Equilibrium (non-)Existence in Games with Competing Principals[*]

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Abstract

We study competing-mechanism games, in which multiple principals contract with multiple agents. We reconsider the issue of non-existence of an equilibrium as first raised by Myerson (1982). In the context of his example, we establish the existence of a perfect Bayesian equilibrium. We clarify that Myerson (1982)'s non-existence result is an implication of the additional requirement he imposes, that each principal selects his preferred continuation equilibrium in the agents' game.

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1 Introduction

Competition in several market settings is modeled as an extensive-form game in which principals post mechanisms to deal with multiple agents. The competing auctions (McAfee (1993); Peters and Severinov (1997)), and competitive search (Wright et al. (2021)) models offer prominent examples of this approach.

Despite the increased economic relevance of competing-mechanism approaches, we still lack a comprehensive characterization of the corresponding market equilibria. While, following Epstein and Peters (1999), the literature has extended the revelation principle to these contexts, the general issue of equilibrium existence remains largely unexplored. Indeed, existence of an equilibrium in games with multiple principals has only been established for the particular case of a single agent (Carmona and Fajardo (2009)). With several agents, the celebrated example of Myerson (1982) provides an instance of equilibrium non-existence. In Myerson's approach, the presence of multiple principals is at the root of the non-existence, since it generates a fundamental discontinuity in the optimal choice of a principal's mechanism.[1] His analysis is framed in terms of competition between principal-agent (manufacturer-retailer) hierarchies, which makes the non-existence result potentially problematic for applications.

We propose a reinterpretation of this result. We argue that, in the example, the non-existence is implied by the requirement that, at equilibrium, each principal chooses his optimal incentive-compatible mechanism. This guarantees that he does not have a profitable deviation *regardless* of the continuation equilibrium selected by agents. Such requirement need not be satisfied by any perfect bayesian equilibrium (PBE) of the competing-mechanism game. We eventually show that existence of a PBE can be established in the example.

We next formalize the competitive game between principal-agent hierarchies (Section 2) and illustrate our result (Section 3).

2 Competing Hierarchies

We consider multiple principals (indexed by $k \in \mathcal{N} = \{1, \ldots, N\}$) contracting with multiple agents (indexed by $i \in \mathcal{I} = \{1, \ldots, I\}$) as in Myerson (1982). Each agent i has a private type t_i in the finite set T_i, with $T = \underset{i \in \mathcal{I}}{\times} T_i$. In contrast with Myerson (1982), agents take no physical actions.

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[1] "[...] the set of feasible (incentive-compatible) mechanisms for principal j varies upper-semicontinuously in the other (mechanism), rather than continuously as is required by the existence theorem of Debreu (1952)" (Myerson, 1982, p. 78).

We denote $d^k \in D^k$ a decision for principal k, with D^k finite and $D = \underset{k \in \mathcal{N}}{\times} D^k$. We let $v^k : D \times T \to \mathbb{R}$ and $u_i : D \times T \to \mathbb{R}$ be the payoffs of principal k and of agent i, respectively. An allocation $\phi : T \to \Delta(D)$ is a mapping associating to each state $t \in T$ a probability distribution over D.

Communication occurs via the public mechanisms posted by principals, and via the messages that agents privately send them. Each agent contracts with one principal only. Letting $\mathcal{I}^k$ be the set of agents dealing with principal k, we require that $\underset{k \in \mathcal{N}}{\cup} \mathcal{I}^k = \mathcal{I}$, and $\mathcal{I}^k \cap \mathcal{I}^{k'} = \emptyset$ for each $k \neq k'$. We call *hierarchy* k the collection of principal k and his agents. Each $i \in \mathcal{I}^k$ sends a report r_i^k in the finite set $R_i^k \supseteq T_i$ to principal k.[2]

Formally, a mechanism for principal k is a mapping $\pi^k : R^k \to \Delta\left(D^k\right)$, with $R^k = \underset{i \in \mathcal{I}^k}{\times} R_i^k$.[3] We let Π^k be the set of mechanisms available to principal k, with $\Pi = \underset{k \in \mathcal{N}}{\times} \Pi^k$. The corresponding game, denoted G^Π, unfolds as follows. First, principals simultaneously commit to mechanisms. Given the observed mechanisms $\left(\pi^1, \pi^2, \ldots, \pi^N\right)$, and their private types $(t_1, \ldots, t_I)$, agents simultaneously report to the principal of their hierarchy. Finally, decisions are implemented, lotteries realize, and payoffs accrue. A (pure) strategy for principal k is a mechanism $\pi^k \in \Pi^k$. A strategy ρ_i^k for agent $i \in \mathcal{I}^k$ associates to every profile of mechanisms and each realized type a probability distribution over R_i^k.

Following the standard approach to competing mechanisms (Epstein and Peters (1999)), we focus on the PBE of G^Π. The strategies $\pi^* = (\pi^{k*}, \pi^{-k*})$ and $\rho^* = (\rho_i^{k*}, \rho_{-i}^{k*})_{k \in \mathcal{N}}$, constitute a PBE if:

1. ρ^* is a continuation equilibrium. That is, for every $\pi \in \Pi$, the strategies $(\rho_i^{k*}, \rho_{-i}^{k*})_{k \in \mathcal{N}}$ form a Bayes-Nash equilibrium of the subgame π;
2. Given ρ^*, the strategies (π^{k*}, π^{-k*}) form a Nash equilibrium of the principals' game.

As in Myerson (1982), a mechanism for principal k is *direct* if agents can only report types. That is, $R_i^k = T_i$ for each $i \in \mathcal{I}^k$.

For a given array of direct mechanisms π^{-k}, a direct mechanism π^k is *incentive compatible* if it induces a continuation equilibrium in which the agents of each hierarchy k are truthful to principal k, under the belief that the same occurs in each hierarchy $-k$. Thus, π^k can be incentive compatible given π^{-k}, but not relative to $\tilde{\pi}^{-k} \neq \pi^{-k}$.[4]

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[2] The analysis extends to situations in which all relevant sets are infinite (Attar et al. (2021)).

[3] Observe that a mechanism does not include private communication from principals to agents, which may take the form of "recommendations". Following Myerson (1982), this is an implication of agents taking no physical actions.

[4] This possibility is already acknowledged by Myerson (1982, p.77). See also McAfee (1993, p. 1288).

Myerson (1982) analyses the competition between principals as a *generalized* single-principal problem, and focuses on *principals' equilibria*, in which each principal chooses an optimal incentive-compatible mechanism. Together with the revelation-principle result of his Proposition 2, this amounts to let each principal select his preferred continuation equilibrium. This choice, we shall argue, is key to the non-existence problem he documents.

3 Restoring Existence

We revisit the example in Section 4 of Myerson (1982). Consider two principals, each of them dealing with only one agent. The type set of each agent $i = 1, 2$ is $\{\alpha, \beta\}$; types are independent, with $prob(\alpha) = prob(\beta) = 1/2$. Each principal $k = 1, 2$ takes decisions in $D^k = \{A, B, C\}$. To simplify exposition, we index agents by the hierarchy they belong to, with agent 1 (A1) participating in hierarchy 1. In addition, we let $t^k \in \{\alpha, \beta\}$ be the realized type of the agent of hierarchy $k = 1, 2$.

In the matrix below, the first number denotes principal k's payoff, and the second one that of his agent of type t^k:

	$t^k = \alpha$	$t^k = \beta$
A	$6, 1$	$0, z^k$
B	$0, z^k$	$6, 1$
C	$5, 0$	$5, 0$

with z^k determined by principal $-k$'s decision. Specifically,

$$z^1 = \begin{cases} 2 & \text{if } d^2 \in \{A, B\} \\ 1 & \text{if } d^2 = C \end{cases}$$

and

$$z^2 = \begin{cases} 1 & \text{if } d^1 \in \{A, B\} \\ 2 & \text{if } d^1 = C. \end{cases}$$

The non-existence result. Proposition 3 in Myerson (1982) shows that, in this example, there is no *principals' equilibrium.* Consider principal 1 (P1): if principal 2 (P2)'s equilibrium mechanism implements C with probability one, P1's optimal incentive-compatible mechanism selects A if A1 reports α, and B if she reports β, which gives him a payoff of 6. In all other cases, P1 cannot achieve a payoff above 5, and his optimal mechanism is to select C regardless of A1's message. P2's optimal mechanism is exactly reversed, that is, it makes available A and B when P1 plays C, and viceversa. This implies the non-existence of a principals' equilibrium.

Existence can however be restored if one focuses on PBE, in line with the competing-mechanism literature. We establish the result in the simple game in which principals post direct mechanisms.

Proposition 1 *Let* $R^k = \{\alpha, \beta\}$ *for* $k = 1, 2$. *There is a PBE* $(\pi^{1*}, \pi^{2*}, \rho_1^*, \rho_2^*)$ *that supports the allocation* $\phi(t) = (C, C)$ *for each* $t \in \{\alpha, \beta\}^2$.

Proof Let principals post the same direct mechanism $\pi^{1*} = \pi^{2*}$, which implements C for every agent's message. Thus, regardless of the agents' reports in these mechanisms, the desired allocation obtains. Agents' equilibrium strategies (ρ_1^*, ρ_2^*) are such that: for each $\pi = (\pi^1, \pi^2)$, $t^k \in \{\alpha, \beta\}$ and k=1,2, and given ρ_{-k}^*, $\rho_k^*(t^k, \pi)$ maximizes type t^k of agent k's payoff. Specifically, let $R_k(t^k, \pi)$ be the set of reports in the mechanism π^k that, given ρ_{-k}^*, are optimal for type t^k of agent $k = 1, 2$. If, for a given $t^k \in \{\alpha, \beta\}$, $R_k(t^k, \pi)$ is not a singleton, then **let**

(i) $\rho_k^*(\alpha, \pi)$ select a message $r_k(\alpha, \pi) \in \underset{r \in R_k(\alpha,\pi)}{argmax}\ \pi^k(B|r)$

(ii) $\rho_k^*(\beta, \pi)$ select a message $r_k(\beta, \pi) \in \underset{r \in R_k(\beta,\pi)}{argmax}\ \pi^k(A|r)$,

with $\pi^k(d^k|r)$ being the probability that π^k implements $d^k \in \{A, B\}$ upon receiving the report r. It is easy to check that (ρ_1^*, ρ_2^*) induce a (Bayes-Nash) equilibrium in any subgame (π^1, π^2).

We now show that no principal has a profitable deviation. Let P1 deviate to some $\pi^1 \neq \pi^{1*}$ and consider the behavior of A1 in the subgame (π^1, π^{2*}). Since P2 sticks to π^{2*}, one has $z^1 = 1$, which implies that, for each $\pi^1 \in \Pi^1$, both types of A1 have the same incentives. That is, $R_1(t^1, \pi^1, \pi^{2*}) = \{r \in \{\alpha, \beta\} \in \underset{\tilde{r} \in \{\alpha,\beta\}}{argmax}\ \ \pi^1(A|\tilde{r}) + \pi^1(B|\tilde{r})\}$ is independent of t^1.

If neither type of A1 is indifferent over her reports, that is, $R_1(t^1, \pi^1, \pi^{2*})$ is a singleton, both types report the same $r \in \{\alpha, \beta\}$ to P1. His payoff is

$$\frac{1}{2}\big(6\pi^1(A|r) + 5\pi^1(C|r)\big) + \frac{1}{2}\big(6\pi^1(B|r) + 5\pi^1(C|r)\big) = \frac{1}{2}\big(6 + 4\pi^1(C|r)\big) \leq 5, \tag{1}$$

since $\pi^1(A|r) + \pi^1(B|r) + \pi^1(C|r) = 1$ for each $r \in \{\alpha, \beta\}$.

If, instead, each type of A1 is indifferent between reporting α and β to P1, then:

$$\pi^1(A|\alpha) + \pi^1(B|\alpha) = \pi^1(A|\beta) + \pi^1(B|\beta). \tag{2}$$

We consider two cases in turn.

Case 1.

$$\pi^1(A|\alpha) \leq \pi^1(A|\beta) \quad \Longleftrightarrow \quad \pi^1(B|\alpha) \geq \pi^1(B|\beta). \tag{3}$$

Given $(i)-(ii)$, both types of A1 are truthful to P1. His payoff is:

$$\begin{aligned} &\frac{1}{2}\big(6\pi^1(A|\alpha)+5\pi^1(C|\alpha)\big)+\frac{1}{2}\big(6\pi^1(B|\beta)+5\pi^1(C|\beta)\big)= \\ &=5+\frac{1}{2}\big(\pi^1(A|\alpha)-5\pi^1(A|\beta)\big)+\frac{1}{2}\big(\pi^1(B|\beta)-5\pi^1(B|\alpha)\big). \end{aligned} \tag{4}$$

Given (3), the deviation is unprofitable.

Case 2.

$$\pi^1(A|\alpha) > \pi^1(A|\beta) \quad \Longleftrightarrow \quad \pi_1(B|\alpha) < \pi_1(B|\beta). \tag{5}$$

Given $(i)-(ii)$, no type of A1 is truthful to P1. His payoff is

$$\begin{aligned} &\frac{1}{2}\big(6\pi^1(A|\beta)+5\pi^1(C|\beta)\big)+\frac{1}{2}\big(6\pi^1(B|\alpha)+5\pi^1(C|\alpha)\big)= \\ &=5+\frac{1}{2}\big(\pi^1(A|\beta)-5\pi^1(A|\alpha)\big)+\frac{1}{2}\big(\pi^1(B|\beta)-5\pi^1(B|\alpha)\big). \end{aligned} \tag{6}$$

Given (5), the deviation is unprofitable.

Consider now P2 deviating to some $\pi^2 \neq \pi^{2*}$. A2's payoff in the subgame (π^{1*}, π^2) may now be type-dependent, since $z^2 = 2$. Specifically, A2's incentive constraints are:

$$\pi^2(A|\alpha)+2\pi^2(B|\alpha) \geq \pi^2(A|\beta)+2\pi^2(B|\beta) \qquad \text{for type } \alpha \tag{IC$_\alpha$}$$

$$2\pi^2(A|\beta)+\pi^2(B|\beta) \geq 2\pi^2(A|\alpha)+\pi^2(B|\alpha) \qquad \text{for type } \beta \tag{IC$_\beta$}$$

Adding them up, one gets

$$\pi^2(A|\beta)+\pi^2(B|\alpha) \geq \pi^2(A|\alpha)+\pi^2(B|\beta). \tag{7}$$

We consider three situations:

a) IC$_\alpha$ and IC$_\beta$ hold as *strict* inequalities. P2 gets the same payoff as in (4), and, given (7), the deviation is unprofitable.

b) IC$_\alpha$ or IC$_\beta$ holds as an equality. Then, either both types keep reporting truthfully, leading back to case a), or only one of them does so. In the latter case, both types send the same report to P2, who gets the same payoff as in (1), making the deviation unprofitable.

c) IC_α and IC_β hold as equalities. The only remaining case is that of both types behaving untruthfully, which yields P2 the same payoff as in (4).■

In the proof, principals do not have full control over the agents' coordination in the continuation game. Specifically, the PBE notion does not prevent agents from punishing principals via untruthful behaviors.

In the language of competing mechanisms, the notion of principals' equilibrium is recovered by that of *strongly-robust* equilibrium (SRE).[5] In an SRE, each principal believes that, in every subgame, agents will coordinate on his preferred continuation equilibrium. Following the reasoning in the proof of Proposition 4 in Myerson (1982), one can check that an SRE does not exist in the example.

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[5] See Epstein and Peters (1999), and Han (2007) for a general formulation of the SRE concept.

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