Permutative automorphisms of the Cuntz algebras: quadratic cycles, an involution and a box product

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Abstract

Permutative automorphisms of the Cuntz algebra $\mathcal{O}_n$ are in bijection with a class of permutations of n^k elements, that are called stable, and are further partitioned by rank. In this work we mainly focus on stable cycles in the quadratic case (i.e., $k = 2$). More precisely, in such a quadratic case we provide a characterization of the stable cycles of rank one (so proving Conjecture 12.1 in [3]), exhibit a closed formula for the number of stable r-cycles of rank one (valid for all n and r), and characterize and enumerate the stable 3-cycles of any given rank. We also show that the set of stable permutations is equipped with a natural involution that preserves the cycle-type and the rank, and that there is a map that associates to two stable permutations of n^k and m^k elements, respectively, a stable permutation of $(nm)^k$ elements.

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Contents

1 Introduction

Cuntz algebras (first defined in [8]) are a much studied class of C^*-algebras, with lots of connections to several areas of research (see e.g. [2, 10, 11, 12, 13, 14]). Notably, the study of their automorphisms, started by Cuntz himself, is extremely difficult and challenging (see e.g. [15]). The existence of an intriguing connection between the automorphisms of the Cuntz algebras and combinatorics was foreseen by Cuntz in the late seventies [9, p.195]. However, there has been little progress in that direction until, about thirty years later, Conti and Szymanski ([7], [4], [5]) not only exhibited a huge number of new examples, but were actually able to set up a convoluted procedure to spot, in principle, all the so-called permutative automorphisms of the Cuntz algebra $\mathcal{O}_n$ at level k, and then determine their number, for small values of n and k, by further theoretical considerations and subsequent massive computer calculations ([7], [6], [1]). The combinatorial investigation of this convoluted procedure was started by Brenti and Conti in [3], to which we refer for further information.

In order to explain the connection with the main topic of the present paper, we need to introduce some more terminology. Given an integer $n \geq 2$, let $\mathcal{O}_n$ be the Cuntz algebra, $\mathcal{F}_n \subset \mathcal{O}_n$ the so-called UHF core C^*-subalgebra generated by a nested family of subalgebras $\mathcal{F}_n^k$ (each one isomorphic to the algebra of complex matrices M_{n^k}) and $\mathcal{D}_n$ be the C^*-subalgebra generated by the family of subalgebras $\mathcal{D}_n^k$ of $\mathcal{F}_n^k$ (each one isomorphic to the algebra of diagonal matrices in M_{n^k}). Then, following the insight by Cuntz, the *reduced Weyl group* of $\mathcal{O}_n$ is defined as

$$\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)/\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$$

where $\mathrm{Aut}(\mathcal{O}_n, X)$ is the subgroup of $\mathrm{Aut}(\mathcal{O}_n)$ consisting of the automorphisms which leave X invariant, while $\mathrm{Aut}_X(\mathcal{O}_n)$ is the one of those which fix X pointwise. This notion, somewhat inspired by the theory of semisimple Lie groups, turns out to be very useful to make advances in the understanding of the general structure of the group of automorphisms of the Cuntz algebras.

It is well-known that unital $*$-endomorphisms of $\mathcal{O}_n$ are in bijective correspondence with unitaries in $\mathcal{O}_n$, call this bijection $u \mapsto \lambda_u$. With some more work the reduced Weyl group can be further identified with the set of automorphisms λ_u of $\mathcal{O}_n$ induced by the so-called permutative unitaries $u \in \cup_{k \geq 1} \mathcal{F}_n^k$. Moreover, for general unitaries $u \in \mathcal{F}_n^k$, for any k, it was shown in [7], that λ_u is an automorphism precisely when the sequence of unitaries

$$\big(\varphi^r(u^*) \cdots \varphi(u^*) u^* \varphi(u) \cdots \varphi^r(u)\big)_{r \geq 0} \tag{1}$$

in $\mathcal{F}_n$ eventually stabilizes, where the endomorphism φ of $\mathcal{F}_n$ corresponds, in the isomorphism between M_{n^k} and $M_n \otimes \cdots \otimes M_n$ (k factors), to the tensor shift map $x \mapsto 1_{M_n} \otimes x$. By coherently identifying permutative unitaries in $\mathcal{F}_n^k$ with permutation matrices in M_{n^k} and thus with permutations of the set $\{1, \ldots, n^k\}$ and finally (by lexicographic ordering) with permutations of the set $[n]^k$, one is lead to the class of *stable permutations*, as defined in the main text (see Sec. 2) and investigated in this paper. These permutations precisely label the elements of the restricted Weyl group, see [3, Theorem 4.2].

The general problem we face is to find explicit combinatorial conditions characterizing many, if not all, stable permutations in $S([n]^k)$. This has a twofold advantage. On the one hand, identifying more and more such permutations, one would gain the possibility of making concrete computations of products in the reduced Weyl groups of the Cuntz algebras $\mathcal{O}_n$, which might unveil more specific algebraic properties of these groups. On the other hand, the problem of finding closed formulas for enumerating the stable permutations, even at a fixed level k, requires the development of combinatorial tools that, in the long run, will certainly make $\mathrm{Aut}(\mathcal{O}_n)$ more accessible to theoretical investigation.

The study of the stable permutations from a combinatorial point of view was started by Brenti and Conti in [3]. There they characterize the stable transpositions in $S([n]^2)$, provide various ways to produce new stable permutations from old ones, and study the enumeration of stable permutations, thanks to an in-depth analysis of the combinatorics of the sequence (1). The purpose of this paper is to continue the combinatorial investigation of stable permutations.

Our main results are a characterisation of the stable cycles of rank one in $S([n]^2)$ (Theorem 3.1), thereby proving Conjecture 12.1 in [3], a characterisation of the stable 3-cycles of any rank in $S([n]^2)$ (Theorem 5.12), the existence of a rank-preserving involution on stable permutations (Theorem 7.3) and a new construction ("box-product") producing a stable permutation in $S([nm]^k)$ from two stable permutations in $S([n]^k)$ and $S([m]^k)$ respectively (Theorem 8.2).

The organisation of the paper is as follows. In Section 2 we recall definitions and results used in the sequel. In Section 3 we characterise the stable cycles of rank one in $S([n]^2)$. In Section 4, using the main result in the previous section, we enumerate the stable cycles of rank one in $S([n]^2)$ (Proposition 4.5, Corollary 4.6 and Theorem 4.7). In Section 5 we characterise the stable 3-cycles in $S([n]^2)$. In Section 6 we venture into the study of stable 4-cycles in $S([n]^2)$, characterising an important subclass of them (Theorems 6.2 and 6.8). In Section 7 we prove the existence of a rank-preserving involution on the set of stable permutations. In Section 8 we introduce the box-product and examine its properties.

2 Preliminaries

In this section we provide the necessary definitions and results to make the paper self-contained. This background material is explained in detail in [3], to which we refer for further insight. For $n \in \mathbb{N}$ we set $[n] = \{1, \ldots, n\}$ and, for $h \in \mathbb{N}$, $[n]^h = [n] \times \ldots \times [n]$ (the cartesian product of h copies of $[n]$). In the sequel, we will mostly focus on permutations u of the finite set $[n]^2$, $u \in S([n]^2)$, but we need more structure. If $u \in S([n]^h)$ and $v \in S([n]^k)$ we define, following [3], the tensor product of u and v to be the permutation $u \otimes v$ in $S([n]^{h+k})$ defined by

$$(u \otimes v)(x_1, \ldots, x_{h+k}) = (u(x_1, \ldots, x_h), v(x_{h+1}, \ldots, x_{h+k}))$$

where $x_i \in [n]$ for all $i = 1, \ldots, h + k$. These operations can be iterated and composed at will. Moreover, in $S([n]^{h+k})$ one has $(u \otimes w)(v \otimes z) = (uv) \otimes (wz)$, for all $u, v \in S([n]^h)$, and $w, z \in S([n]^k)$ (where the products are taken in the respective permutation groups). We think of the maps $u \mapsto u \otimes 1$ and $u \mapsto 1 \otimes u$ as an embedding of $S([n]^h)$ into $S([n]^{h+1})$ and as the *shift*, respectively. Note that the above operations of tensoring on the right and on the left by the identity $1 \in S([n])$ commute, so that, for instance, $1 \otimes (u \otimes 1) = (1 \otimes u) \otimes 1$ will simply be denoted $1 \otimes u \otimes 1$.

Now, for $u \in S([n]^h)$ and $k \in \mathbb{N}$, we define $\psi_k(u) \in S([n]^{h+k})$ as

$$\begin{aligned}\psi_k(u) \;\; := \;\; & (\underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{k} \otimes u^{-1})(\underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{k-1} \otimes u^{-1} \otimes 1) \cdots \\ & \cdots (1 \otimes u^{-1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k-1})(u^{-1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k})(1 \otimes u \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k-1}) \\ & (1 \otimes 1 \otimes u \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k-2}) \cdots (\underbrace{1 \otimes \ldots \otimes 1}_{k-1} \otimes u \otimes 1)(\underbrace{1 \otimes \ldots \otimes 1}_{k} \otimes u)\end{aligned}$$

or, in a shorthand notation,

$$\psi_k(u) = \prod_{i=0}^{k}(\underbrace{1 \otimes \ldots \otimes 1}_{k-i} \otimes u^{-1} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{i}) \prod_{i=1}^{k}(\underbrace{1 \otimes 1 \ldots \otimes 1}_{i} \otimes u \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k-i}) \; .$$

We also set $\psi_0(u) := u^{-1}$. Note that the following recursive relation holds,

$$\psi_k(u) = (\underbrace{1 \otimes \ldots \otimes 1}_{k} \otimes u^{-1})(\psi_{k-1}(u) \otimes 1)(\underbrace{1 \otimes \ldots \otimes 1}_{k} \otimes u) \qquad (2)$$

for all $k \geq 1$.

Recall (see [3]) that , by definition, $u \in S([n]^h)$ is *stable of rank* $k_0 + 1 \in \mathbb{N}$ if the sequence $\{\psi_k(u)\}_{k \geq 0}$ satisfies

$$\psi_k(u) = \psi_{k_0}(u) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k-k_0}$$

for all $k \geq k_0$ and $k_0 \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$ is the least number for which this holds.

The origin of this notion comes from C^*-algebras, namely the search of (permutative) automorphisms of the Cuntz algebras $\mathcal{O}_n$. Indeed, there is a bijection between the stable permutations (where u and $u \otimes 1$ are identified) and the *reduced Weyl group* of $\mathcal{O}_n$ (see [7]). (Note that, however, the product in this group is not given by the product of the corresponding permutations.)

The following result gives an alternative definition of stability, and is proved in [3, Proposition 4.4].

Proposition 2.1. *Let* $u \in S([n]^h)$, *and* $k \in \mathbb{N}$. *Then*

i) if $\psi_k(u) \in S([n]^{k+1}) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{h-1}$ *then* u *is stable of rank* $\leq k+1$*;*

ii) if u *is stable of rank* $\leq k+1$ *then* $\psi_{k+1}(u) \in S([n]^{k+h}) \otimes 1$.

In particular, u is stable if and only if there exists a positive integer k such that $\psi_k(u) \in S([n]^{k+1}) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{h-1}$.

Simple examples show that the product of two stable permutations is not stable, in general. The next result shows that this is the case, however, if the two permutations satisfy an additional condition.

Theorem 2.2. *Let* $w, z \in S([n]^2)$ *be such that* $(z \otimes 1)(1 \otimes w) = (1 \otimes w)(z \otimes 1)$. *Assume also that* w *is stable of rank* $\leq s$ *and* z *is stable of rank* $\leq t$. *Then* wz *is stable of rank* $\leq s + t$.

If two permutations $w, z \in S([n]^2)$ are such that $(z\otimes 1)(1\otimes w) = (1\otimes w)(z\otimes 1)$ then we say that w is *compatible* with z (in this order). The previous result is proved in [3, Theorem 5.2].

Given the importance of compatibility, it is natural to try to understand when two permutations are compatible. The following two results ([3, Propositions 5.6 and 5.7]) answer this question when the first permutation is a transposition.

Proposition 2.3. *Let* $u = ((a, b), (i, j))$, *where* $a, b, i, j \in [n]$, $b \neq j$, *and* $v \in S([n]^2)$. *Then* $(v \otimes 1)(1 \otimes u) = (1 \otimes u)(v \otimes 1)$ *if and only if there is* $\sigma \in S_n$ *such that* $v(x, k) = (\sigma(x), k)$ *for all* $x \in [n]$ *and all* $k \in \{a, i\}$.

Proposition 2.4. *Let* $u = ((a, b), (i, b))$, *where* $a, b, i \in [n]$, $a \neq i$, *and* $v \in S([n]^2)$. *Then* $(v \otimes 1)(1 \otimes u) = (1 \otimes u)(v \otimes 1)$ *if and only if there are* $\sigma \in S_n$ *and* $(\tau_1, \ldots, \tau_n) \in S(\{a, i\})^n$ *such that*

$$v(x, a) = (\sigma(x), \tau_x(a)), \quad v(x, i) = (\sigma(x), \tau_x(i)),$$

for all $x \in [n]$.

It is hard, and a major goal of this line of research, to characterize the stable permutations. The following result ([3, Theorem 8.1]) achieves this for the transpositions.

Theorem 2.5. *Let* $(i, j), (a, b) \in [n]^2$, $(i, j) \neq (a, b)$, *and* $u := ((i, j), (a, b))$. *Then the following conditions are equivalent:*

i) u *is stable;*

ii) u *is stable of rank 1 (i.e.,* $(u \otimes 1)(1 \otimes u) = (1 \otimes u)(u \otimes 1)$ *in* $S([n]^3)$*);*

iii) $\{a, i\} \cap \{b, j\} = \emptyset$.

One of the goals of this work is to find the analogue of this result for the 3-cycles.

3 Stable cycles of rank 1

In this section we characterize the stable cycles of rank one in $S([n]^2)$, solving in the affirmative Conjecture 12.1 of [3].

For $u \in S([n]^2)$ we set, following [3],

$$R(u) := \{i \in [n] \mid \exists j \in [n] : (i,j) \notin F(u)\},$$

and

$$C(u) := \{j \in [n] \mid \exists i \in [n] : (i,j) \notin F(u)\} \ ,$$

where $F(u)$ is the set of fixed points of u. So $R(u)$, resp. $C(u)$, is the set of rows, resp. columns, containing at least one element that is not in $F(u)$.

Theorem 3.1. *Let $(a_i, b_i) \in [n]^2, i \in [r]$ $(r > 1)$ be distinct pairs. Then the permutation $u := ((a_1, b_1), \ldots, (a_r, b_r)) \in S([n]^2)$ is stable of rank 1 if and only if $a_i \neq b_j$, for any $i, j \in [r]$ (i.e., in short, $R(u) \cap C(u) = \emptyset$).*

Proof. If $R(u) \cap C(u) = \emptyset$ then, by Corollary 5.16 of [3], u is stable of rank 1. Conversely, assume by contradiction that u is stable of rank 1 and $R(u) \cap C(u) \neq \emptyset$. We will consider indexes mod r, i.e., $(a_i, b_i) = (a_{i+r}, b_{i+r})$ for any $i \in \mathbb{Z}$.

We construct two directed graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where

- $V_1 = V_2 = [n]^3$;
- $E_1 = \{(a_i, b_i, z) \to (a_{i+1}, b_{i+1}, z) \mid i \in [r],\ z \in [n]\}$,
  $E_2 = \{(x, a_i, b_i) \to (x, a_{i+1}, b_{i+1}) \mid i \in [r],\ x \in [n]\}$.

In other words, G_1 and G_2 are the functional digraphs of $u \otimes 1 \in S([n]^3)$ and $1 \otimes u \in S([n]^3)$, respectively. Note that each vertex of G_i has degree either 1 or 0 (by degree we mean outdegree, which is also equal to indegree). Let, for brevity, $\omega_1 := (u \otimes 1)$ and $\omega_2 := (1 \otimes u)$. Since u is stable of rank 1, then by Proposition 4.5 of [3] we have

$$\omega_1 \omega_2 = \omega_2 \omega_1. \tag{3}$$

Lemma 3.2. *There are no common directed edges in G_1 and G_2, i.e., $E_1 \cap E_2 = \emptyset$.*

Proof. Assume, to the contrary, that there is $v_0 \to v_1 \in E_1 \cap E_2$. Since G_1 and G_2 are disjoint unions of r-cycles, there are edges $v_1 \to v_2 \in E_1$ and $v_1 \to v_2' \in E_2$. So

$$(\omega_1 \omega_2)(v_0) = \omega_1(v_1) = v_2$$

and

$$(\omega_2\omega_1)(v_0) = \omega_2(v_1) = v_2',$$

hence, by (3), $v_2 = v_2'$. Continuing inductively we get that there is a directed cycle $v_0 \to v_1 \to \cdots \to v_r \to v_0$, which belongs to both G_1 and G_2. However, in any cycle of G_1 the third coordinate doesn't change and the first coordinate takes all the values from $R(u)$, and similarly in any cycle of G_2 the first coordinate doesn't change and the third one takes all the values from $C(u)$. Hence $|R(u)| = |C(u)| = 1$, which is a contradiction. □

By our assumption $R(u) \cap C(u) \neq \emptyset$, so there are $i, j \in [r]$ such that $a_i = b_j$, hence there is a vertex $v_0 \in [n]^3$ such that $deg_{G_1}(v_0) = deg_{G_2}(v_0) = 1$ (namely (a_j, a_i, b_i)). Let V' be the vertex set of the connected component of v_0 in $G_1 \cup G_2$.

Lemma 3.3. *All the vertices in V' have degree 1 in both G_1 and G_2, i.e.,*

$$deg_{G_1}(v) = deg_{G_2}(v) = 1 \text{ for any } v \in V'.$$

Proof. Assume, on the contrary, that there is a vertex $w \in V'$ such that

$$deg_{G_1}(w) + deg_{G_2}(w) = 1.$$

Consider a directed path from v_0 to w, and let $w_1 \to w_2$ be the first edge in this path such that

$$deg_{G_1}(w_2) + deg_{G_2}(w_2) = 1,$$

(so $deg_{G_1}(w_1) + deg_{G_2}(w_1) = 2$). Without loss of generality we can assume that $w_1 \to w_2 \in G_1$, so $deg_{G_1}(w_2) = 1$. Furthermore there is $w_2' \in V'$ such that $w_1 \to w_2' \in G_2$. Note that

$$\omega_2\omega_1(w_1) = \omega_2(w_2) = w_2 = \omega_1(w_1),$$

on the other hand we have

$$\omega_1\omega_2(w_1) = \omega_1(w_2'),$$

we obtain $\omega_1(w_1) = \omega_1(w_2')$. However, ω_1 is a permutation and $w_1 \neq w_2'$, contradiction. Then there is no such vertex w. □

Lemma 3.4. *We have $R(u) = C(u)$.*

Proof. Let S be the set of all possible middle coordinates of V', i.e.,

$$S := \{v_y \mid (v_x, v_y, v_z) \in V'\}.$$

Since $deg_{G_1}(v) = 1$ for any $v \in V'$, we have $S \subset C(u)$. On the other hand, when we act on v repeatedly with ω_1 we obtain all the elements of $C(u)$ in the middle coordinate, so that $C(u) = S$. Similarly $S = R(u)$. $\square$

Consider any vertex $(x_0, y_0, z_0) \in V'$. By Lemma 3.3 above, we have that $deg_{G_2}((x_0, y_0, z_0)) = 1$, hence $(y_0, z_0) = (a_{i_0}, b_{i_0})$ for some $i_0 \in [r]$. Acting repeatedly with ω_2, we get that $(x_0, a_i, b_i) \in V'$ for any $i \in [r]$. Then, by (3)

$$\omega_2(\omega_1((x_0, a_i, b_i))) = \omega_1(\omega_2((x_0, a_i, b_i))) = \omega_1((x_0, a_{i+1}, b_{i+1})).$$

Since ω_2 doesn't change the first coordinate, we get that the first coordinate of $\omega_1((x_0, a_i, b_i))$ is the same for all $i \in [r]$. Hence, the first element of $u(x_0, a_i)$ depends only on x_0. Hence, there is a function $f : R(u) \to R(u)$ such that the first element of $u(x, a)$ is $f(x)$, for any $a \in R(u)$. Similarly, there is a function $g : C(u) \to C(u)$ such that the second element of $u(b, z)$ is $g(z)$, for any $b \in C(u)$.

Since $R(u) = C(u)$, we get that there are functions f, g such that for any $a, b \in R(u)$ we have $u(a, b) = (f(a), g(b))$. In particular, f and g are permutations and

$$f(a_i) = a_{i+1} \text{ and } g(b_i) = b_{i+1}$$

for all $i \in [r]$. Hence, f and g are cycles of length $|R(u)|$. We get that for any $(a, b) \in R(u) \times C(u)$,

$$u^{|R(u)|}(a, b) = (a, b),$$

furthermore $|R(u)|$ is the minimal such number for any (a, b). Hence u is the disjoint union of $|R(u)|$ cycles, each of length $|R(u)|$, which is a contradiction. $\square$

4 Enumeration of stable cycles of rank 1

In this section we enumerate stable cycles of rank 1 in $S([n]^2)$. By Theorem 3.1 this is equivalent to enumerating cycles $u \in S([n]^2)$ such that $R(u) \cap C(u) = \emptyset$. We begin by considering for which cycle-lengths these cycles exist.

Proposition 4.1. *Let $r, n \in \mathbb{N}$ and u be an r-cycle of $S([n]^2)$ such that $R(u) \cap C(u) = \emptyset$. Then $r \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$.*

Proof. Since $R(u) \cap C(u) = \emptyset$ we have that $C(u) \subseteq [n] \setminus R(u)$ so $|C(u)| \leq n - |R(u)|$. On the other hand, by definition of $R(u)$ and $C(u)$ one also has that $r \leq |C(u)||R(u)|$. So $r \leq |R(u)|(n - |R(u)|)$. Since the sequence $\{i(n-i)\}_{i=0,..,n}$ is clearly symmetric and unimodal the claim follows. □

Note that the bound given in the previous proposition is best possible. Indeed, if $r \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ then any r-cycle of $[\lfloor \frac{n}{2} \rfloor] \times \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n\}$ satisfies the hypotheses of Proposition 4.1. In particular, this gives the following lower bound for the number of r-cycles in $S([n]^2)$ that are stable of rank 1. Recall that there are $(n-1)!$ n-cycles in S_n.

Proposition 4.2. *Let $r, n \in \mathbb{N}$. Then the number of r-cycles in $S([n]^2)$ that are stable of rank 1 is at least $\binom{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}{r}(r-1)!$.*

On the other hand, for the maximum possible value of r (namely $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$) one can obtain an exact formula for the number of such cycles.

Proposition 4.3. *Let $n \in \mathbb{N}$. Then the number of $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$-cycles in $S([n]^2)$ that are stable of rank 1 is*

$$\begin{cases} \binom{n}{\lfloor \frac{n}{2} \rfloor}(\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - 1)! & \text{if } n \text{ is even} \\ 2\binom{n}{\lfloor \frac{n}{2} \rfloor}(\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - 1)! & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let u be a stable $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$-cycle in $S([n]^2)$. Then, by Theorem 3.1, $R(u) \cap C(u) = \emptyset$. Also, since $i(n-i) < \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ if $i < \frac{n-1}{2}$, we must have $|R(u)| = \lfloor \frac{n}{2} \rfloor$ and $|C(u)| = \lceil \frac{n}{2} \rceil$, or conversely. So such a u is uniquely determined by $R(u)$ and by a cyclic ordering of $R(u) \times C(u) = R(u) \times ([n] \setminus R(u))$. The result follows. □

The preceding proposition shows, in conjunction with Stirling's formula, that the probability of obtaining a stable $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$-cycle in $S([n]^2)$ by choosing uniformly at random among all $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$-cycles in $S([n]^2)$ goes to 0 as $n \to +\infty$.

We conclude this section by observing that by following the strategy used to enumerate the number of permutations $u \in S([n]^2)$ such that $R(u) \cap C(u) = \emptyset$ in [3, Sec. 11.2], one can obtain an exact formula for the number of cycles in $S([n]^2)$ that are stable of rank 1, as well as the following lower bound.

Proposition 4.4. *Let $n \in \mathbb{N}$, then the number of cycles in $S([n]^2)$ that are stable of rank 1 is at least*

$$\sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} (ij-1)! \; .$$

In particular, this number is larger or equal to $\binom{n}{\lfloor \frac{n}{2} \rfloor}(\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - 1)!$.

We denote by $\mathcal{C}_i(r)$ the set of all compositions of r into i parts (see, e.g., [16, Sec. 1.2]).

Proposition 4.5. *Let $r, n \in \mathbb{N}$. Then the number of r-cycles in $S([n]^2)$ that are stable of rank 1 is*

$$(r-1)! \sum_{i=1}^{r} \binom{n}{i} \sum_{(a_1,\ldots,a_i) \in \mathcal{C}_i(r)} \prod_{j=1}^{i} \binom{n-i}{a_j}. \tag{4}$$

Proof. We count the number of subsets $S \subseteq [n]^2$ such that $|S| = r$ and $R(S) \cap C(S) = \emptyset$, where $R(S) := \{i \in [n] : (i,j) \in S \text{ for some } j \in [n]\}$ and $C(S)$ is defined similarly. Let $i := |R(S)|$. Then $1 \leq i \leq n$ and $R(S)$ can be chosen in $\binom{n}{i}$ ways. For a given choice of $R(S)$ let a_j be the number of elements of S in the j-th (from the top, say) row of $R(S)$, for $j = 1, \ldots, i$. Then $a_j \geq 1$ for all $j \in [i]$, and $a_1 + \cdots + a_i = r$. Also, for each such choice of $(a_1, \ldots, a_i)$ there are, for each $j \in [i]$, $\binom{n-i}{a_j}$ possibilities for the intersection of S with the j-th row of $R(S)$, so there are $\prod_{j=1}^{i} \binom{n-i}{a_j}$ subsets S with these values of $(a_1, \ldots, a_i)$ and of $R(S)$. There are $(r-1)!$ ways to order the elements of S in a cycle, so the result follows. □

It is not hard to check that for $r = 2$ and $r = 3$ one recovers Corollaries 8.3 and 8.8 in [3], while for $r = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ one obtains Proposition 4.3, and for $r > \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ Proposition 4.1. Also, note that the previous result implies that, for a given $r \in \mathbb{N}$, the number of stable r-cycles in $S([n]^2)$ is a polynomial in n of degree $2r$ and leading coefficient $1/r$, if $n \geq r$. In particular, this shows that, if one chooses an r-cycle uniformly at random in $S([n]^2)$, then the probability that this is stable goes to 1 as $n \to \infty$.

Corollary 4.6. *Let $n \in \mathbb{N}$. Then the number of cycles in $S([n]^2)$, different from the identity, that are stable of rank 1 is*

$$\sum_{r=2}^{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil} (r-1)! \sum_{i=1}^{r} \binom{n}{i} \sum_{(a_1,\ldots,a_i) \in \mathcal{C}_i(r)} \prod_{j=1}^{i} \binom{n-i}{a_j}. \tag{5}$$

So, for example, the number of cycles that are stable of rank 1 in $S([n]^2)$ is 0, 6, 136, 7640, 2948208, and 8389599806, for $n = 2, \ldots, 7$, respectively. More precisely, for $4 \leq n \leq 6$, these numbers arise as $136 = 36 + 64 + 36$, $7640 = 120 + 560 + 1680 + 2880 + 2400$, and $2948208 = 300 + 2640 + 17460 +$

$83808 + 288000 + 691200 + 1058400 + 806400$, where the i-th summand from the left is the number of stable $(i+1)$-cycles of rank 1 in the relevant $S([n]^2)$.

One can compute the number in Corollary 4.6 also by suitably adapting the argument used in Section 11.2 of [3]. Indeed, the argument in Sec. 11.2 of [3] can be repeated exactly but now using the number of k-cycles in S_k rather that the number of derangements in S_k, so using $(k-1)!$ rather than d_k for $k \geq 2$ (notation as in Sec. 11.2 of [3]). The only difference is that the identity permutation can also be obtained as a permutation that has $n^2 - 1$ fixed points and the remaining point as a 1-cycle (while this cannot happen with derangements). In this way, after simplifications, one obtains the following formula for the number of such cycles, which can also be verified directly.

Theorem 4.7. *Let $n \in \mathbb{N}$, then there are*

$$\sum_{a=1}^{n-1}\sum_{b=1}^{n-a}\sum_{r=2}^{ab}(-1)^{n-a-b}\binom{n}{a, b, n-a-b}\frac{(ab)!}{(ab-r)!\,r} \tag{6}$$

stable cycles of rank 1 *in* $S([n]^2)$ *that are different from the identity.*

Proof. Note that if $X \cap Y = \emptyset$, then there are exactly $\frac{(|X||Y|)!}{(|X||Y|-r)!r}$ stable r-cycles u of rank 1 such that $R(u) \subseteq X$ and $C(u) \subseteq Y$. Variables a and b in the summation (6) correspond to the sizes of X and Y respectively. Now we compute coefficients of stable r-cycles of rank 1 in (6). Let u be a stable r-cycle of rank 1. Then its coefficient is equal to

$$\sum_{R(u)\subseteq X}\ \sum_{C(u)\subseteq Y, X\cap Y=\emptyset}(-1)^{n-|X|-|Y|} = \sum_{R(u)\subseteq X}\delta_{C(u),[n]\setminus X}(-1)^{n-|X|-|C(u)|} = 1$$

where we have used the well known fact that $\sum_{A\subseteq X\subseteq B}(-1)^{|X|-|A|} = \delta_{A,B}$. □

We point out a simple consequence of [3, Corollary 5.5]. Recall (see, e.g., [7]) that for each stable permutation $u \in S([n]^k)$ there is associated an automorphism λ_u of the Cuntz algebra $\mathcal{O}_n$.

Proposition 4.8. *Let $u \in S([n]^2)$ be a stable cycle of rank one, then the order of $\lambda_u \in \mathrm{Aut}(\mathcal{O}_n)$ coincides with the order of u.*

In particular, for each $2 \leq r \leq \lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil$, the number $A_n(r)$ of permutative automorphisms in $\mathrm{Aut}(\mathcal{O}_n)$ at level two of order r is bounded from below by $(r-1)!\sum_{i=1}^{r}\binom{n}{i}\sum_{(a_1,\ldots,a_i)\in\mathcal{C}_i(r)}\prod_{j=1}^{i}\binom{n-i}{a_j}$.

5 Stable 3-cycles

In this section we characterize the stable 3-cycles of any given rank, and enumerate them. In particular, we show that any stable 3-cycle is a compatible product of two stable transpositions.

We need the following result, which is a consequence of Proposition 2.3 in the case of two transpositions.

Proposition 5.1. *Let* $u = ((a,b),(i,j))$*,* $b \neq j$ *and* $v = ((c,d),(k,l))$ $(a,b,c,d,i,j,k,l \in [n])$*,* $(a,b) \neq (i,j)$*,* $(c,d) \neq (k,l)$*. Then* u *is compatible with* v*, that is* $(v \otimes 1)(1 \otimes u) = (1 \otimes u)(v \otimes 1)$*, if and only if either*

$$\{a,i\} \cap \{d,l\} = \emptyset \tag{7}$$

or

$$a = i = l = d \ . \tag{8}$$

Proof. Assume first that u is compatible with v. Then, by Proposition 2.3, there is $\sigma \in S_n$ such that $v(x,k) = (\sigma(x),k)$ for all $x \in [n]$ and all $k \in \{a,i\}$. Suppose that $\{a,i\} \cap \{d,l\} \neq \emptyset$. We may assume that $a = d$. Then we have that

$$(k,l) = v(c,d) = v(c,a) = (\sigma(c),a)$$

so that $\sigma(c) = k$, $a = l$ and hence $l = d$. Therefore $c \neq k$. Similarly, $v(c,i) = (\sigma(c),i) = (k,i)$ so $v(c,i) \neq (c,i)$ and thus $(c,i) = (c,d)$ so $i = d$.
The converse follows immediately from Proposition 2.3. ☐

The next result is the "first half" of our characterization.

Theorem 5.2. *Let* $(a_1,b_1),(a_2,b_2),(a_3,b_3) \in [n]^2$ *be distinct, and* $w := ((a_1,b_1),(a_2,b_2),(a_3,b_3))$ *be such that* $R(w) \cap C(w) \neq \emptyset$*. Then* w *is a compatible product of two stable transpositions if and only if*

$$b_{i+1} = a_i, \quad \{b_i, b_{i-1}\} \cap \{a_1,a_2,a_3\} = \emptyset, \quad b_{i+1} \notin \{a_{i+1}, a_{i-1}\} \tag{9}$$

for some $i \in [3]$ *(where indices are taken modulo* 3*). In this case* $|R(w) \cap C(w)| = 1$ *and* w *is stable of rank* 2*.*

Proof. Suppose first that the conditions in (9) are satisfied. We may assume that $i = 1$. Then we have that $w = u\,v$ where $u := ((a_2,b_2),(a_3,b_3))$, and $v := ((a_3,b_3),(a_1,b_1))$. But, by (9) and Theorem 2.5, u and v are stable, and by Proposition 2.3 u is compatible with v.

Conversely. Since $R(w) \cap C(w) \neq \emptyset$ there is $i \in [3]$ such that $a_i \in C(w)$. We may assume that $i = 1$. Let u, v be two stable transpositions, u compatible with v, such that $w = u\,v$. Then we have three possible cases, namely

$$u = ((a_1, b_1), (a_2, b_2)), \quad v = ((a_2, b_2), (a_3, b_3)) \tag{10}$$

or

$$u = ((a_2, b_2), (a_3, b_3)), \quad v = ((a_3, b_3), (a_1, b_1)) \tag{11}$$

or

$$u = ((a_3, b_3), (a_1, b_1)), \quad v = ((a_1, b_1), (a_2, b_2)). \tag{12}$$

If $a_1 = b_1$ then, by Theorem 2.5, either u or v are not stable, which is a contradiction. So $a_1 \neq b_1$.

If $a_1 = b_3$ then v is not stable in the second case, and u is not stable in the third one. So (10) holds. Since u and v are stable we conclude from Theorem 2.5 that $\{a_1, a_2\} \cap \{b_1, b_2\} = \emptyset$ and $\{a_2, a_3\} \cap \{b_2, b_3\} = \emptyset$. If $b_1 \neq b_2$ then, since u is compatible with v, we have by Proposition 5.1 that $a_1 = a_2 = b_2 = b_3$, a contradiction. So $b_1 = b_2$. Then, by Proposition 2.4, since u is compatible with v, there exists $\sigma \in S_n$ and $(\tau_1, \ldots, \tau_n) \in S(\{a_1, a_2\})^n$ such that $v(x, a_1) = (\sigma(x), \tau_x(a_1))$ and $v(x, a_2) = (\sigma(x), \tau_x(a_2))$ for all $x \in [n]$. Hence $(a_2, b_2) = v(a_3, b_3) = v(a_3, a_1) = (\sigma(a_3), \tau_{a_3}(a_1))$ so $b_2 = \tau_{a_3}(a_1)$ and therefore $b_2 \in \{a_1, a_2\}$, which is a contradiction.

So $a_1 = b_2$. Then, by Theorem 2.5, u is not stable in the first case, and v is not stable in the third one, so (11) holds. Then, again by Theorem 2.5, $b_1 \notin \{a_1, a_3\}$, $b_3 \notin R(w)$, and $b_2 \notin \{a_2, a_3\}$. In particular, $b_2 \neq b_3$. But u is compatible with v so, by Proposition 5.1, either $\{a_2, a_3\} \cap \{b_1, b_3\} = \emptyset$ or $a_2 = a_3 = b_1 = b_3$. But $b_1 \neq a_3$, so $\{a_2, a_3\} \cap \{b_1, b_3\} = \emptyset$, and hence $b_1 \neq a_2$. This proves (9). The second statement follows immediately from Theorems 2.2 and 3.1. □

Note that the 3-cycles that move a point on the diagonal do not comply with the necessary and sufficient conditions of the last theorem (of course, they cannot have rank one as well).

We now analyze the rank of the stable 3-cycles. Let $u = ((i, a), (j, b), (k, c)) \in S([n]^2)$ be a 3-cycle. We already know that there are 3-cycles of ranks 1 and 2 (see Theorems 3.1 and 5.2). We wish to know if other values of the rank are possible, and if there are 3-cycles of rank 2 which do not arise from Theorem 5.2. Since u is a 3-cycle, $|R(u)|, |C(u)| \leq 3$, and we know that u is stable of rank 1 if and only if $R(u) \cap C(u) = \emptyset$. Therefore u is not stable of rank 1 if and only if $|R(u) \cap C(u)| \geq 1$. We analyze the 3-cycles according to the values of $(|R(u)|, |C(u)|, |R(u) \cap C(u)|)$.

We begin with some preliminary results. The first one concerns the case in which one of the elements moved by u is on the diagonal (i.e., if either $i = a$ or $j = b$ or $k = c$, so we may assume that $i = a$).

Lemma 5.3. *Let* $(i,i),(j,b),(k,c) \in [n]^2$*, distinct, and* $u := ((i,i),(j,b),(k,c)) \in S([n]^2)$ *be such that* $c \notin R(u)$ *and* $b \neq i$*. Then* u *is not stable.*

Proof. Let $h \in \mathbb{N}$. Then, using our hypotheses, one computes that

$$\psi_{2h}(u)(j,b,\underbrace{k,c,\ldots,k,c}_{2h}) = (i,\underbrace{k,c,\ldots,k,c}_{2h},i)$$

so u is not stable since $c \neq i$. □

The next result does not assume that u is a cycle.

Lemma 5.4. *Let* $i,j,b \in [n]$*,* $j \neq b$*, and* $u \in S([n]^2)$ *be such that* $u(b,b) = (b,b)$ *and* $u(i,j) = (j,b)$*. Then* u *is not stable.*

Proof. Let $h \in \mathbb{N}$. Then, using our hypotheses, one computes that

$$\psi_h(u)(j,\underbrace{b,\ldots,b}_{h+1}) = (\underbrace{i,\ldots,i}_{h+1},j)$$

so u is not stable. □

We need a further preliminary result.

Lemma 5.5. *Let* $u \in S([n]^2)$ *and* $i \in [n]$ *be such that* $u(i,i) \neq (i,i)$*,* $u(i,a) = (i,a)$*, and* $u(d,a) = (d,a)$*, where* $(a,b) := u(i,i)$ *and* $(c,d) := u^{-1}(i,i)$*. Then* u *is not stable.*

Proof. One can show that

$$\psi_{2p-2}(u)(\underbrace{i,i,\ldots,i}_{2p}) = (c,d,\underbrace{i,i,\ldots,i}_{2p-2})$$

while

$$\psi_{2p-1}(u)(\underbrace{i,i,\ldots,i}_{2p+1}) = (\underbrace{i,i,\ldots,i}_{2p+1})$$

for all $p \in \mathbb{N}$. This shows that $\psi_{2p-1}(u) \neq \psi_{2p-2}(u) \otimes 1$ for all $p \in \mathbb{N}$ so u is not stable. □

We can now easily analyze the 3-cycles such that $|R(u)| = 1$.

Corollary 5.6. *Let $u \in S([n]^2)$ be a 3-cycle such that $|R(u)| = 1$. Then either u is not stable or u is stable of rank 1.*

Proof. Let $u = ((i,a),(i,b),(i,c))$ ($(i,a),(i,b),(i,c) \in [n]^2$, distinct). If $R(u) \cap C(u) = \emptyset$ (i.e., if $i \notin \{a,b,c\}$) then, by Theorem 3.1 u is stable of rank 1. Else $i \in \{a,b,c\}$ and we may assume that $i = a$. Therefore $u(i,i) = (i,b)$, and since $i \neq b$, this implies that $u(b,b) = (b,b)$ so the result follows from Lemma 5.4. □

The previous result shows that we may assume that $(|R(u)|,|C(u)|) \in \{(2,2),(2,3),(3,1),(3,2),(3,3)\}$. We begin with the case $(|R(u)|,|C(u)|) = (2,2)$.

Proposition 5.7. *Let $u \in S([n]^2)$ be a 3-cycle such that $|R(u)| = |C(u)| = 2$. Then either u is stable of rank ≤ 2 or u is not stable.*

Proof. Let $u = ((i,a),(j,b),(k,c))$. Since $|R(u)| = 2$ we may assume that $u = ((i,a),(j,b),(j,c))$. Since $|C(u)| = 2$ we have that either $u = ((i,a),(j,b),(j,a))$, or $u = ((i,a),(j,a),(j,b))$. By Theorem 3.1 we may assume that $|R(u) \cap C(u)| \geq 1$.

Assume first that $u = ((i,a),(j,b),(j,a))$. If $|R(u) \cap C(u)| = 1$ then either $a = i$, or $a = j$, or $b = i$, or $b = j$. If $a = i$ then $u = ((i,i),(j,b),(j,i))$ so u is not stable by Lemma 5.5. If $a = j$ then $u = ((i,j),(j,b),(j,j))$ so u is not stable by Lemma 5.4. If $b = i$ then $u = ((i,a),(j,i),(j,a))$ so u satisfies the equivalent conditions of Theorem 5.2. Finally, if $b = j$ then $u = ((i,a),(j,j),(j,a))$ so u is not stable by Lemma 5.4. If $|R(u) \cap C(u)| = 2$ then either $(a,b) = (i,j)$ or $(a,b) = (j,i)$. In the first case $u = ((i,i),(j,j),(j,i))$ so u is not stable by Lemma 5.5. In the second one $u = ((i,j),(j,i),(j,j))$ so u is not stable by Lemma 5.4.

Assume now that $u = ((i,a),(j,a),(j,b))$. If $|R(u) \cap C(u)| = 1$ then either $a = i$, or $a = j$, or $b = i$, or $b = j$. If $a = i$ then $u = ((i,i),(j,i),(j,b))$ so u is not stable by Lemma 5.5. If $a = j$ then $u = ((i,j),(j,j),(j,b))$ so u is not stable by Lemma 5.4. If $b = i$ then $u = ((i,a),(j,a),(j,i))$ so u is not stable by Lemma 5.4. Finally, if $b = j$ then $u = ((i,a),(j,a),(j,j))$ so u is not stable by Lemma 5.3. If $|R(u) \cap C(u)| = 2$ then either $(a,b) = (i,j)$ or $(a,b) = (j,i)$. In the first case $u = ((i,i),(j,i),(j,j))$ so

$$\psi_{2p-2}(u)(\underbrace{i,i,\ldots,i}_{2p}) = (\underbrace{j,j,\ldots,j}_{2p})$$

for all $p \in \mathbb{N}$, so u is not stable. In the second one $u = ((i,j),(j,j),(j,i))$ so u is not stable by Lemma 5.4. □

We next analyze the case $(|R(u)|,|C(u)|) = (2,3)$. This means that $|\{i,j,k\}| = 2$, and that a,b,c are distinct, and we may assume that $i = j$.

Proposition 5.8. *Let $u \in S([n]^2)$ be a 3-cycle such that $|R(u)| = 2$ and $|C(u)| = 3$. Then either u is not stable or u is stable of rank ≤ 2.*

Proof. As remarked, we may assume that $u = ((i,a),(i,b),(k,c))$ where a,b,c are distinct, and $i \neq k$. If $R(u) \cap C(u) = \emptyset$ then by Theorem 3.1 u is stable of rank 1, so we may assume that $|R(u) \cap C(u)| \geq 1$. If $c = i$ then $u(k,i) = (i,a)$ and $u(a,a) = (a,a)$ so by Lemma 5.4 u is not stable. Similarly, if $b = k$ then $u(i,k) = (k,c)$ and $u(c,c) = (c,c)$ so again by Lemma 5.4 u is not stable. We may henceforth assume that $c \neq i$ and $b \neq k$.

Suppose first that $|R(u) \cap C(u)| = 1$. If u moves at least one element of the diagonal of $[n]^2$ then either $i = a$, or $i = b$, or $k = c$, and one can verify that the hypotheses of Lemma 5.3 are satisfied, so u is not stable. If $i \neq a$, $i \neq b$, and $k \neq c$, then u leaves the diagonal fixed and thus $k = a$. Then u satisfies the equivalent conditions of Theorem 5.2.

Suppose now that $|R(u) \cap C(u)| = 2$. Then, recalling that $c \neq i$ and $b \neq k$, we have that either $(i,k) = (a,c)$, or $(i,k) = (b,c)$, or $(i,k) = (b,a)$. In the first two cases one can verify that the hypotheses of Lemma 5.3 are satisfied and the result follows. Otherwise $u = ((i,k),(i,i),(k,c))$ so

$$\psi_{2h-1}(u)(i,\underbrace{i,k,\ldots,i,k}_{2h}) = (\underbrace{i,k,\ldots,i,k}_{2h},i)$$

for all $h \in \mathbb{N}$ and u is not stable. □

We now turn our attention to the 3-cycles such that $|R(u)| = 3$.

Proposition 5.9. *Let $u \in S([n]^2)$ be a 3-cycle such that $|R(u)| = 3$, $|C(u)| = 1$. Then either u is stable of rank 1 or u is not stable.*

Proof. If $R(u) \cap C(u) = \emptyset$ then u is stable of rank 1 by Theorem 3.1. If $|R(u) \cap C(u)| = 1$ then we may assume that $i = a = b = c$, so $u = ((i,i),(j,i),(k,i))$ and u is not stable by Lemma 5.5. □

We next analyze the case $(|R(u)|,|C(u)|) = (3,2)$.

Proposition 5.10. *Let $u \in S([n]^2)$ be a 3-cycle such that $|R(u)| = 3$, $|C(u)| = 2$. Then either u is not stable or u is stable of rank ≤ 2.*

Proof. Let $u = ((i,a),(j,b),(k,c))$. Since $|C(u)| = 2$ we may assume that $b = c$ so $u = ((i,a),(j,b),(k,b))$. By Theorem 3.1 we may assume that $|R(u) \cap C(u)| \geq 1$.

If $|R(u) \cap C(u)| = 1$ then either $a \in R(u)$ or $b \in R(u)$. If $a = i$ then $u = ((i,i),(j,b),(k,b))$ so u is not stable by Lemma 5.3. If $a = j$ then

$u = ((i,j),(j,b),(k,b))$ so u is not stable by Lemma 5.4. If $a = k$ then $u = ((i,k),(j,b),(k,b))$ so u satisfies the equivalent conditions of Theorem 5.2. If $b = i$ then $u = ((i,a),(j,i),(k,i))$ so u is not stable by Lemma 5.4. If $b = j$ then $u = ((i,a),(j,j),(k,j))$ so u is not stable by Lemma 5.5. If $b = k$ then $u = ((i,a),(j,k),(k,k))$ so u is not stable by Lemma 5.5.

If $|R(u) \cap C(u)| = 2$ then $\{a,b\} \subseteq R(u)$. If $(a,b) = (i,j)$ then $u = ((i,i),(j,j),(k,j))$ so u is not stable by Lemma 5.5. If $(b,a) = (i,j)$ then $u = ((i,j),(j,i),(k,i))$ and u is not stable by Lemma 5.4. If $(a,b) = (i,k)$ then $u = ((i,i),(j,k),(k,k))$ so u is not stable by Lemma 5.5. If $(b,a) = (i,k)$ then $u = ((i,k),(j,i),(k,i))$ so u is not stable by Lemma 5.4. If $(a,b) = (j,k)$ then $u = ((i,j),(j,k),(k,k))$ and u is not stable by Lemma 5.5. Finally, if $(b,a) = (j,k)$ then $u = ((i,k),(j,j),(k,j))$ so u is not stable by Lemma 5.5. □

Finally, we analyze the case $(|R(u)|,|C(u)|) = (3,3)$. For $u \in S([n]^2)$ and $i \in [n+1]$ recall (see [3, Sec. 7]) that the i-th immersion of u is the permutation $u^{(i)} \in S([n+1]^2)$ defined by

$$u^{(i)}(x_1,x_2) := \begin{cases} (x_1,x_2), & \text{if } x_j = i \text{ for some } j \in [2], \\ u((x_1,x_2)^{(i)})^{<i>}, & \text{otherwise}, \end{cases}$$

for all $x_1, x_2 \in [n+1]$, where $a^{(i)} = a - \chi(a \geq i)$ and $a^{<i>} = a + \chi(a \geq i)$ for $a \in \mathbb{N}$, and $(x_1,\ldots,x_r)^{<i>} := (x_1^{<i>},\ldots,x_r^{<i>})$ for $(x_1,\ldots,x_r) \in \mathbb{N}^r$ and similarly for $(x_1,\ldots,x_r)^{(i)}$.

Proposition 5.11. *Let $u \in S([n]^2)$ be a 3-cycle such that $|R(u)| = |C(u)| = 3$. Then either u is not stable or u is stable of rank ≤ 2.*

Proof. We may assume that $u = ((i,a),(j,b),(k,c))$ where a,b,c are distinct, and so are i,j,k. If $R(u) \cap C(u) = \emptyset$ then by Theorem 3.1 u is stable of rank 1, so we may assume that $|R(u) \cap C(u)| \geq 1$.

Suppose first that $|R(u) \cap C(u)| = 3$. Then $R(u) = C(u)$ so u is the immersion of a 3-cycle v in $S([3]^2)$ such that $R(v) = C(v) = [3]$. But one can check that there are only 6 stable 3-cycles in $S([3]^2)$ (namely the ones listed in of [3, Theorem 10.2]) and none of these has $R(v) = C(v) = [3]$. So v is not stable, and hence, by [3, Theorem 7.8], u is not stable.

Suppose now that $|R(u) \cap C(u)| = 2$. We claim that in this case u is not stable. To see this one may assume that $i,j \in \{a,b,c\}$. If $c = i$ then $u(k,i) = (i,a)$ and $u(a,a) = (a,a)$ so by Lemma 5.4 u is not stable. Similarly, if $a = j$ then again by Lemma 5.4 u is not stable. We may henceforth assume that $a \neq j$, and $c \neq i$. We therefore have three cases to consider, namely $(i,j) = (a,b)$, $(i,j) = (a,c)$, and $(i,j) = (b,c)$. If $(i,j) = (a,b)$ then $u(i,i) = (j,j)$,

$j \neq i$, and $c \notin R(u)$, so by Lemma 5.3 u is not stable. If $(i,j) = (a,c)$ then $u = ((i,i),(j,b),(k,j))$ and one can compute that

$$\psi_{2p-1}(u)(i,i,b,\underbrace{k,j,\ldots,k,j}_{2p-2}) = (k,i,\underbrace{k,j,\ldots,k,j}_{2p-2},i)$$

for all $p \in \mathbb{N}$, so u is not stable. Finally, if $(i,j) = (b,c)$, then $u = ((i,a),(j,i),(k,j))$, and one can compute that

$$\psi_{4p-2}(u)(\underbrace{k,i,a,a,\ldots,k,i,a,a}_{4p}) = (j,i,\underbrace{k,i,a,a,\ldots,k,i,a,a}_{4p-4},k,j)$$

for all $p \in \mathbb{N}$, so u again is not stable.

Finally, suppose that $|R(u) \cap C(u)| = 1$. We may assume that $i \in \{a,b,c\}$. If $i = a$ then u is not stable by Lemma 5.3. If $i = b$ then u is stable of rank 2 by Theorem 5.2. If $i = c$ then u is not stable by Lemma 5.4. $\square$

The analysis carried out so far enables us to conclude that the following holds.

Theorem 5.12. *Let $u \in S([n]^2)$ be a 3-cycle. Then the following conditions are equivalent:*

i) u is stable;

ii) u is stable of rank ≤ 2;

iii) u is a compatible product of two stable transpositions.

Moreover, u is stable of rank 1 if and only if $R(u) \cap C(u) = \emptyset$, and u is stable of rank 2 if and only if the conditions in (9) of Theorem 5.2 hold. In particular, if u is stable of rank 2 then $|R(u) \cap C(u)| = 1$.

Proof. It is clear that ii) implies i), and it follows from Theorems 2.2 and 2.5 that iii) implies ii). So assume that i) holds. Let $u := ((a_1,b_1),(a_2,b_2),(a_3,b_3))$ ($(a_1,b_1),(a_2,b_2),(a_3,b_3) \in [n]^2$, distinct). If $R(u) \cap C(u) = \emptyset$ then u is stable of rank 1 by Theorem 3.1. Furthermore, we have that

$$u = ((a_1,b_1),(a_2,b_2))\,((a_2,b_2),(a_3,b_3))$$

and these two transpositions are stable of rank 1 by our hypotheses and Theorem 2.5, and are compatible by [3, Proposition 5.15]. If $|R(u) \cap C(u)| \geq 1$ then the analysis carried out in this section shows that u always satisfies the conditions of Theorem 5.2 so iii) holds. $\square$

The preceding result enables us to enumerate the stable 3-cycles in $S([n]^2)$.

Corollary 5.13. *In $S([n]^2)$ there are*

$$n(n-1)(n-2)(n^2-5n+7)$$

stable 3*-cycles of rank* 2*. Thus, there are*

$$n(n-1)(n-2)(n^3-3n^2-2n+9)/3$$

stable 3*-cycles.*

Proof. Note first that the conditions in (9) of Theorem 5.2, for any given $i \in [3]$, imply that $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ are all distinct. Furthermore these conditions are mutually exclusive for $i \in [3]$. So assume that they hold for $i = 1$. We then have n choices for $a_1 = b_2$. If $a_2 = a_3$ then, in order to satisfy the condition that $b_2 \notin \{a_2, a_3\}$ we have $n-1$ choices for this common value, and, since $b_1, b_3 \notin \{a_1, a_2, a_3\}$, $(n-2)^2$ choices for the pair (b_1, b_3). If $a_2 \neq a_3$ then, in order to satisfy the condition that $b_2 \notin \{a_2, a_3\}$ we have $(n-1)(n-2)$ choices for the pair (a_2, a_3), and, since $b_1, b_3 \notin \{a_1, a_2, a_3\}$, $(n-3)^2$ choices for the pair (b_1, b_3). Therefore there are in total $n(n-1)(n-2)^2 + n(n-1)(n-2)(n-3)^2$ possibilities.

The second statement follows from the first one and [3, Corollary 8.8] (see also Proposition 4.5). □

6 Beyond 3-cycles

In this section we begin the analysis of the stable 4-cycles of $S([n]^2)$, building upon the results in the previous section. More precisely, we characterize the 4-cycles of $S([n]^2)$ that are a compatible product of a stable transposition and a stable 3-cycle in the two possible orders.

Proposition 6.1. *Let $r \in \mathbb{N}$, $r \geq 3$, $(a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r) \in [n]^2$ be distinct, and $w := ((a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r))$ be such that there is $i \in [r]$ so that $a_k = b_j$ if and only if $j = i+1$, and $k = i$, for all $k, j \in [r]$ (where indices are taken modulo r). Then w is stable of rank $\leq r$, and w is a compatible product of $r-1$ stable transpositions.*

Proof. Let, for brevity $P_j := (a_j, b_j)$ for $j \in [r]$. We may clearly assume that $a_2 = b_3$. We proceed by induction on r. If $r = 3$ then we have that $w = uv$ where $u = (P_1, P_3)$, and $v = (P_1, P_2)$. Furthermore, by our hypotheses and [3, Theorem 8.1], u and v are stable and, by [3, Proposition 5.6], u is compatible with v, so the claim follows from [3, Theorem 5.2].

Assume now that $r \geq 4$. Then we have that $w = uv$ where $u = (P_1, P_r)$, and $v = (P_1, P_2, \ldots, P_{r-1})$. Since $r \geq 4$, we have by induction, our hypotheses, and [3, Theorem 8.1] that u is stable, v is stable of rank $\leq r-1$, and v is a compatible product of $r-2$ stable transpositions. Furthermore, again by our hypotheses, $v(x, a_1) = (x, a_1)$ and $v(x, a_r) = (x, a_r)$ for all $x \in [n]$, so by Propositions 5.6 and 5.7 of [3] u is compatible with v, and the result follows from [3, Theorem 5.2]. □

Theorem 6.2. *Let $(a_1, b_1), \ldots, (a_4, b_4) \in [n]^2$ be distinct, and $w := ((a_1, b_1), \ldots, (a_4, b_4))$ be a 4-cycle such that $R(w) \cap C(w) \neq \emptyset$. Then w is a compatible product of a stable transposition and a stable 3-cycle, in this order, if and only if $a_i, a_{i+1} \notin C(w)$, and either*

$$\{a_{i+2}, a_{i+3}\} \cap \{b_{i+1}, b_{i+2}, b_{i+3}\} = \emptyset, \tag{13}$$

or

$$b_{i+3} = a_{i+2}, \quad b_{i+3} \neq a_{i+3}, \quad \{b_{i+2}, b_{i+1}\} \cap \{a_{i+2}, a_{i+3}\} = \emptyset \tag{14}$$

or

$$b_{i+1} = a_{i+3}, \quad b_{i+1} \neq a_{i+2}, \quad \{b_{i+3}, b_{i+2}\} \cap \{a_{i+2}, a_{i+3}\} = \emptyset \tag{15}$$

for some $i \in [4]$ (where indices are taken modulo 4). In particular, if these conditions are satisfied, $a_{i+2} \notin \{b_{i+1}, b_{i+2}\}$, and $a_{i+3} \notin \{b_{i+2}, b_{i+3}\}$. Also, if (13) holds, then $b_i \in \{a_{i+2}, a_{i+3}\}$.

Proof. Let $u, v \in S([n]^2)$ be such that u is a stable transposition, v is a stable 3-cycle, u is compatible with v, and $w = uv$. Say $v = (P_2, P_3, P_4)$ ($P_2, P_3, P_4 \in [n]^2$, distinct), and $u = (A, B)$ ($A, B \in [n]^2$, distinct). Since uv is a 4-cycle $|\{A, B\} \cap \{P_2, P_3, P_4\}| = 1$. We may assume that $B = P_2$, so $w = (A, P_2)(P_2, P_3, P_4)$. Let $P_1 := A$, so $w = (P_1, P_2, P_3, P_4) = (P_1, P_2)(P_2, P_3, P_4)$. Without loss of generality it is enough to consider the case $(a_i, b_i) := P_i$ for $i \in [4]$.

By [3, Theorem 8.1] u is stable if and only if

$$\{a_1, a_2\} \cap \{b_1, b_2\} = \emptyset. \tag{16}$$

Similarly, by Theorem 5.12, v is stable if and only if either

$$\{a_2, a_3, a_4\} \cap \{b_2, b_3, b_4\} = \emptyset, \tag{17}$$

or

$$b_3 = a_2, \quad \{b_2, b_4\} \cap \{a_2, a_3, a_4\} = \emptyset, \quad b_3 \notin \{a_3, a_4\} \tag{18}$$

or

$$b_4 = a_3, \quad \{b_3, b_2\} \cap \{a_2, a_3, a_4\} = \emptyset, \quad b_4 \notin \{a_4, a_2\} \tag{19}$$

or

$$b_2 = a_4, \quad \{b_4, b_3\} \cap \{a_2, a_3, a_4\} = \emptyset, \quad b_2 \notin \{a_2, a_3\}. \tag{20}$$

Finally, since u is compatible with v, we have by Propositions 5.6 and 5.7 of [3] that there are $\sigma \in S_n$, and $(\varepsilon_1, \dots, \varepsilon_n) \in S(\{a_1, a_2\})^n$ such that

$$v(x, a_1) = (\sigma(x), \varepsilon_x(a_1)), \quad v(x, a_2) = (\sigma(x), \varepsilon_x(a_2)) \tag{21}$$

for all $x \in [n]$ (where $\varepsilon_1 = \cdots = \varepsilon_n = Id$ if $b_1 \neq b_2$). In particular, v must leave the union of the columns indexed by a_1 and a_2 invariant. However, by (16), P_2 is not in this union, so, since v is a 3-cycle, v leaves every element of this union fixed. So we conclude that

$$v(x, a_1) = (x, a_1), \quad v(x, a_2) = (x, a_2) \tag{22}$$

for all $x \in [n]$. This, in turn, is equivalent to

$$b_2, b_3, b_4 \notin \{a_1, a_2\}. \tag{23}$$

Conversely, if (22) holds then, by Propositions 5.6 and 5.7 of [3], u is compatible with v. The result follows noting that (18) is impossible by (23).

Note that, if any one of (17), (19) or (20) is satisfied then $a_3 \notin \{b_2, b_3\}$ and $a_4 \notin \{b_3, b_4\}$. Finally, suppose that (17) holds. Then, since $R(w) \cap C(w) \neq \emptyset$ we have that either $a_1 \in \{b_3, b_4\}$ or $b_1 \in \{a_3, a_4\}$. So, by (23), we have that $b_1 \in \{a_3, a_4\}$. □

It is possible to enumerate the 4-cycles satisfying the conditions in Theorem (6.2). To this end, it is convenient to introduce the following sets

$$D_n := \Big\{(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) \in [n]^8 \mid (a_1, b_1), \dots, (a_4, b_4) \text{ mutually distinct}, \{a_1, \dots, a_4\} \cap \{b_1, \dots, b_4\} \neq \emptyset\Big\}$$

$$U_i = \Big\{(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) \in D_n : a_i, a_{i+1} \notin \{b_1, b_2, b_3, b_4\}, \{a_{i+1}, a_{i+2}, a_{i+3}\} \cap \{b_{i+1}, b_{i+2}, b_{i+3}\} = \emptyset\Big\}$$

$$V_i = \Big\{(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) \in D_n : a_i, a_{i+1} \notin \{b_1, b_2, b_3, b_4\}, a_{i+2} = b_{i+3}, \{b_{i+2}, b_{i+1}\} \cap \{a_{i+1}, a_{i+2}, a_{i+3}\} = \emptyset, \ b_{i+3} \notin \{a_{i+3}, a_{i+1}\}\Big\}$$

$$W_i = \Big\{(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) \in D_n : a_i, a_{i+1} \notin \{b_1, b_2, b_3, b_4\}, a_{i+3} = b_{i+1}, \{b_{i+3}, b_{i+2}\} \cap \{a_{i+1}, a_{i+2}, a_{i+3}\} = \emptyset, \ b_{i+1} \notin \{a_{i+1}, a_{i+2}\}\Big\}$$

for $i \in [4]$ (of course, indices are taken modulo 4). For brevity, we let $(n)_k := n(n-1)\cdots(n-k+1)$ for $k \in \mathbb{N}$.

Proposition 6.3. *The number of 4-cycles in $S([n]^2)$ satisfying the conditions in Theorem 6.2 is equal to*

$$2(n)_4(n^3 - 7n^2 + 17n - 13).$$

Proof. First observe that for a fixed i the three sets U_i, V_i and W_i are mutually disjoint. We set $Z_i := U_i \cup V_i \cup W_i$. Hence, the cardinality of Z_i is equal to the sum of the cardinalities of U_i, V_i and W_i. Moreover, one has $Z_i \cap Z_{i+2} = \emptyset$. It is clear that the cycles as in the statement arise from the elements in the union of $\cup_{i=1}^4 Z_i$ and, taking into account cyclic order, their number is given by the cardinality of $\cup_{i=1}^4 Z_i$ divided by 4.

For $i, j, l, m \in [4]$, let

$$L_{ij} = \Big\{ (a_1, \ldots, b_4) \in D_n \ : \ a_i = b_j \text{ and } a_h \neq b_k \text{ for all } (h,k) \neq (i,j) \ \Big\}$$

$$L_{ij,lm} = \Big\{ (a_1, \ldots, b_4) \in D_n \ : \ a_i = b_j, \quad a_l = b_m, \quad \text{and } a_h \neq b_k \text{ for all } (h,k) \notin \{(i,j),(l,m)\} \ \Big\}.$$

Note that $L_{ij,lm} = L_{lm,ij}$. Then $U_1 = L_{31} \cup L_{41} \cup L_{31,41}$, $U_2 = L_{12} \cup L_{42} \cup L_{12,42}$, $U_3 = L_{13} \cup L_{23} \cup L_{13,23}$, $U_4 = L_{24} \cup L_{34} \cup L_{24,34}$, $V_1 = L_{34} \cup L_{31,34} \cup L_{41,34}$, $V_2 = L_{41} \cup L_{41,42} \cup L_{41,12}$, $V_3 = L_{12} \cup L_{12,13} \cup L_{12,23}$, $V_4 = L_{23} \cup L_{23,24} \cup L_{23,34}$, $W_1 = L_{42} \cup L_{42,41} \cup L_{42,31}$, $W_2 = L_{13} \cup L_{13,12} \cup L_{13,42}$, $W_3 = L_{24} \cup L_{24,23} \cup L_{24,13}$, $W_4 = L_{31} \cup L_{31,34} \cup L_{31,24}$ (all disjoint unions).

We begin by computing the cardinality of L_{23}. The computation of the cardinality of L_{ij} $(i \neq j)$ is identical. Here, we have that b_1, b_2, b_4, and a_1, a_3, a_4 are all different from $a_2 = b_3$. There are five cases. If b_1, b_2, b_4 are mutually distinct, we have $(n)_4\,(n-4)^3$ different choices for $a_1, \ldots, b_4$. Indeed there are $(n)_4$ possibilities for $b_1, \ldots, b_4$ and, for each one of these, $(n-4)^3$ possibilities for a_1, a_3, a_4 since they must be different from the b's; as the b's are all distinct, it is also clear that the four pairs (a_i, b_i) are mutually distinct. If $b_1 = b_2, b_4$ are distinct, we similarly have $(n)_3\,(n-3)^3$ choices; as $a_1 \neq a_2$, again the (a_i, b_i) are automatically mutually distinct. If $b_2 = b_4, b_1$ are distinct, we conclude as above that there are $(n)_3\,(n-3)^3$ choices. If $b_1 = b_4, b_2$ are distinct, there are $(n)_3(n-3)^2(n-4)$ choices; the computation is similar except that now a_1 must be different from a_4. If $b_1 = b_2 = b_4$ there are $(n)_2\ (n-2)^2(n-3)$ choices since, again, a_1 and a_4 must be different. All in all, the cardinality of L_{23} is

$$(n)_4\,(n-4)^3 + 2(n)_3\,(n-3)^3 + (n)_3(n-3)^2(n-4) + (n)_2\ (n-2)^2(n-3)\ .$$

Next, we compute the cardinality of $L_{23,24}$ (and, more generally, of $L_{ij,il}$). Here, we have $a_2 = b_3 = b_4$ and a_1, a_3, a_4, b_1, b_2 are all different from this value. There are two cases. If $b_1 \neq b_2$, there are $(n)_3\ (n-3)^2(n-4)$ choices; indeed, we must have $a_3 \neq a_4$. If $b_1 = b_2$ there are similarly $(n)_2\ (n-2)^2(n-3)$ choices. All in all, the cardinality of $L_{23,24}$ is

$$(n)_3\ (n-3)^2(n-4) + (n)_2\ (n-2)^2(n-3)\ .$$

Note that a simple bijection shows that $|L_{ij,il}| = |L_{ji,li}|$.

Finally, we compute the cardinality of $L_{41,34}$ (and, more generally, of $L_{ij,lm}$ with $i \neq l$, $j \neq m$). Here, $b_4 = a_3$, $b_1 = a_4$, $b_1 \neq b_4$, and a_1, a_2, b_2, b_3 are different from b_1, b_4. There are two cases. If $b_2 = b_3$ there are $(n)_3\ (n-3)^2$ possibilities. If $b_2 \neq b_3$ there are $(n)_4\ (n-4)^2$ possibilities. Thus, the cardinality of $L_{41,34}$ is

$$(n)_3\ (n-3)^2 + (n)_4\ (n-4)^2\ .$$

It readily follows that the cardinality of U_i is

$$2\Big((n)_4\,(n-4)^3 + 2(n)_3\,(n-3)^3 + (n)_3(n-3)^2(n-4) + (n)_2\ (n-2)^2(n-3)\Big) \\ + (n)_3\ (n-3)^2(n-4) + (n)_2\ (n-2)^2(n-3)$$

while that of V_i and W_i is

$$(n)_4\,(n-4)^3 + 2(n)_3\,(n-3)^3 + (n)_3(n-3)^2(n-4) + (n)_2\ (n-2)^2(n-3) \\ + (n)_3\ (n-3)^2(n-4) + (n)_2\ (n-2)^2(n-3) + (n)_3\ (n-3)^2 + (n)_4\ (n-4)^2$$

for all $i \in [4]$. Therefore, for each i the cardinality of Z_i is $|U_i| + 2|V_i|$. Note that, since $Z_1 \cap Z_3 = Z_2 \cap Z_4 = \emptyset$, the intersection of any three or more distinct Z_i's is empty. Hence, by the Principle of Inclusion-Exclusion, $|\cup_{i=1}^4 Z_i| = 4|Z_1| - |Z_1 \cap Z_2| - |Z_1 \cap Z_4| - |Z_2 \cap Z_3| - |Z_3 \cap Z_4|$. These four intersections all have the same cardinality, so we compute $|Z_1 \cap Z_2|$. We have, from our definitions of Z_i that $Z_1 \cap Z_2 = (U_1 \cap U_2) \cup (U_1 \cap V_2) \cup (U_1 \cap W_2) \cup (V_1 \cap U_2) \cup (V_1 \cap V_2) \cup (V_1 \cap W_2) \cup (W_1 \cap U_2) \cup (W_1 \cap V_2) \cup (W_1 \cap W_2)$. It is not hard to check that all of these pairwise intersections are empty except that $U_1 \cap V_2 = L_{41}$, $W_1 \cap U_2 = L_{42}$, and $W_1 \cap V_2 = L_{41,42}$, and these three sets are mutually disjoint. Hence we conclude that $|\cup_{i=1}^4 Z_i| = 4|Z_1| - 4|Z_1 \cap Z_2| = 8|L_{1,2}| + 8|L_{41,42}| + 8|L_{12,34}|$. The conclusion follows at once. □

Theorem 3.1 shows that there are stable 4-cycles of rank 1. The preceding result implies that other values of the rank are also possible.

Corollary 6.4. *Let $(a_1, b_1), \ldots, (a_4, b_4) \in [n]^2$ be distinct, and $w := ((a_1, b_1), \ldots, (a_4, b_4))$ be a 4-cycle satisfying the hypotheses of Theorem 6.2. Then the rank of w is 2 if and only if $w \in U_i \cup L_{i+2\,i,i+2\,i+3}$ for some $i \in [4]$ (indices modulo 4), and is 3 otherwise.*

Proof. Let $u, v \in S([n]^2)$ be such that u is a stable transposition, v is a stable 3-cycle, u is compatible with v, and $w = uv$. Then by [3, Theorem 8.1] u has rank 1, and by Theorem 5.12 v has rank ≤ 2. So, by [3, Theorem 5.2] w has rank ≤ 3.

If $w \in U_i$ for some $i \in [4]$ then by Theorem 3.1, v has rank 1, so by [3, Theorem 5.2] w has rank ≤ 2. But, again by Theorem 3.1, w has rank > 1.

Suppose now that $w \in L_{i+2\,i,i+2\,i+3}$ for some $i \in [4]$. We assume, for simplicity, that $i = 1$. Then the statement would follow by a straightforward, albeit lengthy, computation by considering several cases. To avoid this long checking we use a powerful result from the next section. Namely, $w^{\#} = ((b_1, a_1)(b_4, a_4)(b_3, a_3)(b_2, a_2))$ is stable of rank r if and only if w is stable of rank r, see Theorem 7.3. Since $w \in L_{3\,1,3\,4}$, we have $w^{\#} \in L_{1\,2,4\,2} \subset U_4$. Hence, $w^{\#}$ and w both have rank 2 by the previous case.

Conversely, suppose that $w \notin U_1 \cup U_2 \cup U_3 \cup U_4 \cup L_{31,34} \cup L_{42,41} \cup L_{13,12} \cup L_{24,23}$. Since w satisfies the conditions in Theorem 6.2 there is $i \in [4]$ such that either $w \in V_i$ or $w \in W_i$. But

$$V_i \setminus (U_1 \cup U_2 \cup U_3 \cup U_4) = V_i \setminus U_{i+3} = L_{i+2\,i,i+2\,i+3} \cup L_{i+3\,i,i+2\,i+3}$$

and similarly

$$W_i \setminus (U_1 \cup U_2 \cup U_3 \cup U_4) = W_i \setminus U_{i+3} = L_{i+3\,i,i+3\,i+1} \cup L_{i+2\,i,i+3\,i+1}.$$

Hence, in order to finish the proof, it is sufficient to show that if $w \in L_{i+3\,i,i+2\,i+3} \cup L_{i+2\,i,i+3\,i+1}$ for some $i \in [4]$ then w has rank three. We may assume that $i = 1$.

In the first case, the 4-cycle takes the form

$$w = \big((a_1, a_4), (a_2, b_2), (a_3, b_3), (a_4, a_3)\big) \ ;$$

now, if we compute $\psi_2(w)(a_1, a_4, a_2, b_2)$ and $(\psi_1(w) \otimes 1)(a_1, a_4, a_2, b_2)$, we easily get (a_1, a_4, a_2, b_2) and (a_4, a_3, a_2, b_2), respectively, thus showing that w has rank larger than two as $a_1 \neq a_4$.

In the second case, the 4-cycle takes the form

$$w = \big((a_1, a_3), (a_2, a_4), (a_3, b_3), (a_4, b_4)\big) \ ;$$

we check that $\psi_2(w)(a_1, a_2, a_3, b_3) = (a_4, b_4, b_3, b_4)$ while $(\psi_1(w) \otimes 1)(a_1, a_2, a_3, b_3)$ is equal to (a_1, a_1, a_3, b_3) when $a_1 = a_2$ and to (a_1, a_2, a_3, b_3) when $a_1 \neq a_2$; since $a_1 \neq a_4$, w has rank larger than two. $\square$

There is a version of the previous result also in the case that the 4-cycle w is a compatible product of a stable 3-cycle and a stable transposition, in this order. To see this note first the following "symmetric" versions of Propositions 5.6 and 5.7 of [3]. Their proofs are exactly analogous to those of Propositions 5.6 and 5.7 in [3] and are therefore omitted.

Proposition 6.5. *Let $u = ((a,b),(i,j))$, where $a,b,i,j \in [n]$, $a \neq i$, and $v \in S([n]^2)$. Then $(u \otimes 1)(1 \otimes v) = (1 \otimes v)(u \otimes 1)$ if and only if there is $\sigma \in S_n$ such that $v(k,x) = (k,\sigma(x))$ for all $x \in [n]$ and all $k \in \{b,j\}$.*

Proposition 6.6. *Let $u = ((a,b),(a,j))$, where $a,b,j \in [n]$, $b \neq j$, and $v \in S([n]^2)$. Then $(u \otimes 1)(1 \otimes v) = (1 \otimes v)(u \otimes 1)$ if and only if there are $\sigma \in S_n$ and $\varepsilon \in S(\{b,j\})^n$ such that*

$$v(b,x) = (\varepsilon_x(b),\sigma(x)), \quad v(j,x) = (\varepsilon_x(j),\sigma(x))$$

for all $x \in [n]$.

Corollary 6.7. *Let $u = ((a,b),(i,j))$, where $(a,b),(i,j) \in [n]^2$, distinct, and $v \in S([n]^2)$ be a 3-cycle. Then $(u \otimes 1)(1 \otimes v) = (1 \otimes v)(u \otimes 1)$ if and only if*

$$v(k,x) = (k,x)$$

for all $x \in [n]$ and all $k \in \{b,j\}$.

Theorem 6.8. *Let $(a_1,b_1),\dots,(a_4,b_4) \in [n]^2$ be distinct, and $w := ((a_1,b_1),\dots,(a_4,b_4))$ be a 4-cycle such that $R(w) \cap C(w) \neq \emptyset$. Then w is a compatible product of a stable 3-cycle, and a stable transposition in this order, if and only if $b_i, b_{i+3} \notin R(w)$, and either*

$$\{a_{i+1},a_{i+2},a_{i+3}\} \cap \{b_{i+1},b_{i+2},b_{i+3}\} = \emptyset, \tag{24}$$

or

$$b_{i+2} = a_{i+1}, \quad b_{i+2} \notin \{a_{i+2},a_{i+3}\}, \quad b_{i+1} \notin \{a_{i+1},a_{i+2},a_{i+3}\} \tag{25}$$

or

$$b_{i+1} = a_{i+3}, \quad b_{i+2} \notin \{a_{i+1},a_{i+2},a_{i+3}\}, \quad b_{i+1} \notin \{a_{i+1},a_{i+2}\}, \tag{26}$$

for some $i \in [4]$ (where indices are taken modulo 4). In particular, if these conditions are satisfied, $b_{i+1} \notin \{a_{i+1},a_{i+2}\}$, and $b_{i+2} \notin \{a_{i+2},a_{i+3}\}$. Also, if (24) holds, then $a_i \in \{b_{i+1},b_{i+2}\}$.

Proof. Let $w = vu$ with u a stable transposition, v a stable 3-cycle, and v compatible with u. Then reasoning as in the proof of Theorem 6.2 we conclude that $v = ((a_2, b_2), (a_3, b_3), (a_4, b_4))$, and $u = ((a_1, b_1), (a_4, b_4))$. Since u and v are stable, and v is compatible with u we conclude from [3, Theorem 8.1], Theorem 6.2, and Corollary 6.7 that

$$\{a_1, a_4\} \cap \{b_1, b_4\} = \emptyset,$$
$$b_1, b_4 \notin \{a_2, a_3, a_4\} \tag{27}$$

and either

$$\{a_2, a_3, a_4\} \cap \{b_2, b_3, b_4\} = \emptyset, \tag{28}$$

or

$$b_3 = a_2, \quad \{b_2, b_4\} \cap \{a_2, a_3, a_4\} = \emptyset, \quad b_3 \notin \{a_3, a_4\} \tag{29}$$

or

$$b_4 = a_3, \quad \{b_3, b_2\} \cap \{a_2, a_3, a_4\} = \emptyset, \quad b_4 \notin \{a_4, a_2\} \tag{30}$$

or

$$b_2 = a_4, \quad \{b_4, b_3\} \cap \{a_2, a_3, a_4\} = \emptyset, \quad b_2 \notin \{a_2, a_3\}. \tag{31}$$

Note that , by (27), equation (30) cannot hold. The conclusion follows easily from these equations. □

Note that a 4-cycle $((a_1, b_1), \ldots, (a_4, b_4)) \in S([n]^2)$ satisfies the conditions in Theorem 6.2 if and only if the 4-cycle $((b_4, a_4), \ldots, (b_1, a_1))$ satisfies those in Theorem 6.8. Indeed, if the 4-cycle $w \in S([n]^2)$ is such that $w = uv$ with u a stable transposition, v a stable 3-cycle, and u compatible with v, then ${}^t(w^{-1}) =^t (v^{-1})^t(u^{-1})$ and $(v \otimes 1)(1 \otimes u) = (1 \otimes u)(v \otimes 1)$. Therefore, by [3, Proposition 5.13], $({}^t(u^{-1}) \otimes 1)(1 \otimes {}^t(v^{-1})) = (1 \otimes {}^t(v^{-1}))({}^t(u^{-1}) \otimes 1)$, so ${}^t(v^{-1})$ is compatible with ${}^t(u^{-1})$. But, by [3, Theorem 8.1], ${}^t(u^{-1}) =^t u$ is a stable transposition and, by Theorem 7.3, ${}^t(v^{-1})$ is a stable 3-cycle.

We are confident that a "symmetric" version of Corollary 6.4 holds also for the 4-cycles satisfying the conditions in Theorem 6.8. We omit carrying this out in order to keep the paper to a reasonable size.

To continue the investigation of stable 4-cycles one would now need to see if a stable 4-cycle necessarily satisfies either the conditions of Theorem 6.2 or those of Theorem 6.8.

By our results, the following conjecture has been verified for $r = 2, 3$, and partially for $r = 4$.

Conjecture 6.9. *Let $u \in S([n]^2)$ be a stable r-cycle, then the rank of u belongs to the set $\{1, 2, \ldots, r-1\}$.*

7 An involution on stable permutations

In this section we show that the set of stable permutations possesses a natural and non-trivial involution, which preserves the rank and the cycle-type (so the conjugacy class). This involution extends the symmetries of the stable permutations discussed in [3, Sec. 6] and moreover fits well with the content of the previous section.

For $u \in S([n]^r)$ write

$$(u_1(x_1, \ldots, x_r), \ldots, u_r(x_1, \ldots, x_r)) := u(x_1, \ldots, x_r)$$

for all $(x_1, \ldots, x_r) \in [n]^r$. We let ${}^t u \in S([n]^r)$ be the transposed permutation defined by

$$ {}^t u(x_1, \ldots, x_r) := (u_r(x_r, \ldots, x_1), \ldots, u_1(x_r, \ldots, x_1)), \tag{32}$$

for all $(x_1, \ldots, x_r) \in [n]^r$. Note that ${}^t u = u$ if $r = 1$. Equivalently, ${}^t u = w_0^{(r)} u w_0^{(r)}$, where $w_0^{(r)} \in S([n]^r)$ is defined by $w_0^{(r)}(x_1, \ldots, x_r) := (x_r, \ldots, x_1)$ for all $(x_1, \ldots, x_r) \in [n]^r$. Note that this concept coincides (for $r = 2$) with the one by the same name defined in [3, Sec. 3] (for $n = m$).

We note the following simple properties of the transpose.

Proposition 7.1. *Let $u, v \in S([n]^r)$. Then*

i) ${}^t(uv) = {}^t u {}^t v$*;*

ii) ${}^t(u \otimes v) = {}^t v \otimes {}^t u$*.*

Proof. The first identitiy follows readily from the fact that $w_0^{(r)}$ is an involution. The second one is a straightforward computation using the definition. □

Note that, if $u \in S([n]^r)$, then ${}^t(u^{-1}) = ({}^t u)^{-1}$. Indeed, by Proposition 7.1 $uv = 1$ if and only if ${}^t u {}^t v = 1$. In particular, the map $u \mapsto {}^t(u^{-1})$ is an involution. For $u \in S([n]^r)$ let $u^\# := {}^t(u^{-1}) \in S([n]^r)$.

We then have the following consequence of the previous result.

Proposition 7.2. *Let $u, v \in S([n]^2)$. Then*

i) $(uv)^\# = v^\# u^\#$*;*

ii) u is compatible with v if and only if $v^\#$ is compatible with $u^\#$.

Proof. The first point is clear from Proposition 7.1. If u is compatible with v then $(v \otimes 1)(1 \otimes u) = (1 \otimes u)(v \otimes 1)$. Therefore, by Proposition 7.1, $({}^t(u^{-1}) \otimes 1)(1 \otimes {}^t(v^{-1})) = (1 \otimes {}^t(v^{-1}))({}^t(u^{-1}) \otimes 1)$, so $v^\#$ is compatible with $u^\#$. □

We can now prove the main result of this section.

Theorem 7.3. *Let $u \in S([n]^t)$. Then u is stable of rank r if and only if $u^{\#}$ is stable of rank r.*

Proof. Let, for brevity, $v = u^{\#}$. We begin by noting that

$$\begin{aligned}(\psi_k(u))^{\#} \quad = \quad & (v \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k})(1 \otimes v \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k-1}) \cdots (\underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes v \otimes 1) \\ & (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes v^{-1})(\underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes v^{-1} \otimes 1) \cdots (v^{-1} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k}).\end{aligned}$$

Recall that a permutation $u \in S([n]^t)$ is stable of rank r if and only if r is the minimum positive integer k such that $\psi_k(u) = \psi_{k-1}(u) \otimes 1$ for all $k \geq r$. Equivalently, by (2), r is the minimum positive integer k such that, for all $k \geq r$,

$$(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes u)(\psi_{k-1}(u) \otimes 1) = (\psi_{k-1}(u) \otimes 1)(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes u). \tag{33}$$

We will show that (33) holds for u if and only if it holds for v, and this will prove the result. Let

$$A_k(u) := (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes u^{-1} \otimes 1)(\underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes u^{-1} \otimes 1 \otimes 1) \cdots (1 \otimes u^{-1} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k}).$$

Note that $\psi_{k-1}(u) \otimes 1 = A_{k-1}(u)\,(u^{-1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k})\,A_{k-1}(u)^{-1}$, and that $A_k(u)^{\#} = A_k(v)$. Therefore we obtain that u satisfies (33) if and only if

$$\begin{aligned}&(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes u) A_{k-1}(u)\,(u^{-1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k})\,A_{k-1}(u)^{-1} = \\ &A_{k-1}(u)\,(u^{-1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k})\,A_{k-1}(u)^{-1}(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes u)\end{aligned}$$

and similarly for v. Taking the transpose of this equation we obtain that it holds if and only if

$$\begin{aligned}&(v^{-1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k})\,A_{k-1}(v)^{-1}(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes v)\,A_{k-1}(v) = \\ &A_{k-1}(v)^{-1}\,(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes v) A_{k-1}(v)\,(v^{-1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k})\end{aligned}$$

which in turn is equivalent to

$$A_{k-1}(v)\,(v^{-1}\otimes\underbrace{1\otimes\cdots\otimes 1}_{k})\,A_{k-1}(v)^{-1}(\underbrace{1\otimes\cdots\otimes 1}_{k}\otimes v) =$$
$$(\underbrace{1\otimes\cdots\otimes 1}_{k}\otimes v)A_{k-1}(v)\,(v^{-1}\otimes\underbrace{1\otimes\cdots\otimes 1}_{k})\,A_{k-1}(v)^{-1}$$

which concludes the proof. □

8 The box product

In this section we introduce a new binary operation on permutations that preserves stability.

Let $n, m, r \in \mathbb{N}$, $u \in S([n]^r)$, and $v \in S([m]^r)$. We let the *box product* of u and v be the permutation $u \boxtimes v \in S([nm]^r)$ defined by

$$(u \boxtimes v)((a_1, b_1), \ldots, (a_r, b_r)) := ((c_1, d_1), \ldots, (c_r, d_r))$$

for all $(a_1, b_1), \ldots, (a_r, b_r) \in [n]\times[m]$ (where we identify $[n]\times[m]$ and $[nm]$ lexicographically), where $(c_1, \ldots, c_r) := u(a_1, \ldots, a_r)$, and $(d_1, \ldots, d_r) := v(b_1, \ldots, b_r)$. Note that $u \boxtimes v = L(u \otimes v)L^{-1}$, where $L : [n]^r \times [m]^r \to [nm]^r$ is the bijection defined by $L(a_1, \ldots, a_r, b_1, \ldots, b_r) := ((a_1, b_1), \ldots, (a_r, b_r))$.

Our first result lists some useful properties of the box product.

Proposition 8.1. *Let $u \in S([n]^r)$, $v \in S([m]^r)$, $w \in S([n]^t)$, and $z \in S([m]^t)$ $(n, m, r, t \in \mathbb{N})$. Then*

i) $(u \boxtimes v)(w \boxtimes z) = (uw) \boxtimes (vz)$, *if* $t = r$;

ii) $(u \boxtimes v)^{-1} = (u^{-1} \boxtimes v^{-1})$;

iii) $(u \otimes w) \boxtimes (v \otimes z) = (u \boxtimes v) \otimes (w \boxtimes z) \in S([nm]^{r+t})$.

In particular, $(u \otimes 1_n) \boxtimes (v \otimes 1_m) = (u \boxtimes v) \otimes 1_{nm} \in S([nm]^{r+1})$, *and* $(1_n \otimes u) \boxtimes (1_m \otimes v) = 1_{nm} \otimes (u \boxtimes v) \in S([nm]^{r+1})$, *where 1_i denotes the identity of S_i.*

Proof. The first two identities are clear. To verify the third one let $(a_1, b_1), \ldots, (a_{r+t}, b_{r+t}) \in [nm]^{r+t}$ (recall that we identify $[n] \times [m]$ and $[nm]$ lexicographically). Then we have that

$$(u\otimes w)\boxtimes(v\otimes z)((a_1, b_1), \ldots, (a_{r+t}, b_{r+t})) = ((c_1, d_1), \ldots, (c_r, d_r), (f_1, g_1), \ldots, (f_t, g_t))$$

where $(c_1,\dots,c_r) := u(a_1,\dots,a_r)$, $(d_1,\dots,d_r) := v(b_1,\dots,b_r)$, $(f_1,\dots,f_t) := w(a_{r+1},\dots,a_{r+t})$, and $(g_1,\dots,g_t) := z(b_{r+1},\dots,b_{r+t})$, and

$$(u\boxtimes v)\otimes(w\boxtimes z)((a_1,b_1),\dots,(a_{r+t},b_{r+t})) = ((c_1,d_1),\dots,(c_r,d_r),(f_1,g_1),\dots,(f_t,g_t)),$$

as claimed. □

We can now prove that the box product preserves stability.

Theorem 8.2. *Let $u \in S([n]^r)$, $v \in S([m]^r)$ $(n,m,r \in \mathbb{N})$ be stable of ranks s and t, respectively. Then $u \boxtimes v \in S([nm]^r)$ is stable of rank $\leq \max\{s,t\}$.*

Proof. We claim that, for any $k \in \mathbb{N}_0$,

$$\psi_k(u \boxtimes v) = \psi_k(u) \boxtimes \psi_k(v). \tag{34}$$

Indeed, this is clear for $k = 0$ by Proposition 8.1. If $k \geq 1$ then by induction, (2), and repeated use of Proposition 8.1 we have that

$$\begin{aligned}
\psi_k(u \boxtimes v) &= (\underbrace{1_{nm} \otimes \dots \otimes 1_{nm}}_{k} \otimes (u^{-1} \boxtimes v^{-1}))(\psi_{k-1}(u \boxtimes v) \otimes 1_{nm}) \\
&\quad \times (\underbrace{1_{nm} \otimes \dots \otimes 1_{nm}}_{k} \otimes (u \boxtimes v)) \\
&= \big((\underbrace{1_n \otimes \dots \otimes 1_n}_{k} \otimes u^{-1}) \boxtimes (\underbrace{1_m \otimes \dots \otimes 1_m}_{k} \otimes v^{-1})\big) \\
&\quad \times \big((\psi_{k-1}(u) \otimes 1_n) \boxtimes (\psi_{k-1}(v) \otimes 1_m)\big) \\
&\quad \times \big((\underbrace{1_n \otimes \dots \otimes 1_n}_{k} \otimes u) \boxtimes (\underbrace{1_m \otimes \dots \otimes 1_m}_{k} \otimes v)\big) \\
&= \big((\underbrace{1_n \otimes \dots \otimes 1_n}_{k} \otimes u^{-1})(\psi_{k-1}(u) \otimes 1_n)(\underbrace{1_n \otimes \dots \otimes 1_n}_{k} \otimes u)\big) \\
&\quad \boxtimes \big((\underbrace{1_m \otimes \dots \otimes 1_m}_{k} \otimes v^{-1})(\psi_{k-1}(v) \otimes 1_m)(\underbrace{1_m \otimes \dots \otimes 1_m}_{k} \otimes v)\big) \\
&= \psi_k(u) \boxtimes \psi_k(v).
\end{aligned}$$

Now, by our hypotheses,

$$\psi_k(u) = \psi_{s-1}(u) \otimes \underbrace{1_n \otimes \dots \otimes 1_n}_{k-s+1}$$

for all $k \geq s-1$, and

$$\psi_h(v) = \psi_{t-1}(v) \otimes \underbrace{1_m \otimes \dots \otimes 1_m}_{h-t+1}$$

for all $h \geq t-1$. Say $s \geq t$. Then, if $k \geq s-1$, by (34) and Proposition 8.1

$$\psi_k(u \boxtimes v) = \psi_k(u) \boxtimes \psi_k(v) = (\psi_{s-1}(u) \boxtimes (\psi_{t-1} \otimes \underbrace{1_m \otimes \ldots \otimes 1_m}_{s-t})) \otimes \underbrace{1_{nm} \otimes \ldots \otimes 1_{nm}}_{k-s+1},$$

if $k \geq s-1$. Therefore,

$$\psi_k(u \boxtimes v) = \psi_{s-1}(u \boxtimes v) \otimes \underbrace{1_{nm} \otimes \ldots \otimes 1_{nm}}_{k-s+1},$$

so, by our definitions, $u \boxtimes v$ is stable of rank $\leq s$. Similarly if $s < t$. □

By the correspondence between permutative automorphisms of the Cuntz algebras and stable permutations the previous result gives a way to produce, from two permutative automorphisms of $\mathcal{O}_n$ and $\mathcal{O}_m$ a permutative automorphism of $\mathcal{O}_{nm}$.

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