Homogenization of discrete thin structures

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Abstract

We consider graphs parameterized on a portion $X \subset \mathbb{Z}^d \times \{1, \dots, M\}^k$ of a cylindrical subset of the lattice $\mathbb{Z}^d \times \mathbb{Z}^k$, and perform a discrete-to-continuum dimension-reduction process for energies defined on X of quadratic type. Our only assumptions are that X be connected as a graph and periodic in the first d-directions. We show that, upon scaling of the domain and of the energies by a small parameter ε, the scaled energies converge to a d-dimensional limit energy. The main technical points are a dimension-reducing coarse-graining process and a discrete version of the p-connectedness approach by Zhikov.

1 Introduction

The object of the investigation in this paper is the analysis of discrete thin objects through, at the same time, a discrete-to-continuum and dimension-reduction process. The main focus of our work is the great generality of the geometry of our discrete systems, which we essentially require to be a connected graph periodic in the dimensions that are maintained after a discrete-to-continuum passage.

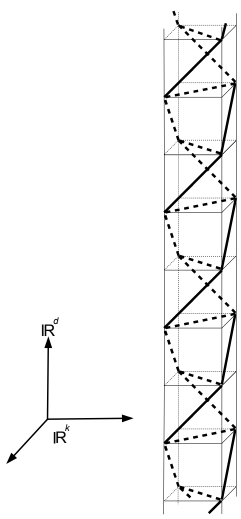

Figure 1: A discrete thin object in three dimensions with a one-dimensional behaviour

An example of the structure that we have in mind is pictured in Fig. 1; in this case the 'macroscopic dimension' is one. The thicker black lines (both the solid ones and the ones dashed for graphic purpose) represent connections between nodes of a cubic lattice in $\mathbb{R}^3$. Equivalently, we may think of the same structure as a network of conducting rods. Note that this object cannot trivially be parameterized as a 'subgraph' of a function depending of the vertical variable, as it consists of a double helix connected through horizontal bonds. Nevertheless, it can be included in a 'regular' thin object; in this case, the cylindrical part of $\mathbb{Z}^3$ whose projection on the two-dimensional horizontal plane are the four vertices of a square. Even if no connection is purely vertical, the overall behaviour of such a structure is expected to be that of a vertical one-dimensional object. Similar examples can be thought of when the macroscopic dimension is two, for example stacking copies of this structure in a planar configurations.

With these examples in mind, we are going to look at graphs whose nodes are a periodic subset X of $\mathbb{Z}^{d+k}$ of period T in the first d directions (in the example pictured $d = 1$, corresponding to the vertical direction, with period $T = 2$), bounded in the last k directions (in the example, $k = 2$, corresponding to the horizontal directions), so that we may always think that it is contained in $\mathbb{Z}^d \times \{0, \dots, M-1\}^k$ for some $M \in \mathbb{N}$. This graph is equipped with a set of edges $\mathcal{E} \subset X \times X$ which makes it connected. This set of edges is supposed to be invariant by the same translations as X.

We are going to show that we may define a continuous d-dimensional approximation of this set. In order to maintain technicalities to a minimum, we consider only quadratic interactions. The Dirichlet energy of such a set is defined as

$$F(u) = \sum_{(i,j)\in\mathcal{E}} (u(i) - u(j))^2$$

on functions $u : X \to \mathbb{R}$. A discrete-to-continuum and dimensionally reduced limit is then obtained by considering a scaled version of the energies

$$F_\varepsilon(u) = \sum_{(i,j)\in\mathcal{E}} \varepsilon^{d-2}(u_i - u_j)^2$$

defined on functions $u : \varepsilon X \to \mathbb{R}$, where we use the notation $u_i = u(\varepsilon i)$, and taking their limit in a suitable sense as $\varepsilon \to 0$. Note that we may interpret

$$\varepsilon^{-1}(u_i - u_j) = |i-j|\Big(\frac{u_i - u_j}{\varepsilon|i-j|}\Big)$$

as an inhomogeneous difference quotient, so that F_ε represent discrete versions of an (inhomogeneous) Dirichlet integral, whose general continuous counterpart is of the form

$$\frac{1}{\varepsilon^k}\int_{\varepsilon E} f(\nabla u)\, dx, \tag{1}$$

with E a subset of $\mathbb{R}^d \times \mathbb{R}^k$ uniformly bounded in the last k variables. Energies of the form (1) are the prototype of thin-structure energies on the continuum (see *e.g.* [8, 9]), which have been treated extensively in the last thirty years. Among the many contribution to the subject we recall the seminal paper by Le Dret and Raoult [25] which gives a general dimension-reduction formula when $E = \mathbb{R}^d \times [0, 1]$ through a lower-dimensional quasiconvexification process. Moreover, a general compactness and integral-representation theorem has been proved by Braides, Fonseca and Francfort [15], who interpret lower-dimensional quasiconvexification through a homogenization formula, and extend the analysis to general thin films with varying profiles. In their approach they deal with E that can be seen as a subgraph of a function defined on $\mathbb{R}^d$. In our case, even if a continuum set E corresponding to X can be constructed, it may not be a subgraph, as it might have holes or even possess a more complex topology. Note that the assumption that the integration be performed on the scaled εE cannot be extended to arbitrary E_ε since in that case the limit might not be simply d-dimensional if the complexity of the topology increased as $\varepsilon \to 0$ (see the example by Braides and Bhattacharya [5]). We note that asymptotic analysis of thin objects can be interpreted as an intermediate step in the study of structures with very fast oscillating profile [11] (see also [4] for an example of application in a continuum geometry). For other aspects of dimension reduction in variational problems we refer *e.g.* to [6, 7, 23, 24, 26, 27, 29].

Discrete-to-continuum analyses for lattice energies are usually performed after identification of functions defined on (portions of) lattices with their piecewise-constant interpolations. This identification allows to embed families of energies in a common Lebesgue-space environment (see the seminal paper by Alicandro and Cicalese [3]). Using this approach, a discrete-to-continuum analog for thin films of the Braides, Fonseca and Francfort theory, has been studied by Alicandro, Braides and Cicalese [2] (see also [28], and the work [18] for a connection with aperiodic lattices). Due to the great generality of our discrete set X, we will not extend functions defined on εX but follow a dimension-lowering coarse-graining approach: to each function $u_\varepsilon : \varepsilon X \to \mathbb{R}$ we associate the function $\overline{u}_\varepsilon : \varepsilon T\mathbb{Z}^d \to \mathbb{R}$ where $\overline{u}_\varepsilon(\varepsilon Tl)$ is obtained by averaging u_ε on $\varepsilon X \cap \big((\varepsilon Tl + \varepsilon\{0,\dots,T-1\}^d)\times\varepsilon\{0,\dots,M-1\}^k\big)$. We find coarse graining convenient in that it directly gives a function defined in a d-dimensional set, without further scaling arguments. Moreover, this approach helps to separate the definition of a dimensionally reduced parameter, which is easily obtained from $\overline{u}_\varepsilon$ from analysis of the finer behaviour of the functions u_ε at the 'microscopic' level, which is needed to use technical arguments for the modification of boundary data. More precisely, we prove that energy bounds on u_ε imply that the piecewise-constant interpolations of the corresponding $\overline{u}_\varepsilon$ are precompact in $L^2_{\rm loc}(\mathbb{R}^d)$ and their limit is in $H^1_{\rm loc}(\mathbb{R}^d)$. In this way the dimensionally reduced continuum parameter can be defined. In order to relate the original u_ε to this limit, a Poincaré inequality must be used at scale ε, which shows that the original u_ε converge to u in a 'perforated domain' fashion (see *e.g.* [13]). Both the coarse-graining and the Poincaré-type inequality are very reminiscent of the p-connectedness approach by Zhikov [30], and of its use in the homogenization of singular structures by Braides and Chiadò Piat [10], even though in those papers p-connectedness is stated for local functionals depending on the gradient. Here we deal with non-local interactions (that is, the energy densities depend on finite differences of the parameter, and not on its gradient), even though the non-locality weakens as $\varepsilon \to 0$, and some additional care has to be taken, similarly to the case of the homogenization of convolution-type energies (see [1, 12, 17]).

The paper is organized as follows. In Section 2 we introduce the notation for the environment $X \subset \mathbb{R}^d \times \mathbb{R}^k$ and for the energies that we consider, which are a little more general than those described above in that a more general inhomogeneity is allowed by introducing interactions coefficients a_{ij}, and the energies are localized by considering interactions parameterized on a set $\Omega \subset \mathbb{R}^d$. Section 3 is devoted to the definition of coarse-grained functions, and to the statement and proof of the two-connectedness property and of a Poincaré-Wirtinger's inequality. Section 4 contains a compactness result for coarse-grained functions whose proof relies on the two-connectedness property and the corresponding convergence of the original functions. The limit defines a function on a subset of $\mathbb{R}^d$. Section 5 contains a result that allows to consider boundary-values on 'lateral boundaries' of thin films. A homogenization theorem for quadratic energies defined on εX is stated in Section 6. Its proof is subdivided into a lower bound by blow-up and an upper bound by a direct construction. Moreover, an application to the description of the asymptotic behaviour of boundary-value problems is also described. Finally, Section 7 contains some simple examples illustrating some possible non-trivial shapes of the thin structures we consider.

Notation

- The letter C denotes a generic strictly positive constant not depending on the parameters of the problem considered, whose value may be different at every its appearance.
- If $x, y \in \mathbb{R}^d$ then $x \cdot y$ denotes their scalar product. If $t \in \mathbb{R}$ then $\lfloor t \rfloor$ is its integer part.
- The characteristic function of a set A is denoted by χ_A.
- For $T \in \mathbb{N}$, we denote by $Q_{T,d}$ the d-dimensional semi-open cube of side length T; *i.e.*, $Q_{T,d} := [0,T)^d$. If $T = 1$, we simply write $Q_d = Q_{1,d}$. For $l \in \mathbb{Z}^d$, $Q^l_{T,d} := lT + [0,T)^d$ and for $T = 1$, we write $Q^l_d = Q^l_{1,d}$.
- $Q_{T,k}$ denotes the k-dimensional semi-open cube of side length T; *i.e.*, $Q_{T,k} := [0,T)^k$. For $T = 1$, we set $Q_k = Q_{1,k}$ and $Q^n_k = n + Q_k$ if $n \in \mathbb{Z}^k$.

- For any measurable set Ω and $u \in L^1(\Omega)$, $\fint_\Omega u(x)dx$ denotes the average of u on Ω; *i.e.*,
$$\fint_\Omega u(x)dx := \frac{1}{|\Omega|}\int_\Omega u(x)dx,$$
where $|\cdot|$ stands for the Lebesgue measure.
- For any open set $\Omega \in \mathbb{R}^d$ and for any $\delta > 0$, we let $\Omega(\delta) := \{x^d \in \Omega : \mathrm{dist}(x^d, \partial\Omega) > \delta\}$.

2 Setting of problem

In the following X will be a fixed subset of $\mathbb{Z}^d \times \{0, \dots, T-1\}^k$, with $d, k \geq 1$ and $T \in \mathbb{N}$. We assume that

(i) X is T-periodic in $e_1, \dots, e_d$;

(ii) X is connected in the following sense: there exists $\mathcal{E} \subset X \times X$ such that for all $i, j \in X$ there exists a sequence $\{i_n\}_{n=0}^N$ of points of X, with $i_0 = i$ and $i_N = j$, such that the segment $(i_n, i_{n+1}) \in \mathcal{E}$. Moreover, the set $\mathcal{E}$ is T-periodic; *i.e.*, if the segment (i, j) belongs to $\mathcal{E}$, then, for any $m = 1, \dots, d$, the segment $(i + Te_m, j + Te_m)$ belongs to $\mathcal{E}$;

(iii) the set $\mathcal{E}$ is equi-bounded; *i.e.*, there exists $R > 0$ such that
$$\max\{|i - j| : (i, j) \in \mathcal{E}\} \leq R.$$

Assumption (iii) can be also seen as an hypothesis on the energy, telling which bonds are active (the other bonds having zero energy).

Note that it is not restrictive to assume that $R \leq T$, upon taking a larger period.

Remark 2.1. In the notation above, we can include also the case of
$$X \subset \mathbb{Z}^d \times \prod_{n=1}^{k}\{0, \dots, M_n - 1\},$$
with $T_m \geq 1$, $m = 1, \dots, d$, and $M_n \geq 1$, $n = 1, \dots, k$, and X T_m-periodic in e_m, for any $m = 1, \dots, d$. In this case, we take T equal to the lowest common multiple of $T_1, \dots, T_d, M_1, \dots, M_k$.

Let a_{ij} be T-periodic coefficients in $e_1, \dots, e_d$; *i.e.*,
$$a_{i+Te_m\, j+Te_m} = a_{ij} \text{ for all } i, j \in X,\ m \in \{1, \dots, d\},$$
such that $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$. It is not restrictive to suppose that $a_{ij} = a_{ji}$. For $\varepsilon > 0$, we introduce the family of functionals F_ε defined on functions $u : \varepsilon X \to \mathbb{R}$ by
$$F_\varepsilon(u) := \sum_{i,j \in X} \varepsilon^d a_{ij}\left(\frac{u_i - u_j}{\varepsilon}\right)^2,$$
where $u_i := u(\varepsilon i)$. Note that also the case $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$ is non trivial. Note moreover that, by the periodicity of a_{ij}, there exists a positive constant C such that $C \leq a_{ij} \leq 1/C$ if $(i, j) \in \mathcal{E}$, so that F_ε is estimated from above and below by the energy corresponding to $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$.

More in general, we will consider 'localized' versions of energies F_ε, limiting interactions to $i, j \in X$ such that $\varepsilon i, \varepsilon j \in \Omega \times \varepsilon Q_{T,k}$ for some Lipschitz open subset Ω of $\mathbb{R}^d$; *i.e.*,
$$F_\varepsilon(u) := \sum_{i,j \in (\frac{1}{\varepsilon}\Omega \times Q_{T,k}) \cap X} \varepsilon^{d-2} a_{ij}(u_i - u_j)^2. \tag{2}$$

In order to study the asymptotic behaviour of F_ε as $\varepsilon \to 0$, we need to identify real-valued functions u defined on εX with piecewise-constant interpolations. To that end, let Ω be an open subset of $\mathbb{R}^d$ with Lipschitz boundary. For $\varepsilon > 0$, let u_ε be a family of functions $u_\varepsilon : (\Omega \times \varepsilon Q_{T,k}) \cap \varepsilon X \to \mathbb{R}$. Setting $I_\varepsilon = I_\varepsilon(\Omega) := \{l \in \mathbb{Z}^d : \varepsilon Q^l_{T,d} \subset \Omega\}$, we define a piecewise-constant function $\overline{u}_\varepsilon$ in $L^2(\Omega)$ by

$$\overline{u}_\varepsilon(x^d) := \sum_{l \in I_\varepsilon} \tilde{u}^l_\varepsilon \chi_{\varepsilon Q^l_{T,d}}(x^d), \tag{3}$$

where $\tilde{u}^l_\varepsilon$ is given by

$$\tilde{u}^l_\varepsilon := \frac{1}{\#[(Q_{T,d} \times Q_{T,k}) \cap X]} \sum_{i \in (Q^l_{T,d} \times Q_{T,k}) \cap X} u_i, \tag{4}$$

with $u_i = u^\varepsilon_i := u_\varepsilon(\varepsilon i)$. The set of functions $\mathcal{C}_\varepsilon(\Omega)$ is defined by

$$\mathcal{C}_\varepsilon(\Omega) := \Big\{ u : \mathbb{R}^d \times Q_{T,k} \to \mathbb{R} : u \text{ is constant on } \varepsilon Q^l_d \times \varepsilon Q^n_k \\ \text{for } (l, n) \in \big(\mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega\big) \times \{0, \dots, T-1\}^k \Big\}, \tag{5}$$

so that a function $u : (\varepsilon \mathbb{Z}^d \cap \Omega) \times \varepsilon\{0, \dots, T-1\}^k \cap \varepsilon X \to \mathbb{R}$ can be identified with its extension belonging to $\mathcal{C}_\varepsilon(\Omega)$. Note that any interpolation is well-defined since we consider half-open cubes $\varepsilon Q^l_d \times \varepsilon Q^n_k$.

We say that the family of function u_ε in $\mathcal{C}_\varepsilon(\Omega)$ *converges to* $u \in H^1(\Omega)$ if

$$\overline{u}_\varepsilon \to u \quad \text{in } L^2_{\mathrm{loc}}(\Omega). \tag{6}$$

Now, we state the main result of this paper regarding the limit analysis as $\varepsilon \to 0$ of the family of functionals F_ε given by (2). This is done through the computation of the corresponding Γ-limit with respect to convergence (6).

Theorem 2.2. *The family of functionals $F_\varepsilon : \mathcal{C}_\varepsilon(\Omega) \to [0, \infty)$ given by* (2) *Γ-converges with respect to convergence* (6) *to the functional $F_{\mathrm{hom}} : H^1(\Omega) \to \mathbb{R}$ defined by*

$$F_{\mathrm{hom}}(u) := \int_\Omega A_{\mathrm{hom}} \nabla u \cdot \nabla u dx,$$

where

$$A_{\mathrm{hom}} z \cdot z \quad := \quad \frac{1}{T^d} \min \Big\{ \sum_{i \in (Q_{T,d} \times Q_{T,k}) \cap X} \sum_{j \in (\mathbb{R}^d \times Q_{T,k}) \cap X} a_{ij}(u_i - u_j)^2 : \\ u_i - z \cdot i^d \quad \text{is } T\text{-periodic in } e_1, \dots, e_d \Big\}. \tag{7}$$

In this formula we interpret $u_i = z \cdot i^d$ as the discrete interpolation of the affine function $z \cdot x^d$, with $x^d \in \mathbb{R}^d$.

The proof of Theorem 2.2 will be given Section 6 after proving some technical results.

3 Two-connectedness and Poincaré-Wirtinger's inequality

In this section we prove two technical lemmas, which will allow to use some compactness results for systems of nearest-neighbour interactions. To that end, for any real-valued function u defined on X, we introduce a coarse-grained lower-dimensional variable $\tilde{u}^l$, with $l \in \mathbb{Z}^d$, given by (4) with $u_i = u(i)$. In other words,

$$\tilde{u}^l := \frac{1}{\#[(Q_{T,d} \times Q_{T,k}) \cap X]} \sum_{i \in (Q^l_{T,d} \times Q_{T,k}) \cap X} u_i.$$

The first result of this section states that a nearest-neighbour interaction energy on the coarse-grained variable is (locally) dominated by the energy on X.

Proposition 3.1. *There exist $C = C(X) > 0$ and $M > 0$ such that*

$$|\tilde{u}^l - \tilde{u}^{l'}|^2 \le C \sum_{i,j \in [(Q^l_{T,d} \cup Q^{l'}_{T,d} + (-M,M)^d) \times Q_{T,k}] \cap X} |u_i - u_j|^2 \tag{8}$$

for any $l, l' \in \mathbb{Z}^d$ such that $|l - l'| = 1$.

Proof. Using definition (4) of $\tilde{u}^l$ and the change of indices $j = i + Te_m$, for some $m = 1, \dots, d$, combined with the Hölder inequality, we deduce that

$$\begin{aligned}
|\tilde{u}^l - \tilde{u}^{l'}|^2 &= \frac{1}{[\#((Q_{T,d} \times Q_{T,k}) \cap X)]^2} \Bigg| \sum_{i \in (Q^l_{T,d} \times Q_{T,k}) \cap X} u_i - \sum_{j \in (Q^{l'}_{T,d} \times Q_{T,k}) \cap X} u_j \Bigg|^2 \\
&= \frac{1}{[\#((Q_{T,d} \times Q_{T,k}) \cap X)]^2} \Bigg| \sum_{i \in (Q^l_{T,d} \times Q_{T,k}) \cap X} (u_i - u_{i+Te_m}) \Bigg|^2 \\
&\le \frac{1}{\#[(Q_{T,d} \times Q_{T,k}) \cap X]} \sum_{i \in (Q^l_{T,d} \times Q_{T,k}) \cap X} |u_i - u_{i+Te_m}|^2. \qquad (9)
\end{aligned}$$

The connectedness of X ensures that for all $i \in (Q^l_{T,d} \times Q_{T,k}) \cap X$, there exists a sequence $\{j_n\}_{n=0}^{N_i}$ of points in X with $j_0 = i$ and $j_{N_i} = i + Te_m$ such that $(j_n, j_{n+1}) \in \mathcal{E}$. Let γ be path joining i and $i + Te_m$ through the points $j_1, \dots, j_{N_i - 1}$. Such a path γ is contained in $[(Q^l_{T,d} \cup Q^{l'}_{T,d} + (-M,M)^d) \times Q_{T,k}] \cap X$, for some $M > 0$ large enough independent of the point i. For any $i \in (Q^l_{T,d} \times Q_{T,k}) \cap X$, we write that

$$u_i - u_{i+Te_m} = \sum_{n=1}^{N_i} (u_{j_{n-1}} - u_{j_n}),$$

so that, due to (9) combined with the Hölder inequality, we have that

$$\begin{aligned}
|\tilde{u}^l - \tilde{u}^{l'}|^2 &\le \frac{1}{\#[(Q_{T,d} \times Q_{T,k}) \cap X]} \sum_{i \in (Q^l_{T,d} \times Q_{T,k}) \cap X} \Bigg| \sum_{n=1}^{N_i} (u_{j_{n-1}} - u_{j_n}) \Bigg|^2 \\
&\le \frac{1}{\#[(Q_{T,d} \times Q_{T,k}) \cap X]} \sum_{i \in (Q^l_{T,d} \times Q_{T,k}) \cap X} N_i \sum_{n=1}^{N_i} |u_{j_{n-1}} - u_{j_n}|^2 \\
&\le \frac{\max\{N_i : i \in (Q^l_{T,d} \times Q_{T,k}) \cap X\}}{\#[(Q_{T,d} \times Q_{T,k}) \cap X]} \sum_{i,j \in [(Q^l_{T,d} \cup Q^{l'}_{T,d} + (-M,M)^d) \times Q_{T,k}] \cap X} |u_i - u_j|^2,
\end{aligned}$$

where in the last inequality we have used the fact that $[(Q^l_{T,d} \cup Q^{l'}_{T,d} + (-M,M)^d) \times Q_{T,k}] \cap X$ contains the path γ joining i and $i + Te_m$ for all $i \in (Q^l_{T,d} \times Q_{T,k}) \cap X$. This proves the desired inequality. □

Remark 3.2. In order to reduce the number of parameters, we can choose $M = T$, up to substituting T with a multiple and taking a slightly larger M.

We point out that in the following (4) and (8) will be applied to functions $u : \varepsilon X \to \mathbb{R}$, where u_i stands for $u(\varepsilon i)$ as in the notation introduced above.

Now, we show a Poincaré-Wirtinger inequality. This will be used to recover information on the original functions u from their coarse-grained versions.

Proposition 3.3. (i) *There exists $C = C(X) > 0$ such that, for any $l \in \mathbb{Z}^d$,*

$$\sum_{i \in (Q^l_{T,d} \times Q_{T,k}) \cap X} |u_i - \tilde{u}^l|^2 \le C \sum_{i,j \in (Q^l_{T,d} \times Q_{T,k}) \cap X} |u_i - u_j|^2;$$

(ii) *there exist positive constants C and M such that, for any $l \in \mathbb{Z}^d$,*

$$\sum_{i\in(Q^l_{T,d}\times Q_{T,k})\cap X} |u_i - \tilde{u}^l|^2 \leq C \sum_{i,j\in[(Q^l_{T,d}+(-M,M)^d)\times Q_{T,k}]\cap X} a_{ij}|u_i - u_j|^2. \tag{10}$$

Proof. (i) Using definition (4) of $\tilde{u}^l$ and thanks to the Hölder inequality, we deduce that

$$\begin{aligned}\sum_{i\in(Q^l_{T,d}\times Q_{T,k})\cap X} |u_i - \tilde{u}^l|^2 &= \frac{1}{[\#((Q_{T,d}\times Q_{T,k})\cap X)]^2} \sum_{i\in(Q^l_{T,d}\times Q_{T,k})\cap X} \left| \sum_{j\in(Q^l_{T,d}\times Q_{T,k})\cap X} (u_i - u_j) \right|^2 \\ &\leq \frac{1}{\#[(Q_{T,d}\times Q_{T,k})\cap X]} \sum_{i,j\in(Q^l_{T,d}\times Q_{T,k})\cap X} |u_i - u_j|^2, \end{aligned} \tag{11}$$

which concludes the proof.

(ii) Since X is connected and due to the boundedness and periodicity properties of the coefficient a_{ij}, there exists $M > 0$ large enough such that if $i, j \in (Q^l_{T,d} \times Q_{T,k}) \cap X$, then there exists a path γ joining i and j which is contained in $[(Q^l_{T,d} + (-M, M)^d) \times Q_{T,k}] \cap X$. From (11), we deduce (10) as desired. □

4 A compactness result

In this section, we show that sequences with equi-bounded energy are compact in L^2 with limit in $H^1_{\text{loc}}(\mathbb{R}^d)$. More specifically, we show that from convergence (6) we obtain that

$$\int_{\widetilde{\Omega}_\varepsilon} |u_\varepsilon - u|^2 \chi_{(\cup_{i\in X}\varepsilon Q^{i^d}_d \times Q^{i^k}_k)} \to 0, \tag{12}$$

where we have set $i = (i^d, i^k) \in \mathbb{Z}^d \times \{0, \ldots, T-1\}^k$ and $\widetilde{\Omega}_\varepsilon$ is given by

$$\widetilde{\Omega}_\varepsilon := \bigcup_{l\in\mathcal{L}_\varepsilon} \varepsilon Q^l_{T,d} \times Q_{T,k}, \tag{13}$$

and $\mathcal{L}_\varepsilon := \{l \in \mathbb{Z}^d : \text{dist}(\varepsilon l, \partial\Omega) > 2\varepsilon\sqrt{d}T\}$. The next proposition provides a compactness result for $\overline{u}_\varepsilon$ given by (3) using the analysis of nearest-neighbour interactions in [3].

Proposition 4.1 (Compactness)**.** *Let Ω be an open set of $\mathbb{R}^d$ with Lipschitz boundary. Let u_ε be a family of functions defined on $(\Omega \times \varepsilon Q_{T,k}) \cap \varepsilon X$ such that*

$$\sum_{l\in\mathbb{Z}^d} \sum_{i\in(Q^l_{T,d}\times Q_{T,k})\cap X} \varepsilon^d |u^\varepsilon_i|^2 \leq C \tag{14}$$

for all $\varepsilon > 0$, where $u^\varepsilon_i = 0$ if $i \notin [(\frac{1}{\varepsilon}\Omega \cap Q^l_{T,d}) \times Q_{T,k}] \cap X$ and

$$\sum_{l,l'\in\mathcal{L}_\varepsilon,|l-l'|=1} \sum_{i,j\in[(Q^l_{T,d}\cup Q^{l'}_{T,d}+(-M,M)^d)\times Q_{T,k}]\cap X} \varepsilon^{d-2}|u^\varepsilon_i - u^\varepsilon_j|^2 \leq C \tag{15}$$

for all $\varepsilon > 0$. Then, up to a subsequence, the family $\overline{u}_\varepsilon$, given by (3)*, strongly converges in $L^2_{\text{loc}}(\Omega)$ to some $u \in H^1(\Omega)$.*

Proof. First, we show that $\overline{u}_\varepsilon$ weakly converges in L^2_{loc} to some u. Indeed, from (14), we deduce that the norm $\|\overline{u}_\varepsilon\|_{L^2(\Omega)}$ is bounded which implies the weak convergence of $\overline{u}_\varepsilon$. Moreover, thanks to assumption (15), an application of [3, Proposition 3.4] ensures that $u \in H^1(\Omega)$.

Now, we prove the strong convergence in $L^2_{\text{loc}}(\Omega)$ of $\overline{u}_\varepsilon$. To this end, the key tool is the Compactness Criterion by Fréchet and Kolmogorov (see, *e.g.* [19, Theorem 4.26]). In other words, we have to prove that, for any $\Omega'' \subset\subset \Omega'$ and for any $\eta > 0$, there exists $\delta > 0$, with $\mathrm{dist}(\Omega'', \mathbb{R}^d \setminus \Omega') > \delta$, such that for every $h \in \mathbb{R}^d$, with $|h| < \delta$, then

$$\|\tau_h \overline{u}_\varepsilon - \overline{u}_\varepsilon\|_{L^2(\Omega'')} < \eta, \tag{16}$$

where $\tau_h \overline{u}_\varepsilon(x) := \overline{u}_\varepsilon(x+h)$. Assume that $h = \lambda e_m$, for some $m = 1, \dots, d$. The inequality (16) for every $h \in \mathbb{R}^d$ is obtained by triangle inequality. Fix $\Omega'' \subset\subset \Omega'$ and set

$$\mathcal{I}_\varepsilon := \{l \in \mathcal{L}_\varepsilon : \Omega' \subset \cup_l \varepsilon Q^l_{T,d} \subset \Omega\}.$$

Take $x \in \varepsilon Q^l_{T,d}$. Hence, we have that $x \in \varepsilon Q^l_{T,d}$ and $(x+h) \in \varepsilon Q^{l'}_{T,d}$, for some $l, l' \in \mathcal{I}_\varepsilon$. By definition of $\overline{u}_\varepsilon$ given by (3), we deduce that

$$|\tau_h \overline{u}_\varepsilon(x) - \overline{u}_\varepsilon(x)|^2 = |\overline{u}_\varepsilon(x+h) - \overline{u}_\varepsilon(x)|^2 = |\tilde{u}^l_\varepsilon - \tilde{u}^{l'}_\varepsilon|^2 \tag{17}$$

Since l and l' are not necessarily such that $|l - l'| = 1$, we need to re-write the two-connectedness inequality in terms of non-neighbouring cubes. In order to show this, let $S_{ll'}$ be union of neighbouring cubes joining $\varepsilon Q^l_{T,d}$ and $\varepsilon Q^{l'}_{T,d}$ such that each two consecutive cubes have one face in common; *i.e.*, $S_{ll'} = \bigcup_{n=0}^{N_\varepsilon} \varepsilon Q^{l_n}_{T,d}$, with $|l_n - l_{n-1}| = 1$, $l_0 = l$ and $l_{N_\varepsilon} = l'$. Note that the number N_ε of the cubes $\varepsilon Q^l_{T,d}$ contained in stripes of cubes joining $\varepsilon Q^l_{T,d}$ and $\varepsilon Q^{l'}_{T,d}$ is of order $|h| T^{-1} \varepsilon^{-1}$. Hence, thanks to inequality (8), we deduce that

$$\begin{aligned}
|\tilde{u}^l_\varepsilon - \tilde{u}^{l'}_\varepsilon|^2 &= \left| \sum_{n=1}^{N_\varepsilon} (\tilde{u}^{l_n}_\varepsilon - \tilde{u}^{l_{n-1}}_\varepsilon) \right|^2 \\
&\le |h| T^{-1} \varepsilon^{-1} \sum_{n=1}^{N_\varepsilon} |\tilde{u}^{l_n}_\varepsilon - \tilde{u}^{l_{n-1}}_\varepsilon|^2 \\
&\le C |h| T^{-1} \varepsilon^{-1} \sum_{n=1}^{N} \sum_{i,j \in [(Q^{l_n}_{T,d} \cup Q^{l_{n-1}}_{T,d} + (-M,M)^d) \times Q_{T,k}] \cap X} |u^\varepsilon_i - u^\varepsilon_j|^2 \\
&\le C |h| T^{-1} \varepsilon^{-1} \sum_{i,j \in [(S_{ll'} + (-M,M)^d) \times Q_{T,k}] \cap X} |u^\varepsilon_i - u^\varepsilon_j|^2.
\end{aligned}$$

Plugging the above inequality in (17), we obtain that

$$\begin{aligned}
|\tau_h \overline{u}_\varepsilon(x) - \overline{u}_\varepsilon(x)|^2 &\le C |h| T^{-1} \varepsilon^{-1} \sum_{i,j \in [(S_{ll'} + (-M,M)^d) \times Q_{T,k}] \cap X} |u^\varepsilon_i - u^\varepsilon_j|^2 \\
&= C |h| T^{-1} \varepsilon^{-1} \sum_{i,j \in (S(x,h) \times Q_{T,k}) \cap X} |u^\varepsilon_i - u^\varepsilon_j|^2,
\end{aligned}$$

where we have set $S(x,h) := S_{ll'} + (-M,M)^d$ which depends on x and h. Now, an integration with respect to $x \in \varepsilon Q^l_{T,d}$ yields

$$\begin{aligned}
\int_{\varepsilon Q^l_{T,d}} |\tau_h \overline{u}_\varepsilon(x) - \overline{u}_\varepsilon(x)|^2 dx &\le C |h| T^{-1} \varepsilon^{-1} \int_{\varepsilon Q^l_{T,d}} \left(\sum_{i,j \in (S(x,h) \times Q_{T,k}) \cap X} |u^\varepsilon_i - u^\varepsilon_j|^2 \right) dx \\
&\le C |h| T^{d-1} \varepsilon^{d-1} \sum_{i,j \in (S(\varepsilon Q^l_{T,d}, h) \times Q_{T,k}) \cap X} |u^\varepsilon_i - u^\varepsilon_j|^2,
\end{aligned}$$

where we have used the fact that

$$S(x,h) \subset S(\varepsilon Q^l_{T,d}, h) := \bigcup \{S(x,h) : x \in \varepsilon Q^l_{T,d}\}.$$

Summing over $\mathcal{I}_\varepsilon$, we have that

$$\sum_{l \in \mathcal{I}_\varepsilon} \int_{\varepsilon Q^l_{T,d}} |\tau_h \overline{u}_\varepsilon(x) - \overline{u}_\varepsilon(x)|^2 dx \le C |h| T^{d-1} \varepsilon^{d-1} \sum_{l \in \mathcal{I}_\varepsilon} \sum_{i,j \in (S(\varepsilon Q^l_{T,d}, h) \times Q_{T,k}) \cap X} |u^\varepsilon_i - u^\varepsilon_j|^2,$$

which implies that

$$\int_{\Omega''} |\tau_h \overline{u}_\varepsilon(x) - \overline{u}_\varepsilon(x)|^2 dx \le \sum_{l\in\mathcal{I}_\varepsilon} \int_{\varepsilon Q^l_{T,d}} |\tau_h \overline{u}_\varepsilon(x) - \overline{u}_\varepsilon(x)|^2 dx \le C|h|,$$

where we have used assumption (15) and the fact that the number of indices l and l' such that $S(\varepsilon Q^l_{T,d}, h) \cap S(\varepsilon Q^{l'}_{T,d}, h) \neq \emptyset$ is of the order of the ratio between the size of $S(\varepsilon Q^l_{T,d}, h)$ and the size of $\varepsilon Q^l_{T,d}$; that is, of $|h|T^{-1}\varepsilon^{-1}$. This concludes the proof of (16).

Finally, applying the compactness criterion, it follows that, up to a subsequence, $\overline{u}_\varepsilon \to v$. Since we already know that $\overline{u}_\varepsilon \rightharpoonup u$, we conclude that $v = u$, which is the desired claim. □

The next proposition provides a convergence result in the sense of (12).

Proposition 4.2. *Let Ω be an open set of $\mathbb{R}^d$ with Lipschitz boundary. Let u_ε be a sequence of functions defined on εX such that*

$$\sup_{\varepsilon>0}\Bigg(\sum_{l\in\mathcal{L}_\varepsilon}\ \sum_{i\in(Q^l_{T,d}\times Q_{T,k})\cap X} \varepsilon^d |u^\varepsilon_i|^2 + F_\varepsilon(u_\varepsilon)\Bigg) \le C. \tag{18}$$

Then, up to a subsequence, we have that

$$\int_{\widetilde{\Omega}_\varepsilon} |u_\varepsilon - u|^2 \chi_{\cup_{i\in X}\varepsilon Q^{i^d}_d \times Q^{i^k}_{T,k}} \to 0,$$

where $u \in H^1(\Omega)$ is the strong limit in $L^2_{\rm loc}(\Omega)$ of the sequence $\overline{u}_\varepsilon$ and $\widetilde{\Omega}_\varepsilon$ is given by (13).

Proof. Set $x = (x^d, x^k) \in \varepsilon Q^l_{T,d} \times Q_{T,k}$ and recall that u_ε is defined on $\varepsilon Q^l_{T,d} \times \varepsilon Q_{T,k}$. Hence,

$$\begin{aligned}\int_{\widetilde{\Omega}_\varepsilon} |(u_\varepsilon - u)(x)\chi_{\cup_{i\in X}\varepsilon Q^{i^d}_d \times Q^{i^k}_k}(x)|^2 dx &\le \int_{\widetilde{\Omega}_\varepsilon} |(u_\varepsilon - \overline{u}_\varepsilon)(x)\chi_{\cup_{i\in X}\varepsilon Q^{i^d}_d \times Q^{i^k}_k}(x)|^2 dx \\ &\quad + \int_{\widetilde{\Omega}_\varepsilon} |(\overline{u}_\varepsilon - u)(x)\chi_{\cup_{i\in X}\varepsilon Q^{i^d}_d \times Q^{i^k}_k}(x)|^2 dx.\end{aligned} \tag{19}$$

With a slight abuse of notation here we let $\overline{u}_\varepsilon(x) = \overline{u}_\varepsilon(x^d)$. From Proposition 4.1, we know that $\overline{u}_\varepsilon$ strongly converges to u in $L^2_{\rm loc}(\Omega)$, so that the second integral in (19) vanishes as $\varepsilon \to 0$.

In order to estimate the first integral of (19), the key tool is the Poincaré-Wirtinger inequality given by (10). Indeed, due to the fact that u_ε is constant on $\varepsilon Q^l_d \times \varepsilon Q_k$ and $\overline{u}_\varepsilon$ is constant on $\varepsilon Q^l_{T,d} \times Q_k$, we deduce that

$$\begin{aligned}\int_{\tilde{\Omega}_\varepsilon} |(u_\varepsilon - \overline{u}_\varepsilon)(x)\chi_{\cup_{i\in X}\varepsilon Q^{i^d}_d \times Q^{i^k}_k}(x)|^2 dx &= \varepsilon^d \sum_{l\in\mathcal{L}_\varepsilon}\ \sum_{i\in(Q^l_{T,d}\times Q_{T,k})\cap X} |u^\varepsilon_i - \tilde{u}^l_\varepsilon|^2 \\ &\le \varepsilon^d \sum_{l\in\mathcal{L}_\varepsilon}\ \sum_{i,j\in[(Q_{T,d}+(-T,T)^d)\times Q_{T,k}]\cap X} a_{ij}|u^\varepsilon_i - u^\varepsilon_j|^2 \\ &\le \varepsilon^2 F_\varepsilon(u_\varepsilon).\end{aligned}$$

From this, combined with assumption (18), we have that also the first integral of (19) goes to 0 as $\varepsilon \to 0$, which concludes the proof. □

5 Treatment of boundary data

In this section we prove a technical result which allows to match boundary conditions. The proof is close in spirit to the method introduced by De Giorgi (see [21], [14, Chapter 11], [20, Chapter 18], and [15] in the context of dimension reduction). For future reference we prove it in a general form.

For any $u \in H^1(\Omega)$, we define the sequence v_ε on εX by

$$v_i^\varepsilon = v_\varepsilon(\varepsilon i) := \fint_{\varepsilon i^d + \varepsilon Q_d} u(x)dx. \tag{20}$$

We have that v_ε converges to u with respect to convergence (6). For any bounded open set A and for $\delta > 0$, we define $A(\delta) := \{x \in A : \mathrm{dist}(x, \partial A) > \delta\}$.

Lemma 5.1. *Let A be a bounded and open set of Ω with Lipschitz boundary. Let u_ε be a sequence converging to $u \in H^1(\Omega)$ with respect to convergence (6). For any $\delta > 0$, there exists a sequence w_ε converging to u with respect convergence (6) such that*

$$\begin{aligned} w_\varepsilon &= u_\varepsilon, && \text{if } i \in (A(2\delta) \times Q_{T,k}) \cap X, \\ w_\varepsilon &= v_\varepsilon, && \text{if } i \in (A \setminus A(\delta) \times Q_{T,k}) \cap X, \end{aligned}$$

and

$$\limsup_{\varepsilon \to 0} (F_\varepsilon(w_\varepsilon) - F_\varepsilon(u_\varepsilon)) \le o(1) \tag{21}$$

as $\delta \to 0$.

Proof. Fixed $N \in \mathbb{N}$ and $\delta \in (0, 1/4)$. For $h \in \{0, \dots, N\}$, we set

$$A_h := \Big\{ x \in A : \mathrm{dist}(x, A(\delta)) < h\frac{\delta}{N} \Big\}.$$

For $h \in \{0, \cdots, N-1\}$, let ϕ_d^h be a cut-off function between A_h and A_{h+1} with $|\nabla \phi_d^h| \le 2N/\delta$ and let w_ε be a function defined by

$$w_i^\varepsilon = w_\varepsilon(\varepsilon i^d, \varepsilon i^k) := \phi_d^h(\varepsilon i^d) u_i^\varepsilon + (1 - \phi_d^h(\varepsilon i^d)) v_i^\varepsilon. \tag{22}$$

Since both u_ε and v_ε converge to u with respect to convergence given by (6), we also deduce that w_ε converges to u with respect to (6). By adding and subtracting the term $\phi_d^h(\varepsilon i^d) u_j^\varepsilon + (1 - \phi_d^h(\varepsilon i^d)) v_j^\varepsilon$, we get that

$$w_i^\varepsilon - w_j^\varepsilon = \phi_d^h(\varepsilon i^d)(u_i^\varepsilon - u_j^\varepsilon) + (1 - \phi_d^h(\varepsilon i^d))(v_i^\varepsilon - v_i^\varepsilon) + (\phi_d^h(\varepsilon i^d) - \phi_d^h(\varepsilon j^d))(u_j^\varepsilon - v_j^\varepsilon). \tag{23}$$

For $h \in \{1, \dots, N-2\}$, we set

$$S_h^d := A_{h+1} \setminus A_h,$$

so that $A = A_h \cup A \setminus \overline{A}_{h+1} \cup S_h^d$. In order to estimate the energy

$$\sum_{i,j \in (\frac{1}{\varepsilon} A \times Q_{T,k}) \cap X} \varepsilon^{d-2} a_{ij} (w_i^\varepsilon - w_j^\varepsilon)^2,$$

we separately evaluate the following cases

i) $i, j \in (\frac{1}{\varepsilon} A_h \times Q_{T,k}) \cap X$;

ii) $i, j \in (\frac{1}{\varepsilon}(A \setminus \overline{A}_{h+1}) \times Q_{T,k}) \cap X$;

iii) $i \in (\frac{1}{\varepsilon} S_h^d \times Q_{T,k}) \cap X$ and $j \in (\frac{1}{\varepsilon} A \times Q_{T,k}) \cap X$;

iv) $i \in (\frac{1}{\varepsilon}(A_h \cup (A \setminus \overline{A}_{h+1})) \times Q_{T,k}) \cap X$ and $j \in (\frac{1}{\varepsilon} S_h^d \times Q_{T,k}) \cap X$;

v) $i \in (\frac{1}{\varepsilon} A_h \times Q_{T,k}) \cap X$ and $j \in (\frac{1}{\varepsilon}(A \setminus \overline{A}_{h+1}) \times Q_{T,k}) \cap X$;

vi) $i \in (\frac{1}{\varepsilon}(A \setminus \overline{A}_{h+1}) \times Q_{T,k}) \cap X$ and $j \in (\frac{1}{\varepsilon} A_h \times Q_{T,k}) \cap X$

as follows

i) In view of definition (22), we deduce that

$$\sum_{i,j\in(\frac{1}{\varepsilon}A_h\times Q_{T,k})\cap X}\varepsilon^{d-2}a_{ij}(w_i^\varepsilon-w_j^\varepsilon)^2=\sum_{i,j\in(\frac{1}{\varepsilon}A_h\times Q_{T,k})\cap X}\varepsilon^{d-2}a_{ij}(u_i^\varepsilon-u_j^\varepsilon)^2$$
$$\leq\sum_{i,j\in(\frac{1}{\varepsilon}A\times Q_{T,k})\cap X}\varepsilon^{d-2}a_{ij}(u_i^\varepsilon-u_j^\varepsilon)^2. \quad (24)$$

ii) We have that

$$\sum_{i,j\in(\frac{1}{\varepsilon}(A\setminus\overline{A}_{h+1})\times Q_{T,k})\cap X}\varepsilon^{d-2}a_{ij}(w_i^\varepsilon-w_j^\varepsilon)^2=\sum_{i,j\in(\frac{1}{\varepsilon}(A\setminus\overline{A}_{h+1})\times Q_{T,k})\cap X}\varepsilon^{d-2}a_{ij}(v_i^\varepsilon-v_j^\varepsilon)^2$$
$$\leq\sum_{i,j\in(\frac{1}{\varepsilon}(A\setminus\overline{A(\delta)})\times Q_{T,k})\cap X}\varepsilon^{d}a_{ij}(v_i^\varepsilon-v_j^\varepsilon)^2. \quad (25)$$

In view of definition of v_i^ε given by (20) and since $\varepsilon i^d+\varepsilon Q_d=\varepsilon(i^d-j^d)+\varepsilon j^d+\varepsilon Q_d$, we deduce that

$$|v_i^\varepsilon-v_j^\varepsilon|^2=\left|\fint_{\varepsilon i^d+\varepsilon Q_d}u(x)dx-\fint_{\varepsilon j^d+\varepsilon Q_d}u(x)dx\right|^2$$
$$=\left|\fint_{\varepsilon j^d+\varepsilon Q_d}(u(x+\varepsilon(i^d-j^d))-u(x))dx\right|^2. \quad (26)$$

Since $u\in H^1(\Omega)$, we have that

$$u(x+\varepsilon(i^d-j^d))-u(x)=\int_0^1\frac{\partial u}{\partial t}(x+\varepsilon t(i^d-j^d))dt$$
$$=\int_0^1\nabla u(x+\varepsilon t(i^d-j^d))\cdot\varepsilon(i^d-j^d)dt.$$

This, combined with (26) and the Fubini theorem, implies that

$$|v_i^\varepsilon-v_j^\varepsilon|^2=\left|\fint_{\varepsilon j^d+\varepsilon Q_d}\int_0^1\nabla u(x+\varepsilon t(i^d-j^d))\cdot\varepsilon(i^d-j^d)dtdx\right|^2$$
$$=\frac{1}{\varepsilon^d}\left|\int_0^1\int_{\varepsilon j^d+\varepsilon Q_d}\nabla u(x+\varepsilon t(i^d-j^d))\cdot\varepsilon(i^d-j^d)dxdt\right|^2$$
$$\leq\varepsilon^{2-d}|i^d-j^d|^2\int_0^1\int_{\varepsilon j^d+\varepsilon Q_d}|\nabla u(x+\varepsilon t(i^d-j^d))|^2dxdt$$
$$\leq\varepsilon^{2-d}T^2\int_0^1\int_{\varepsilon i^d+\varepsilon[0,T+1)^d}|\nabla u(x))|^2dxdt, \quad (27)$$

where in the last inequality we have used the fact that the nodes i and j interact at most at distance T. In view of the assumption of finite range along with estimate above, from (25), it follows that

$$\sum_{i,j\in(\frac{1}{\varepsilon}(A\setminus\overline{A}_{h+1})\times Q_{T,k})\cap X}\varepsilon^{d-2}a_{ij}(w_i^\varepsilon-w_j^\varepsilon)^2\leq C\sum_{i\in(\frac{1}{\varepsilon}(A\setminus\overline{A(\delta)})\times Q_{T,k})\cap X}\int_{\varepsilon i^d+\varepsilon[0,T-1)^d}|\nabla u(x)|^2dx$$
$$\leq C\int_{A\setminus A(2\delta)}|\nabla u(x)|^2dx, \quad (28)$$

where the constant C is due to the fact that a fixed node $i\in\frac{1}{\varepsilon}(A\setminus\overline{A(\delta)})\times Q_{T,k})\cap X$ interacts with a finite number of nodes $j\in\frac{1}{\varepsilon}(A\setminus\overline{A(\delta)})\times Q_{T,k})\cap X$.

iii) First note that due to the assumption of finite range, if $\varepsilon i^d \in S_h^d$, then $\varepsilon j^d \in \widehat{S}_h^d := S_{h-1}^d \cup S_h^d \cup S_{h+1}^d$. This combined with (23) and the Jensen inequality implies that

$$\begin{aligned}
&\sum_{\substack{i\in(\frac{1}{\varepsilon}S_h^d\times Q_{T,k})\cap X\\ j\in(\frac{1}{\varepsilon}A\times Q_{T,k})\cap X}} \varepsilon^{d-2}a_{ij}(w_i^\varepsilon - w_j^\varepsilon)^2 = \sum_{\substack{i\in(\frac{1}{\varepsilon}S_h^d\times Q_{T,k})\cap X\\ j\in(\frac{1}{\varepsilon}\widehat{S}_h^d\times Q_{T,k})\cap X}} \varepsilon^{d-2}a_{ij}(w_i^\varepsilon - w_j^\varepsilon)^2\\
&\leq C \sum_{\substack{i\in(\frac{1}{\varepsilon}S_h^d\times Q_{T,k})\cap X\\ j\in(\frac{1}{\varepsilon}\widehat{S}_h^d\times Q_{T,k})\cap X}} \varepsilon^{d-2}a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2 + C \sum_{\substack{i\in(\frac{1}{\varepsilon}S_h^d\times Q_{T,k})\cap X\\ j\in(\frac{1}{\varepsilon}\widehat{S}_h^d\times Q_{T,k})\cap X}} \varepsilon^{d-2}a_{ij}(v_i^\varepsilon - v_j^\varepsilon)^2\\
&\quad + C \sum_{\substack{i\in(\frac{1}{\varepsilon}S_h^d\times Q_{T,k})\cap X\\ j\in(\frac{1}{\varepsilon}\widehat{S}_h^d\times Q_{T,k})\cap X}} \varepsilon^{d-2}a_{ij}(\phi_d^h(\varepsilon i^d) - \phi_d^h(\varepsilon j^d))^2(u_j^\varepsilon - v_j^\varepsilon)^2. \qquad (29)
\end{aligned}$$

Due to the fact that $|\nabla\phi_d^h| \leq 2N/\delta$, the last integral in (29) can be estimated as follows

$$\begin{aligned}
&\sum_{\substack{i\in(\frac{1}{\varepsilon}S_h^d\times Q_{T,k})\cap X\\ j\in(\frac{1}{\varepsilon}\widehat{S}_h^d\times Q_{T,k})\cap X}} \varepsilon^{d-2}a_{ij}(\phi_d^h(\varepsilon i^d) - \phi_d^h(\varepsilon j^d))^2(u_j^\varepsilon - v_j^\varepsilon)^2\\
&\leq \frac{N^2}{\delta^2} \sum_{\substack{i\in(\frac{1}{\varepsilon}\widehat{S}_h^d\times Q_{T,k})\cap X\\ j\in(\frac{1}{\varepsilon}\widehat{S}_h^d\times Q_{T,k})\cap X}} \varepsilon^{d-2}a_{ij}(u_j^\varepsilon - v_j^\varepsilon)^2 \leq 3\frac{N^2}{\delta^2} \sum_{i,j\in(\frac{1}{\varepsilon}A\times Q_{T,k})\cap X} \varepsilon^{d}a_{ij}(u_j^\varepsilon - v_j^\varepsilon)^2.
\end{aligned}$$

In order to estimate the first two integrals in (29), we may choose $h \in \{1, \dots, N-2\}$ such that

$$\begin{aligned}
&\sum_{\substack{i\in(\frac{1}{\varepsilon}S_h^d\times Q_{T,k})\cap X\\ j\in(\frac{1}{\varepsilon}\widehat{S}_h^d\times Q_{T,k})\cap X}} \varepsilon^{d-2}a_{ij}\left[(u_i^\varepsilon - u_j^\varepsilon)^2 + (v_i^\varepsilon - v_j^\varepsilon)^2\right]\\
&\leq \frac{1}{N-2} \sum_{i,j\in(\frac{1}{\varepsilon}A\times Q_{T,k})\cap X} \varepsilon^{d-2}a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2 + \frac{1}{N-2} \sum_{i,j\in(\frac{1}{\varepsilon}A\times Q_{T,k})\cap X} \varepsilon^{d-2}a_{ij}(v_i^\varepsilon - v_j^\varepsilon)^2\\
&\leq \frac{1}{N-2} \sum_{i,j\in(\frac{1}{\varepsilon}A\times Q_{T,k})\cap X} \varepsilon^{d-2}a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2 + \frac{C}{N-2}\int_{A\setminus A(2\delta)} |\nabla u(x)|^2 dx,
\end{aligned}$$

where we have used (27) and the assumption of finite range. This, combined with (29), leads us to

$$\begin{aligned}
\sum_{\substack{i\in(\frac{1}{\varepsilon}S_h^d\times Q_{T,k})\cap X\\ j\in(\frac{1}{\varepsilon}A\times Q_{T,k})\cap X}} \varepsilon^{d-2}a_{ij}(w_i^\varepsilon - w_j^\varepsilon)^2 &\leq C\frac{1}{N-2} \sum_{i,j\in(\frac{1}{\varepsilon}A\times Q_{T,k})\cap X} \varepsilon^{d-2}a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2\\
&+ \frac{C}{N-2}\int_{A\setminus A(2\delta)} |\nabla u(x)|^2 dx + C\frac{N^2}{\delta^2} \sum_{\substack{i\in(\frac{1}{\varepsilon}S_h^d\times Q_{T,k})\cap X\\ j\in(\frac{1}{\varepsilon}A\times Q_{T,k})\cap X}} \varepsilon^{d}a_{ij}(u_j^\varepsilon - v_j^\varepsilon)^2. \qquad (30)
\end{aligned}$$

iv) Note that

$$\sum_{\substack{i\in(\frac{1}{\varepsilon}(A_h\cup A\setminus\overline{A}_{h+1})\times Q_{T,k})\cap X\\ j\in(\frac{1}{\varepsilon}S_h^d\times Q_{T,k})\cap X}} \varepsilon^{d-2}a_{ij}(w_i^\varepsilon - w_j^\varepsilon)^2 \leq \sum_{\substack{i\in(\frac{1}{\varepsilon}A\times Q_{T,k})\cap X\\ j\in(\frac{1}{\varepsilon}S_h^d\times Q_{T,k})\cap X}} \varepsilon^{d-2}a_{ij}(w_i^\varepsilon - w_j^\varepsilon)^2,$$

so that, the same argument as for iii) can be performed, obtaining estimate (30).

In view of the finite-range assumption, the points belonging to sets of items (v) and (vi) do not have any interaction since $\delta/N >> \varepsilon T$.

Gathering estimates (24), (28) and (30), we obtain that, for $h \in \{1, \dots, N-2\}$,

$$\begin{aligned}\sum_{i,j\in(\frac{1}{\varepsilon}A\times Q_{T,k})\cap X} \varepsilon^{d-2} a_{ij}(w_i^\varepsilon - w_j^\varepsilon)^2 \leq & \sum_{i,j\in(\frac{1}{\varepsilon}A\times Q_{T,k})\cap X} \varepsilon^{d-2} a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2 \\ & + C\int_{A\setminus A(2\delta)} |\nabla u(x)|^2 dx + C\frac{1}{N-2} \sum_{i,j\in(\frac{1}{\varepsilon}A\times Q_{T,k})\cap X} \varepsilon^{d-2} a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2 \\ & + \frac{C}{N-2}\int_{A\setminus A(2\delta)} |\nabla u(x)|^2 dx + C\frac{N^2}{\delta^2} \sum_{\substack{i\in(\frac{1}{\varepsilon}S_h^d\times Q_{T,k})\cap X \\ j\in(\frac{1}{\varepsilon}A\times Q_{T,k})\cap X}} \varepsilon^{d} a_{ij}(u_j^\varepsilon - v_j^\varepsilon)^2. \end{aligned} \tag{31}$$

Note that the last sum vanishes as $\varepsilon \to 0$ since both u_ε and v_ε converge to u with respect to convergence (6). Hence, taking the limit as $\varepsilon \to 0$ of (31), we obtain that

$$\begin{aligned}\limsup_{\varepsilon\to 0} F_\varepsilon(w_\varepsilon) - F_\varepsilon(u_\varepsilon) \leq & C\int_{A\setminus A(2\delta)} |\nabla u(x)|^2 dx + \frac{C}{N-2}\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \\ & + \frac{C}{N-2}\int_{A\setminus A(2\delta)} |\nabla u(x)|^2 dx.\end{aligned}$$

Letting first $N \to \infty$ and then $\delta \to 0$, we get inequality (41) as desired. □

6 Homogenization

This section is devoted to the proof of Theorem 2.2. We adopt a direct approach proving separately the lower and the upper bound inequalities for the family F_ε given by (2).

6.1 Proof of the lower bound

We prove the lower-bound inequality for the family F_ε using the *blow-up method* introduced by Fonseca and Müller [22] (see also [16]).

Let u_ε be a sequence with equi-bounded energy $F_\varepsilon(u_\varepsilon)$ and such that u_ε converge to $u \in H^1(\Omega)$. Let the sequence of positive measures λ_ε be defined as

$$\lambda_\varepsilon := \sum_{i\in(\frac{1}{\varepsilon}\Omega\times Q_{T,k})\cap X} \Bigg(\sum_{j\in(\frac{1}{\varepsilon}\Omega\times Q_{T,k})\cap X} \varepsilon^{d-2} a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2 \Bigg) \delta_{\varepsilon i},$$

where δ_x is the Dirac measure concentrated at x. The d-dimensional measure μ_ε is defined by

$$\mu_\varepsilon(B) := \lambda_\varepsilon(B\times \varepsilon Q_{T,k}) = \sum_{i\in(\frac{1}{\varepsilon}B\times Q_{T,k})\cap X} \sum_{j\in(\frac{1}{\varepsilon}\Omega\times Q_{T,k})\cap X} \varepsilon^{d-2} a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2$$

for Borel sets B of $\mathbb{R}^d$. Note that $\mu_\varepsilon(B)$ takes into account interactions between the nodes with projection in B and the ones in all X, but, in view of the equi-boundedness of $\mathcal{E}$, which is a finite-range assumption, we can limit the interactions between the nodes with projection in B and those with projection in an εR-neighbourhood of B.

Since $\mu_\varepsilon(\Omega) = F_\varepsilon(u_\varepsilon)$ and thanks to the equi-boundedness of $F_\varepsilon(u_\varepsilon)$, the measures μ_ε are also equi-bounded, so that, up to subsequences, we deduce that

$$\mu_\varepsilon \overset{*}{\rightharpoonup} \mu,$$

where μ is a d-dimensional positive measure on Ω. The Radon-Nikodym decomposition of the limit measure μ with respect to the d-dimensional Lebesgue measure $\mathcal{L}^d$ enables us to write that

$$\mu = \frac{d\mu}{dx}\mathcal{L}^d + \mu^s,$$

with $\mu^s \perp \mathcal{L}^d$. Note that the positiveness of μ ensures that its singular part μ^s is positive as well.

Now, we perform a local analysis. Let $x_0 \in \Omega$ be a Lebesgue point for μ with respect to $\mathcal{L}^d$; *i.e.*,

$$\frac{d\mu}{dx}(x_0) = \lim_{\rho\to 0} \frac{\mu(Q_{\rho,d}(x_0))}{\mathcal{L}^d(Q_{\rho,d}(x_0))} = \lim_{\rho\to 0} \frac{\mu(Q_{\rho,d}(x_0))}{\rho^d}, \tag{32}$$

with $Q_{\rho,d}(x_0) := x_0 + [0,\rho)^d$. Thanks to the Besicovitch Derivation Theorem, $\mathcal{L}^d$-almost every $x_0 \in \Omega$ is a Lebesgue point for μ with respect to $\mathcal{L}^d$. Moreover, in view of [31, Theorem 3.4.2], we have that, up to a set of zero Lebesgue measure, x_0 is a point such that

$$\lim_{\rho\to 0} \frac{1}{\rho}\left(\frac{1}{\rho^d}\int_{Q_{\rho,d}(x_0)} |u(x) - u(x_0) - \nabla u(x_0)\cdot(x-x_0)|^2 dx\right)^{1/2} = 0. \tag{33}$$

In other words, performing the change of variables $x = \rho y + x_0$ in the above integral, we have that

$$\frac{u(\rho y + x_0) - u(x_0)}{\rho} \to \nabla u(x_0)\cdot y \quad \text{in } L^2(Q_d).$$

For all $\rho \to 0$ but a countable set, we have that $\mu(\partial Q_{\rho,d}(x_0)) = 0$ and hence for such ρ we have that

$$\mu(Q_{\rho,d}(x_0)) = \lim_{\varepsilon\to 0} \mu_\varepsilon(Q_{\rho,d}(x_0)). \tag{34}$$

Therefore, from (32), it follows that

$$\frac{d\mu}{dx}(x_0) = \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \frac{\mu_\varepsilon(Q_{\rho,d}(x_0))}{\rho^d}.$$

Now, we perform the blow-up argument. Since $x_0 \in \Omega$ is a Lebesgue point and due to a diagonalization argument on (32) and (34), there exists a sequence $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$ such that $\rho_\varepsilon >> \varepsilon$ and the following equalities

$$\frac{d\mu}{dx}(x_0) = \lim_{\varepsilon\to 0} \frac{\mu_\varepsilon(Q_{\rho_\varepsilon,d}(x_0))}{\rho_\varepsilon^d}, \tag{35}$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{\rho_\varepsilon}\int_{Q_{\rho_\varepsilon,d}(x_0)\times Q_{T,k}} |u_\varepsilon - u|(x)\chi_{\cup_{i\in X}\varepsilon Q_d^{i^d}\times Q_k^{i^k}}(x)dx = 0, \tag{36}$$

hold. Thanks to the link between the measure μ_ε and the energy F_ε, equality (35) can be re-written as

$$\frac{d\mu}{dx}(x_0) = \lim_{\varepsilon\to 0}\frac{1}{\rho_\varepsilon^d} \sum_{i\in(Q_{\frac{\rho_\varepsilon}{\varepsilon},d}(\frac{x_0}{\varepsilon})\times Q_{T,k})\cap X} \;\sum_{j\in(\frac{1}{\varepsilon}\Omega\times Q_{T,k})\cap X} \varepsilon^{d-2} a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2.$$

Now, the aim is to estimate the limit above. First, note that since the coefficients a_{ij} are positive, we can consider only interactions taking place between nodes inside the cube $Q_{\frac{\rho_\varepsilon}{\varepsilon},d}(x_0/\varepsilon)\times Q_{T,k}$, so that

$$\frac{\mu_\varepsilon(Q_{\rho_\varepsilon,d}(x_0))}{\rho_\varepsilon^d} \geq \frac{1}{\rho_\varepsilon^d} \sum_{i,j\in(Q_{\frac{\rho_\varepsilon}{\varepsilon},d}(\frac{x_0}{\varepsilon})\times Q_{T,k})\cap X} \varepsilon^{d-2} a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2. \tag{37}$$

We need to modify u_ε in order to define a function v_ε converging to the affine function $\nabla u(x_0)\cdot x^d$ in $L^2(Q_d)$. To that end, let $\eta_\varepsilon = \frac{\varepsilon}{\rho_\varepsilon}$, and let $X_{\eta_\varepsilon,\varepsilon}$ be the set X rescaled to $\eta_\varepsilon\mathbb{Z}^d \times \varepsilon\{0,\dots,T-1\}^k$. We define v_ε^ρ on $(Q_d \times Q_{T,k})\cap X_{\eta_\varepsilon,\varepsilon}$ by

$$v_\varepsilon^\rho(\eta_\varepsilon i^d, \varepsilon i^k) := \frac{u_\varepsilon(\varepsilon i^d + x_0, \varepsilon i^k) - u(x_0)}{\rho_\varepsilon}, \tag{38}$$

where u_ε is defined on $(Q_{\rho_\varepsilon,d}(x_0)\times Q_{T,k})\cap \varepsilon X$. Note that since u_ε is a function in $\mathcal{C}_\varepsilon(\Omega)$, v_ε^ρ can be identified with a piecewise-constant function on $\eta_\varepsilon Q_d^{i^d}\times \varepsilon Q_k^{i^k}$ if $(i^d, i^k)\in X$. For $x = (x^d, x^k)\in\mathbb{R}^d\times Q_{T,k}$, we set $w_0(x) := \nabla u(x_0)\cdot x^d$ and we show that

$$\lim_{\varepsilon\to 0}\int_{Q_d\times Q_{T,k}} |v_\varepsilon^\rho(x) - w_0(x)|^2 \chi_{\cup_{i\in X}\eta_\varepsilon Q_d^{i^d}\times Q_k^{i^k}}(x)dx = 0. \tag{39}$$

To this end, we introduce the function u_0 given by $u_0(x^d) := u(x_0) + w_0(x)$. Hence,

$$u(x_0) = u_0(\rho x^d) - \rho \nabla u(x_0) \cdot x^d = u_0(\rho x^d) - \rho w_0(x).$$

This, combined with (38), implies that

$$\begin{aligned}
&\int_{Q_d \times Q_{T,k}} |v_\varepsilon^\rho(x) - w_0(x)|^2 \chi_{\cup_{i\in X} \eta_\varepsilon Q_d^{i^d} \times Q_k^{i^k}}(x)dx \\
&\quad = \int_{Q_d \times Q_{T,k}} \left| \frac{u_\varepsilon(\rho_\varepsilon x^d + x_0, \varepsilon x^k) - u(x_0)}{\rho_\varepsilon} - w_0(x) \right|^2 \chi_{\cup_{i\in X} \eta_\varepsilon Q_d^{i^d} \times Q_k^{i^k}}(x)dx \\
&\quad = \int_{Q_d \times Q_{T,k}} \left| \frac{u_\varepsilon(\rho_\varepsilon x^d + x_0, \varepsilon x^k) - u_0(\rho_\varepsilon x^d)}{\rho_\varepsilon} \right|^2 \chi_{\cup_{i\in X} \eta_\varepsilon Q_d^{i^d} \times Q_k^{i^k}}(x)dx \\
&\quad \leq \int_{Q_d \times Q_{T,k}} \left| \frac{u_\varepsilon(\rho_\varepsilon x^d + x_0, \varepsilon x^k) - u(\rho_\varepsilon x^d + x_0)}{\rho_\varepsilon} \right|^2 \chi_{\cup_{i\in X} \eta_\varepsilon Q_d^{i^d} \times Q_k^{i^k}}(x)dx \\
&\qquad + \int_{Q_d \times Q_{T,k}} \left| \frac{u(\rho_\varepsilon x^d + x_0) - u_0(\rho_\varepsilon x^d)}{\rho_\varepsilon} \right|^2 \chi_{\cup_{i\in X} \eta_\varepsilon Q_d^{i^d} \times Q_k^{i^k}}(x)dx. \qquad (40)
\end{aligned}$$

The first integral in (40) goes to 0 as $\varepsilon \to 0$. Indeed, due to the change of variables $y^d = \rho_\varepsilon x^d + x_0$, we deduce that

$$\begin{aligned}
&\int_{Q_d \times Q_{T,k}} \left| \frac{u(\rho_\varepsilon x^d + x_0) - u_0(\rho_\varepsilon x^d)}{\rho_\varepsilon} \right|^2 \chi_{\cup_{i\in X} \eta_\varepsilon Q_d^{i^d} \times Q_k^{i^k}}(x)dx^d dx^k \\
&\qquad = \frac{1}{\rho_\varepsilon^{d+2}} \int_{Q_{\rho_\varepsilon,d}(x_0) \times Q_{T,k}} |u_\varepsilon(y^d, \varepsilon x^k) - u(y^d)|^2 \chi_{\cup_{i\in X} \varepsilon Q_d^{i^d} \times Q_k^{i^k}}(x)dy^d dx^k,
\end{aligned}$$

which vanishes as $\varepsilon \to 0$ thanks to (36). We evaluate the second integral in (40). Using again the change of variables $y^d = \rho_\varepsilon x^d + x_0$ and the definition of u_0, we have that

$$\begin{aligned}
&\int_{Q_{\rho_\varepsilon,d}(x_0) \times Q_{T,k}} |u(y^d, \varepsilon x^k) - u(y^d)|^2 \chi_{\cup_{i\in X} \varepsilon Q_d^{i^d} \times Q_k^{i^k}}(x)dy^d dx^k \\
&\qquad \leq T^k \frac{1}{\rho_\varepsilon^2} \frac{1}{\rho_\varepsilon^d} \int_{Q_{\rho_\varepsilon,d}(x_0)} |u(y^d) - u_0(y - x_0)|^2 dy^d \\
&\qquad = T^k \frac{1}{\rho_\varepsilon^2} \frac{1}{\rho_\varepsilon^d} \int_{Q_{\rho_\varepsilon,d}(x_0)} |u(y^d) - u(x_0) - \nabla u(x_0) \cdot (y - x_0)|^2 dy^d.
\end{aligned}$$

Thanks to (33), it follows that also the integral above vanishes as $\varepsilon \to 0$ so that we can conclude that (39) holds. Set

$$v_\varepsilon(\eta_\varepsilon i^d, \varepsilon i^k) := v_\varepsilon^{\rho_\varepsilon}(\eta_\varepsilon i^d, \varepsilon i^k).$$

Now, using Lemma 5.1, we may modify the sequence v_ε to get a new sequence $\tilde{v}_\varepsilon$ which is equal to $\nabla u(x_0) \cdot \eta_\varepsilon i^d$ near the boundary $(\partial Q_d \times Q_{T,k}) \cap X_{\eta_\varepsilon}$, where X_{η_ε} is the set X rescaled to $\eta_\varepsilon \mathbb{Z}^d \times \{0, \dots, T-1\}^k$, and

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \sum_{(\eta_\varepsilon i^d, i^k), (\eta_\varepsilon j^d, j^k) \in (Q_d \times Q_{T,k}) \cap X_{\eta_\varepsilon}} \eta^{d-2} a_{ij} (\tilde{v}_\varepsilon(\eta_\varepsilon i^d, \varepsilon i^k) - (\tilde{v}_\varepsilon(\eta_\varepsilon j^d, \varepsilon j^k)) \\
&\leq \limsup_{\varepsilon\to 0} \sum_{(\eta_\varepsilon i^d, i^k), (\eta_\varepsilon j^d, j^k) \in (Q_d \times Q_{T,k}) \cap X_{\eta_\varepsilon}} \eta^{d-2} a_{ij} (v_\varepsilon(\eta_\varepsilon i^d, \varepsilon i^k) - (v_\varepsilon(\eta_\varepsilon j^d, \varepsilon j^k)) + o(1).
\end{aligned} \qquad (41)$$

In order to simplify the notation, we may assume that $x_0 \in \varepsilon T\mathbb{Z}^d$ so that we avoid the translation of the coefficients a_{ij}. In view of (37) and thanks estimate (41), we have that

$$\begin{aligned}
\frac{d\mu}{dx}(x_0) &\geq \limsup_{\varepsilon\to 0} \sum_{(\eta_\varepsilon i^d, i^k),(\eta_\varepsilon j^d, j^k)\in(Q_d(\frac{x_0}{\rho_\varepsilon})\times Q_{T,k})\cap X_{\eta_\varepsilon}} \eta_\varepsilon^{d-2} a_{ij}\left(\frac{u_i^\varepsilon - u_j^\varepsilon}{\rho_\varepsilon}\right)^2 \\
&= \limsup_{\varepsilon\to 0} \sum_{(\eta_\varepsilon i^d, i^k),(\eta_\varepsilon j^d, j^k)\in(Q_d\times Q_{T,k})\cap X_{\eta_\varepsilon}} \eta_\varepsilon^{d-2} a_{ij}(v_\varepsilon(\eta_\varepsilon i^d, \varepsilon i^k) - v_\varepsilon(\eta_\varepsilon j^d, \varepsilon j^k))^2 \\
&\geq \limsup_{\varepsilon\to 0} \sum_{(\eta_\varepsilon i^d, i^k),(\eta_\varepsilon j^d, j^k)\in(Q_d\times Q_{T,k})\cap X_{\eta_\varepsilon}} \eta_\varepsilon^{d-2} a_{ij}(\tilde{v}_\varepsilon(\eta_\varepsilon i^d, \varepsilon i^k) - \tilde{v}_\varepsilon(\eta_\varepsilon j^d, \varepsilon j^k))^2 \\
&\geq \limsup_{\varepsilon\to 0} \inf\Big\{ \sum_{(\eta_\varepsilon i^d, i^k),(\eta_\varepsilon j^d, j^k)\in(Q_d\times Q_{T,k})\cap X_{\eta_\varepsilon}} \eta_\varepsilon^{d-2} a_{ij}(w_\varepsilon(\eta_\varepsilon i^d, \varepsilon i^k) - w_\varepsilon(\eta_\varepsilon j^d, \varepsilon j^k))^2 : \\
&\qquad w_\varepsilon(\eta_\varepsilon i^d, \varepsilon i^k) = \nabla u(x_0)\cdot \eta_\varepsilon i^d, \quad \text{if } \operatorname{dist}(\eta_\varepsilon i^d, \partial Q_d) < 2\eta_\varepsilon\sqrt{d}T \Big\}.
\end{aligned}$$

Setting $K_\varepsilon = \lfloor 1/(\eta_\varepsilon T)\rfloor$, we have that

$$\begin{aligned}
\frac{d\mu}{dx}(x_0) &\geq \liminf_{\varepsilon\to 0} \frac{1}{(K_\varepsilon T)^d} \inf\Big\{ \sum_{i,j\in(Q_{K_\varepsilon T,d}\times Q_{T,k})\cap X} \frac{1}{\eta_\varepsilon^2} a_{ij}(w_\varepsilon(\eta_\varepsilon i^d, \varepsilon i^k) - w_\varepsilon(\eta_\varepsilon j^d, \varepsilon j^k))^2 : \\
&\qquad w_\varepsilon(\eta_\varepsilon i^d, \varepsilon i^k) = \nabla u(x_0)\cdot \eta_\varepsilon i^d \text{ if } \operatorname{dist}(\eta_\varepsilon i^d, \partial Q_{K_\varepsilon T,d}) < 2\eta_\varepsilon\sqrt{d}T \Big\} \\
&\geq \liminf_{\varepsilon\to 0} \frac{1}{(K_\varepsilon T)^d} \inf\Big\{ \sum_{i,j\in(Q_{K_\varepsilon T,d}\times Q_{T,k})\cap X} a_{ij}(\tilde{w}_i - \tilde{w}_i)^2 : \\
&\qquad \tilde{w}_i = \nabla u(x_0)\cdot i^d \text{ if } \operatorname{dist}(i^d, \partial Q_{k_\varepsilon T,d}) < 2\sqrt{d}T \Big\} \\
&= f_0(\nabla u(x_0)),
\end{aligned}$$

where we have set $\tilde{w}_i := w_\varepsilon(\eta_\varepsilon i^d, \varepsilon i^k)/\eta_\varepsilon$. Therefore, for $\mathcal{L}^d$-almost every $x_0 \in \Omega$, we have

$$\frac{d\mu}{dx}(x_0) \geq A_{\text{hom}}\nabla u(x_0)\cdot\nabla u(x_0).$$

Integrating on Ω, we conclude that

$$\mu(\Omega) \geq \int_\Omega \frac{d\mu(\Omega)}{dx}dx \geq \int_\Omega A_{\text{hom}}\nabla u(x)\cdot\nabla u(x)dx.$$

Since $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$, we have that $\liminf_{\varepsilon\to 0}\mu_\varepsilon(\Omega) \geq \mu(\Omega)$. This implies that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon\to 0}\mu_\varepsilon(\Omega) \geq \mu(\Omega) \geq \int_\Omega A_{\text{hom}}\nabla u(x)\cdot\nabla u(x)dx = F_{\text{hom}}(u),$$

which concludes the proof of the lower bound.

It remains to prove that f_0 satisfies formula (7). First, we prove the existence of the limit.

Proposition 6.1. *For all $z \in \mathbb{R}$ there exists the limit*

$$f_0(z) = \lim_{K\to\infty} \frac{1}{(KT)^d} \inf\Big\{ \sum_{i,j\in(Q_{KT,d}\times Q_{T,k})\cap X} a_{ij}(u_i - u_j)^2 : \\
u_i = z\cdot i^d \text{ if } \operatorname{dist}(i^d, \partial Q_{KT,d}) < 2\sqrt{d}T \Big\}. \tag{42}$$

Proof. For fixed $K \in \mathbb{N}$ and $z \in \mathbb{R}^d$, we set

$$f_0^K(z) := \frac{1}{(KT)^d} \inf\Big\{ \sum_{i,j\in(Q_{KT,d}\times Q_{T,k})\cap X} a_{ij}(u_i - u_j)^2 : \\
u_i = z\cdot i^d \text{ if } \operatorname{dist}(i^d, \partial Q_{KT,d}) < 2\sqrt{d}T \Big\}.$$

Let u^K be a function such that

$$\frac{1}{(KT)^d}\sum_{i,j\in(Q_{KT,d}\times Q_{T,k})\cap X} a_{ij}(u_i^K-u_j^K)^2\le f_0^K(z)+\frac{1}{K},$$

and $u_i^K=z\cdot i^d$, if $\mathrm{dist}(i^d,\partial Q_{KT,d})<2\sqrt{d}T$. For $H>K$, we introduce the set of indices $\mathcal{I}:=\{l\in\mathbb{Z}^d : 0\le (K+1)l_m<H, m=1,\dots,d\}$. We define

$$u_i^H:=\begin{cases} u^K(i^d-l,i^k)+z\cdot l, & (i^d,i^k)\in Q^l_{KT,d}\times Q_{T,k},\ l\in\mathcal{I},\\ z\cdot i^d, & \text{otherwise.}\end{cases}$$

We have that

$$\begin{aligned} f_0^H(z)&\le \frac{1}{(HT)^d}\sum_{i,j\in(Q_{HT,d}\times Q_{T,k})\cap X} a_{ij}(u_i^H-u_j^H)^2\\ &=\frac{1}{(HT)^d}\sum_{i\in(\cup_{l\in\mathcal{I}}Q^l_{KT,d}\times Q_{T,k})\cap X}\ \sum_{j\in(Q_{HT,d}\times Q_{T,k})\cap X} a_{ij}(u_i^K-u_j^H)^2\\ &\quad+\frac{1}{(HT)^d}\sum_{i\in[(Q_{HT,d}\setminus\cup_{l\in\mathcal{I}}Q^l_{K,d})\times Q_{T,k}]\cap X}\ \sum_{j\in(Q_{HT,d}\times Q_{T,k})\cap X} a_{ij}(z\cdot i^d-u_j^H)^2. \end{aligned}\tag{43}$$

Due to the finite-range assumption there is no interaction between nodes in $(\bigcup_{l\in\mathcal{I}}Q^l_{KT,d}\times Q_{T,k})\cap X$ and those in $[(Q_{HT,d}\setminus\bigcup_{l\in\mathcal{I}}Q^l_{KT,d})\times Q_{T,k}]\cap X$. This implies that the first sum in (43) may be estimated as

$$\begin{aligned} &\frac{1}{(HT)^d}\sum_{i\in(\cup_{l\in\mathcal{I}}Q^l_{KT,d}\times Q_{T,k})\cap X}\ \sum_{j\in(Q_{HT,d}\times Q_{T,k})\cap X} a_{ij}(u_i^K-u_j^H)^2\\ &\quad=\frac{1}{(HT)^d}\sum_{i,j\in(\cup_{l\in\mathcal{I}}Q^l_{KT,d}\times Q_{T,k})\cap X} a_{ij}(u_i^K-u_j^K)^2\le\frac{K^d}{H^d}\sum_{l\in\mathcal{I}}(f_0^K(z)+K^{-1})\\ &\quad\le\frac{K^d}{H^d}\left\lfloor\frac{H}{K+1}\right\rfloor^d(f_0^H(z)+K^{-1})\le\frac{K^d}{(K+1)^d}(f_0^K(z)+K^{-1}). \end{aligned}\tag{44}$$

Using the finite-range assumption, the second sum in (43) may be estimated as

$$\begin{aligned} &\frac{1}{(HT)^d}\sum_{i\in[(Q_{HT,d}\setminus\cup_{l\in\mathcal{I}}Q^l_{KT,d})\times Q_{T,k}]\cap X}\ \sum_{j\in(Q_{HT,d}\times Q_{T,k})\cap X} a_{ij}(z\cdot i^d-u_j^H)^2\\ &\quad\le\frac{1}{(HT)^d}\sum_{i\in[(Q_{HT,d}\setminus\cup_{l\in\mathcal{I}}Q^l_{KT,d})\times Q_{T,k}]\cap X}\Big(\sum_{j\in(Q_{HT,d}\times Q_{T,k})\cap X} a_{ij}|z\cdot i^d-z\cdot j^d|^2\Big)\\ &\quad\le\frac{C}{(HT)^d}\sum_{i\in[(Q_{HT,d}\setminus\cup_{l\in\mathcal{I}}Q^l_{KT,d})\times Q_{T,k}]\cap X}\Big(\sum_{j\in(Q_{HT,d}\times Q_{T,k})\cap X} a_{ij}|i^d-j^d|^2\Big)\\ &\quad\le\frac{C}{(HT)^d}\left\lfloor\frac{H}{K+1}\right\rfloor^d. \end{aligned}\tag{45}$$

Combining (44) and (45), from (43) it follows that

$$f_0^H(z)\le\frac{K^d}{(K+1)^d}(f_0^K(z)+K^{-1})+\frac{1}{T^d(K+1)^d}.$$

Taking first the limsup as $H\to\infty$ and then the lower limit as $K\to\infty$, we obtain

$$\limsup_{H\to\infty} f_0^H(z)\le\liminf_{K\to\infty} f_0^K(z),$$

which concludes the proof. $\square$

Since we deal with convex energies, asymptotic homogenization formula (42) can be reduced to a single periodic minimization problem.

Proposition 6.2. *We have that* $f_0(z)$ *defined by* (42) *coincides with* $f_{\rm hom}(z)$ *defined for* $z\in\mathbb{R}^d$ *as*

$$f_{\rm hom}(z) := \frac{1}{T^d}\inf\Bigg\{\sum_{i\in(Q_{T,d}\times Q_{T,k})\cap X}\ \sum_{j\in(\mathbb{R}^d\times Q_{T,k})\cap X} a_{ij}(u_i-u_j)^2 : \\ u_i - z\cdot i^d \quad \text{is } T\text{-periodic in } e_1,\dots,e_d\Bigg\}. \qquad (46)$$

Proof. Fix $z\in\mathbb{R}^d$. First, we prove that $f_0(z)\le f_{\rm hom}(z)$. To this end, for $\delta>0$, let $u^\#$ be a function satisfying

$$\frac{1}{T^d}\sum_{i\in(Q_{T,d}\times Q_{T,k})\cap X}\ \sum_{j\in(\mathbb{R}^d\times Q_{T,k})\cap X} a_{i,j}(u^\#_i-u^\#_j)^2\le f_{\rm hom}(z)+\delta$$

and $u^\#_i - z\cdot i^d$ is T-periodic in $e_1,\dots,e_d$. We define $u^\varepsilon_i = u_\varepsilon(\varepsilon i) := \varepsilon u^\#(i)$. Note that u^ε_i converges to $z\cdot x^d$ with respect to the convergence given by (6). Set $I^d := \{l\in\mathbb{Z}^d : \varepsilon lT + \varepsilon Q_{T,d}\cap\Omega\neq\emptyset\}$. In view of Theorem 2.2 and the periodicity of a_{ij}, we deduce that

$$\begin{aligned}
|\Omega| f_0(z) &\le \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)\\
&\le \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \le \limsup_{\varepsilon\to 0}\sum_{l\in I^d}\ \sum_{i,j\in(Q^l_{T,d}\times Q_{T,k})\cap X}\varepsilon^d a_{ij}(u^\#_i-u^\#_j)^2\\
&\le \limsup_{\varepsilon\to 0}\sum_{l\in I^d}\varepsilon^d\sum_{i\in(Q_{T,d}\times Q_{T,k})\cap X}\ \sum_{j\in(\mathbb{R}^d\times Q_{T,k})\cap X} a_{ij}(u^\#_i-u^\#_j)^2\\
&\le |\Omega|(f_{\rm hom}(z)+\delta).
\end{aligned}$$

From the arbitrariness of δ, the conclusion follows.

It remains to show that $f_{\rm hom}(z)\le f_0(z)$. Let v be a function defined on $(Q_{KT,d}\times Q_{T,k})\cap X$ such that $v_i = z\cdot i^d$ if $\mathrm{dist}(i^d,\partial Q_{KT,d})\le 2\sqrt{d}T$. We define a function u on $(Q_{T,d}\times Q_{T,k})\cap X$ by

$$u(i) := \frac{1}{K^d}\sum_{l\in\{0,\dots,K-1\}^d} v(i^d+lT, i^k).$$

With the help of Jensen's inequality combined with the assumption of finite range and the periodicity of a_{ij}, we deduce that

$$\begin{aligned}
f_{\rm hom}(z) &\le \frac{1}{T^d}\sum_{i\in(Q_{T,d}\times Q_{T,k})\cap X}\ \sum_{j\in(\mathbb{R}^d\times Q_{T,k})\cap X} a_{ij}(u_i-u_j)^2\\
&\le \frac{1}{(KT)^d}\sum_{i\in(Q_{T,d}\times Q_{T,k})\cap X}\ \sum_{j\in(\mathbb{R}^d\times Q_{T,k})\cap X}\ \sum_{l\in\{0,\dots,K-1\}^d} a_{ij}(v(i^d+lT,i^k)-v(j^d+lT,j^k))^2\\
&= \frac{1}{(KT)^d}\sum_{i,j\in(Q_{KT,d}\times Q_{T,k})\cap X} a_{ij}(v_i-v_j)^2.
\end{aligned}$$

Taking the infimum, we get

$$f_{\rm hom}(z)\le\frac{1}{(KT)^d}\inf\Bigg\{\sum_{i,j\in(Q_{KT,d}\times Q_{T,k})\cap X} a_{ij}(v_i-v_j)^2 : v_i = z\cdot i^d \text{ if } \mathrm{dist}(i^d,\partial Q_{KT,d})<2\sqrt{d}T\Bigg\}.$$

Then, passing to the limit as $K\to\infty$, we have the desired inequality which concludes the proof. □

6.2 Proof of the upper bound

We now prove the Γ-lim sup inequality. The proof is independent of the blow-up result and it relies on the validity of the homogenization formula and a standard density argument by piecewise-affine functions (see [8, Remark 1.29]). We consider the case when the target function u is piecewise-affine and we assume that the gradient of u takes λ values, for some λ positive integer. For fixed $z_1, \dots, z_\lambda \in \mathbb{R}$, we define

$$\Omega_q := \{x^d \in \Omega : u(x^d) = z_q \cdot x^d + c_q\},$$

for $q = 1, \dots, \lambda$ (with c_q some constant).

We fix one such q. We choose $w^q \in \mathcal{C}_\varepsilon(Q_{T,d})$ such that $w_i^q - z \cdot i^d$ is T-periodic in $e_1, \dots, e_d$ and

$$\sum_{i,j \in (Q_{T,d} \times Q_{T,k}) \cap X} a_{ij}(w_i^q - w_j^q)^2 = f_{\hom}(z_q).$$

For any $q = 1, \dots, \lambda$, we define $u_i^{\varepsilon,q} := u_\varepsilon^q(\varepsilon i) = \varepsilon w^q(i) + c_q$. In view of Lemma 5.1, we may modify the sequence u_ε^q to obtain a new sequence $v_\varepsilon^{q,\delta}$ converging to $z_q \cdot x^d + c_q$ with respect to convergence (6) such that

$$\begin{aligned} v_i^{\varepsilon,q,\delta} &= z_q \cdot \varepsilon i^d + c_q, \quad \text{if } i \in ((\Omega_q \setminus \overline{\Omega_q(\delta)}) \times Q_{T,k}) \cap X, \\ v_i^{\varepsilon,q,\delta} &= u_i^{\varepsilon,q}, \quad \text{if } i \in (\Omega_q(2\delta) \times Q_{T,k}) \cap X, \end{aligned}$$

and

$$\limsup_{\varepsilon \to 0} F_\varepsilon^q(v_\varepsilon^{q,\delta}) \leq \limsup_{\varepsilon \to 0} F_\varepsilon^q(u_\varepsilon^q) + o(1) \tag{47}$$

as $\delta \to 0$, where F_ε^q is the functional defined as in (2) with Ω_q in the place of Ω.

Now, we estimate $F_\varepsilon^q(u_\varepsilon^q)$. To that end, for $q = 1, \dots, \lambda$, we introduce the set of indices $\mathcal{I}_{\varepsilon,q}^d := \{l \in \mathbb{Z}^d : \varepsilon Q_{T,d}^l \cap \Omega_q \neq \emptyset\}$, and we deduce that

$$\begin{aligned} F_\varepsilon^q(u_\varepsilon^q) &= \sum_{i,j \in (\frac{1}{\varepsilon}\Omega_q \times Q_{T,k}) \cap X} \varepsilon^{d-2} a_{ij}(u_i^{\varepsilon,q} - u_j^{\varepsilon,q})^2 \\ &\leq \sum_{i,j \in (\cup_{l \in \mathcal{I}_{\varepsilon,q}^d} Q_{T,d}^l \times Q_{T,k}) \cap X} \varepsilon^d a_{ij}(w^q(i) - w^q(j))^2 \\ &\leq \sum_{l \in \mathcal{I}_\varepsilon^d} \varepsilon^d \sum_{i,j \in (Q_{T,d}^l \times Q_{T,k}) \cap X} a_{ij}(w^q(i) - w^q(j)^2 \leq |\Omega_q| f_{\hom}(z_q) + o(1), \end{aligned}$$

as $\varepsilon \to 0$. This combined with (47) implies that

$$\limsup_{\varepsilon \to 0} F_\varepsilon^q(v_\varepsilon^{q,\delta}) \leq |\Omega_q| f_{\hom}(z_q) + o(1) \tag{48}$$

as $\delta \to 0$.

Now, we define the recovery sequence v^ε by

$$v_i^\varepsilon = v_i^{\varepsilon,q,\delta} \qquad \text{if } i \in \Omega_q,$$

for $q = 1, \dots, \lambda$. To conclude the proof, it remains to show that, given $q_1, q_2 \in \{1, \dots, \lambda\}$,

$$\limsup_{\varepsilon \to 0} \sum_{\substack{i \in (\frac{1}{\varepsilon}\Omega_{q_1} \times Q_{T,k}) \cap X \\ j \in (\frac{1}{\varepsilon}\Omega_{q_2} \times Q_{T,k}) \cap X}} \varepsilon^{d-2} a_{ij}(v_i^\varepsilon - v_j^\varepsilon)^2 = o(1) \tag{49}$$

as $\delta \to 0$.

Since $\delta >> \varepsilon T$, the interactions between nodes in $(\frac{1}{\varepsilon}\Omega_{q_1}(2\delta) \times Q_{T,k}) \cap X$ and $(\frac{1}{\varepsilon}(\Omega_{q_2} \setminus \overline{\Omega_{q_2}(\delta)}) \times Q_{T,k}) \cap X$ or nodes in $(\frac{1}{\varepsilon}\Omega_{q_1}(2\delta) \times Q_{T,k}) \cap X$ and $(\frac{1}{\varepsilon}\Omega_{q_2}(2\delta) \times Q_{T,k}) \cap X$ do not take place. This allows to reduce (49) to the following estimate

$$\limsup_{\varepsilon \to 0} \sum_{\substack{i \in [\frac{1}{\varepsilon}(\Omega_{q_1} \setminus \overline{\Omega_{q_1}(\delta)}) \times Q_{T,k}] \cap X \\ j \in [\frac{1}{\varepsilon}(\Omega_{q_2} \setminus \overline{\Omega_{q_2}(\delta)}) \times Q_{T,k}] \cap X}} \varepsilon^d a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2 = o(1) \tag{50}$$

as $\delta \to 0$, where we have set $u_i^\varepsilon = u(\varepsilon i^d)$, and used the fact that $v_i^{\varepsilon,q,\delta} = z_q \cdot \varepsilon i^d + c_d = u_i^\varepsilon$ if $i \in (\Omega_q \setminus \overline{\Omega_q(\delta)} \times Q_{T,k}) \cap X$. By the Lipschitz continuity of u we deduce that

$$\begin{aligned}\sum_{\substack{i\in[\frac{1}{\varepsilon}(\Omega_{q_1}\setminus\overline{\Omega_{q_1}(\delta)})\times Q_{T,k}]\cap X\\ j\in[\frac{1}{\varepsilon}(\Omega_{q_2}\setminus\overline{\Omega_{q_2}(\delta)})\times Q_{T,k}]\cap X}} \varepsilon^{d-2}a_{ij}(u_i^\varepsilon - u_j^\varepsilon)^2 &\leq C \sum_{\substack{i\in[\frac{1}{\varepsilon}(\Omega_{q_1}\setminus\overline{\Omega_{q_1}(\delta)})\times Q_{T,k}]\cap X\\ j\in[\frac{1}{\varepsilon}(\Omega_{q_2}\setminus\overline{\Omega_{q_2}(\delta)})\times Q_{T,k}]\cap X}} \varepsilon^d a_{ij}|i^d - j^d|^2 \\ &\leq C\max\{a_{ij}\}T^k \sum_{q=1}^{\lambda}\Big|\bigcup_{l\in[\frac{1}{\varepsilon}(\Omega_q\setminus\overline{\Omega_q(\delta)})]} \varepsilon Q_d^l\Big| \leq C\delta\end{aligned}$$

(the final C taking into account the bound for a_{ij}, their range, T and the $\mathcal{H}^{d-1}$ measure of the union of $\partial\Omega_q$), which proves (50).

Gathering estimates (48) and (49), we deduce that

$$\limsup_{\varepsilon\to 0} \sum_{i,j\in(\frac{1}{\varepsilon}\Omega\times Q_{T,k})\cap X} \varepsilon^{d-2}a_{ij}(v_i^\varepsilon - v_j^\varepsilon)^2 \leq \sum_{q=1}^{\lambda}|\Omega_q| f_{\mathrm{hom}}(z_q) + o(1).$$

as $\delta \to 0$, which concludes the proof of the upper bound.

Remark 6.3. Recall that the Γ-limit of a family of non-negative quadratic forms is still a non-negative quadratic form (see *e.g.* [20, Theorem 11.10]). Applying this property in our setting, we deduce that the Γ-limit F_{hom} of F_ε is a non-negative quadratic form. In other words, there exists a symmetric matrix A_{hom} such that $f_{\mathrm{hom}}(z) = A_{\mathrm{hom}} z\cdot z$, which finally gives (7).

6.3 Convergence of minimum problems

In this section, we deal with minimum problems with boundary data. To this end, we derive compactness result in the case that the functionals F_ε are subjected to Dirichlet boundary conditions. In the discrete setting, such conditions are imposed by introducing a parameter $r \in \mathbb{N}$ and fixing the value of u in a neighbourhood of the 'lateral boundary' of $\Omega \times Q_{T,k}$, corresponding to i^d in a neighbourhood of the boundary of $\Omega \subset \mathbb{R}^d$, of size εr.

For any $r > 0$ and given $\varphi \in H^1(\mathbb{R}^d)$, we introduce the set

$$\mathcal{C}_\varepsilon^{\varphi,r}(\Omega) := \left\{ u \in \mathcal{C}_\varepsilon(\Omega) : u(\varepsilon i) = \fint_{\varepsilon i^d + \varepsilon Q_d} \varphi(x^d)dx^d \text{ if } (\varepsilon i^d + (-\varepsilon r, \varepsilon r)^d) \cap \mathbb{R}^d \setminus \Omega \neq \emptyset \right\}.$$

We define the functional $F_\varepsilon^{\varphi,r}$ by

$$F_\varepsilon^{\varphi,r}(u) := F_\varepsilon(u), \quad u \in \mathcal{C}_\varepsilon^{\varphi,r}(\Omega).$$

Theorem 6.4. *For any $\varphi \in H^1(\mathbb{R}^d)$, let F^φ be the functional defined by*

$$F^\varphi(u) := \begin{cases} \displaystyle\int_\Omega A_{\mathrm{hom}}\nabla u\cdot\nabla u\,dx, & u - \varphi \in H_0^1(\Omega) \\ \infty, & \textit{otherwise,} \end{cases}$$

where A_{hom} is given by (7)*. Then, for any $r > 0$, the family of functionals $F_\varepsilon^{\varphi,r}$ Γ-converges to the functional F^φ with respect to convergence* (6)*.*

Proof. We prove the Γ-liminf inequality. To that end, we prove that if u_ε converges to u with respect to convergence (6) and $F_\varepsilon^{\varphi,r}(u_\varepsilon)$ is equibounded, then $u - \varphi \in H_0^1(\Omega)$. First, note that if $\sup_{\varepsilon>0} F_\varepsilon^{\varphi,r}(u_\varepsilon) < \infty$, then, thanks to the coerciveness of the coefficients a_{ij}, we deduce that

$$\sup_{\varepsilon\to 0} \sum_{i,j\in(\frac{1}{\varepsilon}\Omega\times Q_{T,k})\cap X} \varepsilon^{d-2}(u_i^\varepsilon - u_j^\varepsilon)^2 < \infty.$$

We denote by $\tilde{u}_\varepsilon$ the extension of u_ε on the whole X defined by $\tilde{u}^\varepsilon_i = \varphi(\varepsilon i^d)$, for any $\varepsilon > 0$ and outside Ω. Analogously, $\tilde{u}$ is the extension of u on $\mathbb{R}^d$ obtained by setting $\tilde{u}(x^d) = \varphi(x^d)$. Let Ω' be an open set such that $\Omega \subset\subset \Omega'$. Hence, we have that

$$\sum_{i,j\in(\frac{1}{\varepsilon}\Omega'\times Q_{T,k})\cap X} \varepsilon^{d-2} a_{ij}(\tilde{u}^\varepsilon_i - \tilde{u}^\varepsilon_j)^2 \leq \sum_{i,j\in(\frac{1}{\varepsilon}\Omega\times Q_{T,k})\cap X} \varepsilon^{d-2} a_{ij}(u^\varepsilon_i - u^\varepsilon_j)^2 + \sum_{i,j\in(\frac{1}{\varepsilon}(\Omega'\setminus\Omega_r)\times Q_{T,k})\cap X} \varepsilon^{d-2} a_{ij}(\varphi(\varepsilon i^d) - \varphi(\varepsilon j^d))^2 \leq C.$$

Repeating similar arguments as the proof of Γ-lim inf inequality of Theorem 2.2 and since $\tilde{u}_\varepsilon$ converges to $\tilde{u}$, we deduce that $\tilde{u} \in H^1(\Omega')$ and hence $u - \varphi \in H^1_0(\Omega)$. Then, invoking again Theorem 2.2, we have that

$$\liminf_{\varepsilon\to 0} F^{\varphi,r}_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq F^\varphi(u),$$

as desired.

Now, we show the Γ-limsup inequality. First, consider the case where $u \in H^1(\Omega)$ such that $\mathrm{supp}(u-\varphi) \subset\subset \Omega$. The general case is obtained by a density argument.

Consider a target function u such that $\mathrm{supp}(u - \varphi) \subset\subset \Omega$. In view of Theorem 2.2, we know that there exists a recovery sequence u_ε converging to u such that

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = \int_\Omega A_{\mathrm{hom}} \nabla u \cdot \nabla u dx.$$

In order to modify the sequence u_ε near the boundary of Ω, we apply Lemma 5.1 with $v_\varepsilon = u$. Hence, there exists a sequence w_ε such that w_ε still converges to u with respect to convergence (6), $w_\varepsilon = u$ is a neighbourhood of Ω and

$$\limsup_{\varepsilon\to 0} F_\varepsilon(w_\varepsilon) \leq \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) + o(1).$$

Since $\mathrm{supp}(u - \varphi) \subset\subset \Omega$, it follows that w_ε is equal to φ is a neighbourhood of Ω, so that $F^{\varphi,r}_\varepsilon(w_\varepsilon) = F_\varepsilon(w_\varepsilon)$. We may conclude that

$$\limsup_{\varepsilon\to 0} F^{\varphi,r}_\varepsilon(w_\varepsilon) \leq \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) + o(1) = F^\varphi(u) + o(1),$$

which concludes the proof. □

Now, we state the following result which deals with convergence of minimum problems with Dirichlet boundary data.

Proposition 6.5. *We have that*

$$\lim_{\varepsilon\to 0} \inf\{F_\varepsilon(u) : u \in \mathcal{C}^{\varphi,r}_\varepsilon(\Omega)\} = \min\{F_{\mathrm{hom}}(u) : u - \varphi \in H^1_0(\Omega)\}.$$

Moreover, if $u_\varepsilon \in \mathcal{C}^{\varphi,r}_\varepsilon(\Omega)$ *converges to* $\tilde{u}$ *with respect to convergence* (6) *and it is such that*

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0} \inf\{F_\varepsilon(u) : u \in \mathcal{C}^{\varphi,r}_\varepsilon(\Omega)\},$$

Then, u *is a minimizer for* $\min\{F_{\mathrm{hom}}(u) : u - \varphi \in H^1_0(\Omega)\}$.

Proof. We have to show the equi-coerciveness of $F^{\varphi,r}_\varepsilon$ with respect the topology defined by (6). To that end, consider $\{u_\varepsilon\} \subset \mathcal{C}^{\varphi,r}_\varepsilon(\Omega)$ such that $\sup_{\varepsilon>0} F^{\varphi,r}_\varepsilon(u_\varepsilon) < \infty$. In view of inequality (51) applied to $u - \varphi$, we deduce that

$$\sum_{l\in\mathbb{Z}^d} \sum_{i,j\in(Q^l_{T,d}\times Q_{T,k})\cap X} \varepsilon^d |u^\varepsilon_i - \varphi_i|^2 \leq C \sum_{i,j\in(\frac{1}{\varepsilon}\Omega\times Q_{T,k})\cap X} \varepsilon^d a_{ij} |(u^\varepsilon_i - \varphi_i) - (u^\varepsilon_j - \varphi_j)|^2 \leq C F_\varepsilon(u_\varepsilon) + C \sum_{i,j\in(\frac{1}{\varepsilon}\Omega\times Q_{T,k})\cap X} \varepsilon^d a_{ij} |\varphi_i - \varphi_j|^2 \leq C.$$

Hence, we may apply Proposition 4.1 to deduce that there exists a subsequence u_ε such that $\overline{u}_\varepsilon$ is converging. This concludes the proof. □

The next proposition shows the Poincaré inequality for functions $u \in \mathcal{C}_\varepsilon^{\varphi,r}(\Omega)$. We prove it assuming that $\varphi = 0$.

Proposition 6.6. *Let Ω be a bounded open set of $\mathbb{R}^d$ and let u be a function in $\mathcal{C}_\varepsilon(\Omega)$ such that $u_i = u(\varepsilon i) = 0$ if $\operatorname{dist}(\varepsilon i^d, \partial\Omega) \leq 2\varepsilon\sqrt{d}T$. Then, there exists a constant $C > 0$ such that*

$$\sum_{i,j\in(\frac{1}{\varepsilon}\Omega\times Q_{T,k})\cap X} \varepsilon^d |u_i|^2 \leq C \sum_{i,j\in(\frac{1}{\varepsilon}\Omega\times Q_{T,k})\cap X} \varepsilon^d a_{ij}|u_i - u_j|^2, \tag{51}$$

where C is of order of $[\operatorname{diam}(\Omega)]^2$.

Proof. We identify u with its extension to $(\mathbb{R}^d \times Q_{T,k}) \cap X$ which is equal to 0 outside $(\Omega\times Q_{T,k})\cap X$. Due to the boundedness of Ω, there exists $M > 0$ such that $\Omega \subset [0, M)^d$ and $(\frac{1}{\varepsilon}[0, M)^d \times Q_{T,k}) \cap X$ contains a path joining two arbitrary nodes $i, j \in (\frac{1}{\varepsilon}\Omega \times Q_{T,k}) \cap X$.

Fix $i \in (\frac{1}{\varepsilon}\Omega \times Q_{T,k}) \cap X$ and let j be a node such that $\operatorname{dist}(\varepsilon j^d, \partial\Omega) \leq 2\varepsilon\sqrt{d}T$ and $i^d - j^d = \lambda T e_1$, where, without lost of generality, we may assume that λ is a positive integer. Note that λ depends on the fixed node i and it is of order $MT^{-1}\varepsilon^{-1}$. Let l_i and l_j be two indices in $\mathbb{Z}^d$ such that $i \in Q_{T,d}^{l_i}$ and $j \in Q_{T,d}^{l_j}$. Let $S_{T,d}^\lambda$ be the union of $(\lambda + 1)$ neighbouring cubes joining $Q_{T,d}^{l_i}$ and $Q_{T,d}^{l_j}$ such that each two consecutive cubes having one face in common. In other words,

$$S_{T,d}^\lambda := \bigcup_{q=0}^{\lambda} Q_{T,d}^{l_q},$$

where $l_q = l_j + qTe_1$, for $q = 1, \ldots, \lambda$ and $l_0 = l_j$. Since X is connected, there exists a path of nodes $\{j_q\}_{q=0}^\lambda$ joining $j_0 = j$ and $j_\lambda = i$ such that it is contained in $S_{T,d}^\lambda + (-T, T)^d$, $j_q \in (Q_{T,d}^{l_q} \times Q_{T,k}) \cap X$ and $(j_q, j_{q+1}) \in \mathcal{E}$. Such a path can be built repeating periodically the path joining $j \in (Q_{T,d}^{l_0} \times Q_{T,k}) \cap X$ and $j + Te_1 \in (Q_{T,d}^{l_1} \times Q_{T,k}) \cap X$. Since $u_j = 0$, we have that

$$u_i = \sum_{q=1}^{\lambda} (u_{j_q} - u_{j_{q-1}}).$$

Hence, an application of the Jensen inequality leads us to

$$|u_i|^2 \leq \lambda \sum_{q=1}^{\lambda} |u_{j_q} - u_{j_{q-1}}|^2.$$

Summing over $i \in (\frac{1}{\varepsilon}\Omega \times Q_{T,k}) \cap X$, we get

$$\begin{aligned}
\sum_{i\in(\frac{1}{\varepsilon}\Omega\times Q_{T,k})\cap X} \varepsilon^d |u_i|^2 &\leq \lambda \sum_{q=1}^{\lambda} \sum_{i,j\in(\frac{1}{\varepsilon}\Omega\times Q_{T,k})\cap X} \varepsilon^d |u_{j_q} - u_{j_{q-1}}|^2 \\
&\leq C\lambda^2 \sum_{i,j\in(\frac{1}{\varepsilon}\Omega\times Q_{T,k})\cap X} \varepsilon^d a_{ij} |u_{j_q} - u_{j_{q-1}}|^2,
\end{aligned}$$

where the constant C takes into account the fact that the possible multiplicity of the paths containing the connection joining j_{q-1} and j_q, which is anyhow uniformly bounded. Recalling that λ is of order $MT^{-1}\varepsilon^{-1}$, we get the inequality (51), as desired. □

7 Examples

In this section, we exhibit some examples of the possible geometries of the set X. We also compute the homogenized matrix A_{hom} given by formula (7). In the examples below, we think of X as a subset of $\mathbb{Z}^{d+k}$ where $d = 1$ is identified with the horizontal direction and $k = 1$ or 2. Since $d = 1$ the homogenized matrix actually reduces to a single coefficient giving the homogenized energy density $A_{\text{hom}} z^2$.

In all the following examples the value of the non-zero coefficients a_{ij} is always 1, and the corresponding connections are represented by solid lines in the figures.

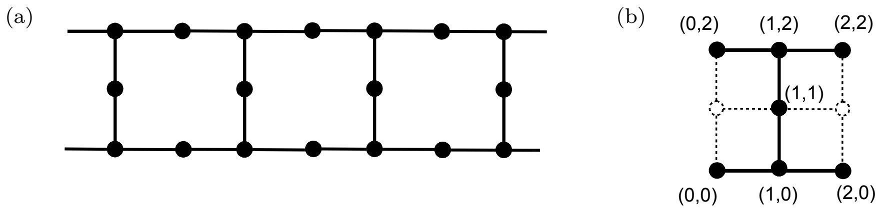

Figure 2: Figure (a) shows X and Figure (b) shows the periodicity cell $(Q_{2,d} \times Q_{2,k}) \cap X$.

Example 7.1. Let X be the set pictured in Figure 2(a). Here, we have that $d = k = 1$ and the period T is equal to 2. Figure 2(b) shows a periodicity cell. The geometry of the set X can be thought as the discrete version of a perforated domain. Indeed, note that nodes $(0,1)$ and $(2,1)$ in Figure 2(b) are missing. A minimizer $\tilde{u}$ for (7) is given by $\tilde{u}(0,0) = \tilde{u}(0,2) = 0$, $\tilde{u}(1,0) = \tilde{u}(1,1) = \tilde{u}(1,2) = z$ and $\tilde{u}(2,0) = \tilde{u}(2,2) = 2z$, so that $A_{\rm hom} = 4$.

In the next three examples $d = k = 1$, the set X is always simply $\mathbb{Z} \times \{0,1\}$ and the period T is 1, but the set $\mathcal{E}$ is such that the graph cannot be directly seen as a discretization of a thin film in the continuum parameterized as a subgraph of a function of one real variable.

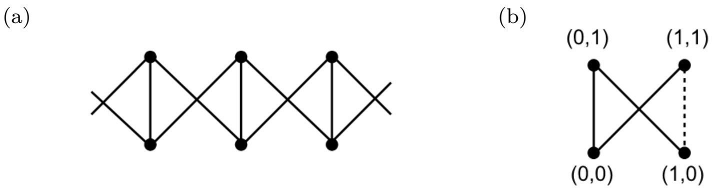

Figure 3: Figure (a) shows X and Figure (b) shows the periodicity cell $(Q_{1,d} \times Q_{1,k}) \cap X$.

Example 7.2. Let X be as drawn in Figure 3(a). In this case $\mathcal{E}$ contains all 'cross-connections' between points of X. The minimizer $\tilde{u}$ for $A_{\rm hom}z^2$ is $\tilde{u}(0,0) = \tilde{u}(1,1) = 0$ and $\tilde{u}(1,0) = \tilde{u}(0,1) = z$, so that $A_{\rm hom} = 4$.

The following two examples can also be reformulated in a square or triangular lattice, respectively, by a change of variables, so that they can be treated as in [2]. Our result make these changed of variable not necessary.

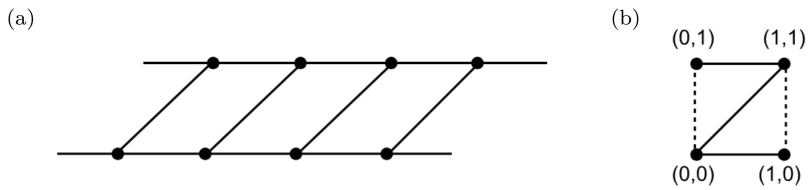

Figure 4: Figure (a) shows X and Figure (b) shows the periodicity cell $(Q_{1,d} \times Q_{1,k}) \cap X$.

Example 7.3. Consider X as drawn in Figure 4(a). Here, the graph is analog to a nearest-neighbour thin film, but with a translation of a unit of one of the two copies of $\mathbb{Z}$, which again makes this geometry not immediately seen as a discretization of a continuum thin film. The 1-periodic minimizer $\tilde{u}$ for $A_{\rm hom}z^2$ is given by $\tilde{u}(0,0) = \tilde{u}(1,1) = 0$, $\tilde{u}(1,0) = z$ and $\tilde{u}(0,1) = -z$ and the homogenized coefficient is $A_{\rm hom} = 4$.

Example 7.4. Consider X as drawn in Figure 5(a). Here the set of connections has the structure of a triangular lattice. The minimizer $\tilde{u}$ for $A_{\rm hom}z^2$ is given by $\tilde{u}(0,0) = 0$, $\tilde{u}(1,0) = z$, $\tilde{u}(0,1) = -1/2z$ and $\tilde{u}(1,1) = 1/2z$. The homogenized coefficient is $A_{\rm hom} = 5/2$.

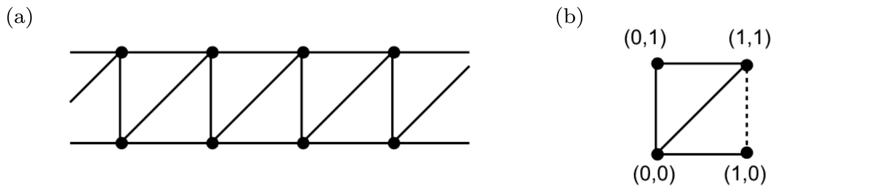

Figure 5: Figure (a) shows X and Figure (b) shows the periodicity cell $(Q_{1,d} \times Q_{1,k}) \cap X$.

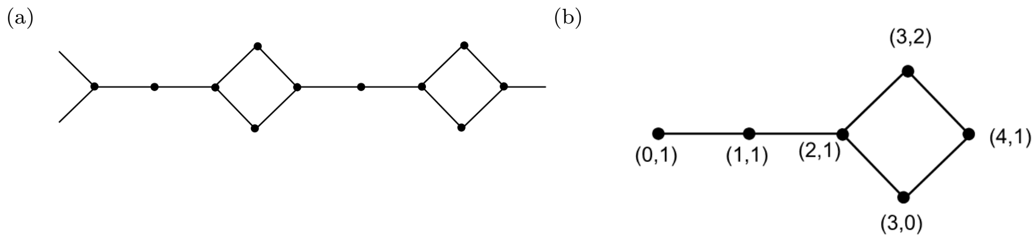

Figure 6: Figure (a) shows X and Figure (b) shows the periodicity cell $(Q_{4,d} \times Q_{4,k}) \cap X$.

Example 7.5. Let X be the set pictured in Figure 6(a), where $d = k = 1$ and the period T is equal to 4. The set X is a subset of $\mathbb{Z} \times \{0, 1, 2\}$. Such a set X can be though as a discrete layered media, whose conductivity is equal to 1 along the straight lines, while in the part corresponding to the rhombus structure the effective conductivity is 2.

The minimizer $\tilde{u}$ for $A_{\rm hom} z^2$ is given by $\tilde{u}(0, 0) = z$, $\tilde{u}(1, 1) = 4z/3$, $\tilde{u}(2, 1) = 8z/3$, $\tilde{u}(3, 0) = \tilde{u}(3, 2) = 10z/3$ and $\tilde{u}(4, 1) = 4z$ and $A_{\rm hom} = 8/3$.

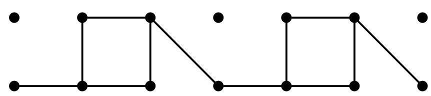

Figure 7: An alternate structure for Example 7.5

Note that the same example can be restated with $X = \mathbb{Z} \times \{0, 1\}$ using the set of connections in Fig. 7.

Example 7.6. We consider the set X drawn in Figure 1. To uniform the notation introduced in this section, we rotate X, obtaining the structure pictured in Figure 8(a). Here $d = 1$ and $k = 2$. The period T is equal to 2 and the periodicity cell is drawn in Figure 5(b). The structure is actually the same as that in Example 7.2 but transposed to a three-dimensional setting, and $A_{\rm hom} = 4$. Note that in this case the solid lines representing the connections do not intersect and they have all the same length, so that they can also be interpreted as a system of homogeneous conducting rods.

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(a) (b)

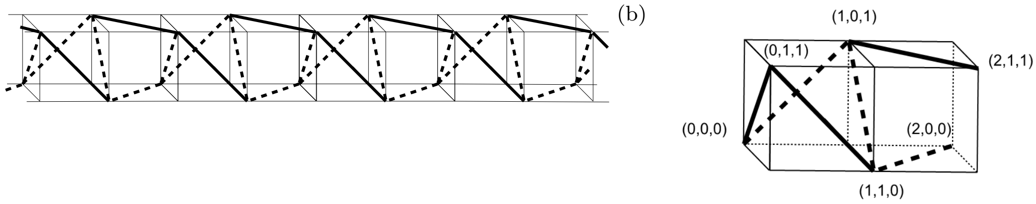

Figure 8: Figure (a) shows X and Figure (b) shows the periodicity cell $(Q_{2,d} \times Q_{2,k}) \cap X$.

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