi_0 \rangle &= \langle \alpha(a)\xi_0, y^* y\xi_0 \rangle \\ &= \langle pa\xi_0, y^* y\xi_0 \rangle \\ &\leq \langle pa\xi_0, y^* y\xi_0 \rangle + \langle p^\perp a\xi_0, y^* y\xi_0 \rangle \\ &= \langle a\xi_0, y^* y\xi_0 \rangle \\ &= \langle ay\xi_0, y\xi_0 \rangle \\ &\leq \|a\| \|y\xi_0\|^2,\end{aligned}$$

where we have used the assumption that $p^\perp$ preserves $\mathcal{P}^\sharp$. □

From this we see that $\alpha(a) \leq a$ as self-adjoint operators. The map α extends to a linear mapping of $\mathcal{M}$.

Lemma 3.2. *The map α maps every projection to a projection.*

Proof. Let e be a projection of $\mathcal{M}$. By the observation above, we have $\alpha(e) \leq e$. Then using the fact $e\alpha(e) = \alpha(e)$ we can calculate

$$\begin{aligned}\langle \alpha(e)^2 \xi_0 \xi_0 \rangle &= \langle \alpha(e)\xi_0, \alpha(e)\xi_0 \rangle \\ &= \langle pe\xi_0, pe\xi_0 \rangle \\ &= \langle pe\xi_0, e\xi_0 \rangle \\ &= \langle \alpha(e), e\xi_0 \rangle \\ &= \langle \alpha(e), \xi_0 \rangle.\end{aligned}$$

We can see that $\alpha(e)^2 = \alpha(e)$ as in the proof of Lemma 2.2. □

Then the mapping α is a normal Jordan homomorphism and there is a central projection g of $\alpha(\mathcal{M})'' \subset \mathcal{M}$ such that $\alpha(\cdot)g$ is homomorphic and $\alpha(\cdot)g^\perp$ is antihomomorphic. The proof is similar to the one for the case of subcones.

Now we have the following.

Theorem 3.3. *Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, ξ_0 be a cyclic separating vector for $\mathcal{M}$ and $\mathcal{P}^\sharp = \overline{\mathcal{M}_+\xi_0}$. Then a projection $p \in \mathcal{B}(\mathcal{H})$ is a central projection of $\mathcal{M}$ if and only if p and $p^\perp$ preserve $\mathcal{P}^\sharp$.*

Proof. The "only if" part is trivial.

Let p be a projection which and whose orthogonal complement preserve $\mathcal{P}^\sharp$. Note that $\alpha(x) \in \mathcal{M}$ and that $\alpha(\alpha(x)) = \alpha(x)$ holds. In fact, we have

$$\alpha(\alpha(x))\xi_0 = p\alpha(x)\xi_0 = ppx\xi_0 = px\xi_0 = \alpha(x)\xi_0,$$

since p is a projection.

As in the situation of subcones, α is a sum of a normal homomorphism and a normal antihomomorphism whose ranges are mutually orthogonal. The kernels of the homomorphism and the antihomomorphism are central projections of $\mathcal{M}$. Thus the support of α is the orthogonal complement of the intersection of these kernels. In particular it is a central projection $e \in \mathcal{M}$.

Recall that $\alpha(e) \leq e$. Take an arbitrary positive element a from $\mathcal{M}$. If we apply α to $ea - \alpha(ea)$, since the composition of α and α equals α itself, we have

$$\alpha\left(ea - \alpha(ea)\right) = \alpha(ea) - \alpha(ea) = 0.$$

The argument of the left hand side is less than the support of α, hence it must vanish. Thus we see that ea is fixed by α. By linearity, this holds for arbitrary element $x \in \mathcal{M}$ instead of positive element a.

Again since e is the support of α, we have $\alpha(x) = \alpha(xe) = xe$. Comparing this with the definition of α we can determine p.

$$\begin{aligned} px\xi_0 &= \alpha(x)\xi_0 \\ &= ex\xi_0 \end{aligned}$$

With the cyclicity of ξ_0 we see that p equals e. In particular, p must be a central projection of $\mathcal{M}$. □

4 Properties of $(\mathcal{P}^\sharp, \xi_0)$

In this section, we study the properties of $\mathcal{P}^\sharp$ coupled with a specified vector ξ_0. We begin with the following lemma.

Let us write $\zeta \leq \eta$ if $\eta - \zeta \in \mathcal{P}^\sharp$.

Lemma 4.1. *Let ζ be a vector in $\mathcal{P}^\sharp$. Then the following hold.*

1. *If $\zeta \leq \xi_0$, then there is a positive contractive operator $a \in \mathcal{M}$ such that $\zeta = a\xi_0$. In this case we say that ζ is contractive.*
2. *If ζ is contractive and if $\zeta \perp (\xi_0 - \zeta)$, then there is a projection $e \in \mathcal{M}$ such that $\zeta = e\xi_0$. When these conditions hold, we call ζ a projective vector.*
3. *If η and ζ are projective and $\zeta \leq \xi_0 - \eta$, then e and f are mutually orthogonal projections where $\eta = e\xi_0$ and $\zeta = f\xi_0$. We say η and ζ are mutually operationally orthogonal.*

Proof. The proofs of the first and the second statements are same as in the proofs of Lemma 2.1 and 2.2 respectively. We do not repeat them here.

Suppose $\eta = e\xi_0$, $\zeta = f\xi_0$ and $\eta \leq \xi_0 - \zeta$. Then according to this order, $e \leq I - f$. When e and f are projections, this shows the mutual orthogonality. □

We denote the set of contractive vectors by $\mathcal{P}^\sharp_1$. By the Lemma above, to each vector in $\mathcal{P}^\sharp_1$ there corresponds a positive contractive operator of $\mathcal{M}$.

Similarly to every vector ζ in $\mathbb{R}_+\mathcal{P}^\sharp_1$ there corresponds a bounded positive operator a of $\mathcal{M}$. Put $\mathcal{P}^\sharp_b = \mathbb{R}_+\mathcal{P}^\sharp_1$ and $\mathcal{K} = \mathbb{R}\mathcal{P}^\sharp_1$.

Lemma 4.2. *For an arbitrary vector ζ in $\mathcal{P}^\sharp_1$ there is a least projective vector such that $\eta \geq \zeta$. Let us call η the support of ζ.*

Proof. As noted above, there is a positive operator a such that $\zeta = a\xi_0$. As we have seen, the order structure of $\mathcal{P}^\sharp_1$ is consistent with this correspondence. Let e be the support projection of a. Then we have $\eta = e\xi_0 \geq a\xi_0 = \zeta$. Hence η is the least projective vector in $\mathcal{P}^\sharp_1$. □

Lemma 4.3. *Every vector ζ in $\mathcal{K}$ is uniquely decomposed as $\zeta = \zeta_+ - \zeta_-$ where ζ_+ and ζ_- are vectors of $\mathcal{P}_b^\sharp$ and supports of ζ_+ and ζ_- are mutually operationally orthogonal.*

Proof. Since every vector in $\mathcal{P}_1^\sharp$ corresponds to a positive contractive operator in $\mathcal{M}$, vectors of $\mathcal{P}_b^\sharp$ (resp. $\mathcal{K}$) correspond to positive operators (resp. self-adjoint operators).

Now the lemma follows from the theory of self-adjoint operators. The self-adjoint operator z corresponding to ζ has the Jordan decomposition $z = z_+ - z_-$ where z_+ and z_- are positive operators of $\mathcal{M}$ whose supports are mutually orthogonal. By Lemma 4.1, ζ has the corresponding decomposition. □

Lemma 4.4. *The cone $\mathcal{P}_b^\sharp$ is dense in $\mathcal{P}^\sharp$.*

Proof. For each vector ζ in $\mathcal{P}^\sharp$ there is a positive self-adjoint linear operator A affiliated to $\mathcal{M}$ such that $\zeta = A\xi_0$[11]. Let E_A be the spectral measure associated to A. Then $AE_A([0,n])$ is bounded positive operator in $\mathcal{M}$. It is well known that $\{AE_A([0,n])\,\xi_0\}$ converges to $A\xi_0$. □

In addition, we can recover the operator norm in terms of $\mathcal{P}_b^\sharp$. For $\zeta \in \mathcal{P}_b^\sharp$ we define the new "sharp" norm $\|\zeta\|_\sharp$ as follows.

$$\|\zeta\|_\sharp = \sup\left\{ c \geq 0 \,\middle|\, \frac{1}{c}\zeta \leq \xi_0 \right\}.$$

Lemma 4.5. *If $a \in \mathcal{M}_+$ and $\zeta = a\xi_0$, then $\|\zeta\|_\sharp = \|a\|$.*

Proof. We only have to note that $ca\xi_0 \leq \xi_0$ if and only if $ca \leq I$. Then the spectral decomposition of a completes the proof. □

The set $\mathcal{K}$ is a real linear subspace of $\mathcal{H}$. To $\mathcal{K}$ we can extend the new norm $\|\cdot\|_\sharp$ as follows. For $\zeta \in \mathcal{K}$ define

$$\|\zeta\|_\sharp = \inf\left\{ \max\left\{ \|\zeta_1\|_\sharp, \|\zeta_2\|_\sharp \right\} \,\middle|\, \zeta_1, \zeta_2 \in \mathcal{P}_b^\sharp, \zeta_1 - \zeta_2 = \zeta \right\}.$$

It is easily seen that if $z \in \mathcal{M}_{sa}$ corresponds to $\zeta \in \mathcal{K}$, we have

$$\max\{\|z_+\|, \|z_-\|\} = \|z\| = \|\zeta\|_\sharp = \max\left\{ \|\zeta_+\|_\sharp, \|\zeta_-\|_\sharp \right\}.$$

5 Jordan structure on $\mathcal{K} + i\mathcal{K}$

First we define the square operation for vectors in $\mathcal{K}$.

Definition 5.1. If ζ is a real linear combination of mutually operationally orthogonal projective vectors, i.e. $\zeta = \sum_k c_k \zeta_k$ where $c_k \in \mathbb{R}$ and $\{\zeta_k\}$ are mutually operationally orthogonal, then we define the square of ζ as follows.

$$\zeta^2 = \sum_k c_k^2 \zeta_k.$$

As we have seen in Lemma 4.1, mutually operationally orthogonal projective vectors $\{\zeta_k\}$ correspond to mutually orthogonal projections $\{e_k\}$. Thus the square of a real linear combination $\sum_k c_k e_k$ equals $\sum_k c_k^2 e_k$ and for these vectors the definition of square is consistent.

The set of vectors which are real linear combinations of mutually operationally orthogonal projective vectors is dense in $\mathcal{K}$ in the sharp norm defined in Section 4. In fact, these vectors correspond to real linear combinations of mutually orthogonal projections in $\mathcal{M}$, i.e. self-adjoint operators with finite spectra.

Since the sharp norm on $\mathcal{K}$ is consistent with the operator norm on $\mathcal{M}$, we can extend the definition of square to $\mathcal{K}$ by continuity. We have the following.

$$\text{If } \zeta = z\xi_0 \text{ for } z \in \mathcal{M}_{sa}, \text{ then } \zeta^2 = z^2\xi_0.$$

Once we have defined the square operation on $\mathcal{K}$, we can define Jordan polynomials as follows. For η and ζ in $\mathcal{K}$ let us define

$$\eta\zeta + \zeta\eta = (\eta + \zeta)^2 - \eta^2 - \zeta^2.$$

Using this, for $\zeta = \zeta_1 + i\zeta_2 \in \mathcal{K} + i\mathcal{K}$ we put

$$\zeta^2 = \zeta_1^2 + i(\zeta_1\zeta_2 + \zeta_2\zeta_2) - \zeta_2^2.$$

As for vectors in $\mathcal{K}$, we define the "Jordan product" on $\mathcal{K} + i\mathcal{K}$ by

$$\eta\zeta + \zeta\eta = (\eta + \zeta)^2 - \eta^2 - \zeta^2.$$

Using this, finally we define

$$\zeta\eta\zeta = \frac{1}{2}\left[(\zeta\eta + \eta\zeta)\zeta + \zeta(\zeta\eta + \eta\zeta)\right] - \frac{1}{2}\left(\zeta^2\eta + \eta\zeta^2\right).$$

If $\eta = y\xi_0$ and $\zeta = z\xi_0$ for $y, z \in \mathcal{M}$, then it follows that $\zeta\eta\zeta = zyz\xi_0$. This follows because we have defined square and Jordan polynomials on $\mathcal{K}$ consistently.

If we fix ζ, we give names to the following mappings.

$$\begin{aligned} \mathrm{c}_\zeta &: \mathcal{K} + i\mathcal{K} \ni \eta \longmapsto \zeta\eta\zeta \in \mathcal{K} + i\mathcal{K}, \\ \mathrm{od}_\zeta &: \mathcal{K} + i\mathcal{K} \ni \eta \longmapsto \eta - \mathrm{c}_\zeta(\eta) - \mathrm{c}_{\zeta^\perp}(\eta) \in \mathcal{K} + i\mathcal{K}. \end{aligned}$$

Let $\eta = y\xi_0$ and $\zeta = e\xi_0$ where e is a projection. Then we see that

$$\begin{aligned} \mathrm{c}_\zeta(\eta) &= eye\xi_0, \text{ and} \\ \mathrm{od}_\zeta(\eta) &= y\xi_0 - eye\xi_0 - e^\perp ye^\perp\xi_0 = \left[eye^\perp + e^\perp ye\right]\xi_0 \end{aligned}$$

correspond to the corner of y and the off-diagonal part of y, respectively.

6 Recovery of projections in $\mathcal{M}$ in the case when $\mathcal{M}^\sigma = \mathbb{C}I$

Let p be a projection of $\mathcal{B}(\mathcal{H})$. We seek a necessary and sufficient condition for p to be a projection of $\mathcal{M}$.

We need a criterion for a projection in $\mathcal{M}$ to be fixed by the modular automorphism.

Lemma 6.1. *Let e be a projection in $\mathcal{M}$. If $px\xi_0 = xe\xi_0$ holds for all $x \in \mathcal{M}$, then we have $e \in \mathcal{M}^\sigma$ and $p = JeJ$.*

Proof. Note that we get $p\xi_0 = e\xi_0$ if we use the assumption with $x = I$.

Again by the assumption it follows that

$$\begin{aligned}\langle xe\xi_0, \xi_0\rangle &= \langle px\xi_0, \xi_0\rangle \\ &= \langle x\xi_0, p\xi_0\rangle \\ &= \langle x\xi_0, e\xi_0\rangle \\ &= \langle ex\xi_0, \xi_0\rangle.\end{aligned}$$

This implies that $e \in \mathcal{M}^\sigma$[11]. In particular, we have

$$e\xi_0 = Se\xi_0 = J\Delta^{\frac{1}{2}}e\xi_0 = Je\xi_0.$$

Now the equality $JeJx\xi_0 = xJeJ\xi_0 = xe\xi_0 = px\xi_0$ and the cyclicity of ξ_0 complete the proof. □

Recall that $S = J\Delta^{\frac{1}{2}}$ can be defined in terms of $\overline{\mathcal{K}}$ [10].

Theorem 6.2. *Let p be a projection in $\mathcal{B}(\mathcal{H})$. There is a projection $e \in \mathcal{M}$ and a central projection $q \in \mathcal{M}$ such that $q^\perp e \in \mathcal{M}^\sigma$ and $p = qe + Jq^\perp eJ$ if and only if the following hold:*

1. *$p\xi_0 \le \xi_0$.*
2. *If $\zeta \le p\xi_0$, then $p\zeta = \zeta$.*
3. *If $\zeta \le p^\perp\xi_0$, then $p^\perp\zeta = \zeta$.*
4. *For every vector $\xi \in \mathcal{K} + i\mathcal{K}$ we have $p\xi \in \mathcal{K} + i\mathcal{K}$ and*
   - *(a)* $\mathrm{c}_{p\xi_0}\left(p \ \mathrm{od}_{p\xi_0}(\xi)\right) = 0,$
   - *(b)* $\mathrm{c}_{p^\perp\xi_0}\left(p \ \mathrm{od}_{p\xi_0}(\xi)\right) = 0,$
   - *(c)* $\left(p \ \mathrm{od}_{p\xi_0}(\xi)\right)^2 = 0,$
   - *(d)* $\left(p^\perp \ \mathrm{od}_{p\xi_0}(\xi)\right)^2 = 0,$
   - *(e)* $Sp \ \mathrm{od}_{p\xi_0}(\xi) = p^\perp S \ \mathrm{od}_{p\xi_0}(\xi).$

Proof. First let us show the "only if" part. In this case, we have

$$p\xi_0 = qe\xi_0 + Jq^\perp eJ\xi_0 = qe\xi_0 + q^\perp e\xi_0 = e\xi_0 \le \xi_0,$$

hence the first part of the conditions is satisfied. For the second condition, if $\zeta = z\xi_0 \le p\xi_0 = e\xi_0$, then the support of z is less than or equal to e and we have

$$p\zeta = qez\xi_0 + zJeq^\perp J\xi_0 = qez\xi_0 + zeq^\perp\xi_0 = z\xi_0 = \zeta.$$

Similar proof works for the third. To see the conditions of the fourth, let $\xi = x\xi_0 \in \mathcal{K} + i\mathcal{K}$. We note that

$$\begin{aligned}
\mathrm{c}_{p\xi_0}(\xi) &= \mathrm{c}_{e\xi_0}(x\xi_0) = exe\xi_0, \\
\mathrm{od}_{p\xi_0}(\xi) &= \mathrm{od}_{e\xi_0}(x\xi_0) = \left[exe^\perp + e^\perp xe\right]\xi_0, \\
p\ \mathrm{od}_{p\xi_0}(\xi) &= \left[qexe^\perp + q^\perp e^\perp xe\right]\xi_0, \\
p^\perp\ \mathrm{od}_{p\xi_0}(\xi) &= \left[qe^\perp xe + q^\perp exe^\perp\right]\xi_0, \\
Sp\ \mathrm{od}_{p\xi_0}(\xi) &= \left[qe^\perp x^* e + q^\perp ex^* e^\perp\right]\xi_0, \\
p^\perp S\ \mathrm{od}_{p\xi_0}(\xi) &= (qe^\perp + Jq^\perp e^\perp J)\left[e^\perp x^* e + ex^* e^\perp\right]\xi_0 \\
&= \left[qe^\perp x^* e + q^\perp ex^* e^\perp\right]\xi_0.
\end{aligned}$$

Thus it is easy to see that each of the conditions is valid.

We turn to the "if" part. Let p satisfy the conditions of the statement.

Take $x \in \mathcal{M}$ satisfying $x = exe^\perp$. If we use the matrix, x takes the following form.

$$\begin{array}{c} \\ \mathrm{Ran}(e) \\ \mathrm{Ran}(e^\perp) \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{Ran}(e) & \mathrm{Ran}(e^\perp) \end{array} \\ \left(\begin{array}{cc} 0 & X \\ 0 & 0 \end{array}\right) \end{array}.$$

Then it holds that $\mathrm{od}_{p\xi_0}(x\xi_0) = x\xi_0$.

By assumption 4, there exists $y \in \mathcal{M}$ such that $px\xi_0 = y\xi_0$. In addition, by assumptions 4a and 4b, we have $eye = e^\perp ye^\perp = 0$, i.e. y has trivial corners. By assumption 4c, it follows $y^2 = 0$. Hence y takes the following form.

$$y = \left(\begin{array}{cc} 0 & \begin{array}{ccc} y_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \\ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & 0 \end{array} & 0 \end{array}\right),$$

where we decomposed $\mathrm{Ran}(e)$ and $\mathrm{Ran}(e^\perp)$ as follows.

$$\begin{aligned}
\mathrm{Ran}(e) &= \mathrm{Dom}(e^\perp ye) \oplus \mathrm{Ran}(eye^\perp) \oplus \left(\mathrm{Ran}(e) \ominus \mathrm{Dom}(e^\perp ye) \ominus \mathrm{Ran}(eye^\perp)\right), \\
\mathrm{Ran}(e^\perp) &= \mathrm{Dom}(eye^\perp) \oplus \mathrm{Ran}(e^\perp ye) \oplus \left(\mathrm{Ran}(e^\perp) \ominus \mathrm{Dom}(eye^\perp) \ominus \mathrm{Ran}(e^\perp ye)\right).
\end{aligned}$$

Subspaces which appear here are mutually orthogonal because the square of y vanishes.

According to this, we further decompose x.

$$x = \left(\begin{array}{cc} 0 & \begin{array}{ccc} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{array} \\ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} & 0 \end{array}\right).$$

By assumption 4d, the square of $p^\perp x\xi_0 = (x-y)\xi_0$ must vanish.

$$x - y = \begin{pmatrix} 0 & \begin{matrix} x_1 - y_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & -y_2 & 0 \\ 0 & 0 & 0 \end{matrix} & 0 \end{pmatrix},$$

$$(x-y)^2 = \begin{pmatrix} \begin{matrix} 0 & -x_2y_2 & 0 \\ 0 & -x_5y_2 & 0 \\ 0 & -x_8y_2 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 0 & 0 \\ -y_2x_4 & -y_2x_5 & -y_2x_6 \\ 0 & 0 & 0 \end{matrix} \end{pmatrix}.$$

Then it follows that $x_2 = x_4 = x_5 = x_6 = x_8 = 0$.

If we use assumption 4e, then we get

$$px^*\xi_0 = pSx\xi_0 = Sp^\perp x\xi_0 = (x^* - y^*)\xi_0.$$

Applying assumption 4c to $\xi = (x + x^*)\xi_0$, the square of $p(x + x^*)\xi_0 = (y + x^* - y^*)\xi_0$ vanishes.

$$y + x^* - y^* = \begin{pmatrix} 0 & \begin{matrix} y_1 & 0 & 0 \\ 0 & -y_2^* & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} x_1^* - y_1^* & 0 & x_7^* \\ 0 & y_2 & 0 \\ x_3^* & 0 & x_9^* \end{matrix} & 0 \end{pmatrix},$$

$$(y + x^* - y^*)^2 = \begin{pmatrix} \begin{matrix} y_1(x_1^* - y_1^*) & 0 & y_1x_7^* \\ 0 & -y_2^*y_2 & 0 \\ 0 & 0 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} (x_1^* - y_1^*)y_1 & 0 & 0 \\ 0 & -y_2y_2^* & 0 \\ x_3^*y_1 & 0 & 0 \end{matrix} \end{pmatrix}.$$

Thus it follows that $y_2 = x_3 = x_7 = 0$ and $x_1 = y_1$.

Summing up, for every $x = exe^\perp \in \mathcal{M}$ we have

$$x = \begin{pmatrix} \text{\Large 0} & \begin{matrix} x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x_9 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \text{\Large 0} \end{pmatrix},$$

$$y\xi_0 = px\xi_0 = \begin{pmatrix} \text{\Large 0} & \begin{matrix} x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \text{\Large 0} \end{pmatrix} \xi_0.$$

The point is that $\mathrm{Dom}(y)$ and $\mathrm{Dom}(x-y)$, $\mathrm{Ran}(y)$ and $\mathrm{Ran}(x-y)$ are mutually orthogonal, respectively.

If we take another element $z = eze^\perp \in \mathcal{M}$ and put $w\xi_0 = pz\xi_0$, then by the same argument we see that $\mathrm{Dom}(w)$ and $\mathrm{Dom}(z-w)$, $\mathrm{Ran}(w)$ and $\mathrm{Ran}(z-w)$ are mutually orthogonal, respectively. In addition, by noting that $w + x - y = e(w + x - y)e^\perp$ and $p(w + x - y)\xi_0 = w\xi_0$, it follows that $\mathrm{Dom}(x-y) \perp \mathrm{Dom}(w)$ and $\mathrm{Ran}(x-y) \perp \mathrm{Ran}(w)$. Similarly it holds that $\mathrm{Dom}(z-w) \perp \mathrm{Dom}(y)$ and $\mathrm{Ran}(z-w) \perp \mathrm{Ran}(y)$. Then let us define f_1 (resp. f_3) to be the projection onto the supremum of such $\mathrm{Ran}(x-y)$'s (resp. $\mathrm{Dom}(x-y)$'s) where $x = exe^\perp$ runs all the elements of this form in $\mathcal{M}$ and put $f_2 = e - f_1$, $f_4 = e^\perp - f_3$. They are mutually orthogonal projections of $\mathcal{M}$.

Using them every $x = exe^\perp \in \mathcal{M}$ is decomposed as follows.

$$\begin{array}{r} \\ \mathrm{Ran}(f_1) \\ \mathrm{Ran}(f_2) \\ \mathrm{Ran}(f_3) \\ \mathrm{Ran}(f_4) \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{Ran}(f_1) & \mathrm{Ran}(f_2) & \mathrm{Ran}(f_3) & \mathrm{Ran}(f_4) \end{array} \\ \begin{pmatrix} 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

According to this decomposition, it is easy to see that every $x \in \mathcal{M}$ must have the following form.

$$x = \begin{pmatrix} x_1 & 0 & x_3 & 0 \\ 0 & x_2 & 0 & x_4 \\ x_5 & 0 & x_7 & 0 \\ 0 & x_6 & 0 & x_8 \end{pmatrix}.$$

Put $q = f_1 + f_3$. This is clearly a central projection.

Since p preserves vectors of the set $\{\zeta \,|\, \zeta \le p\xi_0 = e\xi_0\}$ by assumption 2, it holds that $p\ exe\xi_0 = exe\xi_0$ for $x \in \mathcal{M}$. Similarly, by assumption 3, we see $p^\perp\ e^\perp xe^\perp\xi_0 = e^\perp xe^\perp\xi_0$, hence $p\ e^\perp xe^\perp\xi_0 = 0$.

Now, letting x be an arbitrary element of $\mathcal{M}$, p acts on $x\xi_0$ as follows.

$$\begin{aligned} px\xi_0 &= p\begin{pmatrix} x_1 & 0 & x_3 & 0 \\ 0 & x_2 & 0 & x_4 \\ x_5 & 0 & x_7 & 0 \\ 0 & x_6 & 0 & x_8 \end{pmatrix}\xi_0 = \begin{pmatrix} x_1 & 0 & x_3 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x_6 & 0 & 0 \end{pmatrix}\xi_0 \\ &= (qex + q^{\perp}xe)\xi_0. \end{aligned}$$

Then using the cyclicity of ξ_0 and Lemma 6.1, we arrive at the conclusion that $p = qe + Jq^{\perp}eJ$. □

Corollary 6.3. *If $\mathcal{M}^{\sigma} = \mathbb{C}I$, then the conditions in Theorem 6.2 assure that p is a projection of $\mathcal{M}$.*

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