

Representation theory of the stabilizer subgroup of the point at infinity in $\text{Diff}(S^1)$

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Abstract

The group $\text{Diff}(S^1)$ of the orientation preserving diffeomorphisms of the circle S^1 plays an important role in conformal field theory. We consider a subgroup B_0 of $\text{Diff}(S^1)$ whose elements stabilize “the point at infinity”. This subgroup is of interest for the actual physical theory living on the punctured circle, or the real line.

We investigate the unique central extension \mathcal{K} of the Lie algebra of that group. We determine the first and second cohomologies, its ideal structure and the automorphism group. We define a generalization of Verma modules and determine when these representations are irreducible. Its endomorphism semigroup is investigated and some unitary representations of the group which do not extend to $\text{Diff}(S^1)$ are constructed.

1 Introduction

In this paper we study a certain subalgebra of the Virasoro algebra defined below. The Virasoro algebra is a fundamental object in conformal quantum field theory.

The symmetry group of the chiral component of a conformal field theory in 1+1 dimension is B_0 , the group of all orientation-preserving diffeomorphisms of the real line which are smooth at the point at infinity (for example, see [15]). Instead of working on \mathbb{R} , it is customary to consider a chiral model on the compactified line S^1 with the symmetry group $\text{Diff}(S^1)$. In a quantum theory, we are interested in its projective representations.

With positivity of the energy, which is a physical requirement, the representation theory of the central extension of $\text{Diff}(S^1)$ has been well studied [15]. In any irreducible unitary projective representation of $\text{Diff}(S^1)$, the central element acts as a scalar c . The (central extension of the) group $\text{Diff}(S^1)$ has a subgroup S^1 of rotations and by positivity of energy the subgroup has the lowest eigenvalue $h \geq 0$. It is known for which values of c and h there exist irreducible, unitary, positive-energy, projective representations of $\text{Diff}(S^1)$. All such representations are classified by c and h .

The Lie algebra of $\text{Diff}(S^1)$ is the algebra of all the smooth vector fields on S^1 [14]. It is sometimes convenient to study its polynomial subalgebra, the Witt algebra. The Witt algebra has a unique central extension [15] called the Virasoro algebra Vir . In a similar way

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as above, we can define lowest energy representations of Vir with parameters c, h and it is known when these representations are unitary [6]. On the other hand, for any positive energy, unitary lowest weight representation of Vir there is a corresponding projective representation of $\text{Diff}(S^1)$ [4].

In a physical context, conformal field theory in 1+1 dimensional Minkowski space can be decomposed into its chiral components on two lightlines. Thus it is mathematically useful to study the subgroup B_0 of stabilizers of one point (“the point at infinity”) of $\text{Diff}(S^1)$. We can construct nets of von Neumann algebras on \mathbb{R} from representations of B_0 , and nets on \mathbb{R}^2 by tensor product. The theory of local quantum physics are extensively studied with techniques of von Neumann algebras [5][1][10][7]. In the case of nets on S^1 , the nets generated by $\text{Diff}(S^1)$ play a key role in the classification of diffeomorphism covariant nets [8]. This gives a strong motivation for studying the representation theory of B_0 , since for nets on \mathbb{R} the group B_0 should play a similar role to that of $\text{Diff}(S^1)$ for nets on S^1 .

Some properties of the restrictions of representations of $\text{Diff}(S^1)$ to B_0 have been studied. For example, the restriction to B_0 of every irreducible unitary positive energy representation of $\text{Diff}(S^1)$ is irreducible [17]. Different values of c, h may correspond to equivalent representations [17]. Unfortunately little is known about representations which are not restrictions. In this paper we address this problem.

1.1 Preliminaries

We identify the real line with the punctured circle through the Cayley transformation:

$$x = i \frac{1+z}{1-z} \iff z = \frac{x-i}{x+i}, x \in \mathbb{R}, z \in S^1 \subset \mathbb{C}.$$

The group $\text{Diff}(S^1)$ contains the following important one-parameter subgroups. They are called respectively the groups of rotations, translations and dilations:

$$\begin{aligned} \rho_s(z) &= e^{is}z, \text{ for } z \in S^1 \subset \mathbb{C} \\ \tau_s(x) &= x + s, \text{ for } x \in \mathbb{R} \\ \delta_s(x) &= e^s x, \text{ for } x \in \mathbb{R}, \end{aligned}$$

where rotations are defined in the circle picture, on the other hand translations and dilations are defined in the real line picture. Here we see that the point $z = e^{2\pi i\theta} = 1$ or $\theta = 0$ on the circle is identified with the point at infinity in the real line picture.

The positivity of the energy for $\text{Diff}(S^1)$ is usually defined as the boundedness from below of the generator of the group of rotation (since we consider projective representations, the generator of a one-parameter subgroup is defined only up to an addition of a real scalar multiple of the identity). It is well known that this is equivalent to the boundedness from below of the generator of the group of translations (see [11]). The latter definition is the one having its origin in physics. Concerning the group B_0 , as it does not include the group of rotations, the positivity of energy is defined by boundedness from below of the generator of the group of translations.

In the rest of this section we explain our notation regarding some infinite dimensional Lie algebras (see [15]).

The Witt algebra (we denote it by Witt) is the Lie algebra generated by L_n for $n \in \mathbb{Z}$ with the following commutation relations:

$$[L_m, L_n] = (m - n)L_{m+n}.$$

The Witt algebra has a central extension with a central element C , unique up to isomorphisms, with the following commutation relations:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{C}{12}m(m^2 - 1)\delta_{m,-n}.$$

This algebra is called the Virasoro algebra Vir . On Witt and Vir we can define a $*$ -operation by

$$(L_n)^* = L_{-n}, C^* = C.$$

The Witt algebra is a subalgebra of the Lie algebra $\text{Vect}(S^1)$ of smooth complex functions on the circle S^1 with the following commutation relations:

$$[f, g] = fg' - f'g,$$

and the correspondence $L_n \mapsto ie^{in\theta}$. Its real part is the Lie algebra of the group of diffeomorphisms of S^1 [14]. This algebra is equipped with the smooth topology, namely, a net of functions f_n converges to f if and only if the k -th derivatives $f_n^{(k)}$ converge to $f^{(k)}$ uniformly on S^1 for all $k \geq 0$. The central extension above extends continuously to this algebra. As the group $\text{Diff}(S^1)$ is a manifold modelled on $\text{Vect}(S^1)$, it is equipped with the induced topology of the smooth topology of $\text{Vect}(S^1)$.

We consider a subspace \mathcal{K}_0 of the Witt algebra spanned by $K_n = L_n - L_0$ for $n \neq 0$. By a straightforward calculation this subspace is indeed a $*$ -subalgebra with the following commutation relations:

$$[K_m, K_n] = \begin{cases} (m - n)K_{m+n} - mK_m + nK_n & (m \neq -n) \\ -mK_m - mK_{-m} & (m = -n) \end{cases}.$$

We denote $\text{Vect}(S^1)_0 \subset \text{Vect}(S^1)$ the subalgebra of smooth functions which vanish on $\theta = 0$. This is the Lie algebra of the group B_0 of all the diffeomorphisms of S^1 which stabilize $\theta = 0$. The algebra \mathcal{K}_0 is a $*$ -subalgebra of $\text{Vect}(S^1)_0$.

We will show that \mathcal{K}_0 has a unique (up to isomorphisms) central extension which is a subalgebra of Vir . The central extension is denoted by \mathcal{K} and has the following commutation relations:

$$[K_m, K_n] = \begin{cases} (m - n)K_{m+n} - mK_m + nK_n & (m \neq -n) \\ -mK_m - mK_{-m} + \frac{C}{12}m(m^2 - 1) & (m = -n) \end{cases}. \quad (1)$$

In section 2, we determine the first and second cohomologies of the algebra \mathcal{K}_0 . The first cohomology corresponds to one dimensional representations and the second cohomology corresponds to central extensions. It will be shown that the only possible central extension is the natural inclusion into the Virasoro algebra. On the other hand the first cohomology is one dimensional and does not extend to Vir .

In section 3, we determine the ideal structure of \mathcal{K}_0 and calculate their commutator subalgebras. It will be shown that all of these ideals can be defined by the vanishing of certain derivatives at the point at infinity.

In section 4, we determine the automorphism group of the central extension \mathcal{K} of \mathcal{K}_0 . This group turns out to be very small but contains some elements not extending to automorphisms of the Virasoro algebra.

In section 5, we construct several representations of \mathcal{K} . Each of these representations has an analogue of a lowest weight vector and has the universal property. Thanks to the result of Feigin and Fuks [3], we can completely determine which of these representations are irreducible.

In section 6, we investigate the endomorphism semigroup of \mathcal{K} . Compositions of these endomorphisms with known unitary representations give rise some strange kinds of representations. Corresponding representations of the group B_0 are studied in section 7.

2 First and Second cohomologies of \mathcal{K}_0

We will discuss the following cohomology groups of \mathcal{K}_0 [15]:

$$\begin{aligned} H^1(\mathcal{K}_0, \mathbb{C}) &:= \{\phi : \mathcal{K}_0 \rightarrow \mathbb{C} \mid \phi \text{ is linear and vanishes on } [\mathcal{K}_0, \mathcal{K}_0].\} \\ Z^2(\mathcal{K}_0, \mathbb{C}) &:= \{\omega : \mathcal{K}_0 \times \mathcal{K}_0 \rightarrow \mathbb{C} \mid \omega \text{ is bilinear and} \\ &\quad \text{for } a, b, c \in \mathcal{K}_0 \text{ satisfies } \omega(a, b) = -\omega(b, a), \\ &\quad \omega([a, b], c) + \omega([b, c], a) + \omega([c, a], b) = 0\} \\ B^2(\mathcal{K}_0, \mathbb{C}) &:= \{\omega : \mathcal{K}_0 \times \mathcal{K}_0 \rightarrow \mathbb{C} \mid \text{there is } \mu \text{ s.t } \omega(a, b) = \mu([a, b]).\} \\ H^2(\mathcal{K}_0, \mathbb{C}) &:= Z^2/B^2. \end{aligned}$$

Elements in the (additive) group H^1 correspond to one dimensional representations of \mathcal{K}_0 . The group H^2 corresponds to the set of all central extensions of \mathcal{K}_0 . We call H^1 and H^2 the first and the second cohomology groups of \mathcal{K}_0 , respectively.

Lemma 2.1. $[\mathcal{K}_0, \mathcal{K}_0]$ has codimension one in \mathcal{K}_0 .

Proof. Let us define a linear functional ϕ on \mathcal{K}_0 by the following:

$$\phi(K_n) = n.$$

As K_n 's form a basis of \mathcal{K}_0 , this defines a linear functional. By the commutation relation above, we have

$$\phi([K_m, K_n]) \begin{cases} \text{(for the case } m \neq -n) \\ = (m-n)\phi(K_{m+n}) - m\phi(K_m) + n\phi(K_n) \\ = (m-n)(m+n) - m^2 + n^2 \\ = 0 \\ \text{(for the case } m = -n) \\ = -m\phi(K_m) - m\phi(K_{-m}) \\ = -m^2 - m(-m) \\ = 0 \end{cases} .$$

Hence this vanishes on the commutator. The linear functional ϕ is nontrivial and the commutator subalgebra $[\mathcal{K}_0, \mathcal{K}_0]$ is in the nontrivial kernel of ϕ . In particular, $[\mathcal{K}_0, \mathcal{K}_0]$ is not equal to \mathcal{K}_0 .

To see that the commutator subalgebra of \mathcal{K}_0 has codimension one, we will show that all the element of \mathcal{K}_0 can be obtained as the linear combination of K_1 and elements of $[\mathcal{K}_0, \mathcal{K}_0]$.

Let us note that

$$\begin{aligned} [K_1, K_{-1}] &= -K_1 - K_{-1} \\ [K_2, K_{-1}] &= 3K_1 - 2K_2 - K_{-1} \\ [K_{-2}, K_1] &= -3K_{-1} + 2K_{-2} + K_1. \end{aligned}$$

So K_{-1}, K_2, K_{-2} can be obtained. For other elements in the basis, we only need to see

$$\begin{aligned} [K_n, K_1] &= (n-1)K_{n+1} - nK_n + K_1 \\ [K_{-n}, K_{-1}] &= -(n-1)K_{-n-1} + nK_{-n} - K_{-1}, \end{aligned}$$

and to use mathematical induction. \square

Remark 2.2. In proposition 3.2 of [17] it is claimed that $[\mathcal{K}, \mathcal{K}] = \mathcal{K}$ where \mathcal{K} is the central extension of \mathcal{K}_0 defined in the introduction of the present paper. It is wrong, as seen in lemma 2.1: \mathcal{K} , as well as \mathcal{K}_0 , is not perfect. In the proof of [17], there is a sentence “confronting what we have just obtained with (14), we get that ...”, which does not make sense. In accordance with this, the remark after proposition 3.6 and corollary 3.8 in that article should be corrected as to allow the difference by scalar. On the other hand, what is used in corollary 3.3 is only the fact that $\phi(C) = 0$ and the conclusion is not changed. The main results of the paper are not at all affected.

Corollary 2.3. *$H^1(\mathcal{K}_0, \mathbb{C})$ is one dimensional. In particular, there is a unique (up to scalar) one dimensional representation of \mathcal{K}_0 .*

Next we will determine the second cohomology group of \mathcal{K}_0 .

Lemma 2.4. *The following set forms a basis of the commutator subalgebra of \mathcal{K}_0 .*

$$[K_n, K_1], [K_{-n}, K_{-1}] \text{ for } n > 1, [K_{-2}, K_1], [K_2, K_{-1}], [K_1, K_{-1}].$$

Proof. As we have seen, the commutator subalgebra is the kernel of the functional of lemma 2.1. The last three elements in the set are linearly independent and contained in the subspace spanned by K_{-2}, K_{-1}, K_1 and K_2 . The elements $[K_n, K_1]$ (respectively the elements $[K_{-n}, K_{-1}]$.) contain K_{n+1} terms (respectively $K_{-(n+1)}$ terms,) hence they are independent and form the basis of the commutator subalgebra. \square

Theorem 2.5. *$H^2(\mathcal{K}_0, \mathbb{C})$ is one dimensional.*

Proof. Take an element ω of $Z^2(\mathcal{K}_0, \mathbb{C})$. Let $\omega_{m,n} := \omega(K_m, K_n)$ for $m, n \in \mathbb{Z} \setminus \{0\}$ be complex numbers. From the definition of $Z^2(\mathcal{K}_0, \mathbb{C})$, the following holds:

$$\begin{aligned} \omega_{m,n} &= -\omega_{n,m} \\ 0 &= \omega(K_l, [K_m, K_n]) + \omega(K_n, [K_l, K_m]) + \omega(K_m, [K_n, K_l]) \\ &= (m-n)\omega_{l,m+n} - m\omega_{l,m} + n\omega_{l,n} \\ &\quad + (l-m)\omega_{n,l+m} - l\omega_{n,l} + m\omega_{n,m} \\ &\quad + (n-l)\omega_{m,n+l} - n\omega_{m,n} + l\omega_{m,l}, \end{aligned} \tag{2}$$

where this holds also for the cases $l+m=0, m+n=0$, or $n+l=0$ if we define $w_{k,0} = w_{0,k} = 0$ for $k \in \mathbb{Z}$.

Let α be a linear functional on the commutator subalgebra defined by

$$\begin{aligned}\alpha([K_n, K_1]) &= \omega_{n,1} \text{ for } n > 1 \\ \alpha([K_{-n}, K_{-1}]) &= \omega_{-n,-1} \text{ for } n > 1 \\ \alpha([K_{-2}, K_1]) &= \omega_{-2,1} \\ \alpha([K_2, K_{-1}]) &= \omega_{2,-1} \\ \alpha([K_1, K_{-1}]) &= \omega_{1,-1}.\end{aligned}$$

This definition is legitimate by lemma 2.4.

If we define $\omega'_{m,n} = \omega_{m,n} - \alpha([K_m, K_n])$, there is a corresponding element ω' in $Z^2(\mathcal{K}_0, \mathbb{C})$ and belongs to the same class in $Z^2/B^2(\mathcal{K}_0)$. To keep the brief notation, we assume from the beginning the following:

$$\omega_{n,1} = \omega_{-n,-1} = \omega_{-2,1} = \omega_{2,-1} = \omega_{1,-1} = 0 \text{ for } n > 1$$

and we will show that $\omega_{m,n} = 0$ if $m \neq -n$.

Now we set $l = 2, m = 1, n = -1$ in (2) to get:

$$0 = 2\omega_{2,0} - \omega_{2,1} - \omega_{2,-1} + \omega_{-1,3} - 2\omega_{-1,2} + \omega_{-1,1} - 3\omega_{1,1} + \omega_{1,-1} + 2\omega_{1,2}.$$

From this we see that $\omega_{-1,3}$ vanishes because by assumption all the other terms are zero. Similarly if we let $l = -2, m = 1, n = 1$, we have $\omega_{1,-3} = 0$.

Furthermore, setting $l > 1, m = 1, n = -1$ we get

$$\begin{aligned}0 = 2\omega_{l,0} - \omega_{l,1} - \omega_{l,-1} + (l-1)\omega_{-1,l+1} - l\omega_{-1,l} + \omega_{-1,1} \\ - (l+1)\omega_{1,l-1} + \omega_{1,-1} + l\omega_{1,l}.\end{aligned}$$

This implies $\omega_{-1,l+1} = 0$ by induction for $l > 1$. Similarly, letting $l < -1, m = 1, n = -1$ we see $\omega_{1,l-1} = 0$ for $l < -1$.

Next we use formula (2) substituting $l = 1, n = -m$ to get

$$\begin{aligned}0 = 2m\omega_{1,0} - m\omega_{1,m} - m\omega_{1,-m} + (1-m)\omega_{-m,m+1} - \omega_{-m,1} + m\omega_{-m,m} \\ + (-m-1)\omega_{m,1-m} + m\omega_{m,-m} + \omega_{m,1}.\end{aligned}$$

Since $\omega_{1,m} = \omega_{-1,m} = 0$, as we have seen above, and by the antisymmetry $\omega_{-m,m} = -\omega_{m,-m}$, we have

$$(1-m)\omega_{-m,1+m} + (-m-1)\omega_{m,1-m} = 0.$$

By assumption, we have $\omega_{-1,2} = 0$. By induction on m , we observe $\omega_{-m,m+1} = 0$. Similarly it holds $\omega_{-m,m-1} = 0$.

Finally we fix $k \in \mathbb{N}$ and let $l = 1, n = k - m$ to get

$$\begin{aligned}0 = (2m-k)\omega_{1,k} - m\omega_{1,m} + (k-m)\omega_{1,k-m} + (1-m)\omega_{k-m,m+1} - \omega_{k-m,1} \\ + m\omega_{k-m,m} + (k-m-1)\omega_{m,k-m+1} - (k-m)\omega_{m,k-m} + \omega_{m,1}.\end{aligned}$$

By assumption, as before, the preceding equation becomes the following:

$$\begin{aligned}0 &= (1-m)\omega_{k-m,m+1} + k\omega_{k-m,m} + (k-m-1)\omega_{m,k-m+1} \\ &= (1-m)\omega_{(k+1)-(m+1),m+1} + k\omega_{k-m,m} + (k-m-1)\omega_{m,(k+1)-m}\end{aligned}\tag{3}$$

If we let $k = 1$, the second term vanishes by the observation above and we see

$$(1 - m)\omega_{1-m, m+1} - m\omega_{m, 2-m} = 0$$

Again by induction on m , we see $\omega_{2-m, m}$ vanishes for all m . Then by induction on k and using (3), we can conclude $\omega_{k-m, m}$ vanishes for all $k \in \mathbb{N}, m \in \mathbb{Z}$. Similar argument applies for $k < 0$.

Summarizing, if we have an element in $Z^2(\mathcal{K}_0, \mathbb{C})$, we may assume that all the off-diagonal parts vanish. Letting $l = -m - n$ in (2), we see that there is a possibility of one (and only) dimensional second cohomology as in the case of Virasoro algebra (see [15]). \square

This theorem shows that there is a unique central extension (up to isomorphism) of \mathcal{K}_0 . We denote the central extension by \mathcal{K} . By fixing a cocycle $\omega \in Z^2(\mathcal{K}_0, \mathbb{C}) \setminus B^2(\mathcal{K}_0, \mathbb{C})$ the algebra \mathcal{K} is formally defined as $\mathcal{K}_0 \oplus \mathbb{C}$ with the commutation relations

$$[(x, a), (y, b)] := ([x, y], \omega(x, y)) \text{ for } x, y \in \mathcal{K}_0, a, b \in \mathbb{C}.$$

Equivalently, in this article and in literature, using a formal central element C , one writes:

$$[x + aC, y + bC] = [x, y] + \omega(x, y)C.$$

Proposition 2.6. *Let us fix a real number λ . On \mathcal{K} , there is a $*$ -automorphism Λ defined by $\Lambda(K_n) = K_n + in\lambda C$ and $\Lambda(C) \mapsto C$.*

Proof. It is clear that this preserves the $*$ -operation. Since the change by this mapping is just an addition of a scalar multiple of the central element, this does not change the commutator. On the other hand, as seen in lemma 2.1, the map $K_n \mapsto n$ vanishes on the commutator subalgebra, hence the linear map in question preserves the commutators. \square

Proposition 2.7. *The $*$ -automorphism in Proposition 2.6 does not extend to the Virasoro algebra unless $\lambda = 0$.*

Proof. Assume the contrary, namely that Λ extends to Vir. Since \mathcal{K} has codimension one in the Virasoro algebra, we only have to determine where L_0 is mapped. The algebra Vir is the linear span of K_n 's, C and L_0 , hence $\Lambda(L_0)$ takes the following form.

$$\Lambda(L_0) = \sum_{n \neq 0} a_n K_n + a_0 L_0 + bC,$$

where a_n 's and b are complex numbers and a_n 's vanish except for finitely many n .

On the other hand, in Vir, we have

$$[K_n, L_0] = [L_n - L_0, L_0] = nL_n = nK_n + nL_0.$$

Since in the sum of $\Lambda(L_0)$ only finitely many terms appear, let N be the largest integer with which a_N does not vanish. If $N > 1$, recalling $[K_1, L_0] = K_1 + L_0$, we have

$$\begin{aligned} \Lambda([K_1, L_0]) &= [K_1 + i\lambda C, \Lambda(L_0)] \\ &= \Lambda(K_1) + \Lambda(L_0), \end{aligned}$$

which is impossible because the second expression contains K_{N+1} term but the last expression does not. Hence N must be less than 2. By the same argument replacing K_1 by K_2 , we

have that N must be less than 1. Similarly replacing K_1 by K_{-1} or K_{-2} , it can be shown that $\Lambda(L_0)$ must be of the form

$$\Lambda(L_0) = a_0L_0 + bC.$$

We need to note that a_0 and b must be real as Λ is a *-automorphism.

Now let us calculate again

$$\begin{aligned} [\Lambda(K_1), \Lambda(L_0)] &= [K_1 + i\lambda C, a_0L_0 + b \cdot C] \\ &= a_0K_1 + a_0L_0, \end{aligned}$$

by assumption this must be equal to

$$\begin{aligned} \Lambda([K_1, L_0]) &= \Lambda(K_1 + L_0) \\ &= K_1 + a_0L_0 + (b + i\lambda)C, \end{aligned}$$

which is impossible since b is real, except the case $\lambda = 0$ (and in this case $b = 0, a_0 = 1$). \square

Remark 2.8. When we make compositions of these automorphisms with a representation of \mathcal{K} , we might obtain inequivalent representations of \mathcal{K} . However these representations integrate to equivalent projective unitary representations of the group B_0 , since with these automorphisms the changes of self-adjoint elements in \mathcal{K} are only scalars and the changes of their exponentials are only phases, therefore equivalent as projective representations of B_0 .

3 Derived subalgebras and groups

3.1 A sequence of ideals in \mathcal{K}_0

We will investigate the derived subalgebras of \mathcal{K}_0 . The derived subalgebra (or the commutator subalgebra) of a Lie algebra is, by definition, the subalgebra generated by all the commutators of the given Lie algebra.

The easiest and most important property of the commutator subalgebra is that it is an ideal. This is clear from the definition. If a Lie algebra is simple, then the commutator subalgebra must coincide with the Lie algebra itself. This is the case for the Virasoro algebra.

On the other hand, the algebra \mathcal{K}_0 and its unique nontrivial central extension \mathcal{K} are not simple. This can be seen from lemma 2.1: the commutator subalgebra (which we denote by $\mathcal{K}_0^{(1)}$) has codimension 1 in \mathcal{K}_0 and it is the kernel of a homomorphism of the Lie algebra.

Let us denote $\text{Vect}(S^1)_0$ the subalgebra of $\text{Vect}(S^1)$ whose element vanish at $\theta = 0$. We remind that the commutator on $\text{Vect}(S^1)$ is the following.

$$[f, g] = fg' - f'g. \tag{4}$$

Now it is easy to see that $\text{Vect}(S^1)_0$ is a subalgebra. Let us recall that we embed \mathcal{K}_0 in $\text{Vect}(S^1)_0$ by the correspondence $K_n \mapsto i(\exp(in \cdot) - 1)$. We clarify the meaning of the homomorphism ϕ by considering the larger algebra $\text{Vect}(S^1)_0$.

Lemma 3.1. *The homomorphism $\phi : K_n \mapsto -n$ on K_0 continuously extends to $\text{Vect}(S^1)_0$ and the result is*

$$\begin{aligned}\phi : \text{Vect}(S^1)_0 &\rightarrow \mathbb{R} \\ f &\mapsto f'(0).\end{aligned}$$

Proof. It is easy to see that ϕ and the derivative on 0 coincide. The latter is clearly continuous on $\text{Vect}(S^1)_0$ in its smooth topology.

To see that the extension is still a homomorphism of $\text{Vect}(S^1)_0$, we only have to calculate the derivative of $[f, g]$ on $\theta = 0$:

$$\begin{aligned}\frac{d}{dt}[f, g](0) &= \left. \frac{d}{dt} (fg' - f'g) \right|_{t=0} \\ &= (f'g' + fg'' - f''g - f'g')(0) \\ &= (f''g - fg'')(0) \\ &= 0,\end{aligned}$$

since f and g are elements of $\text{Vect}(S^1)_0$. □

We set $\phi_1 := \phi$ and we define similarly,

$$\begin{aligned}\phi_k : \text{Vect}(S^1)_0 &\rightarrow \mathbb{R} \\ f &\mapsto f^{(k)}(0),\end{aligned}$$

where $f^{(k)}$ is the k -th derivative of the function f . Again these maps are continuous in the topology of smooth vectors.

We show the following.

Lemma 3.2. *Let f and g be in $\text{Vect}(S^1)_0$. Suppose $\phi_m(f) = \phi_m(g) = 0$ for $m = 1, \dots, k$. Then $\phi_m([f, g]) = \phi_m(fg' - f'g) = 0$ for $m = 1, \dots, 2k + 1$.*

Proof. First we recall the general Leibniz rule:

$$(F \cdot G)^{(k)}(\theta) = \sum_{m=0}^k {}_k C_m F^{(m)}(\theta) G^{(k-m)}(\theta),$$

where ${}_k C_m$ denotes the choose function $\frac{k!}{m!(k-m)!}$. Then, in each term of the m -th derivatives of $[f, g] = fg' - f'g$ where $m \leq 2k$, there appears a factor which is a derivative f or g of order $m \leq k$ and the term vanishes by assumption. To consider the $(2k + 1)$ -th derivative, the only nonvanishing terms are

$$\begin{aligned}[f, g]^{(2k+1)}(\theta) &= {}_{2k+1} C_{k+1} f^{(k+1)} g^{(k+1)} - {}_{2k+1} C_k f^{(k+1)} g^{(k+1)} \\ &= 0.\end{aligned}$$

□

Proposition 3.3. *The subspace $\text{Vect}(S^1)_k = \{f \in \text{Vect}(S^1)_0 : \phi_1(f) = \dots = \phi_k(f) = 0\}$ is an ideal of $\text{Vect}(S^1)_0$ and it holds that*

$$[\text{Vect}(S^1)_k, \text{Vect}(S^1)_k] \subset \text{Vect}(S^1)_{2k+1}$$

Proof. The latter part follows directly from lemma 3.2. To show that $\text{Vect}(S^1)_k$ is an ideal, we only have to take $f \in \text{Vect}(S^1)_0$ and $g \in \text{Vect}(S^1)_k$ and to calculate derivatives of $[f, g]$. By the Leibniz rule above, for $m \leq k$, in each term of the m -th derivative of $[f, g]$ there is a factor which is a derivative of g of order less than m or f itself and they must vanish at $\theta = 0$ by assumption. \square

Note that if we restrict ϕ_m to \mathcal{K}_0 , it acts like $\phi_m(K_k) = i(ik)^m$. Defining $\mathcal{K}_k = \{x \in \mathcal{K}_0 : \phi_1(x) = \cdots \phi_k(x) = 0\}$, we can see similarly that $\{\mathcal{K}_k\}$ are ideals of \mathcal{K}_0 and that $[\mathcal{K}_k, \mathcal{K}_k] \subset \mathcal{K}_{2k+1}$.

3.2 Basis for \mathcal{K}_k

Our next task is to determine the derived subalgebras of $\{\mathcal{K}_k\}$. For this purpose, it is appropriate to take a new basis for each \mathcal{K}_k .

The following observation is easy.

Lemma 3.4. *If V is the vector space spanned by a countable basis $\{B_n\}_{n \in \mathbb{Z}}$, then $\{B_n - B_{n+1}\}_{n \in \mathbb{Z}}$ is a linearly independent set and the vector space spanned by them has codimension 1 in V .*

We set recursively,

$$\begin{aligned} M_n^0 &:= L_n - L_{n+1} \\ M_n^1 &:= M_n^0 - M_{n+1}^0 \\ M_n^{k+1} &:= M_n^k - M_{n+1}^k, \end{aligned}$$

where $\{L_n\}$ is the basis of the Witt algebra. By lemma 3.4, we have a sequence of subspaces of Witt. We will see that they coincide with $\{\mathcal{K}_n\}$. For this purpose we need the combinatorial formula in lemma 3.7.

Remark 3.5. We use the convention that a polynomial of degree -1 is 0.

Lemma 3.6. *If $k \geq 0$ and if $p(x)$ is a polynomial of x of degree k , then $p(x) - p(x+1)$ is a polynomial of degree $k-1$.*

Proof. We just have to consider the terms of the highest and the second highest degrees. \square

We fix a natural number k . Let us define a sequence of polynomials recursively by

$$\begin{aligned} p_k(x) &= x^k, \\ p_{m-1}(x) &= p_m(x) - p_m(x+1) \text{ for } 0 \leq m \leq k. \end{aligned}$$

Lemma 3.7. *We have the explicit formulae for $-1 \leq m \leq k$.*

$$p_m(x) = \sum_{l=0}^{k-m} (x+l)^k (-1)^l {}_{k-m}C_l.$$

Proof. We show this lemma by induction. If $m = k$, $p_m(x) = x^k$ and the lemma holds.

Let us assume that the formula holds for m . We use the well-known combinatorial fact that if $1 < j \leq i$ then ${}_i C_{j-1} + {}_i C_j = {}_{i+1} C_j$. Now let us calculate

$$\begin{aligned}
p_{m-1}(x) &= p_m(x) - p_m(x+1) \\
&= \sum_{l=0}^{k-m} (x+l)^k (-1)^l {}_{k-m} C_l - \sum_{l=0}^{k-m} (x+1+l)^k (-1)^l {}_{k-m} C_l \\
&= \sum_{l=0}^{k-m} (x+l)^k (-1)^l {}_{k-m} C_l - \sum_{l'=1}^{k-m+1} (x+l')^k (-1)^{l'-1} {}_{k-m} C_{l'-1} \\
&= \sum_{l=0}^{k-m} (x+l)^k (-1)^l {}_{k-m} C_l + \sum_{l=1}^{k-m+1} (x+l)^k (-1)^l {}_{k-m} C_{l-1} \\
&= (x+k-m+1)^k (-1)^{k-m+1} \\
&\quad + \sum_{l=1}^{k-m} (x+l)^k (-1)^l ({}_{k-m} C_{l-1} + {}_{k-m} C_l) + x^k \\
&= (x+k-m+1)^k (-1)^{k-m+1} + \sum_{l=1}^{k-m} (x+l)^k (-1)^l {}_{k-m+1} C_l + x^k \\
&= \sum_{l=0}^{k-m+1} (x+l)^k (-1)^l {}_{k-m} C_l.
\end{aligned}$$

□

Proposition 3.8. *For $k \geq 0$, as a polynomial of x , it holds*

$$\sum_{l=0}^{k+1} (x+l)^k (-1)^l {}_{k+1} C_l = 0.$$

Proof. If we put $m = -1$ in lemma 3.7, we get the left hand side of this formula. On the other hand, by definition of p_{-1} and by lemma 3.6, it must be a polynomial of degree -1 , in other words, it vanishes. □

We want to apply this formula to the calculation of the functionals ϕ_k . For this purpose we need formulae for $\{M_n^k\}$ (which are defined at the beginning of this subsection) in terms $\{L_n\}$.

Proposition 3.9. *It holds that*

$$M_n^k = \sum_{l=0}^{k+1} (-1)^l {}_{k+1} C_l L_{n+l}$$

Proof. Again we show this by induction. If $k = 0$, then $M_n^0 = L_n - L_{n+1}$ and this case is proved.

Assume it holds $M_n^k = \sum_{l=0}^{k+1} (-1)^l {}_{k+1}C_l L_{n+l}$. Again using the combinatorial formula ${}_i C_{j-1} + {}_i C_j = {}_{i+1} C_j$, let us calculate

$$\begin{aligned}
M_n^{k+1} &= M_n^k - M_{n+1}^k \\
&= \sum_{l=0}^{k+1} (-1)^l {}_{k+1}C_l L_{n+l} - \sum_{l=0}^{k+1} (-1)^l {}_{k+1}C_l L_{n+1+l} \\
&= \sum_{l=0}^{k+1} (-1)^l {}_{k+1}C_l L_{n+l} - \sum_{l'=1}^{k+2} (-1)^{l'-1} {}_{k+1}C_{l'-1} L_{n+l'} \\
&= \sum_{l=0}^{k+1} (-1)^l {}_{k+1}C_l L_{n+l} + \sum_{l=1}^{k+2} (-1)^l {}_{k+1}C_{l-1} L_{n+l} \\
&= (-1)^{k+2} L_{n+k+2} + \sum_{l=1}^{k+1} (-1)^l ({}_{k+1}C_l + {}_{k+1}C_{l-1}) L_{n+l} + L_0 \\
&= \sum_{l=0}^{k+2} (-1)^l {}_{k+2}C_l L_{n+l}.
\end{aligned}$$

And this is what we had to prove. \square

Corollary 3.10. *For fixed $k \geq 0$, $\{M_n^k | n \in \mathbb{Z}\}$ is a basis of \mathcal{K}_k .*

Proof. We can extend ϕ_k to the Witt algebra by $\phi_k(L_n) = i(in)^k$ (for $k = 0$, $\phi_0(L_n) = i$ by definition).

Then, it is immediate that we have the following.

$$\begin{aligned}
\mathcal{K}_0 &= \{x \in \text{Witt} : \phi_0(x) = 0\} \\
\mathcal{K}_k &= \{x \in \text{Witt} : \phi_0(x) = \phi_1(x) = \dots = \phi_k(x) = 0\}.
\end{aligned}$$

Clearly $\{\phi_k\}$ are independent and each \mathcal{K}_{k+1} has codimension 1 in \mathcal{K}_k .

We will prove the corollary by induction. The set $\{M_n^0\}$ spans a subspace of Witt with codimension 1 by lemma 3.4 and it is immediate to see that $\phi_0(M_n^0) = 0$. On the other hand \mathcal{K}_0 is the kernel of ϕ_0 and has codimension one in Witt. Hence they must coincide.

Assume that $\{M_n^{k-1}\}$ is the basis of \mathcal{K}_{k-1} . Then it is obvious that $M_n^k = M_n^{k-1} - M_{n+1}^{k-1} \in \mathcal{K}_{k-1}$. Now, by proposition 3.9 and proposition 3.7, we see easily that for $n \in \mathbb{Z}$

$$\begin{aligned}
\phi_k(M_n^k) &= \sum_{l=0}^{k+1} (-1)^l {}_{k+1}C_l \phi_k(L_{n+l}) \\
&= \sum_{l=0}^{k+1} (-1)^l {}_{k+1}C_l (n+l)^k \\
&= 0.
\end{aligned}$$

This means that $M_n^k \in \mathcal{K}_k$.

The linear span of $\{M_n^k\}_{n \in \mathbb{Z}}$ must have codimension 1 by lemma 3.4 in \mathcal{K}_{k-1} , therefore it must coincide with \mathcal{K}_k , since \mathcal{K}_k has codimension 1 in \mathcal{K}_{k-1} . \square

3.3 Commutator subalgebras of \mathcal{K}_k

Now we can completely determine all the commutator subalgebras of \mathcal{K}_k . The key fact is that we can easily calculate the commutator in the basis we have obtained in the previous section.

Proposition 3.11. *Let $k \geq 0$ and $m, n \in \mathbb{Z}$. It holds that*

$$[M_m^k, M_n^k] = (m - n)M_{m+n}^{2k+1}$$

Proof. We prove the proposition by induction. The case for $k = 0$ is shown as follows.

$$\begin{aligned} [M_m^0, M_n^0] &= [L_m - L_{m+1}, L_n - L_{n+1}] \\ &= (m - n)L_{m+n} - (m + 1 - n)L_{m+1+n} \\ &\quad - (m - n - 1)L_{m+n+1} + (m - n)L_{m+1+n+1} \\ &= (m - n)(L_{m+n} - L_{m+n+1}) - (m - n)(L_{m+n+1} - L_{m+n+2}) \\ &= (m - n)(M_{m+n}^0 - M_{m+n+1}^0) \\ &= (m - n)M_{m+n}^1 \end{aligned}$$

Let us assume that the formula holds for k . We calculate

$$\begin{aligned} [M_m^{k+1}, M_n^{k+1}] &= [M_m^k - M_{m+1}^k, M_n^k - M_{n+1}^k] \\ &= (m - n)M_{m+n}^{2k+1} - (m + 1 - n)M_{m+1+n}^{2k+1} \\ &\quad - (m - n - 1)M_{m+n+1}^{2k+1} + (m - n)M_{m+1+n+1}^{2k+1} \\ &= (m - n) \left((M_{m+n}^{2k+1} - M_{m+n+1}^{2k+1}) - (M_{m+n+1}^{2k+1} - M_{m+n+2}^{2k+1}) \right) \\ &= (m - n)(M_{m+n}^{2k+2} - M_{m+n+1}^{2k+2}) \\ &= (m - n)M_{m+n}^{2k+3}. \end{aligned}$$

This completes the induction. □

Remark 3.12. The Witt algebra can be treated as \mathcal{K}_{-1} in this context, in the sense that the formula of the proposition holds for $k = -1$.

Theorem 3.13. *It holds that $\mathcal{K}_{2k+1} = \mathcal{K}_k^{(1)}$, where $\mathcal{K}_k^{(1)}$ is the derived subalgebra of \mathcal{K}_k .*

Proof. It is clear from corollary 3.10 and proposition 3.11 that the derived subalgebra of \mathcal{K}_k is included in \mathcal{K}_{2k+1} and the commutators of elements in the basis of \mathcal{K}_k exhaust the basis of \mathcal{K}_{2k+1} . □

3.4 The ideal structure of \mathcal{K}_0

The basis obtained in the previous subsection is suitable to determine all the ideals of \mathcal{K}_0 . In fact, we will see that any ideal of \mathcal{K}_0 must coincide with one of $\{\mathcal{K}_k\}$ or $\ker \phi_1 \cap \ker \phi_3$.

Lemma 3.14. *If \mathcal{I} is a nontrivial ideal of \mathcal{K}_0 , then it includes \mathcal{K}_k for some k .*

Proof. Let x be a nontrivial element of \mathcal{I} . It has an expansion $x = \sum_{j=1}^N a_j M_{n_j}^0$ and we may assume $a_j \neq 0$ for all j . Since \mathcal{I} is an ideal of \mathcal{K}_0 , any commutator with x must be in \mathcal{I} again. In particular,

$$[M_{n_N}^0, x] = \sum_{j=1}^N a_j (n_N - n_j) M_{n_j+n_N}^1 = \sum_{j=1}^{N-1} b_j M_{n_j+n_N}^1,$$

where each of $b_j = a_j(n_N - n_j)$, for $j = 1, 2, \dots, N-1$, is nonzero, must be an element of \mathcal{I} .

Similarly $[M_{n_N+n_{N-1}}^1, [M_{n_N}^0, x]] = \sum_{j=1}^{N-2} c_j M_{n_j+n_N+n_{N-1}}^1$ is also an element of \mathcal{I} . Repeating this procedure, we see that \mathcal{I} contains some M_m^l . Then, using the commutation relation in \mathcal{K}_l , we see that \mathcal{I} contains $\{M_n^{2l+1}\}_{n \neq 2m}$ and $\{M_n^{4l+3}\}_{n \in \mathbb{Z}}$. This implies that \mathcal{I} includes \mathcal{K}_{4l+3} . \square

To prove the next lemma, we need to recall that \mathcal{K}_0 is a subalgebra of smooth vector fields on S^1 and all the functionals $\{\phi_k\}_{k \in \mathbb{N}}$ have analytic interpretations as in subsection 3.1. There, we have identified the real line with the punctured circle, the point at infinity with the point $\theta = 0$. The algebra \mathcal{K}_0 is realized as a subalgebra of smooth functions on the circle vanishing at $\theta = 0$. Seen as the algebra of functions, their commutation relations are $[x, y] = xy' - x'y$.

Lemma 3.15. *Let \mathcal{I} be a nontrivial ideal of \mathcal{K}_0 and let k be the smallest number such that \mathcal{K}_k is included in \mathcal{I} (this exists by lemma 3.14). If $k \geq 4$, then $\mathcal{I} = \mathcal{K}_k$.*

Proof. We will prove this lemma by contradiction. Let us assume that $\mathcal{I} \neq \mathcal{K}_k$ and that $x \in \mathcal{I} \setminus \mathcal{K}_k$. Possible cases are (1) $x \in \mathcal{K}_2$ (2) $x \in \mathcal{K}_0 \setminus \mathcal{K}_1$ (3) $x \in \mathcal{K}_1 \setminus \mathcal{K}_2$. We treat these cases in this order.

If $x \in \mathcal{K}_2$, then there is l such that $2 \leq l < k$ and $x \in \mathcal{K}_l \setminus \mathcal{K}_{l+1}$. Let us take an element y from $\mathcal{K}_1 \setminus \mathcal{K}_2$. Then, since \mathcal{K}_l is an ideal of \mathcal{K}_0 by the remark after proposition 3.3, we see $[x, y] \in \mathcal{K}_l$ and we calculate the derivatives at $\theta = 0$. By the assumption on x and y , the derivatives vanish up to certain orders and we have the following:

$$\begin{aligned} [x, y]^{(l+1)}(0) &= \sum_{k=0}^{l+1} l_{+1} C_k \left(y^{(k)}(0) x^{(l+1-k+1)}(0) - y^{(k+1)}(0) x^{(l+1-k)}(0) \right) \\ &= 0, \end{aligned}$$

$$\begin{aligned} [x, y]^{(l+2)}(0) &= \sum_{k=0}^{l+2} l_{+2} C_k \left(y^{(k)}(0) x^{(l+2-k+1)}(0) - y^{(k+1)}(0) x^{(l+2-k)}(0) \right) \\ &= (l_{+2} C_2 - l_{+2} C_1) y^{(2)}(0) x^{(l+1)}(0) \\ &= \frac{(l+2)(l-1)}{2} y^{(2)}(0) x^{(l+1)}(0). \end{aligned}$$

The latter cannot be zero by assumption and the fact $2 \leq l$. This means $[x, y]$ is in $\mathcal{K}_{l+1} \setminus \mathcal{K}_{l+2}$. Repeating this procedure, we obtain an element of \mathcal{I} in $\mathcal{K}_{k-1} \setminus \mathcal{K}_k$. Therefore \mathcal{I} contains \mathcal{K}_{k-1} because by definition \mathcal{I} contains \mathcal{K}_k and \mathcal{K}_{k-1} has codimension 1 in \mathcal{K}_{k-1} . But this contradicts the definition of k and we see that $x \in \mathcal{K}_2$ is impossible.

Next we $x \in \mathcal{K}_0 \setminus \mathcal{K}_1$. Then we can expand $x = a_0M_0^0 + a_1M_0^1 + y$ (here we use same symbols as before to save the number of characters) where $y \in \mathcal{K}_2$, hence a_0 is nonzero. If $a_1 \neq 0$ we have $[M_0^0, x] = a_1M_1^1 + [M_0^0, y]$. If $a_1 = 0$ we have $[M_1^0, x] = a_0M_1^1 + [M_1^0, y]$. Therefore at least one of these is in $\mathcal{K}_1 \setminus \mathcal{K}_2$ and we may assume that $x \in \mathcal{K}_1 \setminus \mathcal{K}_2$.

Let us assume that $x \in \mathcal{K}_1 \setminus \mathcal{K}_2$. Here we consider the following two cases, namely (3-1) $\phi_3(x) \neq 0$ (3-2) $\phi_3(x) = 0$. If $\phi_3(x) \neq 0$ and $y \in \mathcal{K}_0 \setminus \mathcal{K}_1$, then we see that $[x, y] + y'(0)x \in \mathcal{K}_2 \setminus \mathcal{K}_3$ (and this element is clearly in \mathcal{I}). In fact, by a direct calculation or by the Leibniz rule, we see

$$\begin{aligned} ([x, y] + y'(0)x)^{(2)}(0) &= -y'(0)x^{(2)}(0) + y'(0)x^{(2)}(0) \\ &= 0, \\ ([x, y] + y'(0)x)^{(3)}(0) &= -y'(0)x^{(2)}(0) + y'(0)x^{(2)}(0) \\ &= -2y'(0)x^{(3)}(0) + y'(0)x^{(3)}(0) \\ &= -y'(0)x^{(3)}(0). \end{aligned}$$

This implies that there is an element of \mathcal{I} in $\mathcal{K}_2 \setminus \mathcal{K}_3$. By repeating the argument in the paragraph for the case $x \in \mathcal{K}_2$, we see again a contradiction. Hence we must have $\phi_3(x) = 0$.

By the calculation above, this time $[x, y] + y'(0)x \in \mathcal{K}_3$, but using $\phi_3(x) = 0$ we see

$$\begin{aligned} ([x, y] + y'(0)x)^{(4)}(0) &= 2y^{(3)}(0)x^{(2)}(0) - 3y'(0)x^{(4)}(0) + y'(0)x^{(4)}(0) \\ &= 2y^{(3)}(0)x^{(2)}(0) - 2y'(0)x^{(4)}(0). \end{aligned}$$

Hence with an appropriate element y this does not vanish. That means $[x, y] + y'(0)x$ is an element of $\mathcal{K}_3 \setminus \mathcal{K}_4$. By the same argument as in the case of $x \in \mathcal{K}_2$, we see that this contradicts the definition of k and this completes the proof. \square

We state now the final result of this subsection.

Theorem 3.16. *If \mathcal{I} is an ideal of \mathcal{K}_0 , then the possibilities are*

- $\mathcal{I} = \{0\}$
- $\mathcal{I} = \mathcal{K}_k$ for some $k \geq 0$
- $\mathcal{I} = \ker \phi_1 \cap \ker \phi_3$.

Proof. As before, we can define a number $k \geq 0$ as the smallest number such that \mathcal{K}_k is included in \mathcal{I} .

If $k = 0$ or $k = 1$, then there is nothing to do because the former case means $\mathcal{I} = \mathcal{K}_0$ and in the latter case \mathcal{K}_1 has already codimension 1 and \mathcal{I} must coincide with it.

Next we consider the case $k = 2$. Since \mathcal{K}_2 has codimension 2 in \mathcal{K}_0 , it holds $\mathcal{I} = \mathcal{K}_2$ or \mathcal{I} has an extra element. But the latter case cannot happen because if $x \in \mathcal{I} \setminus \mathcal{K}_2$ we can expand $x = a_0M_0^0 + a_1M_0^1 + y$ (the same symbols again, but the coefficients of a different element) where $y \in \mathcal{K}_2$ and $a_0 \neq 0$ (since otherwise $x \in \mathcal{K}_1$ and contradicts the assumption that $k = 2$). If $a_1 \neq 0$ then $[M_0^0, x] = a_1M_1^1 + [M_0^0, y] \in \mathcal{K}_1 \setminus \mathcal{K}_2$. If $a_1 = 0$ then $[M_1^0, x] = a_0M_1^1 + [M_1^0, y] \in \mathcal{K}_1 \setminus \mathcal{K}_2$. In both cases they contradict the assumption $k = 2$.

Let us assume $k = 3$ and $\mathcal{I} \neq \mathcal{K}_3$. We can take an element $x \in \mathcal{I} \setminus \mathcal{K}_3$ and expand it as

$$x = a_0M_0^0 + a_1M_0^1 + a_2M_0^2 + y$$

(same symbols again to different coefficients) where $y \in \mathcal{K}_3$. By straightforward calculations we see that:

$$\begin{aligned} [M_0^0, x] &= a_1 M_1^1 + 2a_2 M_1^2 + [M_0^0, y] \\ [M_1^0, x] &= (a_0 + a_1) M_1^1 + a_2 (M_1^2 + M_2^2) [M_1^0, y] \\ [M_1^0, [M_0^0, x]] &= a_1 M_3^1 + 4a_2 M_3^2 + [M_1^0, [M_0^0, y]] \\ [M_1^0, [M_1^0, x]] &= (a_0 + a_1) M_3^1 + a_2 (M_3^2 - 3M_4^2). \end{aligned}$$

We note that all these elements are in \mathcal{I} since it is an ideal. By comparing the first and third equations, we see

$$\begin{aligned} [M_0^0, x] - [M_1^0, [M_0^0, x]] &= a_1 (M_1^2 + M_2^2) + 2a_2 (M_1^2 - 2M_3^2) + z \\ &= 2(a_1 - a_2) M_3^2 + a_1 (M_1^3 + 2M_2^3) + 2a_2 (M_1^3 + M_2^3) + z, \end{aligned}$$

where z is the sum of commutators of y and hence again in \mathcal{K}_3 . Now it is easy to see that this element is in \mathcal{K}_3 if and only if $a_1 = a_2$. And this must be in \mathcal{K}_3 , since otherwise it is in $\mathcal{K}_2 \setminus \mathcal{K}_3$ and contradicts the assumption that $k = 3$. Therefore we have $a_1 = a_2$.

Next we consider the difference of the second and fourth equations with $a_1 = a_2$ above and we get

$$\begin{aligned} [M_1^0, x] - [M_1^0, [M_1^0, x]] &= (a_0 + a_1) (M_1^2 + M_2^2) + \\ &= a_1 (M_1^2 + M_2^2 - M_3^2 - 3M_4^2) + z' \\ &= a_0 (M_1^2 + M_2^2) \\ &+ a_1 (2M_1^2 + 2M_2^2 - M_3^2 - 3M_4^2) + z' \\ &= a_0 (M_1^2 + M_2^2) + a_1 (2M_1^3 + 4M_2^3 + 3M_3^3) + z', \end{aligned}$$

where z' is again an element of \mathcal{K}_3 . As before it is in \mathcal{I} . By the assumption $k = 3$ it is contained in \mathcal{K}_3 , therefore $a_0 = 0$. This indicates that an extra element of \mathcal{I} must have the form

$$x = a_1 (M_0^1 + M_0^2)$$

and it is immediate to see this is in $\ker \phi_1 \cap \ker \phi_3$. Since \mathcal{K}_3 has codimension 1 in this intersection, \mathcal{I} must be equal to $\ker \phi_1 \cap \ker \phi_3$.

By calculating derivatives, we can see that $\ker \phi_1 \cap \ker \phi_3$ is surely an ideal of $\text{Vect}(S^1)_0$ and it is also the case even when restricted to \mathcal{K}_0 .

The case $k \geq 4$ is already done in lemma 3.15. □

3.5 The derived subgroup of B_0

As mentioned in the introduction, $\text{Diff}(S^1)$ is the group of smooth, orientation preserving diffeomorphisms of S^1 . The group B_0 is the subgroup of $\text{Diff}(S^1)$ whose elements fix the point $\theta = 0$. Identifying S^1 and $\mathbb{R}/2\pi\mathbb{Z}$, we can think of an element of B_0 as a smooth function g on \mathbb{R} , satisfying $g(\theta + 2\pi) = g(\theta) + 2\pi$, $g(0) = 0$ and $g'(\theta) > 0$. The last condition comes from the fact that g has a smooth inverse. On the other hand, a function on \mathbb{R} with the conditions above can be considered as an element of B_0 . And it is easy to see that the composition operation of the group coincides with the composition of functions. In what follows we identify the group B_0 with the set of smooth functions with these conditions.

Under this identification, Lie algebra $\text{Vect}(S^1)$ of B_0 is seen as the space of smooth functions f such that $f(0) = 0$ and $f(\theta + 2\pi) = f(\theta)$.

Proposition 3.17. $B_1 := \{g \in B_0 : g'(0) = 1\}$ is a subgroup of B_0 .

Proof. By a simple calculation. □

Proposition 3.18. The derived group $[B_0, B_0]$ is included in B_1 .

Proof. Take elements g, h from B_0 . It holds that

$$\begin{aligned} \frac{d}{d\theta}[g, h](0) &= \frac{d}{d\theta} (g \circ h \circ g^{-1} \circ h^{-1})(0) \\ &= g'(h(g^{-1}(h^{-1}(0)))) \times h'(g^{-1}(h^{-1}(0))) \\ &\quad \times (g^{-1})'(h^{-1}(0)) \times (h^{-1})'(0) \\ &= g'(0) \times h'(0) \times (g^{-1})'(0) \times (h^{-1})'(0) \\ &= 1, \end{aligned}$$

where the last equality holds since the derivative of the inverse function on the corresponding point is the inverse number. □

We need the following well-known result [16] [13] [2].

Theorem 3.19. The group $\text{Diff}(\mathbb{R})_c$ is simple, where $\text{Diff}(\mathbb{R})_c$ is the group of smooth orientation-preserving diffeomorphisms of \mathbb{R} whose supports are compact.

Here, a support of a diffeomorphism means the closure of the set on which the given diffeomorphism is not equal to the identity map.

Corollary 3.20. Let B_c be the subgroup of B_0 whose elements have supports not containing $\theta = 0$. Then B_c is simple.

Proof. There is a smooth diffeomorphism between \mathbb{R} and $S^1 \setminus \{0\}$, for example, the stereographic projection. This diffeomorphism induces an isomorphism between $\text{Diff}(\mathbb{R})_c$ and B_c . □

The following is a result similar to the fact $[\mathcal{K}_0, \mathcal{K}_0] = K_1$ which we have proved in theorem 3.13.

Theorem 3.21. $[B_0, B_0]$ is dense in B_1 .

Proof. By corollary 3.20, B_0 has a simple subgroup B_c . The simplicity of B_c implies $[B_0, B_0]$ includes $[B_c, B_c] = B_c$, since any commutator subgroup is normal. Hence we can freely use compactly supported diffeomorphisms.

Let g be an element of B_1 . By the observation above, there is an element h of $[B_0, B_0]$ such that $g \circ h$ has compact support around 0. In other words, we may assume that g has a compact support around 0 and we only have to approximate g with elements in $[B_0, B_0]$.

By the stereographic projection in corollary 3.20, we can consider g as a diffeomorphism of \mathbb{R} . It is well-known that dilations of \mathbb{R} are mapped by this isomorphism to elements of B_0 . Let δ_t be the dilation by t . For $x \in \mathbb{R}$, it holds

$$\delta_t^{-1} \circ g^{-1} \circ \delta_t(x) = \frac{1}{t} g^{-1}(tx).$$

By assumption $g'(0) = 1$. It is easy to see that for $t \rightarrow 0$ the functions $\frac{1}{t}g^{-1}(tx)$, its first derivative and higher-order derivatives converge to x , 1 and 0 respectively, uniformly on each compact set of \mathbb{R} . This means $\frac{1}{t}g^{-1}(tx)$ approximates the identity map around $x = 0$.

Let ϵ be a positive number. Let γ be a smooth positive function on \mathbb{R} such that it is 1 on $[-\epsilon, \epsilon]$ and 0 on $x \leq -2\epsilon$ or $x \geq 2\epsilon$. And let us consider the following functions parametrized by t .

$$h_t(x) = x + \left(\frac{1}{t}g^{-1}(tx) - x \right) \gamma(x).$$

It is easy to see that h_t 's are smooth, $h_t(0) = 0$, h_t 's are equal to x outside a compact set and if t is sufficiently small then each of h_t has the first derivative which is strictly larger than 0. Hence we can consider h_t as a diffeomorphism of \mathbb{R} with a compact support. From the observation above it is clear that h_t and its derivatives converge to x , 1, and 0 uniformly on \mathbb{R} , namely h_t converge to the identity element in the smooth topology.

An important fact is that h_t is equal to $\frac{1}{t}g^{-1}(t \cdot)$ on $[-\epsilon, \epsilon]$. The map $\delta_t^{-1} \circ g \circ \delta_t \circ h_t$ has a compact support which does not contain 0, hence it corresponds to an element of B_c . We denote it by f_t .

Now it is evident that $(g \circ \delta_t^{-1} \circ g^{-1} \circ \delta_t) \circ f_t = g \circ (\delta_t^{-1} \circ g^{-1} \circ \delta_t \circ f_t)$ is in $[B_0, B_0]$ because it is a product of a commutator and a diffeomorphism with compact support. It is equal to $g \circ h_t$ which converges to g with all its derivatives. This shows $[B_0, B_0]$ is dense in B_1 . \square

Remark 3.22. The Lie group $\text{Diff}(S^1)$ is simple [16] [13] [2], but the Lie algebra $\text{Vect}(S^1)$ is not simple. This is easy to see: for example, we only have to consider the subalgebra of vector fields with compact supports in some fixed proper subinterval of S^1 . By the commutation relation (4) this subalgebra is an ideal. This is closed in the smooth topology, hence $\text{Vect}(S^1)$ is not even topologically simple.

On the other hand, the Witt algebra is simple. This can be seen by observing that the linear map $[L_0, \cdot]$ is diagonalized on the standard basis of Witt with no degeneration and that we can raise or lower the elements by commuting with L_n or L_{-n} . From this it is easy to see that any ideal containing nontrivial element must contain Witt.

4 The automorphism group of \mathcal{K}

In this section we will completely determine the *-automorphism group of \mathcal{K} , the unique central extension of \mathcal{K}_0 defined in section 2. However, this group is not necessarily a natural object. As we have seen in the introduction, the algebra \mathcal{K}_0 is a subalgebra of $\text{Vect}(S^1)_0$, the Lie algebra of vector fields on S^1 which vanish at $\theta = 0$. On this algebra of vector fields the stabilizer subgroup B_0 of $\theta = 0$ of $\text{Diff}(S^1)$ acts as automorphisms, but when we restrict these actions to \mathcal{K}_0 , it does not necessarily globally stabilize \mathcal{K}_0 . In fact, the group of *-automorphisms turns out to be very small. The situation is similar for the Virasoro algebra [18].

We will study this problem only for the interest of representation theory. Many things are known about the representation theory of Virasoro algebra. In particular, all the irreducible unitary highest weight representations are completely classified [6]. But for the algebra \mathcal{K} the situation is different. Of course we can restrict any unitary representation of the Virasoro algebra to \mathcal{K} to obtain a unitary representation of \mathcal{K} . But it is not known if there are other unitary representations which are not localized at the point at infinity.

On the other hand, if we make a composition of a (known) unitary representation with an endomorphism of \mathcal{K} then we obtain a (possibly new) unitary representation. The result will show that this method is not productive and, in fact, all the representations obtained by this method are already known.

The algebra \mathcal{K} has a natural decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- \oplus \mathbb{C}C$ where $\mathcal{K}_+ = \text{span}\{K_n : n \geq 1\}$, $\mathcal{K}_- = \text{span}\{K_n : n \leq -1\}$. Each of these direct summands is a subalgebra and it holds $\mathcal{K}_+^* = \mathcal{K}_-$.

Lemma 4.1. *Let K and K' be elements of \mathcal{K} . We expand them in the standard basis:*

$$\begin{aligned} K &= a_0C + a_{n_1}K_{n_1} + a_{n_2}K_{n_2} + \cdots + a_{n_k}K_{n_k}, \\ K' &= b_0C + b_{m_1}K_{m_1} + b_{m_2}K_{m_2} + \cdots + b_{m_l}K_{m_l}. \end{aligned}$$

We assume here that all a_{n_i} and b_{m_j} but a_0 and b_0 are not zero and that $n_1 < n_2 < \cdots < n_k$ and $m_1 < m_2 < \cdots < m_l$. Suppose the expansion of $[K, K']$ in the standard basis does not contain terms K_i where $i > \max\{n_k, m_l\}$. If we decompose $K = K_+ + K_- + a_0C$ and $K'_+ + K'_- + b_0C$ according to the decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- \oplus \mathbb{C}C$, then K_+ and K'_+ are proportional.

Proof. We take a look of the commutation relations (1) of \mathcal{K} . It is easy to see that in $[K_m, K_n]$ the term with index higher than m and n appears if and only if m and n are positive. And in such a case, the term K_{m+n} appears if $m \neq n$.

We may assume n_k and m_l are positive, since otherwise the statement would be trivial.

From the observation above, we see that n_k must be equal to m_l . Otherwise, the term $K_{n_k+m_l}$ (which is larger than $\max\{n_k, m_l\}$) appears in $[K, K']$ and cannot be cancelled, but this contradicts the assumption that there is no term with index higher than n_k and m_l in the commutator.

Now K and K' have the following form:

$$\begin{aligned} K &= a_{n_1}K_{n_1} + a_{n_2}K_{n_2} + \cdots + a_{n_{k-1}}K_{n_{k-1}} + a_{n_k}K_{n_k}, \\ K' &= b_{m_1}K_{m_1} + b_{m_2}K_{m_2} + \cdots + b_{m_{l-1}}K_{m_{l-1}} + b_{m_l}K_{n_k}. \end{aligned}$$

In the commutator $[K, K']$, the terms with the highest indices are now $K_{n_k+n_{k-1}}$ and $K_{n_k+m_{l-1}}$ which appear from the commutators of K_{n_k} and $K_{n_{k-1}}$ or $K_{m_{l-1}}$. If one of n_{k-1} and m_{l-1} is still positive, then again by the assumption, the highest term in the commutator must be cancelled. This implies that again $n_{k-1} = m_{l-1}$ and $a_{n_k}b_{m_{l-1}} = b_{m_l}a_{n_{k-1}}$. This means that the last two terms of K and K' are proportional.

Next steps go similarly: we know the last two terms are proportional and their commutator vanishes. Again by considering the terms with highest indices which appear from the commutator $[K, K']$, we see also that the last three terms are proportional. Continuing this procedure, we can see that all the positive part of K and K' must be proportional. \square

Note that with a completely analogous proof we can show a similar lemma for the negative parts.

Lemma 4.2. *If ρ is a $*$ -endomorphism of \mathcal{K} , then there is an element K of \mathcal{K}_+ and $\lambda, \mu, \nu \in \mathbb{C}$ such that $\rho(K_1)$ takes the form*

$$\rho(K_1) = \lambda K + \mu K^* + \nu C.$$

Proof. Since ρ is a $*$ -endomorphism, it holds that $\rho(K_{-1}) = \rho(K_1)^*$ and from (1)

$$[\rho(K_1), \rho(K_1)^*] = -\rho(K_1) - \rho(K_1)^*.$$

We can apply lemma 4.1 to see that the positive part of $\rho(K_1)$ is proportional to the positive part of $\rho(K_1)^*$. This is the statement of the lemma. \square

With an analogous argument we have the following:

Lemma 4.3. *If ρ is a $*$ -endomorphism of \mathcal{K} , then there is an element K^* of \mathcal{K}_+ and $\lambda', \mu', \nu' \in \mathbb{C}$ such that $\rho(K_2)$ takes the form:*

$$\rho(K_2) = \lambda'K' + \mu'K'^* + \nu'C.$$

By a direct calculation we see that the map τ defined by

$$\tau(K_n) = -K_{-n}, \tau(C) = -C$$

is a $*$ -automorphism of \mathcal{K} (it extends also to the Virasoro algebra).

It is also immediate that C is the unique central element up to a scalar. This means that any automorphism must map $\mathbb{C}C$ to $\mathbb{C}C$.

Lemma 4.4. *If ρ is a $*$ -automorphism of \mathcal{K} , then there are two possibilities.*

1. *There are elements K, K' of \mathcal{K}_+ and $\nu, \nu' \in \mathbb{C}$ such that $\rho(K_1) = K + \nu C$ and $\rho(K_2) = K' + \nu' C$.*
2. *There are elements K, K' of \mathcal{K}_- and $\nu, \nu' \in \mathbb{C}$ such that $\rho(K_1) = K + \nu C$ and $\rho(K_2) = K' + \nu' C$.*

Proof. By lemma 4.2, $\rho(K_1)$ takes the form $\rho(K_1) = \lambda K + \mu K^* + \nu C$ where $K \in \mathcal{K}_+$ and $\lambda, \mu, \nu \in \mathbb{C}$. By lemma 4.3 we have that $\rho(K_2) = \lambda'K' + \mu'K'^* + \nu'$. Let us recall that the following commutation relation holds.

$$[\rho(K_2), \rho(K_1)^*] = 3\rho(K_1) - 2\rho(K_2) - \rho(K_1)^*.$$

Note that $\rho(K_1)^* = \bar{\mu}K + \bar{\lambda}K^* + \bar{\nu}C$.

By considering the composition with τ , we may assume that $\lambda \neq 0$ ($\lambda = \mu = 0$ is impossible because it would mean that K_1 is mapped to a central element and ρ would not be an automorphism). We show that $\mu = 0$. If not, applying lemma 4.1 we see that K' must be proportional to K . But this is impossible because we would have

$$\begin{aligned} \rho(K_1) &= \lambda K + \mu K^* + \nu C, \\ \rho(K_1)^* &= \bar{\mu}K + \bar{\lambda}K^* + \bar{\nu}C, \\ \rho(K_2) &= \lambda'K + \mu'K^* + \nu' C, \\ \rho(K_2)^* &= \bar{\mu}'K + \bar{\lambda}'K^* + \nu' C, \end{aligned}$$

which are linearly dependent. The map ρ is an automorphism and this is a contradiction. Similarly we have $\mu' = 0$ applying lemma 4.1 to the negative parts of $\rho(K_1)^*$ and $\rho(K_2)$. This concludes the lemma. \square

Now we can determine all the elements of the *-automorphism group of \mathcal{K} . Recall there is a family Λ of *-automorphisms parametrized by $\lambda \in \mathbb{R}$ defined in proposition 2.6.

Theorem 4.5. *If ρ is a *-automorphism of \mathcal{K} , then $\rho = \Lambda$ for some $\lambda \in \mathbb{R}$ or $\rho = \Lambda \circ \tau$.*

Proof. By lemma 4.4 and possibly a composition with τ , we may assume that $\rho(K_1) = K + \nu C$ and $\rho(K_2) = K' + \nu' C$ where K and K' are in \mathcal{K}_+ .

Let us expand K and K' in the standard basis of \mathcal{K} ,

$$K = \sum_{i=1}^N a_i K_i, K' = \sum_{j=1}^M b_j K_j,$$

and assume $a_N \neq 0 \neq b_M$.

Since ρ is a *-automorphism, it must hold that

$$[\rho(K_1), \rho(K_1)^*] = -\rho(K_1) - \rho(K_1)^*, \quad (5)$$

$$[\rho(K_2), \rho(K_2)^*] = -2\rho(K_2) - 2\rho(K_2)^* + \frac{C}{2}, \quad (6)$$

$$[\rho(K_2), \rho(K_1)^*] = 3\rho(K_1) - 2\rho(K_2) - \rho(K_1)^*. \quad (7)$$

Considering the terms K_N in the first equation, we see that $\sum_{i=1}^N a_i = \frac{1}{N}$. Similarly, considering the terms K_M in the second equation, we obtain $\sum_{j=1}^M b_j = \frac{2}{M}$. On the other hand, by comparing the terms K_{-N} in the third equation, it turns out that $-N\bar{a}_N \sum_{j=1}^M b_j = -\bar{a}_N$. Since we have the assumption that a_N is not zero, this implies that $2N = M$.

The subalgebra \mathcal{K}_+ is generated by K_1 and K_2 with the recursive formula

$$K_{n+1} = \frac{1}{n-1} ([K_n, K_1] + nK_n - K_1).$$

From this formula we see by induction that the term with the highest index of $\rho(K_k)$ is K_{kN} . If N was larger than 1, these terms would not span all of \mathcal{K}_+ and ρ could not be surjective. Thus N must be 1.

Again, by equation (5) and by a direct calculation, we obtain $a_1 = 1$, namely:

$$\rho(K_1) = K_1 + \nu_1 C,$$

where ν is a pure imaginary number. Similarly we have two solutions for equation (6):

$$\rho(K_2) = \begin{cases} K_2 + \nu_2 C, \\ -\frac{1}{3}K_1 + \frac{4}{3}K_2 + \nu_2 C. \end{cases}$$

The second solution does not satisfy equation (7). Then again by (7) we see $2\nu_1 = \nu_2$.

We have seen in proposition 2.6 that this ρ can surely be extended to a *-automorphism of \mathcal{K} . Since K_1 and K_2 are the generators of \mathcal{K} as a *-Lie algebra, this determines ρ uniquely. \square

Corollary 4.6. $Aut(\mathcal{K}) \cong \mathbb{R} \rtimes \mathbb{Z}_2$.

Remark 4.7. It is also possible to determine the automorphism group of the Virasoro algebra [18]: it is generated by the extension of τ and one-parameter subgroup of rotation:

$$\begin{aligned} \rho_t(L_n) &= e^{itn} L_n, \\ \rho_t(C) &= C. \end{aligned}$$

It is again isomorphic to $\mathbb{R} \rtimes \mathbb{Z}_2$, but the action of the \mathbb{R} part is different.

5 Generalized Verma modules

As we have seen in section 2, \mathcal{K}_0 has the unique (up to isomorphism) central extension which is a subalgebra of Vir . We denote it by \mathcal{K} . This section is an attempt to construct a family of unitary representations of \mathcal{K} .

We are going to construct modules $V_{h+ih',c,\lambda}$ parametrized by three complex numbers $h+ih', c, \lambda$, where $h, h' \in \mathbb{R}$ and $c, \lambda \in \mathbb{C}$. Every module has a “lowest weight vector” which satisfies $K_nv = (h+ih'+n\lambda)v$ for $n \geq 1$ and $Cv = cv$. If we restrict to the case $\lambda = 0$, this module reduces to the restriction of the Virasoro module to \mathcal{K} .

Recall that \mathcal{K} is a $*$ -Lie algebra. A sesquilinear form $\langle \cdot, \cdot \rangle$ on a module V is said to be contravariant if for any $v, w \in V$ and $x \in \mathcal{K}$ it holds $\langle xv, w \rangle = \langle v, x^*w \rangle$. In addition if this sesquilinear form is positive definite, then the representation of \mathcal{K} on V is said to be unitary.

It turns out that for any set of values of h, h', c, λ we can construct a corresponding module. In addition, if c is real, there exists a contravariant sesquilinear form on the module. Then we arrive at natural problems, for example, when the contravariant form is unitary, when the representation of \mathcal{K} integrates to the (projective unitary) representation of B_0 and when these representations are inequivalent, etc. These problems will be addressed in further publications of the author.

Here we make some remarks. It is easy to see that these modules are inequivalent as representations of the Lie algebra \mathcal{K} , however, as we saw in the remark 2.8 (after proposition 2.7), the imaginary part of λ does not make difference for the corresponding projective representation of the group B_0 . In addition, in [17] it has been proved that there are modules which integrate to equivalent projective representations of the group for some different values of h . Furthermore, as we will see in section 7, there exist true (non projective) representation of B_0 whose naturally corresponding representations of \mathcal{K} are not lowest weight modules. In the case of $\text{Diff}(S^1)$ there is a one-to-one correspondence between irreducible unitary positive energy projective representations of the group and irreducible lowest weight unitary representations of the Virasoro algebra. But for B_0 and \mathcal{K} we cannot expect such a correspondence.

5.1 General construction of modules

We start with general notions. Let \mathcal{L}_0 be a Lie algebra, $U(\mathcal{L}_0)$ the universal enveloping algebra of \mathcal{L}_0 , ψ_0 a nontrivial linear functional on \mathcal{L}_0 which vanishes on the commutator subalgebra $[\mathcal{L}_0, \mathcal{L}_0]$. In particular, we assume that \mathcal{L}_0 is not semisimple (otherwise ψ_0 would be trivial). Later \mathcal{L}_0 will be a upper-triangular subalgebra of a Lie algebra.

Lemma 5.1. *The linear functional ψ_0 extends to a homomorphism of the universal algebra $U(\mathcal{L}_0)$.*

Proof. Clearly ψ_0 extends to a homomorphism of the tensor algebra of \mathcal{L}_0 . Now we only have to recall that $U(\mathcal{L}_0)$ is the quotient algebra by the two-sided ideal generated by elements of the form $a \otimes b - b \otimes a - [a, b]$ where $a, b \in \mathcal{L}_0$. By assumption, ψ_0 vanishes on these elements, hence on the ideal generated by them. This implies ψ_0 is well-defined on $U(\mathcal{L}_0)$. \square

Lemma 5.2. *Let \mathcal{J}_0 be the left ideal of $U(\mathcal{L}_0)$ (the subspace invariant under the multiplication from the left) generated by elements of the form $\psi_0(a) - a$ for $a \in \mathcal{L}_0$.*

Then $U(\mathcal{L}_0)/\mathcal{J}_0$ is nontrivial if and only if ψ_0 vanishes on $[\mathcal{L}_0, \mathcal{L}_0]$. In this case $\mathcal{J}_0 = \ker \psi_0$ and the quotient space is one-dimensional.

Proof. If ψ_0 vanishes on $[\mathcal{L}_0, \mathcal{L}_0]$, then by lemma 5.1 ψ_0 extends to $U(\mathcal{L}_0)$ and \mathcal{J}_0 is included in $\ker \psi_0$. Since ψ_0 is nontrivial, $\ker \psi_0$ is nontrivial.

On the other hand, if ψ_0 doesn't vanish at $[\mathcal{L}_0, \mathcal{L}_0]$, then take $x, y \in \mathcal{L}_0$ such that $\psi_0([x, y]) \neq 0$. Then it holds that

$$\begin{aligned} [(\psi_0(x) - x), (\psi_0(y) - y)] &= [x, y] \in \mathcal{J}_0, \\ \psi_0([x, y]) - [x, y] &\in \mathcal{J}_0. \end{aligned}$$

Hence \mathcal{J}_0 contains a nontrivial scalar and generates all.

To complete the proof, we only have to show that $\mathcal{J}_0 \supset \ker \psi_0$ since the other inclusion has been done. Therefore it is enough to show that \mathcal{J}_0 has codimension 1 in $U(\mathcal{L}_0)$. This is a rephrasing of the claim that any element of $U(\mathcal{L}_0)$ is equivalent to a scalar modulo \mathcal{J}_0 . This is easy to see since any element of $U(\mathcal{L}_0)$ is a linear combination of tensor products $a_1 \otimes a_2 \otimes \cdots \otimes a_n$. By definition there is an element $a_1 \otimes a_2 \otimes \cdots \otimes (a_n - \psi_0(a_n))$ in \mathcal{J}_0 , therefore $a_1 \otimes a_2 \otimes \cdots \otimes a_n \equiv_{\mathcal{J}_0} a_1 \otimes a_2 \otimes \cdots \otimes \psi_0(a_n)$. By repeating this procedure, we see that every element of $U(\mathcal{L}_0)$ is equivalent to a scalar. \square

In the following we assume that \mathcal{L} is a *-Lie algebra with a decomposition into Lie subalgebras $\mathcal{L} = \mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+$, where $(\mathcal{N}_+)^* = \mathcal{N}_-$, $(\mathcal{H})^* = \mathcal{H}$, and \mathcal{H} is commutative.

Let ψ be a linear functional on $\mathcal{H} \oplus \mathcal{N}_+$ which vanishes on its commutator subalgebra. In other words, ψ is an element of $H^1(\mathcal{N}_+ \oplus \mathcal{H}, \mathbb{C})$. We will show that for any such ψ we have a left module on \mathcal{L} . Again let $U(\mathcal{L})$ be the universal enveloping algebra of \mathcal{L} . It is naturally a left module on \mathcal{L} .

Proposition 5.3. *Let \mathcal{J} be the left ideal of $U(\mathcal{L})$ generated by elements of the form $\psi(l_+) - l_+$, where $l_+ \in \mathcal{H} \oplus \mathcal{N}_+$. The subspace \mathcal{J} is a nontrivial submodule on $U(\mathcal{L})$.*

Proof. By the theorem of Poincarè-Birkhoff-Witt, it holds that $U(\mathcal{L}) = U(\mathcal{N}_-) \otimes U(\mathcal{H}) \otimes U(\mathcal{N}_+)$. By lemma 5.2, $\ker \psi$ has codimension one in $U(\mathcal{H}) \otimes U(\mathcal{N}_+)$. It is easy to see that \mathcal{J} takes the form $U(\mathcal{N}_-) \otimes \ker \psi$, hence it is nontrivial. \square

For a fixed ψ we define the quotient module $V = U(\mathcal{L})/\mathcal{J}$. Since $U(\mathcal{H}) \oplus U(\mathcal{N}_+)/\ker \psi$ is one dimensional, the module V is linearly isomorphic to $U(\mathcal{N}_-)$ and we identify them. There is a specified vector v which corresponds to $1 \in \mathbb{C} \subset U(\mathcal{N}_-)$ and, on v , an element x of $\mathcal{H} \otimes \mathcal{N}_+$ acts as $xv = \psi(x)v$.

Example 1. The Virasoro algebra has the following decomposition:

$$\text{Vir} = \mathcal{V}_- \oplus \mathcal{H} \oplus \mathcal{V}_+,$$

where $\mathcal{V}_+ = \text{span}\{L_n : n > 0\}$ and $\mathcal{H} = \text{span}\{L_0, C\}$. It is easy to see that the commutator subalgebra $[\mathcal{H} \oplus \mathcal{V}_+, \mathcal{H} \oplus \mathcal{V}_+]$ is equal to \mathcal{V}_+ . According to proposition 5.3, we obtain a module of Vir for any linear functional ψ on $\mathcal{H} \oplus \mathcal{V}_+$ vanishing on \mathcal{V}_+ . The linear functional ψ is determined by the two values $c := \psi(C)$ and $h := \psi(L_0)$. It is well known that for some values of c and h we can define inner products on these modules and these representations integrate to representations of the group $\text{Diff}(S^1)$ [4].

Example 2. The *-Lie algebra \mathcal{K} has the decomposition

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{H} \oplus \mathcal{K}_-,$$

where $\mathcal{K}_+ = \text{span}\{K_n : n > 0\}$ and $\mathcal{H} = \text{span}\{C\}$. It can be shown that $H^1(\mathcal{K}_+ \oplus \mathcal{H}, \mathbb{C})$ is three dimensional and an element ψ in $H^1(\mathcal{K}_+ \oplus \mathcal{H}, \mathbb{C})$ takes the form

$$\psi(C) = c, \psi(K_n) = h + ih' + \lambda n \text{ where } c, \lambda \in \mathbb{C}, h, h', \in \mathbb{R}.$$

We denote this module on \mathcal{K} by $V_{h+ih',c,\lambda}$. If $c \in \mathbb{C}$, $\psi(K_n) = h + ih' \in \mathbb{C}$ and $\lambda = 0$ then the modules $V_{h+ih',c,0}$ reduce to Verma modules on the Virasoro algebra (see proposition 5.7).

Let us return to general cases. From now on we assume that ψ is self-adjoint on \mathcal{H} (namely, $\psi(h^*) = \overline{\psi(h)}$ for $h \in \mathcal{H}$). Recall that V is the quotient module $U(\mathcal{L})/\mathcal{J}$ as in the remark after proposition 5.3. Our next task is to define a contravariant sesquilinear form on V . Note that the *-operation extends naturally to $U(\mathcal{L})$.

We define a sesquilinear map on $V \times V (= U(\mathcal{N}_+) \times U(\mathcal{N}_+))$ into $U(\mathcal{L})$ by

$$\alpha(L_1^-, L_2^-) = (L_2^-)^* \otimes L_1^-, \text{ for } L_1^-, L_2^- \in U(\mathcal{N}_-) = V.$$

On the other hand, we can define a linear form β on $U(\mathcal{L})$ using the decomposition $U(\mathcal{N}_-) \otimes U(\mathcal{H}) \otimes U(\mathcal{N}_+)$, by

$$\beta(L_- \otimes H \otimes L_+) = \overline{\psi((L_-)^*)} \psi(H) \psi(L_+).$$

It is easy to see that β is self-adjoint since ψ is self-adjoint on \mathcal{H} .

Theorem 5.4. $\beta \circ \alpha := \gamma$ is contravariant.

Proof. We have to show that for any $L \in \mathcal{L}$ it holds

$$\gamma(L \otimes L_1^-, L_2^-) = \gamma(L_1^-, L^* \otimes L_2^-).$$

As elements of $U(\mathcal{L})$, we have the following decompositions by the Poincarè-Birkhoff-Witt theorem:

$$\begin{aligned} L \otimes L_1^- &= \sum_k L_k^- \otimes H_k \otimes L_k^+ \\ (L_2^-)^* \otimes L_k^- &= \sum_l L_{k,l}^- \otimes H_{k,l} \otimes L_{k,l}^+ \\ H_k \otimes L_k^+ \otimes H_{k,l} \otimes L_{k,l}^+ &= \sum_m H_{k,l,m} \otimes L_{k,l,m}^+ \end{aligned} \tag{8}$$

where elements in the decompositions are $L_k^-, L_{k,l}^- \in U(\mathcal{N}_-)$, $H_k, H_{k,l}, H_{k,l,m} \in U(\mathcal{H})$ and $L_k^+, L_{k,l}^+, L_{k,l,m}^+ \in U(\mathcal{N}_+)$.

Now we calculate

$$\begin{aligned} \gamma(L \otimes L_1^-, L_2^-) &= \gamma\left(\sum_k L_k^- \psi(H_k \otimes L_k^+), L_2^-\right) \\ &= \sum_k \psi(H_k \otimes L_k^+) \beta((L_2^-)^* \otimes L_k^-) \end{aligned}$$

By substituting the expression in (8) to $(L_2^-)^* \otimes L_k^-$, we have

$$\begin{aligned}\gamma(L \otimes L_1^-, L_2^-) &= \sum_{k,l} \psi(H_k \otimes L_k^+) \overline{\psi\left((L_{k,l}^-)^*\right)} \psi(H_{k,l} \otimes L_{k,l}^+) \\ &= \sum_{k,l} \overline{\psi\left((L_{k,l}^-)^*\right)} \psi(H_{k,l} \otimes L_{k,l}^+ \otimes H_k \otimes L_k^+)\end{aligned}$$

By substituting the expression in (8) to $H_k \otimes L_k^+ \otimes H_{k,l} \otimes L_{k,l}^+$,

$$\begin{aligned}\gamma(L \otimes L_1^-, L_2^-) &= \sum_{k,l,m} \overline{\psi\left((L_{k,l}^-)^*\right)} \psi(H_{k,l,m} \otimes L_{k,l,m}^+) \\ &= \beta \left(\sum_{k,l,m} L_{k,l}^- \otimes H_{k,l,m} \otimes L_{k,l,m}^+ \right) \\ &= \beta \left(\sum_{k,l} L_{k,l}^- \otimes H_{k,l} \otimes L_{k,l}^+ \otimes H_k \otimes L_k^+ \right) \\ &= \beta \left(\sum_k (L_2^-)^* \otimes L_k^- \otimes H_k \otimes L_k^+ \right) \\ &= \beta \left((L_2^-)^* \otimes L \otimes L_1^- \right).\end{aligned}$$

Similarly, in order to see $\beta\left((L_2^-)^* \otimes L \otimes L_1^-\right) = \gamma(L_1^-, (L)^* \otimes L_2^-)$ we need the following decompositions (we use same notations to save number of letters.).

$$\begin{aligned}L^* \otimes L_2^- &= \sum_k L_k^- \otimes H_k \otimes L_k^+ \\ (L_k^-)^* \otimes L_1^- &= \sum_l L_{k,l}^- \otimes H_{k,l} \otimes L_{k,l}^+ \\ (L_{k,l}^-)^* \otimes H_k \otimes L_k^+ &= \sum_m H_{k,l,m} \otimes L_{k,l,m}^+, \end{aligned}$$

where elements in the decompositions are $L_k^-, L_{k,l}^- \in U(\mathcal{N}_-)$, $H_k, H_{k,l}, H_{k,l,m} \in U(\mathcal{H})$ and $L_k^+, L_{k,l}^+, L_{k,l,m}^+ \in U(\mathcal{N}_+)$. Now the final computation goes as follows.

$$\begin{aligned}\gamma(L_1^-, L^* \otimes L_2^-) &= \gamma\left(L_1^-, \sum_k L_k^- \otimes H_k \otimes L_k^+\right) \\ &= \sum_k \overline{\psi(H_k \otimes L_k^+)} \beta\left((L_k^-)^* \otimes L_1^-\right) \\ &= \sum_{k,l} \overline{\psi(H_k) \psi(L_k^+) \psi\left((L_{k,l}^-)^*\right)} \psi(H_{k,l}) \psi(L_{k,l}^+) \\ &= \sum_{k,l} \overline{\psi\left((L_{k,l}^-)^* \otimes H_k \otimes L_k^+\right)} \psi(H_{k,l}) \psi(L_{k,l}^+) \\ &= \sum_{k,l,m} \overline{\psi(H_{k,l,m}) \psi(L_{k,l,m}^+)} \psi(H_{k,l}) \psi(L_{k,l}^+).\end{aligned}$$

In the next step (and only here) we need the self-adjointness of ψ on \mathcal{H} . Continuing,

$$\begin{aligned}
\gamma(L_1^-, L^* \otimes L_2^-) &= \sum_{k,l,m} \psi((H_{k,l,m})^*) \overline{\psi(L_{k,l,m}^+)} \psi(H_{k,l}) \psi(L_{k,l}^+) \\
&= \beta \left(\sum_{k,l,m} (L_{k,l,m}^+)^* \otimes (H_{k,l,m})^* \otimes H_{k,l} \otimes L_{k,l}^+ \right) \\
&= \beta \left(\sum_{k,l} (L_k^+)^* \otimes (H_k)^* \otimes L_{k,l}^- \otimes H_{k,l} \otimes L_{k,l}^+ \right) \\
&= \beta \left(\sum_k (L_k^+)^* \otimes (H_k)^* \otimes (L_k^-)^* \otimes L_1^- \right) \\
&= \beta((L_2^-)^* \otimes L \otimes L_1^-).
\end{aligned}$$

This completes the proof. \square

In the case of Vir, $c = \psi(C)$ and $h = \psi(L_0)$ must be real for the sesquilinear form to be defined. For such ψ it has been completely determined when the sesquilinear forms are positive definite thanks to the Kac determinant formula [6].

In the case of \mathcal{K} , the only condition for the existence of sesquilinear form is that $\psi(C) \in \mathbb{R}$. Hence there are additional parameters $h' \in \mathbb{R}$, $\lambda \in \mathbb{C}$ for generalized Verma modules $V_{h+ih',c,\lambda}$ on \mathcal{K} .

5.2 Irreducibility of generalized Verma modules on \mathcal{K}

In this section, we completely determine for which values of $h + ih'$, c , λ the corresponding generalized Verma modules on \mathcal{K} are irreducible. The proof heavily relies on the result of Feigin and Fuks [3] which has determined when the Verma modules on the Virasoro algebra are irreducible. To utilize their result, we extend the generalized Verma modules on \mathcal{K} to (non-unitary) representations of the Virasoro algebra.

Let $V_{h+ih',c,\lambda}$ be a generalized Verma module on \mathcal{K} and v be the corresponding lowest weight vector such that

$$K_n v = (h + ih' + n\lambda)v \text{ for } n \geq 1 \text{ and } Cv = cv. \quad (9)$$

First we observe that

$$K_n \mapsto K_n - n\lambda I, C \mapsto C,$$

where I is the identity operator on $V_{h+ih',c,\lambda}$, extends by linearity to a well-defined (non *-) representation (on the same space $V_{h+ih'+n\lambda}$) of \mathcal{K} (the proof is the same as that of proposition 2.6). On the other hand, it is straightforward to see that this new representation is equivalent to $V_{h+ih',c,0}$. Irreducibility of a representation of an algebra is not changed even if we add the identity operator to the set of operators. Therefore the irreducibility of $V_{h+ih',c,\lambda}$ is equivalent to that of $V_{h+ih',c,0}$ and we may restrict the consideration to the latter case. We denote it $V_{h+ih',c}$.

Lemma 5.5. *For any $w \in V_{h+ih',c}$ there is $N \in \mathbb{N}$ such that $K_m w = K_n w$ for $m, n \geq N$.*

Proof. The module $V_{h+ih',c}$ is spanned by vectors $K_{n_1} \cdots K_{n_k} v$. We will show the lemma by induction with respect to k . If $w = v$, the lowest weight vector, then the lemma obviously holds with $N = 1$, hence the case $k = 0$ is done.

Assume that the lemma holds for w and put $\lim_m K_m w = w'$ (here \lim has nothing to do with any topology, but simply means that “the equality holds for sufficiently large m ”). We will show that it also holds for $K_n w$. Let us calculate

$$\begin{aligned} K_m K_n w &= ([K_m, K_n] + K_n K_m) w \\ &= ((m-n)K_{m+n} - mK_m + nK_n + K_n K_m) w, \end{aligned}$$

and for sufficiently large m this is equal to

$$(m-n)w' - mw' + nK_n w + K_n w' = -nw' + nK_n w + K_n w'$$

and this does not depend on m . □

Let us define $Dw = \lim_m K_m w$. Then, it is clear that D is a linear operator on $V_{h+ih',c}$ and it holds $Dv = (h + ih')v$.

Lemma 5.6. *The following commutation relation holds:*

$$[D, K_n] = n(K_n - D). \tag{10}$$

Proof. We only need to calculate

$$\begin{aligned} (DK_n - K_n D)w &= \lim_m (K_m K_n - K_n K_m) w \\ &= \lim_m ((m-n)K_{m+n} - mK_m + nK_n) w \\ &= n(K_n - D)w. \end{aligned}$$

□

The relation (10) can be rewritten as $[K_n - D, -D] = n(K_n - D)$.

Proposition 5.7. *The representation of \mathcal{K} on $V_{h+ih',c,0}$ extends to a representation of Vir . This extension is the Verma module with $-h - ih', c$.*

Proof. We take a correspondence $L_0 \mapsto -D, L_n \mapsto K_n - D, C \mapsto C$. Now that we know all the commutation relations between D and K_n , the confirmation that this correspondence is a representation is straightforward.

It is clear that the lowest weight vector is v and $-Dv = (-h - ih')v, (K_n - D)v = 0$ for $n \geq 0, Cv = cv$. We only have to show that all the vectors of the form $(K_{n_1} - D) \cdots (K_{n_k} - D)v$, where $n_1 \leq \cdots \leq n_k$, are linearly independent. But this is clear from the fact that these vectors are eigenvectors of D and the fact that $\{K_{n_1} \cdots K_{n_k} v\}$ are independent by definition. The former fact is shown by a straightforward induction. □

Here we remark that this extension of the representation does not change the irreducibility. If the module on \mathcal{K} is irreducible, then clearly it is irreducible as a module on Vir . On the other hand the operator D above is defined as the limit of K_n 's, hence if the module on \mathcal{K} is reducible then it is still reducible as a module on Vir .

The following theorem is due to Feigin and Fuks [3].

Theorem 5.8. *For $h, c \in \mathbb{C}$, the Verma module $V_{h,c}$ on the Virasoro algebra is reducible if and only if there are natural numbers α, β such that*

$$\begin{aligned} \Phi_{\alpha,\beta}(h,c) &:= \left(h + \frac{1}{24}(\alpha^2 - 1)(c - 13) + \frac{1}{2}(\alpha\beta - 1) \right) \\ &\quad \times \left(h + \frac{1}{24}(\beta^2 - 1)(c - 13) + \frac{1}{2}(\alpha\beta - 1) \right) \\ &\quad + \frac{(\alpha^2 - \beta^2)^2}{16} = 0. \end{aligned}$$

The application of this to our case is now straightforward.

Corollary 5.9. *For $h, h' \in \mathbb{R}$, $c, \lambda \in \mathbb{C}$, the generalized Verma module $V_{h+ih',c,\lambda}$ on \mathcal{K} is reducible if and only if there are natural numbers α, β such that*

$$\Phi_{\alpha,\beta}(-h - ih', c) = 0.$$

6 Endomorphisms of \mathcal{K}

This section is devoted to the study of *-endomorphisms of the algebra \mathcal{K} . As in the case of automorphisms, endomorphisms of \mathcal{K} are not natural objects, but they are interesting from the viewpoint of representations. We remarked before that any composition of a *-endomorphism and unitary representation provides a unitary representation. In this way, we obtain a strange kind of representations of \mathcal{K} . We will also have a rough classification of endomorphisms.

It is well known (for example, see [12][17]) that the following maps are endomorphisms of the Virasoro algebra and they restrict to \mathcal{K} :

$$\begin{aligned} \delta_r(L_N) &= \frac{1}{r}L_{rn} + \frac{C}{24} \left(r - \frac{1}{r} \right), \\ \delta_r(C) &= rC, \end{aligned}$$

for any integer $r \in \mathbb{Z}$.

We have another type of *-endomorphisms of \mathcal{K} parametrized by a complex number α . In the next section we will see that these endomorphisms are related to some unitary representation of $\text{Diff}(S^1)_0$.

Proposition 6.1. *Let $\alpha \in \mathbb{C}$ and K be an element of \mathcal{K} which satisfies $[K, K^*] = -K - K^*$. Define*

$$\begin{aligned} \sigma_\alpha(K_n) &= \left(\frac{n^2 + n}{2}\alpha + \frac{n^2 - n}{2}\bar{\alpha} - \frac{n^2 - n}{2} \right) K \\ &\quad + \left(\frac{n^2 + n}{2}\alpha + \frac{n^2 - n}{2}\bar{\alpha} - \frac{n^2 + n}{2} \right) K^*, \\ \sigma_\alpha(C) &= 0. \end{aligned}$$

*Then σ_α extends to a *-endomorphism of \mathcal{K} by linearity.*

Remark 6.2. Examples of K in this proposition are $K = K_1, -K_{-1}, -\frac{1}{6}K_2 + \frac{2}{3}K_1$. Since the image of C is 0, σ_α extends also to a *-homomorphism of \mathcal{K}_0 into \mathcal{K} . Therefore, the kernel of σ_α is the direct sum of $\ker \sigma_\alpha$ as a homomorphism of \mathcal{K}_0 and $\mathbb{C}C$.

Proof. It is clear that σ_α preserves the *-operation. We only have to confirm that it preserves commutation relations and this is done by straightforward calculations. However, we will exhibit a clearer procedure.

Let us put $\beta = 3\alpha + \bar{\alpha} - 1$. The definition of σ_α can be rewritten as

$$\begin{aligned}\sigma_\alpha(K_n) &= \left(\frac{n^2 - n}{2}\beta - (n^2 - 2n)\alpha \right) K \\ &\quad + \left(\frac{n^2 - n}{2}\beta - (n^2 - 2n)\alpha - n \right) K^*.\end{aligned}$$

If we put $\gamma_n = \frac{n^2 - n}{2}\beta - (n^2 - 2n)\alpha$, this takes the form $\sigma_\alpha(K_n) = \gamma_n K + (\gamma_n - n)K^*$. Now it is easy to see that

$$\begin{aligned}[\sigma_\alpha(K_n), \sigma_\alpha(K_{-n})] &= [\gamma_n K + (\gamma_n - n)K^*, (\bar{\gamma}_n - n)K + \bar{\gamma}_n K^*] \\ &= (-|\gamma_n|^2 + |\gamma_n - n|^2)(K + K^*) \\ &= -n(2\operatorname{Re}\gamma_n - n)(K + K^*) \\ &= -n(\sigma_\alpha(K_n) + \sigma_\alpha(K_{-n})).\end{aligned}$$

Next we calculate a general commutator, for $m \neq -n$,

$$\begin{aligned}&[\sigma_\alpha(K_m), \sigma_\alpha(K_n)] \\ &= \left(m \left(\frac{n^2 - n}{2}\beta - (n^2 - 2n)\alpha - n \right) - n \left(\frac{m^2 - m}{2}\beta - (m^2 - 2m)\alpha - m \right) \right) \\ &\quad \times (K + K^*) \\ &= \left(\frac{\beta}{2} - \alpha \right) (m^2 n - mn^2)(K + K^*)\end{aligned}$$

On the other hand,

$$\begin{aligned}&(m - n)\sigma_\alpha(K_{m+n}) - m\sigma_\alpha(K_m) + n\sigma_\alpha(K_n) \\ &= ((m - n)\gamma_{m+n} - m\gamma_m + n\gamma_n - (m - n)(m + n) + m^2 - n^2) \\ &\quad \times (K + K^*) \\ &= \left(\frac{\beta}{2} - \alpha \right) (m^2 n - mn^2)(K + K^*)\end{aligned}$$

and this completes the proof. \square

Proposition 6.3. *Let us assume that $K + K^* \neq 0$. If $\alpha \in \frac{1}{2} + i\mathbb{R}$, then $\ker(\sigma_\alpha)$ is $\mathcal{K}_1 \oplus \mathbb{C}C$ (see section 3.1). Otherwise, $\ker(\sigma_\alpha)$ is $\mathcal{K}_2 \oplus \mathbb{C}C$.*

Proof. As we have noted in the remark 6.2, first we think σ_α as a homomorphism of \mathcal{K}_0 .

By direct calculations, we have (see section 3.2),

$$\begin{aligned}\rho(M_n^0) &= (-(n + 1)\alpha - n\bar{\alpha} + n)K + (-(n + 1)\alpha - n\bar{\alpha} + n + 1)K^*, \\ \rho(M_n^1) &= (\alpha + \bar{\alpha} - 1)(K + K^*), \\ \rho(M_n^2) &= 0.\end{aligned}$$

The kernel of σ_α must be one of ideals in theorem 3.16. From this it is clear that $\ker(\sigma_\alpha)$ contains \mathcal{K}_2 and contains \mathcal{K}_1 if and only if $\operatorname{Re}\alpha = \frac{1}{2}$.

By the remark 6.2, the kernel of σ_α as a *-endomorphism is $\mathcal{K}_1 \oplus \mathbb{C}C$ or $\mathcal{K}_2 \oplus \mathbb{C}C$, respectively. \square

We have a partial classification of endomorphisms of \mathcal{K} .

Proposition 6.4. *If ρ is a nontrivial *-endomorphism of \mathcal{K} , then the possibilities are:*

1. $\rho = \sigma_\alpha$ with appropriate K and $\alpha \in \frac{1}{2} + i\mathbb{R}$. In this case, $\ker(\rho) = \mathcal{K}_1 \oplus \mathbb{C}C$ and $\rho(K_1) = \alpha K + (\alpha - 1)K^*$.
2. $\rho = \sigma_\alpha$ with appropriate K and $\alpha \notin \frac{1}{2} + i\mathbb{R}$. In this case, $\ker(\rho) = \mathcal{K}_2 \oplus \mathbb{C}C$ and $\rho(K_1) = \alpha K + (\alpha - 1)K^*$.
3. $\rho(K_1) = \sum_{i=1}^N a_i K_i + a_0 C \in \mathcal{K}_+ \oplus \mathbb{C}C$, $\rho(K_2) = \sum_{i=1}^{2N} b_i K_i + b_0 C \in \mathcal{K}_+ \oplus \mathbb{C}C$, where $\sum_{i=1}^N a_i = \sum_{i=1}^{2N} b_i = \frac{1}{N}$. In this case, $\ker(\rho) = \{0\}$.
4. $\rho(K_1) = \sum_{i=-N}^{-1} a_i K_i + a_0 C \in \mathcal{K}_- \oplus \mathbb{C}C$, $\rho(K_2) = \sum_{i=2N}^{-1} b_i K_i + b_0 C \in \mathcal{K}_+ \oplus \mathbb{C}C$, where $\sum_{i=-N}^{-1} a_i = \sum_{i=2N}^{-1} b_i = -\frac{1}{N}$. In this case, $\ker(\rho) = \{0\}$.
5. $\rho(K_n) = in\lambda C$ for some $\lambda \in \mathbb{R}$.

Proof. By lemma 4.2 and 4.3, it takes the form $\rho(K_1) = \lambda K + \mu K^* + \nu C$, $\rho(K_2) = \lambda' K' + \mu' K'^* + \nu' C$, where K and K' are elements of \mathcal{K}_+ . Also by lemma 4.1 with the commutation relation of K_2 and K_{-1} , K and K'^* must be proportional.

If both of λ and μ are nonzero, then also K and K' must be proportional. By the commutation relation of K_1 and K_{-1} we see that some scalar multiple of K plus a central element (we call it temporarily \tilde{K}) satisfies $[\tilde{K}, \tilde{K}^*] = -\tilde{K} - \tilde{K}^*$. Hence from the beginning we may assume $[K, K^*] = -K - K^* + \kappa C$ for some $\kappa \in \mathbb{C}$. Then again by the commutation relation, $\mu = \lambda - 1$. Similarly, it holds $\mu' = \lambda' - 2$. By the commutation relation of K_2 and K_{-1} we see $\lambda' = 3\lambda + \bar{\lambda} - 1$. Then this is exactly the case (1) or (2). It depends on the value of λ whether it is (1) or (2).

Let one of λ and μ be zero. By composing an automorphism τ , we may assume $\mu = 0$ and we will show that we have the case (3). By the same argument of the beginning of theorem 4.5, $\rho(K_1)$ takes the form $\rho(K_1) = \sum_{i=1}^N a_i K_i + a_0 C$, $\rho(K_2) = \sum_{j=1}^{2N} b_j K_j + b_0 C$ and $\sum_{i=1}^N a_i = \frac{1}{N} = \sum_{i=1}^{2N} b_i$. Any finite set of $\rho(K_i)$'s is linearly independent (by considering the highest or lowest terms of $\rho(K_i)$ in the standard basis of \mathcal{K}) and we see $\ker(\rho) = \{0\}$.

If $\lambda = \mu = 0$, by the commutation relations (1), $\rho(K_2)$ must be mapped to a central element. By the same argument as that of Lemma 2.1, ρ is of the form $\rho(K_n) = in\lambda C$. \square

Let \mathfrak{p} be the Lie algebra of the group generated by translations and dilations in $\operatorname{Diff}(S^1)$. This algebra has a basis $\{T, D\}$ with the relation $[D, T] = T$ [11][9]. Its complexification (which we denote again \mathfrak{p}) is a *-Lie algebra with the *-operation $D^* = -D, T^* = -T$. By setting $K = -D + iT$, we have $[K, K^*] = -K - K^*$.

Lemma 6.5. *Any unitary representation φ' of \mathfrak{p} produces a representation φ'_1 of \mathcal{K}_0 (or a representation of \mathcal{K} with the central charge $c = 0$).*

Proof. It suffices to set

$$\varphi'_1(K_n) = \frac{n^2 + n}{2} \varphi'(K) + \frac{n^2 - n}{2} \varphi'(K^*).$$

We see that φ'_1 preserves the commutation relations by the same computations in the proof of proposition 6.1 with $\alpha = 1$. \square

Remark 6.6. Any composition of a *-endomorphism and a unitary representation of \mathcal{K} is again a unitary representation. As we shall see in the next section, a composition of an endomorphism of type (1) or (2) in proposition 6.4 and a lowest weight representation gives rise to a strange representation (in the sense that they are “localized at the point at infinity”). On the other hand, a composition of the type (3) endomorphism and a lowest weight representation contains at least one lowest weight vector in the sense of subsection 5.2, equation (9) which is the lowest weight vector of the original representation, and the value of $h + ih'$ is changed to $\frac{1}{N}(h + ih')$. If we start with the restriction to \mathcal{K} of a unitary representation of Vir , representations with “complex energy” (namely, $h' \neq 0$) do not arise in this way.

7 Some unitary representations of B_0

In this section we will construct true (not projective) unitary representations of B_0 . Symmetries in physics are in general described by unitary projective representations of a group [15]. From this point of view, one dimensional true representations are trivial, since they are equivalent to the trivial representations as projective representations. Nevertheless, we here exhibit a construction of a one dimensional representation. The author believes that this reveals the big difference between $\text{Diff}(S^1)$ and B_0 . In fact, $\text{Diff}(S^1)$ does not admit any positive energy true representation (see [15]). This difference comes mainly from the fact that $\text{Diff}(S^1)$ is simple but B_0 is not simple.

We identify B_0 with a space of functions on \mathbb{R} as in section 3.5.

Proposition 7.1. *For any $\lambda \in \mathbb{R}$ the map*

$$\begin{aligned} \varphi : B_0 &\rightarrow S^1 \\ f &\mapsto \exp(i\lambda \log f'(0)) \end{aligned}$$

is a (one-dimensional) unitary representation of B_0 .

Proof. Recall that B_0 is the group of orientation preserving, 0-stabilizing diffeomorphisms of S^1 . By the identification with the function space, the derivative of any element is everywhere (in particular at $\theta = 0$) positive, hence the map is properly defined.

By the formula

$$(f \circ g)'(0) = f'(0) \cdot g'(0),$$

we see the map φ above is multiplicative. \square

Remark 7.2. This φ is obviously irreducible and does not extend to $\text{Diff}(S^1)$. In fact, φ is the integration of the one-dimensional representation of corollary 2.3. If $g \in B_0$ is localized on some closed interval which does not include 0, then $\varphi(g) = 1$. In this sense, φ is “localized at the point at infinity”.

Next we need a general lemma.

Lemma 7.3. *Let G be a group, H a normal subgroup of G and π the quotient map $G \rightarrow G/H$. Let F be a subgroup of G such that $F \cap H = \{e\}$ and $\pi(F) = G/H$. Then G/H and F are isomorphic by a canonical isomorphism γ such that $\gamma \circ \pi|_F = \text{id}$. If φ is a representation of F , it extends to a representation $\tilde{\varphi} := \varphi \circ \gamma \circ \pi$ of G .*

Let $B_2 = \{g \in B_0 : f'(0) = 1, f''(0) = 0\}$. It is easy to see that B_2 is a normal subgroup of B_0 .

Let $G = B_0$, $H = B_2$ and $F = P$ be the subgroup generated by dilations and translations. It is obvious that any element of F can be written as a product of a dilation and a translation. The derivative of a translation at point 0 is always 1, whereas a nontrivial dilation has the derivative different from 1 at 0. From this, the intersection of F and H must be pure translations. But then, any element of this intersection must have a vanishing second derivative at 0. This implies that the intersection is trivial.

By a similar consideration, it is not difficult to see that $\pi(P) = \pi(B_0)$. By the previous lemma, the unitary irreducible representation of $F = P$ extends to a unitary irreducible representation of B_0 having B_2 in the kernel.

Also this representation is “localized at the point at infinity”, since if a diffeomorphism is localized in a closed interval which does not contain 0, then it is an element of B_2 and hence mapped to the identity operator.

Summing up, we have the following.

Theorem 7.4. *Any unitary representation φ of P canonically extends to a representation $\tilde{\varphi}$ of B_0 which is localized at the point at infinity.*

We describe the relation between this representation and the endomorphism of \mathcal{K} constructed in section 6. The group P admits a unique irreducible positive energy (which means that the generator of translation is positive) true (not projective) representation [11]. This representation can be considered as the integration of several lowest weight representations of the Lie algebra \mathfrak{p} of \mathcal{P} . In the following, we fix such a representation of \mathfrak{p} and extend it to \mathcal{K} . The representation space of \mathfrak{p} is a dense subspace of the representation space of P and it is the core of any generator of one-parameter subgroup of P (see [11]). Through $\tilde{\varphi}$, any one-parameter subgroup g_t of B_0 is first mapped to P by $\gamma \circ \pi$ and then represented as a one-parameter group of unitary operators. Hence any unbounded operator appearing here is in the representation of \mathfrak{p} explained above and there arise no problems of domains or self-adjointness.

Proposition 7.5. *Let φ be a unitary representation of the Lie group P , φ' be the corresponding representation of the Lie algebra \mathfrak{p} and φ'_1 be the extension to \mathcal{K} in proposition 6.5, then φ'_1 integrates to $\tilde{\varphi}$ in the theorem 7.4.*

Proof. The quotient group B_0/B_2 is isomorphic to $\mathbb{R}_+ \times \mathbb{R}$ with the group operation:

$$(X_1, X_2) \cdot (Y_1, Y_2) = (X_1 Y_1, X_1 Y_2 + Y_1^2 X_2), \text{ for } X_1, Y_1 \in \mathbb{R}_+, X_2, Y_2 \in \mathbb{R}.$$

The isomorphism ρ is given by $f \mapsto (f'(0), f''(0))$.

It's Lie algebra has the structure $\mathbb{R} \oplus \mathbb{R}$ with

$$[(x_1, x_2), (y_1, y_2)] = (0, x_2 y_1 - x_1 y_2) \text{ for } x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

If g^s is a one-parameter subgroup in B_0 with generator v , then the corresponding element in the algebra is $\rho'(v) = (v'(0), v''(0))$, where ρ' is the derivative of ρ .

The generator of the one-parameter subgroup of dilations $D_s(\theta)$ is $\frac{1}{2}(K_1 - K_1^*)(\theta) =: d_1(\theta) = \sin \theta$ and the generator of translations $T_s(\theta)$ is $-\frac{i}{2}(K_1 + K_1^*)(\theta) =: t_1(\theta) = 1 - \cos \theta$. Thus $\rho'(d_1) = (1, 0)$ and $\rho'(t_1) = (0, 1)$. Similarly, the generator $\frac{1}{2}(K_n - K_n^*)(\theta) =: d_n(\theta) = \sin n\theta$ is mapped to $(n, 0)$ and $-\frac{i}{2}(K_n + K_n^*)(\theta) =: t_n(\theta) = 1 - \cos n\theta$ is mapped to $(0, n^2)$. In short, it holds that $\rho'(d_n) = n\rho'(d_1)$, $\rho'(t_n) = n^2\rho'(t_1)$. Hence these relations hold also for the derivative of $\tilde{\varphi}$, namely $\tilde{\varphi}'(d_n) = n\tilde{\varphi}'(d_1)$, $\tilde{\varphi}'(t_n) = n^2\tilde{\varphi}'(t_1)$.

On the other hand, for φ'_1 we have

$$\begin{aligned}\varphi'_1 \left(\frac{1}{2}(K_1 - K_1^*) \right) &= \frac{1}{2}(K - K^*), \\ \varphi'_1 \left(-\frac{i}{2}(K_1 + K_1^*) \right) &= -\frac{i}{2}(K + K^*), \\ \varphi'_1 \left(\frac{1}{2}(K_n - K_n^*) \right) &= \frac{n}{2}(K - K^*) = n\varphi'_1 \left(\frac{1}{2}(K_1 - K_1^*) \right), \\ \varphi'_1 \left(-\frac{i}{2}(K_n + K_n^*) \right) &= -\frac{in^2}{2}(K + K^*) = n^2\varphi'_1 \left(-\frac{i}{2}(K_1 + K_1^*) \right).\end{aligned}$$

From this it is clear that φ'_1 and $\tilde{\varphi}'$ are equivalent, since by definition $\varphi_1(d_1) = \tilde{\varphi}'(d_1)$ and $\varphi_1(t_1) = \tilde{\varphi}'(t_1)$ \square

As remarked before, there is a unique irreducible positive energy representation of P . By the proposition above, it extends to an irreducible positive energy true representation of B_0 .

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