Optimal Tracking for Periodic Linear Hybrid Systems

Corrado Possieri, *Member, IEEE*

***Abstract*—This article provides a comprehensive characterization of the quadratic optimal tracking problem for hybrid systems with linear dynamics undergoing periodic time-driven jumps. Solutions to such a problem are proposed for both the finite horizon and the periodic cases. Furthermore, it is shown that if the reference signals are not known in advance, then the best control strategy to deal with the worst case reference signals is to simply minimize the (scaled) outputs. Finally, the derived optimal solutions are used to solve two relevant control problems, which are the reconstruction of vector fields from noisy measurements of the corresponding flows and the estimation of the time derivatives of a periodic, sampled, and noisy signal.**

***Index Terms*—Hybrid systems, linear systems, optimal control, output tracking.**

I. Introduction

Hybrid systems, i.e., plants characterized by the interplay between continuous-time dynamics (flow) and discrete-time events (jump), have attracted increasing research interest [1], [2], thanks to their ability of modeling complex physical phenomena, such as the dynamics of systems subject to impacts [3] and cyber-physical plants [4]. Thus, it is not surprising that a lot of attention has been devoted to the solution of optimal control problems for this class of systems [5]–[10].

This article focuses the attention on a widely studied class of hybrid systems [11], [12], i.e., those with linear flow and jump dynamics and whose jumps are governed by a timer state variable that imposes a fixed dwell time constraint between two consecutive jumps. Several control problems have been solved for this class of hybrid systems. In [13], a technique was proposed to characterize their structural properties and to design an output feedback controller. In [14] and [15], the problem of output regulation was considered. In [16], the disturbance attenuation problem was dealt with, and in [17], the linear quadratic (LQ) regulation problem was addressed. The latter results were extended to multimodal time-varying hybrid linear systems in [18].

The main contribution of this article is to provide a comprehensive characterization of the solution to the LQ optimal tracking problem for the abovementioned class of hybrid systems. In particular, the optimal solution to such a problem is derived both in the finite horizon and periodic cases, assuming that either the reference signal is known in advance or carrying out a worst case analysis.

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The author is with the Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma Tor Vergata, 00133 Roma, Italy, and also with the Istituto di Analisi dei Sistemi ed Informatica “Antonio Ruberti,” Consiglio Nazionale delle Ricerche (IASI-CNR), 00185 Roma, Italy (e-mail: corrado.possieri@uniroma2.it).

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II. Notation and Preliminaries

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of natural, real, and complex numbers, respectively. Define $\mathbb{C}_g := \{s \in \mathbb{C} : |s| < 1\}$. Given a symmetric positive semidefinite matrix $M \in \mathbb{R}^{\nu\times\nu}$, let $\|v\|_M^2$, $v \in \mathbb{R}^\nu$, be the M-seminorm, i.e., $\|v\|_M^2 := v^\top M v$.

Following the notation used in [2], consider the hybrid system governed by the flow dynamics as

$$\begin{cases} \dot{\tau} = 1, \\ \dot{x} = Ax + Bu, \end{cases} \quad \text{for } (\tau, x) \in [0, \tau_M] \times \mathbb{R}^n \tag{1a}$$

and by the jump dynamics

$$\begin{cases} \tau^+ = 0, \\ x^+ = Ex + Fv, \end{cases} \quad \text{for } (\tau, x) \in \{\tau_M\} \times \mathbb{R}^n \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the flow input, $v \in \mathbb{R}^p$ is the jump input, $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{n\times p}$, $\tau(0,0) = 0$, and $x(0,0) = x_0 \in \mathbb{R}^n$. In (1), the constant $\tau_M > 0$ imposes a fixed dwell time constraint between two consecutive jumps, which occur every τ_M time instants due to the dynamics of the timer variable τ. Namely, complete solutions to system (1) are defined on the *hybrid time domain*

$$\mathcal{T} := \{(t, k), t \in [k\tau_M, (k+1)\tau_M], k \in \mathbb{N}\}$$

which is therefore *a priori* fixed. Given $k \in \mathbb{N}$, the shortcut $t_k := k\tau_M$ is used to denote the jump times.

Letting $\phi(t, k, x_0, u, v)$ denote the solution to system (1) at hybrid time $(t, k) \in \mathcal{T}$ with initial condition $x(0,0) = x_0$, flow input u, and jump input v, define the flow output $y \in \mathbb{R}^s$ and the jump output $z \in \mathbb{R}^\ell$ of the hybrid system (1) as

$$y(t, k) := C\phi(t, k, x_0, u, v) \tag{2a}$$

$$z(k) := G\phi(t_k, k-1, x_0, u, v) \tag{2b}$$

where $C \in \mathbb{R}^{s\times n}$ and $G \in \mathbb{R}^{\ell\times n}$. By (2), it is assumed that no direct input–output connection is present, but the results given in this article can be easily extended to such a case at the expense of a much more complex notation. See [17, Remark 1] for a comparison between the lifting approach proposed in [19] and the framework reviewed in this section and [13] for the characterization of the structural properties of systems (1) and (2).

III. Hybrid LQ Optimal Tracking Over Finite Horizon

The main goal of this section is to solve the LQ optimal tracking problem for systems (1) and (2) over a given finite horizon. Given the initial condition $x_0 \in \mathbb{R}^n$, a continuous reference $\bar{y} \in \mathbb{R}^s$, a discrete reference $\bar{z} \in \mathbb{R}^\ell$, a final hybrid time $(T, K) \in \mathcal{T}$, $T \in \mathbb{R}$, $K \in \mathbb{N}$, and two positive-definite matrices: 1) $R \in \mathbb{R}^{m\times m}$; and 2) $L \in \mathbb{R}^{p\times p}$, the *hybrid LQ optimal tracking problem over finite horizon* aims at determining the flow input u and the jump input v that minimize the cost functional as

$$J_{x_0}^{K,T}(u,v) = \sum_{k=1}^{K}\Bigg(\int_{t_{k-1}}^{t_k} \left(\|y(t,k-1) - \bar{y}(t,k-1)\|^2 + \|u(t,k-1)\|_R^2\right) \mathrm{d}t + \left(\|z(k) - \bar{z}(k)\|^2 + \|v(k)\|_L^2\right)\Bigg) + \int_{t_K}^{T} \left(\|y(t,K) - \bar{y}(t,K)\|^2 + \|u(t,K)\|_R^2\right) \mathrm{d}t + \|y(T,K) - \bar{y}(T,K)\|^2. \quad (3)$$

The cost functional $J_{x_0}^{T,K}$ is the extension of classical quadratic tracking cost functions for continuous-time and discrete-time linear systems [20]. In fact, it penalizes the deviation of the system outputs from the desired reference trajectories and the control effort, both in continuous- and discrete-time.

Theorem 1: Consider systems (1) and (2) and the cost functional (3). Let P, h, and c be the solutions defined on the hybrid time domain $\mathcal{T}$ to the hybrid system with flow dynamics

$$-\dot{P} = C^\top C + PA + A^\top P - PBR^{-1}B^\top P \quad (4a)$$

$$-\dot{h} = -C^\top \bar{y} + A^\top h - PBR^{-1}B^\top h \quad (4b)$$

$$-\dot{c} = \bar{y}^\top \bar{y} - h^\top BR^{-1}B^\top h \quad (4c)$$

and jump dynamics

$$P = G^\top G + E^\top P^+ E - E^\top P^+ F\left(L + F^\top P^+ F\right)^{-1} F^\top P^+ E \quad (4d)$$

$$h = -G^\top \bar{z}^+ + E^\top h^+ - E^\top P^+ F\left(L + F^\top P^+ F\right)^{-1} F^\top h^+ \quad (4e)$$

$$c = \left(\bar{z}^+\right)^\top \bar{z}^+ + c^+ - \left(h^+\right)^\top F\left(L + F^\top P^+ F\right)^{-1} F^\top h^+ \quad (4f)$$

where $\bar{z}^+(t_k, k-1)$ denotes $\bar{z}(k)$, and final condition

$$P(T,K) = C^\top C \quad (4g)$$

$$h(T,K) = C^\top \bar{y}(T,K) \quad (4h)$$

$$c(T,K) = \|\bar{y}(T,K)\|^2. \quad (4i)$$

Then, the solution to the hybrid LQ optimal tracking problem over finite horizon is

$$u^\star = -R^{-1}B^\top(Px + h) \quad (5a)$$

$$v^\star = -\left(L + F^\top P^+ F\right)^{-1} F^\top\left(P^+ E x + h^+\right) \quad (5b)$$

and the corresponding value of the cost $J_{x_0}^{T,K}$ is

$$J_{x_0}^{T,K}(u^\star, v^\star) = x_0^\top P(0,0)x_0 + 2\,h^\top(0,0)x_0 + c(0,0). \quad (6)$$

Proof: The statement follows from the same reasoning used in [17, Sec. 3], but by using the cost-to-go function $V(t,k,x) = x^\top P(t,k)x + 2h^\top(t,k)x + c(t,k)$ and considering the additional terms appearing in the continuous and discrete-time Hamilton–Jacobi equations due to the presence of the reference outputs $\bar{y}$ and $\bar{z}$. ■

Remark 1: The results given in Theorem 1 hold, even if $\tau(0,0) \neq 0$ and $\tau(0,0) = \tau_0 \in (0, \tau_M]$. In fact, by construction, the function $V(t,k,x) = x^\top P(t,k)x + 2h^\top(t,k)x + c(t,k)$ is the cost-to-go function for all $(t,k) \in \mathcal{T}$ and all $x \in \mathbb{R}^n$. Therefore, even in $\tau(0,0) \neq 0$, the control inputs (5) solve the tracking problem and the corresponding value of the cost $J_{x_0}^{T,K}$ is given by $x_0^\top P(\tau_0,0)x_0 + 2h^\top(\tau_0,0)x_0 + c(\tau_0,0)$.

IV. Periodic Hybrid LQ Optimal Tracking

Suppose that the reference flow output $\bar{y}$ is $(K\tau_M, K)$-periodic, i.e., $\bar{y}(t,k) = \bar{y}(t + K\tau_M, k + K)$ for all $t \geqslant 0$ and all $k \in \mathbb{N}$, and that the reference jump output $\bar{z}$ if K-periodic, i.e., $\bar{z}(k+K) = \bar{z}(k)$ for all $k \in \mathbb{N}$. In this case, rather than aiming at determining a solution to the finite horizon optimal control problem given by (1)–(3), one may envision a control policy that minimizes

$$\bar{J}^K(u,v) = \frac{1}{K}\sum_{k=1}^{K}\Bigg(\frac{1}{\tau_M}\int_{t_{k-1}}^{t_k}(\|y(t,k-1) - \bar{y}(t,k-1)\|^2 + \|u(t,k-1)\|_R^2)\mathrm{d}t + \left(\|z(k) - \bar{z}(k)\|^2 + \|v(k)\|_L^2\right)\Bigg). \quad (7)$$

And, such that the closed-loop system admits a periodic trajectory, i.e., the closed-loop solution $x(t,k)$ to the hybrid system (1) starting at x_0 satisfies $x(K\tau_M, K) = x_0$, for some $x_0 \in \mathbb{R}^n$. Note that this periodicity constraint was absent in the optimal tracking problem over finite horizon considered in Section III. As for the cost defined in (3), the one defined in (7) is the extension of classical quadratic tracking cost functionals for continuous- and discrete-time linear systems. This problem is usually referred to as *periodic optimal tracking problem* [21]. In view of the periodicity constraint, an optimal periodic control can obviously be obtained by periodic extension of any solution to problems (1), (2), and (7).

Lemma 1: Consider systems (1) and (2) and the cost (7). If there exists a solution to the boundary value problem defined over the hybrid time domain $\mathcal{T}$, with flow dynamics

$$-\dot{P} = C^\top C + PA + A^\top P - PBR^{-1}B^\top P \quad (8a)$$

$$-\dot{h} = \left(A - BR^{-1}B^\top P\right)^\top h - C^\top \bar{y} \quad (8b)$$

$$\dot{x} = \left(A - BR^{-1}B^\top P\right)x - BR^{-1}B^\top h \quad (8c)$$

and jump dynamics

$$P = G^\top G + E^\top P^+ E - E^\top P^+ F\left(L + F^\top P^+ F\right)^{-1} F^\top P^+ E \quad (8d)$$

$$h = \left(E - F\left(L + F^\top P^+ F\right)^{-1} F^\top P^+ E\right)^\top h^+ - G^\top \bar{z}^+ \quad (8e)$$

$$x^+ = \left(E - F\left(L + F^\top P^+ F\right)^{-1} F^\top P^+ E\right)x - F\left(L + F^\top P^+ F\right)^{-1} F^\top h^+ \quad (8f)$$

and the boundary conditions

$$P(0,0) = P(t_K, K) \quad (8g)$$

$$h(0,0) = h(t_K, K) \quad (8h)$$

$$x(0,0) = x(t_K, K) \quad (8i)$$

then the control inputs

$$u^* = -R^{-1}B^\top(Px + h) \quad (9a)$$

$$v^* = -\left(L + F^\top P^+ F\right)^{-1} F^\top\left(P^+ E x + h^+\right) \quad (9b)$$

solve the periodic hybrid LQ optimal tracking problem.

Proof: Following the same reasoning used in [22, Th. 2.1], if there exists a function $V(t,k,x)$ such that

$$V(0,0,x) - V(t_K, K, x) = \gamma(K) \tag{10}$$

for some function $\sigma : \mathbb{N} \to \mathbb{R}$, which satisfies

$$-\frac{\partial V(t,k,x)}{\partial t} = \min_u \Big\{ (Cx - \bar{y}(t,k))^\top (Cx - \bar{y}(t,k)) + u^\top R u + \frac{\partial V(t,k,x)}{\partial x}(Ax + Bu) \Big\} \tag{11a}$$

$$V(t_k, k-1, x) = \min_v \left\{ (Gx - \bar{z}(k))^\top (Gx - \bar{z}(k)) + v^\top L v + V(t_k, k, Ex + Fv) \right\} \tag{11b}$$

and such that the inputs

$$u^* = \arg\min_u \Big\{ (Cx - \bar{y}(t,k))^\top (Cx - \bar{y}(t,k)) \tag{12a}$$

$$+ u^\top R u + \frac{\partial V(t,k,x)}{\partial x}(Ax + Bu) \Big\} \tag{12b}$$

$$v^* = \arg\min_v \left\{ (Gx - \bar{z}(k))^\top (Gx - \bar{z}(k)) \right. \tag{12c}$$

$$\left. + v^\top L v + V(t_k, k, Ex + Fv) \right\} \tag{12d}$$

are such that $x(t_K, K) = x_0$ for some initial condition $x_0 \in \mathbb{R}^n$, then u^* and v^* solve the hybrid periodic optimal control problem. Letting $c(t,k)$ be the solution to (4c) and (4f) with arbitrary $c(t_K, K)$, consider the candidate solution

$$V(t,k,x) = x^\top P(t,x) x + 2h^\top(t,k)x + c(t,k).$$

Since P and h satisfy (8) and c satisfies (4c) and (4f), the function V satisfies (11). Moreover, since $P(0,0) = P(t_K, K)$ and $h(0,0) = h(t_K, K)$, the function V satisfies

$$\begin{aligned} V(0,0,x) - V(t_K, K, x) &= c(t_K, K) - c(0,0) = \gamma(K) \\ &:= \sum_{k=1}^{K} \Big(h^\top(t_k,k) F (L + F^\top P(t_k,k) F)^{-1} F^\top h(t_k,k) \\ &\int_{t_{k-1}}^{t_k} h^\top(\sigma, k-1) B R^{-1} B^\top h(\sigma, k-1) \mathrm{d}\sigma \Big) \\ &- \sum_{k=1}^{K} \Big(\bar{z}^\top(k) \bar{z}(k) + \int_{t_{k-1}}^{t_k} \bar{y}^\top(\sigma, k-1) \bar{y}(\sigma, k-1) \mathrm{d}\sigma \Big). \end{aligned}$$

Finally, since the inputs given in (9) satisfy (12) and the corresponding solution x to system (1) from the initial condition $x(0,0)$ satisfies $x(T,K) = x(0,0)$, such inputs solve the periodic hybrid LQ optimal tracking problem. ■

In order to determine a solution to the boundary value problem (8), consider the following problem:

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} P_\infty(\sigma) = -A^\top P_\infty(\sigma) - P_\infty(\sigma) A - C^\top C + + P_\infty(\sigma) B R^{-1} B^\top P_\infty(\sigma) \tag{13a}$$

for all $\sigma \in [0, \tau_M]$, and

$$P_\infty(\tau_M) = G^\top G + E^\top P_\infty(0) E + - E^\top P_\infty(0) F (L + F^\top P_\infty(0) F)^{-1} F^\top P_\infty(0) E. \tag{13b}$$

By [17, Lemma 1], if system (1) is stabilizable, then (13) admits a solution $P_\infty : [0, \tau_M] \to \mathbb{R}^{n\times n}$, which can be easily computed by means of the techniques given in [17]. Under such a hypothesis, define $Q(\sigma) = -(A - BR^{-1}B^\top P_\infty(\sigma))^\top$, and let $\Phi(\sigma, t)$ be the state transition matrix [23, p. 598] related to Q. Finally, define, for $k = 1, \ldots, K$,

$$\eta(k) = -\int_0^{\tau_M} \Phi(0,\sigma) C^\top y(t_{k-1} + \sigma, k-1) \mathrm{d}\sigma.$$

By using the abovementioned matrices, the following lemma provides conditions for the existence of a solution to the boundary value problem given by (8a), (8b), (8d), (8e), (8g), and (8h).

Lemma 2: Suppose that system (1) is stabilizable, and let

$$W = \left(E - F \left(L + F^\top P_\infty(0) F \right)^{-1} F P_\infty(0) E \right)^\top$$

$$\zeta = \sum_{i=0}^{K-1} (\Phi(0,\tau_M) W)^i \left(\eta(i+1) - \Phi(0,\tau_M) G^\top \bar{z}(i+1) \right).$$

If the following rank condition holds:

$$\operatorname{rank}\left(I - (\Phi(0,\tau_M) W)^K \right) = \operatorname{rank}\left[I - (\Phi(0,\tau_M) W)^K \ \zeta \right] \tag{14}$$

then the boundary value problem given by (8a), (8b), (8d), (8e), (8g), and (8h) admits a solution.

Proof: By [17, Lemma 1], if system (1) is stabilizable, then there exists a solution P_∞ to (13), which satisfies (8a), (8d), and (8g). Furthermore, any solution to (8b) and (8e) with P substituted by the periodic continuation of P_∞ satisfies

$$\begin{aligned} h(t_k, k-1) &= W h(t_k, k) - G^\top \bar{z}(k) \\ h(t_{k-1}, k-1) &= \Phi(0, \tau_M) h(t_k, k-1) \\ &\quad - \int_0^{\tau_M} \Phi(0,\sigma) C^\top y(t_{k-1} + \sigma, k-1) \mathrm{d}\sigma. \end{aligned}$$

Therefore, one has, for all $k \in \{1, \ldots, K\}$, that

$$h(t_{k-1}, k-1) = \Phi(0,\tau_M) W h(t_k, k) - \Phi(0,\tau_M) G^\top \bar{z}(k) - \int_0^{\tau_M} \Phi(0,\sigma) C^\top y(t_{k-1} + \sigma, k-1) \mathrm{d}\sigma. \tag{15}$$

Note that, (15) is a discrete-time system, whose solution is

$$h(t_{K-j}, K-j) = (\Phi(0,\tau_M) W)^j h(t_K, K) + \sum_{i=0}^{j-1} (\Phi(0,\tau_M) W)^i \Big(\eta(K + i - j + 1) - \Phi(0,\tau_M) G^\top \bar{z}(K + i - j + 1) \Big).$$

Thus, if (14) holds, then there exists h_0 such that (8b), (8e), and (8h) hold with P substituted by the periodic continuation of P_∞ and with $h(0,0) = h(t_K, K) = h_0$. Therefore, the solution to (8a), (8b) (8d), and (8e) with $P(t_K, K) = P_\infty(0)$, and $h(t_K, K) = h_0$ is a solution to the problem given by (8a), (8b), (8d), (8e), (8g), and (8h). ■

Using the results given in Lemmas 1 and 2, the following theorem provides a constructive procedure to determine the solution to the periodic hybrid LQ optimal tracking problem.

Theorem 2: Consider systems (1) and (2) and the cost functional (7). If systems (1) and (2) are stabilizable and detectable, then there exists a solution to the periodic hybrid LQ optimal tracking problem.

Proof: If systems (1) and (2) are stabilizable and detectable, then the hybrid system governed by the flow dynamics is

$$\dot{\tau} = 1, \qquad \dot{x} = \left(A - BR^{-1} B^\top P_\infty(\tau) \right) x$$

when $(\tau, x) \in [0, \tau_M] \times \mathbb{R}^n$, and by the jump dynamics

$$\tau^+ = 0, \quad x^+ = \left(E - F\left(L + F^\top P_\infty(0) F\right)^{-1} F P_\infty(0) E\right) x$$

when $(\tau, x) \in \{\tau_M\} \times \mathbb{R}^n$ is asymptotically stable by [17, Th. 4]. Therefore, letting $\Psi(\sigma, t)$ be the state transition matrix related to the time-varying matrix $V(\sigma) = A - BR^{-1}B^\top P_\infty(\sigma)$, one has that the eigenvalues of the matrix $D = W^\top \Psi(\tau_M, 0)$ are in $\mathbb{C}_g$. Since $V(\sigma) = -Q^\top(\sigma)$, one has that $\Psi(\sigma, t) = \Phi^\top(t, \sigma)$. Therefore, also the eigenvalues of $\Phi(0, \tau_M)W = D^\top$ are in $\mathbb{C}_g$, thus implying that the rank condition given in (14) holds. By Lemma 2, this implies that the boundary value problem given by (8a), (8b), (8d), (8e), (8g), and (8h) admits the solutions P_∞ and h_∞, where, with a slight abuse of notation, P_∞ is the periodic continuation of the solution to (13) and h_∞ is the solution to the system defined over the hybrid time domain $\mathcal{T}$ with flow dynamics

$$-\dot{h} = \left(A - BR^{-1}B^\top P_\infty\right)^\top h - C^\top \bar{y} \tag{16a}$$

and jump dynamics

$$h = \left(E - F\left(L + F^\top P_\infty^+ F\right)^{-1} F^\top P_\infty^+ E\right)^\top h^+ - G^\top \bar{z}^+ \tag{16b}$$

and final conditions $h(t_K, K) = h_0$ and $\tau(t_K, K) = 0$, where h_0 is the solution to the linear system of equalities

$$\left(I - (\Phi(0, \tau_M)W)^K\right) h_0 = \zeta. \tag{17}$$

Then, it remains to prove that there exists $x_0 \in \mathbb{R}^n$ such that the solution to the hybrid system with flows dynamics

$$\dot{\tau} = 1 \tag{18a}$$

$$\dot{x} = \left(A - BR^{-1}B^\top P_\infty(\tau)\right) x - BR^{-1}B^\top h_\infty \tag{18b}$$

when $(\tau, x) \in [0, \tau_M] \times \mathbb{R}^n$, and by the jump dynamics

$$\tau^+ = 0 \tag{18c}$$

$$\begin{aligned} x^+ = &\left(E - F\left(L + F^\top P_\infty(0) F\right)^{-1} F P_\infty(0) E\right) x \\ &- F\left(L + F^\top P_\infty(0) F\right)^{-1} F^\top h_\infty^+ \end{aligned} \tag{18d}$$

when $(\tau, x) \in \{\tau_M\} \times \mathbb{R}^n$, with $x(0,0) = x_0$ satisfies $x(t_K, K) = x_0$. Following the same construction used in the proof of Lemma 2, let $\nu(k) = \int_0^{\tau_M} \Psi(\tau_M, \sigma) h_\infty(t_k + \sigma, k) \mathrm{d}\sigma$. The solution to system (18) with $x(0,0) = x_0$ satisfies $x(t_{k+1}, k+1) = W^\top \Psi(\tau_M, 0) x(t_k, k) + W^\top \nu(k) - F(L + F^\top P_\infty(0) F)^{-1} F^\top h_\infty (t_{k+1}, k+1)$. Therefore, it results that $x(t_K, K) = D^K x_0 + \sum_{i=0}^{K-1} D^i (W^\top \nu(k-i-1) - F(L + F^\top P_\infty(0) F)^{-1} F^\top h_\infty(t_{k-i}, k-i))$. Thus, by considering that the eigenvalues of D are in $\mathbb{C}_g$, the initial condition x_0 leading to periodic trajectories of (18) is

$$\begin{aligned} x_0 = &\left(I - D^K\right)^{-1} \sum_{i=0}^{K-1} D^i \bigg(W^\top \nu(k-i-1) \\ &- F\left(L + F^\top P_\infty(0) F\right)^{-1} F^\top h_\infty(t_{k-i}, k-i)\bigg). \end{aligned} \tag{19}$$

Letting x_∞ be the solution to system (18) with $x(0,0) = x_0$, since P_∞, h_∞, and x_∞ solve the boundary value problem (8), the statement follows by Lemma 1. ■

The proof of Theorem 2 is constructive and allows one to determine a solution to the periodic hybrid optimal tracking problem by explicitly solving the problem given in (8) by determining the solution to a system of linear equations. The following proposition provides an alternative method to determine such a solution.

Proposition 1: Consider systems (1) and (2) and the cost functional (7). Suppose that systems (1) and (2) are stabilizable and detectable. Let h_N be the solution to system (16) with final condition $h_N(t_{NK}, NK) = \bar{h}$, where $\bar{h}$ is an arbitrary vector in $\mathbb{R}^n$. Then, one has that $\lim_{N \to \infty} h_N(0,0) = h_0$, where h_0 is the solution to (17).

Proof: By the proofs of Lemma 2 and Theorem 2, since $\bar{y}$ is $K\tau_M$-periodic and $\bar{z}$ is K-periodic, one has $h_N(t_{(j-1)K}, (j-1)K) = (\Phi(0, \tau_M)W)^K h_N(t_{jK}, jK) + \zeta$. Therefore, since the eigenvalues of $\Phi(0, \tau_M)W$ are in $\mathbb{C}_g$, the statement holds. ■

By using Proposition 1 and [17, Lemma 1], an approximate solution to the boundary value problem (8) is given by the solution to system (4) over the hybrid time domain $\mathcal{T}$ with $P(t_{NK}, NK) = 0$ and $h(t_{NK}, NK) = 0$ for $N \to \infty$, i.e., the solution to system (4) tends to one solution to the boundary value problem (8) as the final time goes to infinity. This additionally shows that the solution to the periodic optimal tracking problem is a good approximate of the solution to the optimal tracking problem over finite horizon, provided that the reference trajectories are periodic and the horizon is sufficiently large. The following corollary shows that such a solution has particularly desirable properties. In particular, if one applies the periodic extension of such inputs to the hybrid system (1), its solution tends to the periodic optimal one.

Corollary 1: Let systems (1) and (2) be stabilizable and detectable and let P_c, h_c, and x_c be the periodic continuations of the solution to the boundary value problem (8) given above. Then, letting x be the solution to system (1) with the control inputs

$$u_c = -R^{-1}B^\top (P_c x + h_c) \tag{20a}$$

$$v_c = -\left(L + F^\top P_c^+ F\right)^{-1} F^\top \left(P_c^+ E x + h_c^+\right) \tag{20b}$$

from any initial condition $x_0 \in \mathbb{R}^n$, one has that

$$\lim_{t+k \to \infty} \|x(t,x) - x_c(t,k)\| = 0.$$

Proof: Let $\varphi(t, k, x_0, \bar{u}, \bar{v})$ be the solution at time $(t,k) \in \mathcal{T}$ to the hybrid system governed by the flow dynamics

$$\dot{\tau} = 1, \qquad \dot{x} = \left(A - BR^{-1}B^\top P_\infty(\tau)\right) x + B\bar{u}$$

when $(\tau, x) \in [0, \tau_M] \times \mathbb{R}^n$, and by the jump dynamics

$$x^+ = \left(E - F\left(L + F^\top P_\infty(0) F\right)^{-1} F P_\infty(0) E\right) x + F\bar{v}$$

$\tau^+ = 0$, when $(\tau, x) \in \{\tau_M\} \times \mathbb{R}^n$, with initial conditions $x(0,0) = x_0$ and $\tau(0,0) = 0$. The closed-loop trajectories of systems (1) and (20) are given by $\varphi(t, k, x_0, \bar{u}, \bar{v})$ with

$$\bar{u} = -R^{-1}B^\top h_c, \qquad \bar{v} = -\left(L + F^\top P_c^+ F\right)^{-1} F^\top h_c^+.$$

By linearity, we have that $\varphi(t, k, x_0, \bar{u}, \bar{v}) = \varphi(t, k, x_c(0,0), \bar{u}, \bar{v}) + \varphi(t, k, x_0 - x_c(0,0), 0, 0)$. Thus, the statement follows by the fact that:

$$\varphi(t, k, x_c(0,0), \bar{u}, \bar{v}) = x_c(t, x)$$

$$\lim_{t+k \to \infty} \varphi(t, k, x_0 - x_c(0,0), 0, 0) = 0$$

by the same reasoning used in the proof of Theorem 2. ■

Using the results stated in Corollary 1, the following remark highlights further properties of the determined optimal solution

Remark 2: Let φ be defined as in the proof of Corollary 1, by [17, Th. 4] there exist $c > 0$ and $\rho \in (0,1)$ such that

$$\|\varphi(t, k, x_0, 0, 0)\| \leqslant c\rho^k \|x_0\|.$$

Therefore, a straightforward consequence of Corollary 1 is that the control inputs given in (20) actually minimize the average cost defined as $\lim_{K\to\infty} \bar{J}^K$, i.e., such inputs constitute a solution to the optimal tracking problem over infinite horizon for periodic reference signals.

The following remark compares the results given in this section with those given in Section III.

Remark 3: Although the problems considered in this and the previous sections seem similar, their solution is in fact rather different. First, in order to determine a solution to the optimal tracking problem over finite horizon defined by (1)–(3), no assumption is required about systems (1) and (2), whereas in order to determine a solution to the periodic optimal tracking problem defined by (1), (2), and (7), one has to assume that systems (1) and (2) are detectable and observable. Furthermore, while the solution of the former problem is given in terms of the solution to a hybrid dynamical equation to be solved backward in time, i.e., the hybrid Riccati equations (4), the solution of the latter is given in terms of the solution to a two-point boundary value problem, i.e., (8). These differences are due to the presence of an additional constraint in the periodic optimal tracking problem, which is the existence of an initial condition that makes the corresponding trajectory of the closed-loop system periodic.

V. Case of Unknown Reference Outputs

In the case that the reference outputs $\bar{y}$ and $\bar{z}$ are not known *a priori*, a possible strategy to deal with the reference tracking problem is to recur to the framework of zero-sum hybrid differential games [16]. Namely, by defining the cost

$$\begin{aligned}\tilde{J}^{T,K}_{x_0}(u,v,\bar{y},\bar{z}) = \sum_{k=1}^{K}\bigg(&\int_{t_{k-1}}^{t_k}(\|y(t,k-1)-\bar{y}(t,k-1)\|^2 \\ &+\|u(t,k-1)\|_R^2 - r\|\bar{y}(t,k-1)\|^2)\,\mathrm{d}t \\ &+\left(\|z(k)-\bar{z}(k)\|^2+\|v(k)\|_L^2 - l\|z(k)\|^2\right)\bigg) \\ &+\int_{t_K}^{T}\left(\|y(t,K)-\bar{y}(t,K)\|^2+\|u(t,K)\|_R^2\right. \\ &\left.-r\|\bar{y}(t,K)\|^2\right)\mathrm{d}t \end{aligned} \tag{21}$$

where $r, l > 0$, one may attempt at determining control inputs $u^\diamond$ and $v^\diamond$ and reference outputs $\bar{y}^\diamond$ and $\bar{z}^\diamond$ that constitute a saddle point of the cost functional $\tilde{J}^{T,K}_{x_0}$, such that

$$\tilde{J}^{T,K}_{x_0}(u^\diamond,v^\diamond,\bar{y}^\diamond,\bar{z}^\diamond) \geqslant \tilde{J}^{T,K}_{x_0}(u^\diamond,v^\diamond,\bar{y},\bar{z})$$
$$\tilde{J}^{T,K}_{x_0}(u^\diamond,v^\diamond,\bar{y}^\diamond,\bar{z}^\diamond) \leqslant \tilde{J}^{T,K}_{x_0}(u,v,\bar{y}^\diamond,\bar{z}^\diamond)$$

for all bounded control inputs u and v and for all bounded reference signals $\bar{y}$ and $\bar{z}$. If such inputs and reference signals exist, then $u^\diamond$, $v^\diamond$ and $\bar{y}^\diamond$, $\bar{z}^\diamond$ constitute a *Nash equilibrium* of the zero-sum hybrid game defined by systems (1) and (2) and the cost functional (21). As for the costs defined in (3) and (7), the one defined in (21) is the extension of classical cost functionals arising in the zero-sum dynamic game for continuous- and discrete-time linear systems.

Theorem 3: Let systems (1) and (2) and the cost functional (21) be given, and suppose that $r, l > 1$. Then, letting Π be the solution defined on the hybrid time domain $\mathcal{T}$ to the hybrid system with flow dynamics

$$-\dot{\Pi} = \frac{r}{r-1}C^\top C + PA + A^\top\Pi - \Pi B R^{-1}B^\top\Pi \tag{22a}$$

jump dynamics

$$\begin{aligned}\Pi = &\frac{l}{l-1}G^\top G + E^\top\Pi^+E \\ &- E^\top\Pi^+F\left(L+F^\top\Pi^+F\right)^{-1}F^\top\Pi^+E\end{aligned} \tag{22b}$$

and final condition

$$\Pi(T,K) = 0, \tag{22c}$$

the Nash equilibrium of the zero-sum hybrid game is

$$u^\diamond = -R^{-1}B^\top\Pi x \tag{23a}$$
$$v^\diamond = -\left(L+F^\top\Pi^+F\right)^{-1}F^\top\Pi^+Ex \tag{23b}$$
$$\bar{y}^\diamond = -\frac{1}{r-1}Cx \tag{23c}$$
$$\bar{z}^\diamond = -\frac{1}{l-1}Gx. \tag{23d}$$

Proof: Following [24, Sec. 6.2 and 6.5] and the reasoning used in the proof of Theorem 1, if there exists a function $V(t,k,x)$, such that for all $t \in [t_{k-1}, t_k]$

$$\begin{aligned}-\frac{\partial V(t,k,x)}{\partial t} &= \max_{\bar{y}}\min_{u}\{H_c(t,k,x,\bar{y},u)\} \\ &= \min_{u}\max_{\bar{y}}\{H_c(t,k,x,\bar{y},u)\}\end{aligned} \tag{24a}$$

where $H_c(t,k,x,\bar{y},u) = \|Cx-\bar{y}\|^2 + \|u\|_R^2 - r\|\bar{y}\|^2 + (\partial V(t,k,x)/\partial x)(Ax+Bu)$ and for all $k \in \{1,\ldots,K\}$

$$\begin{aligned}V(t_k,k-1,x) &= \max_{\bar{z}}\min_{v}\{H_d(k,x,\bar{z},v)\} \\ &= \min_{v}\max_{\bar{z}}\{H_d(k,x,\bar{z},v)\}\end{aligned} \tag{24b}$$

where $H_d(k,x,\bar{z},v) = \|Gx-\bar{z}\|^2 + \|v\|_L^2 - l\|\bar{z}\|^2 + V(t_k,k,Ex+Fv)$, then the solutions to the abovementioned min–max problems constitute a Nash equilibrium of the game. Since the argument of (24a) [respectively, (24b)] is convex in u (respectively, v) and concave in $\bar{y}$ (respectively, $\bar{z}$), if Π satisfies the hybrid dynamics (22), then $V(t,k,x) = x^\top\Pi(t,k)x$ satisfies the Hamilton–Jacobi–Isaacs equations (24), and the corresponding saddle points are given by (23).

Since $r, l > 1$, both $(r/[r-1])C^\top C$ and $(l/[l-1])G^\top G$ are the positive semidefinite matrices. Therefore, the existence of a unique solution to (22) is guaranteed by classical results on the differential and discrete matrix Riccati equations. ∎

The following remark discusses the case $r, l \leqslant 1$.

Remark 4: In the case that $r \leqslant 1$ or $l \leqslant 1$, by inspecting (24), it can be easily derived that there does not exist any Nash equilibrium for the considered zero-sum hybrid game. It can be easily observed that in these two cases, the player choosing the reference signals $\bar{y}$ and $\bar{z}$ is incentivized to let them take arbitrarily large values.

The following remark frames the Nash equilibrium determined in Theorem 3 in terms of an optimal control problem.

Remark 5: By comparing the results stated in Theorem 3 with [17, Th. 1], it appears evident that the control inputs constituting the unique Nash equilibrium of the zero-sum hybrid game (1), (2), and (21) are those minimizing the cost

$$\begin{aligned}\sum_{k=1}^{K}\bigg(&\int_{t_{k-1}}^{t_k}\left(\frac{r}{r-1}\|y(t,k-1)\|^2+\|u(t,k-1)\|_R^2\right)\mathrm{d}t \\ &+\frac{l}{l-1}\|z(k)\|^2+\|v(k)\|_L^2\bigg) \\ &+\int_{t_K}^{T}\left(\frac{r}{r-1}\|y(t,K)\|^2+\|u(t,K)\|_R^2\right)\mathrm{d}t\end{aligned} \tag{25}$$

and that the corresponding reference signals are given by the scaled outputs of the closed-loop system with these inputs.

A similar result can be obtained in the case that the reference signals $\bar{y}$ and $\bar{z}$ are $(K\tau_M, K)$-periodic and K-periodic, respectively, by considering the cost

$$\check{J}^K(u, v, \bar{y}, \bar{z}) = \frac{1}{K}\sum_{k=1}^{K}\Bigg(\frac{1}{\tau_M}\int_{t_{k-1}}^{t_k}\big(\|y(t,k-1) - \bar{y}(t,k-1)\|^2 + \|u(t,k-1)\|_R^2 - r\|\bar{y}(t,k-1)\|^2\big)\,\mathrm{d}t + \big(\|z(k) - \bar{z}(k)\|^2 + \|v(k)\|_L^2 - l\|\bar{z}(k)\|^2\big)\Bigg). \tag{26}$$

In this case, one aims at determining control inputs u° and v° and reference outputs $\bar{y}^\circ$ and $\bar{z}^\circ$ that constitute a saddle point of the cost functional $\check{J}$, such that

$$\check{J}^K(u^\circ, v^\circ, \bar{y}, \bar{z}) \leqslant \check{J}^K(u^\circ, v^\circ, \bar{y}^\circ, \bar{z}^\circ) \leqslant \check{J}^K(u, v, \bar{y}^\circ, \bar{z}^\circ)$$

for all bounded control inputs u and v, such that the closedloop system admits a periodic trajectory for some initial condition $x_0 \in \mathbb{R}^n$, and for all bounded reference signals $\bar{y}$ and $\bar{z}$, which are extended periodically outside $\mathcal{T} \cap ([0, K\tau_M] \times \{0,\ldots,K\})$ by letting $y(t,k) = y(t + K\tau_M, k + K)$ for all $t \geqslant 0$ and $z(k+K) = z(k)$ for all $k \in \mathbb{N}$. The solution to such a problem is given in the following theorem.

Theorem 4: Let systems (1) and (2) and the cost functional (26) be given, and suppose that $r, l > 1$. Let systems (1) and (2) be stabilizable and detectable. Then, letting Π_∞ be the solution to the following two-point boundary value problem:

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\Pi_\infty(\sigma) = -A^\top\Pi_\infty(\sigma) - \Pi_\infty(\sigma)A - \frac{r}{r-1}C^\top C + \Pi_\infty(\sigma)BR^{-1}B^\top\Pi_\infty(\sigma) \tag{27a}$$

for all $\sigma \in [0, \tau_M]$, and

$$\Pi_\infty(\tau_M) = \frac{l}{l-1}G^\top G + E^\top\Pi_\infty(0)E - E^\top\Pi_\infty(0)F\left(L + F^\top\Pi_\infty(0)F\right)^{-1}F^\top\Pi_\infty(0)E. \tag{27b}$$

The solution to the periodic zero-sum hybrid game is

$$u^\circ(t,k) = -R^{-1}B^\top\Pi_\infty(t - t_k)x(t,k) \tag{28a}$$

$$v^\circ(k) = -\left(L + F^\top\Pi_\infty(0)F\right)^{-1}F^\top \times \Pi_\infty(0)Ex(t_k, k-1) \tag{28b}$$

$$\bar{y}^\circ(t,k) = -\frac{1}{r-1}Cx(t,k) \tag{28c}$$

$$\bar{z}^\circ(k) = -\frac{1}{l-1}Gx(t_k, k-1). \tag{28d}$$

Proof: Since system (1) is stabilizable, by [17, Lemma 1] there exists a solution Π_∞ to (27). Furthermore, since systems (1) and (2) are also detectable, the control inputs u° and v° given in (28a) and (28b), respectively, are stabilizing. Thus, following the same reasoning used in [22, Th. 2.1], if there exists a function $V(t,k,x)$ that satisfies (10) and (24), and such that the inputs attaining the minimum on the right-hand side of (24a) and (24b) are such that $x(T,K) = x_0$ for $x(0,0) = x_0$ and some $x_0 \in \mathbb{R}^n$, then such inputs constitute a solution to the periodic zero-sum hybrid game. Note that the function $V(t,k,x) = x^\top\Pi_\infty(t - t_k)x$ satisfies these properties with $\sigma(K) = 0$, and the corresponding inputs, given in (28a) and (28b), are such that if the closed-loop system is initialized at $x_0 = 0$, then $x(T,K) = 0$. ■

The following remark relates the Nash equilibrium determined in Theorem 4 to the solution to an optimal control problem over infinite horizon.

Remark 6: By comparing the results given in Theorem 4 and in [17, Th. 3], the control inputs constituting a solution to the periodic zero-sum game given by (1), (2), and (26) are those minimizing the cost functional given in (25) for $T + K \to \infty$. Furthermore, by [17, Th. 4], if systems (1) and (2) are stabilizable and detectable, then the periodic continuation of the control inputs u° and v° makes the closed-loop system asymptotically stable. Therefore, a direct consequence of Theorem 4 is that the control inputs and the reference signals given in (28) actually constitute a Nash equilibrium of the zero-sum hybrid game over infinite horizon given by (1) and (2), and $\lim_{K\to\infty}\check{J}^K$; see also Remark 2.

VI. Applications to Vector Field Reconstruction and Derivative Estimation

In this section, the results given in the previous sections are used to reconstruct vector fields from noisy measurements of the corresponding flows (see Section VI-A) and to estimate the time derivatives of a periodic signal (see Section VI-B).

A. Reconstruction of Vector Fields

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function and assume that the system

$$\dot{\xi} = f(\xi) \tag{29}$$

admits a unique solution $\xi : [0,T] \to \mathbb{R}^n$ for $\xi(0) = \xi_0$, $\xi_0 \in \mathbb{R}^n$. The main goal of this section is to reconstruct the vector field f given sampled measurements $\xi(k\tau_M)$ of the solution to system (29), $k = 1,\ldots,K$, $K \leqslant \frac{T}{\tau_M}$.

Such a problem can be dealt with letting $x = [\,\xi^\top\ \dot{\xi}^\top\,]^\top$, and

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad C = 0$$
$$E = I, \qquad F = 0, \qquad G = \begin{bmatrix} I & 0 \end{bmatrix}$$

And, $\bar{y}(t,k) = 0$ for all $(t,k) \in \mathcal{T}$ and $\bar{z}(k) = \xi(k\tau_M)$, $k = 1,\ldots,K$, and determining a solution to the optimal tracking problem given by (1)–(3). Then, an estimate of f can be obtained using the pairs

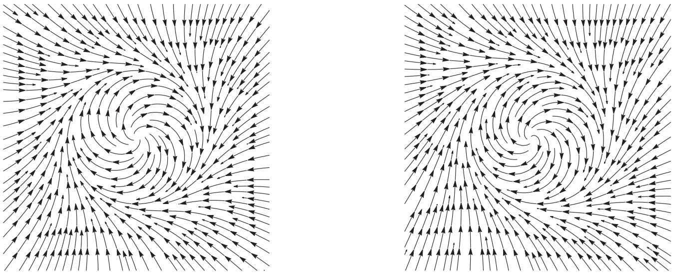
$$\hat{\xi} = \begin{bmatrix} I & 0 \end{bmatrix} x(t,k)$$
$$\hat{f}(\hat{\xi}) = \begin{bmatrix} 0 & I \end{bmatrix} x(t,k)$$

where x denotes the closed-loop solution of the tracking problem initialized at $x(0,0) = [\,\xi_0^\top\ \dot{\xi}_0^\top\,]^\top$, with $\dot{\xi}_0$ given by the solution to the following quadratic problem:

$$\min_{\dot{\xi}_0}\left\{\begin{bmatrix} \xi_0 \\ \dot{\xi}_0 \end{bmatrix}^\top P(0,0)\begin{bmatrix} \xi_0 \\ \dot{\xi}_0 \end{bmatrix} + 2h^\top(0,0)\begin{bmatrix} \xi_0 \\ \dot{\xi}_0 \end{bmatrix}\right\}$$

and using any function approximation technique.

Example 1: In [25], a technique was proposed to design a vector field f so that the trajectories of system (29) tend to a given set with the aim of designing safe paths of motion for a mobile robot. The objective of this example is to solve a converse problem, i.e., given paths of motion of the mobile robot, find the vector field governing its dynamics. Following [25], 100 different robot trajectories have been



(a) (b)

Fig. 1. Stream plots of the vector fields f and $\check{f}$ in Example 1. (a) Vector field f. (b) Vector field $\check{f}$.

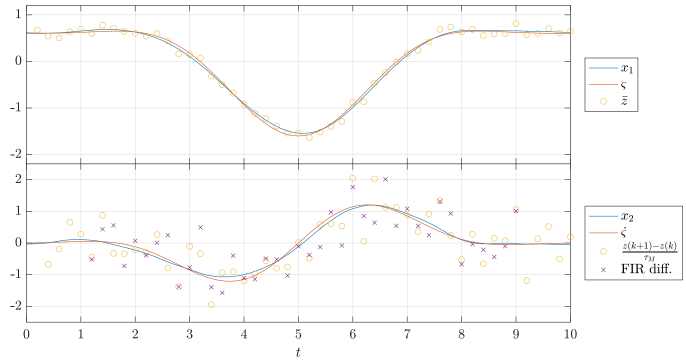


Fig. 2. Closed-loop trajectory of system (1) in Example 2.

simulated initializing ξ_0 at random using a Gaussian distribution and letting

$$f(\xi) = \begin{bmatrix} -\xi_1\left(\xi_1^2+\xi_2^2-1\right)+\xi_2 \\ -\xi_2\left(\xi_1^2+\xi_2^2-1\right)-\xi_1 \end{bmatrix}.$$

A zero-mean disturbance with standard deviation 0.01 has been added to the samples of the trajectories of system (29), so as to account for position measurement errors. Then, the technique outlined above, with $\tau_M = 0.1$, $R = 10^{-6}I$, and $L = I$, has been used to gather pairs $(\hat{\xi}, \hat{f}(\hat{\xi}))$. Such pairs have finally been used to fit a polynomial model $\check{f}$ of total degree 3 for the vector field f using classical linear regression [26]. Fig. 1 shows the stream plots of the vector fields f and $\check{f}$. As shown by such a figure, the vector field obtained by using the proposed technique is a good approximation of the one governing the motion of the mobile robot.

B. Estimation of Time Derivatives

Let $\varsigma : \mathbb{R} \to \mathbb{R}$ be a given T-periodic smooth signal. The solution to the periodic optimal tracking problem given in Section IV can be used to estimate the time derivatives of ς from its sampled measurements. Namely, let $\bar{y}(t,k) = 0$ and $\bar{z}(k) = \varsigma(k\tau_M)$, $k = 1,\ldots,K$ with $K = T/\tau_M$, define

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = 0$$
$$E = I, \qquad F = 0, \qquad G = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and compute a solution to the periodic optimal tracking problem given by (1), (2), and (7). Then, the closed-loop solution of the tracking problem initialized at $x(0,0) = x_0$, where x_0, given by (19), is an estimate of $\dot{\varsigma}$.

Example 2: Consider the signal $\varsigma(t) = \cos([\pi/5]t) - 0.5\cos([2\pi/5]t) + 0.1\cos([3\pi/5]t)$. The objective of this example is to estimate the time derivative of this signal from noisy sampled measurements. Letting $\tau_M = 0.2$ and $K = 50$, the signal $\bar{z}$ and the hybrid systems (1) and (2) have been defined as above. A Gaussian noise with zero mean and standard deviation, i.e., 0.1, has been added to the samples $\bar{z}$, so as to account for measurement noise. Finally, the technique given in Section IV has been used to solve the periodic tracking problem, with $R = 1$ and $L = 1$. Fig. 2 shows the closed-loop trajectory of system (1), the signal ς, its time derivative $\dot{\varsigma}$, the noisy samples $\bar{z}$, the estimate of $\dot{\varsigma}$ obtained as $(z(k+1) - z(k))/\tau_M$, and the one gathered using an finite impulse response (FIR) differentiator of order 10. As shown by such a figure, despite the signal ς is sampled and the samples are affected by measurement noise, the proposed method is capable of determining a good estimate of the time derivatives of such a signal. In particular, the obtained estimates are more reliable than those gathered using dirty derivatives and FIR differentiators in the presence of measurement noise.

VII. Conclusion

A solution to the LQ optimal tracking problem was proposed for the hybrid system with linear dynamics and time-driven periodic jumps both in the finite horizon and periodic cases. By means of a game theoretic formulation, it was shown that if the reference signals are not known in advance, then the best control strategy that allows one to cope with the worst case references is to minimize the (scaled) outputs of the system. Finally, it was shown that how to use the derived solution to solve two relevant control tasks: 1) the reconstruction of vector fields from noisy measurements of the corresponding flows; and 2) the estimation of the time derivatives of a periodic, sampled, and noisy signal.

The results given in this article can be adapted to deal periodic optimal switching control problems [27], by adding an additional state, which is constant during flow and whose postjump values equal the discrete-time input v, and with impulsive optimal tracking problems [28], by letting $B = 0$.

Comparing the results given in this, not with those in [17], note that although the problem considered was different (namely, optimal tracking in the former and optimal regulation to zero in the latter), the solution to the LQ tracking problem inherits the feedback gain from the solution to the LQ regulation with an additional term depending on the reference signal. Similarly, comparing the results given in Section V with those given in [16], the feedback Nash equilibrium of the hybrid LQ zero-sum games defined by system (1) and the cost (21) or (26) have the same form of the solutions given in [16], despite the considered problems are different.

Following the ideas given in [18], the techniques given in Section III can be extended to deal with multimodal time-varying hybrid systems by letting the matrices $A, B, E, F, R,$ and L in (4) be time-varying, and admitting that the dimension of the matrix P and the vector h has varying dimension, depending on the discrete time k.

As in classical LQ tracking, one of the drawbacks of the proposed approach is that all the reference signals have to be known in advance in order to determine the optimal control law. However, if such signals are actually known, then the proposed optimal control can be determined by means of simple linear algebra tools.

Future work will deal with the extension of the proposed results to singular, cheap, and constrained tracking problems and to the cases of hybrid systems with state-driven jumps and of nonperiodic references over infinite horizon.

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