Irreducible Subfactors Derived from Popa's Construction for Non-Tracial States

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Abstract. For an inclusion of the form $\mathbb{C} \subseteq M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ is endowed with a state with diagonal weights $\lambda = (\lambda_1, \dots, \lambda_n)$, we use Popa's construction, for non-tracial states, to obtain an irreducible inclusion of II_1 factors, $N^\lambda(Q) \subseteq M^\lambda(Q)$ of index $\sum \frac{1}{\lambda_i}$. $M^\lambda(Q)$ is identified with a subfactor inside the centralizer algebra of the canonical free product state on $Q \star M_N(\mathbb{C})$. Its structure is described by "infinite" semicircular elements as in [32].

The irreducible subfactor inclusions obtained by this method are similar to the first irreducible subfactor inclusions, of index in $[4, \infty)$ constructed in [24], starting with the Jones' subfactors inclusion $R^s \subseteq R$, $s > 4$. In the present paper, since the inclusion we start with has a simpler structure, it is easier to control the algebra structure of the subfactor inclusions.

If the weights correspond to a unitary, finite-dimensional representation of a Woronowicz's compact quantum group G, then the factor $M^\lambda(Q)$ is contained in the fixed point algebra of an action of the quantum group on $Q \star M_N(\mathbb{C})$, with equality if G is $SU_q(N)$, (or $SO_q(3)$ when $N = 2$). By Takesaki duality, the factor $M^\lambda(L(F_N))$ is Morita equivalent to $\mathcal{L}(F_\infty)$.

This method gives also another approach to find, as also recently proved in [36], irreducible subfactors of $\mathcal{L}(F_\infty)$ for index values bigger than 4.

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0. Introduction and definitions

In this paper we consider the structure of subfactors obtained from Popa's construction, for non-tracial states, for the inclusion $M_N(\mathbb{C}) \subseteq M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$. We fix a diagonal matrix with non-zero weights $\lambda_1, \dots, \lambda_N$. The states on the two algebras are respectively $tr(D\cdot)$ and $tr(D \otimes D^{-1}\cdot)$. This is then the Jones' iterated basic construction ([11], [12]) for $\mathbb{C} \subseteq M_N(\mathbb{C})$, where $M_N(\mathbb{C})$ is endowed with the state $tr(D\cdot)$. The algebra $M_N(\mathbb{C}) \otimes 1$ is invariated by the modular group of the state $tr(D \otimes D^{-1}\cdot)$ on $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$. Hence by [19], there exists a conditional

expectation and a corresponding Jones projection, which is also invariated by the modular group. The value of the state on the Jones projection is $\sum \frac{1}{\lambda_i}$. Let Q be a diffuse finite von Neumann algebra with faithful trace τ.

We apply Popa's construction for the inclusion $M_N(\mathbb{C}) \subseteq M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$. This yields an irreducible inclusion $N^\lambda(Q) \subseteq M^\lambda(Q)$ of type II_1 factors, inside the type III factor $(Q \otimes M_N(\mathbb{C})) *_{M_N(\mathbb{C})} (M_N(\mathbb{C}) \otimes M_N(\mathbb{C}))$, considered with the amalgamated free product state.

By applying the reduction method described in [29], [33], we can reduce this procedure to the case when the algebra over which we amalgamate is abelian. Thus, we end up with a concrete description of the factor and the subfactor inside $Q \star M_N(\mathbb{C})$, where the later algebra is endowed with the free product state $\tau \star tr(D \cdot)$. This factor for $N = 2$, $Q = L^\infty([0,1])$ can be very explicitly described in terms of the "infinite" Voiculescu's type ([40]) free semicircular element used to identify the core of $L^\infty([0,1]) \star M_2(\mathbb{C})$ in [32].

Moreover, by analogy with the case of a trace on $M_N(\mathbb{C})$, it turns out that $M^\lambda(Q)$ is contained in the fixed point algebra for a free product coaction (of the type considered in [37]) on $Q \star M_N(\mathbb{C})$.

Let α be any finite-dimensional unitary selfadjoint coaction of a Woronowicz quantum group G on the finite-dimensional algebra $M_N(\mathbb{C})$. We prove that the algebra $M^\lambda(Q)$ described above is contained in the fixed point algebra of coaction of the quantum group. If G is the quantum group $SU_q(N)$ then the algebra $M^\lambda(Q)$ is exactly the fixed point algebra. This is analogous to the result in [14], [20], where the fixed point algebra of the infinite product action on $M_N(\mathbb{C})$ of $SU_q(N)$ coincides with the hyperfinite factor.

Assume (by analogy with [41]) that $\alpha^{\otimes^n}$ contains any other finite-dimensional unitary representation of G. Then we prove that the fixed point algebra $(Q \star M_N(\mathbb{C}))^G$ is Morita equivalent to the cross product by α. Such a cross product is naturally described ([37]) by a free product with amalgamation. Since the fixed point algebra is a II_1 factor it follows that the fixed point algebra is Morita equivalent with an amalgamated free product of the form $(Q \otimes D) *_D C$, where C, D are (infinite) direct sums of matrix algebras. By the techniques in [29] it follows, for $Q = \mathcal{L}(F_N)$, that the fixed point algebra is the von Neumann algebra of a free group with infinitely many generators. The II_1 factors is invariated by the action of an automorphism scaling the trace on a larger II_∞ factor.

We also reobtain in this way, by a different approach, the result recently proved in the remarkable paper by D. Shlyahtenko and Y. Ueda ([36]) that $L(F_\infty)$ has irreducible subfactors for any index value bigger than 4. One theme in their paper is that one can determine the isomorphism class of a fixed point algebra by looking at the crossed product algebra (via Takesaki's duality). We owe to their paper the use of this philosophy for coactions.

We do not know if the subfactors in this paper coincide the ones constructed in [36]. In both cases the higher relative commutants invariant is A_∞ (though the factors in [36] seem to generate other higher relative commutants invariants) when

used for other compact quantum groups). Both algebra of the factors are obtained by taking the the fixed point algebra of a coaction of a quantum group. In one case the fixed point algebra is a II_1 factor and in the other case the fixed point algebra is of type III.

The construction in this paper proves that the non-tracial version of Popa's construction of subfactors, naturally yields irreducible subfactors, In addition the corresponding algebras are fixed algebras of coactions of quantum groups.

These algebras, although they are very similar to the subfactors in the breakthrough construction in [24], are not yet proven to be isomorphic to the algebras in [24], [26]. Though, our result is strong evidence to the conjecture that the subfactors in [24], [25] are free group factors when Q is $\mathcal{L}(F_\infty)$ (the only case when the subfactors in [24], [25] are known to be free group factors is for index values less than 4, and the higher relative commutants are the Temperly-Lieb algebra, see [29]).

The method in this papers also gives an explicit model of the subfactors in terms of (infinite) semicircular random matrices ([40], [30], [32]).

1. Subfactors derived from Popa's construction for non-tracial states

In this section we describe the structure of the algebra of the subfactors that are obtained, by using Popa's construction, from a finite-dimensional inclusion $B \subseteq A$ with Markov state. We apply this construction to the case of the inclusion $\mathbb{C} \subseteq M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ is endowed with a state. This will give irreducible subfactor of indices from 4 to ∞.

We start with the following lemma which proves that the subfactor associated to $B \subseteq A$ only depends on the inclusion matrix. To do this we use a reduction style procedure that was used in [29], to the case when B is abelian. The following lemma is proved in [33]

Lemma 1.1. *Let $C \subseteq B$ be a finite-dimensional inclusion of matrices, with trivial centers intersection and assume that B is endowed with a λ Markov state (that is there exists a normal conditional expectation from B onto C which preserves the state, there exists a state on the corresponding basic construction extending the given one and the corresponding Jones' projection has expectation λ times the scalars).*

Let Q be a type II_1 factor, let $A = \langle B, e_1 \rangle$ be the first step in the Jones' basic construction of $C \subseteq B$, where e_1 is the Jones projection. Assume that A is endowed with the canonical Markov state. Let $m_i \in B$ be a Pimsner-Popa orthonormal basis for the inclusion $C \subseteq B$. Let $\mathcal{A}$ be the algebra $(Q \otimes B) \star_B A$ and let Φ be the Popa's map associated to the inclusion defined by

$$\Phi(x) = \sum m_i e_1 x e_1 m_i^*, \quad x \in \mathcal{A}.$$

Then as proved in [24], [25], [26], Φ maps B' into A'.

Let f_i be a family of representatives of minimal projections in B and let F_0 be the sum of this projections. Let $\mathcal{A}_0, A_0, B_0$ be the reductions of respectively the algebras $\mathcal{A}, A, B$ by the projection F_0. In particular B_0 is abelian and, as proved in ([29]), $\mathcal{A}_0$ is identified with $(Q \otimes B_0) \star_{B_0} A_0$. Since F_0 has central support 1, Φ induces a map on $\mathcal{A}_0$ which maps B_0' into A_0'. Let $N(Q) \subseteq M(Q) \subseteq B'$ be the minimal algebra (introduced in [24]) in $\mathcal{A}$ that is closed under Φ and contains Q. Correspondingly let $N_0(Q) \subseteq M_0(Q) \subseteq B_0'$ be the minimal algebra in $\mathcal{A}_0$ that is closed under Φ_0 and contains Q. Then

$$M_0(Q) = (M(Q))_{F_0}; \quad N_0(Q) = (N(Q))_{F_0}.$$

Moreover if m_i' is any orthonormal basis for $B_0 \subseteq A_0$, then

$$\Phi_0(x) = \sum m_i' x (m_i')^*, \quad x \in \mathcal{A}_0.$$

Proof. This lemma will be proved in full generality in [33]. We give here in this section an ad-hoc proof for the case $C = \mathbb{C} \subseteq B = M_N(\mathbb{C})$, where $M_N(\mathbb{C})$ has a state $tr(D\cdot)$ (as in the next lemma). Indeed the next step in the basic construction (as proved in the next lemma) is $A = M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$. Here the algebra $\mathcal{A}$ is $(Q \otimes M_N(\mathbb{C})) *_{M_N(\mathbb{C})} (M_N(\mathbb{C}) \otimes M_N(\mathbb{C}))$ and consequently $\mathcal{A}_0$ is $Q * M_N(\mathbb{C})$. By construction Φ_0 is of the form $\Phi_0(x) = \sum_\alpha n_\alpha x (n_\alpha)^*$, for some fixed n_α in $M_N(\mathbb{C})$. Let e_{ij} be a matrix unit diagonalizing D.

By cutting by a minimal projection, it follows that $\Phi_0(x)$ takes values into $(M_N(\mathbb{C}))' \cap \mathcal{A}_0$. Thus necessary, there exists real $\theta_1, \ldots, \theta_N$ numbers such that Φ_0 is of the form

$$\Phi_0(x) = \sum \theta_j e_{ij} x e_{ji}, \quad x \in \mathcal{A}_0.$$

But Φ_0 has also the property that the conditional expectation from $\mathcal{A}_0$ onto $M(Q)$ maps $M_N(\mathbb{C})$ into the scalars. This is only possible if $\theta_j e_{ij}$ is an orthonormal basis for $M_N(\mathbb{C})$, i.e., if θ_i are the inverses of the eigenvalues of D.   □

As will see below, even in the case when the inclusion $B \subseteq A$ is $\mathbb{C} \subseteq M_N(\mathbb{C})$, with a trace on $M_N(\mathbb{C})$ has to be handled by a rather a complicated machinery. Indeed in this case the above subfactor is the fixed algebra under the action of $SU(N)$ on $Q * M_N(\mathbb{C})$ (here $SU(N)$ acts trivially on Q and by conjugation on $M_N(\mathbb{C})$). This is because the element $\sum e_{ij} \otimes e_{ji}$ is invariant under the product action of the group $SU(N)$ and all the others are obtained by intercalation or concatenation from this element ([44]).

We generalize below the above construction to the case when the trace on $M_N(\mathbb{C})$ is replaced by a state. First we record the following well known folklore lemma. It deals with the Jones basic construction for states (see, e.g., [27], [12])

Lemma 1.2. *Let $\phi = tr(D\cdot)$ be the state on $M_n(\mathbb{C})$ where D has diagonal eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the next step in the Jones basic construction with Markov state is $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ with state*

$$\frac{1}{tr D^{-1}} tr((D \otimes D^{-1}\cdot)).$$

This state is so that the Jones projection is invariated by the modular group and projects onto a scalar multiple of the identity. The value of the scalar is $(\sum \frac{1}{\lambda_i})^{-1}$.

Proof. Indeed if (e_{ij}) is a matrix unit, which also diagonalizes D, then the Jones' projection is

$$\sum_{ij} \lambda_i^{-1/2} \lambda_j^{-1/2} e_{ij} \otimes e_{ij}. \qquad \square$$

In the next lemma we describe what Popa's construction ([24]) is in the non-tracial case in a very particular case (for the more general construction see [33]). This construction allows to obtain irreducible subfactors, in factors with non-trivial fundamental group, of index values in $[4, \infty)$ starting from the inclusion $\mathbb{C} \subseteq M_N(\mathbb{C})$. We will describe this factors explicitly.

Lemma 1.3. *We use the notations from the previous lemma and its proof. Let e_1 be the Jones projection for the basic construction in the previous lemma. Let $(M_\alpha)e_1$, in the centralizer algebra of the state ϕ, be a Pimsner-Popa basis for the inclusion $M_N(\mathbb{C}) \subseteq M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$. Note that this is always possible since the centralizer algebra (which contains $e_{ij} \otimes e_{ij}$), is isomorphic to $M_N(\mathbb{C})$ and contains e_1.*

Let Q be a diffuse abelian von Neumann algebra or a type II_1 factor with faithful trace τ and let $\mathcal{A} = (Q \otimes M_N(\mathbb{C})) *_{M_N(\mathbb{C})} (M_N(\mathbb{C}) \otimes M_N(\mathbb{C}))$ be the amalgamated free product factor obtained by taking the GNS-construction corresponding to the given state on $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$ and the trace τ. This is obviously a type III factor if ϕ is not a trace ([8], [35], [5], [39]).

Let as in [24], [25], Φ be the map defined on $\mathcal{A}$ by

$$\Phi(x) = \sum_\alpha M_\alpha e_1 x e_1 M_\alpha^*, \quad x \in \mathcal{A}.$$

Let $M^\lambda(Q)$ be the minimal algebra in $\mathcal{A}$ containing Q that is invariant under the map Φ. Let $N^\lambda(Q)$ be the image of $M^\lambda(Q)$ through Φ.

Then, (as in [24]) $N^\lambda(Q) \subseteq M^\lambda(Q)$ is an irreducible inclusion of type II_1 factors of index $\sum \frac{1}{\lambda_i}$.

Proof. This is basically proved in [24], [25] (see also [33]). The only additional care comes from the fact that we are dealing with a state instead of a trace. But since $M_\alpha e_1$ are chosen in the centralizer of the state it follows that $M^\lambda(Q)$ is a type II_1 factor. The main part of the argument (the Markovianity of the trace) follows from the fact that the conditional expectation of the algebra $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$ onto $M^\lambda(Q)$ is the algebra of scalars, which follows from the properties of the Pimsner-Popa basis. $\qquad \square$

It is not clear if the above factor is the same as the one constructed in [24] starting from the inclusion derived from the Temperly-Lieb algebra for index values bigger than 4. In that case the algebras stay in a much larger amalgamated free product, since the Pimsner-Popa basis sits in hyperfinite factor (and cannot be chosen to belong to a finite-dimensional algebra).

Remark 1.4. To obtain any additional step in the Jones' basic construction of the above inclusion one has to proceed as follows. One starts with the iterated Jones's basic construction of $\mathbb{C} \subseteq M_N(\mathbb{C})$, where $M_N(\mathbb{C})$ is endowed as above with the state $tr(D\cdot)$. Then the kth, $(k+1)$th steps of this basic construction are $B = M_N(\mathbb{C})^{\otimes^k}$, $A = M_N(\mathbb{C})^{\otimes^{k+1}}$. The states on the iterated steps in the basic construction are states with weights involving consecutive products of D and D^{-1}. Let $e_{k-1}, \ldots, e_1$ be the corresponding Jones' projections, indexed so that $e_1 \in A$ is exactly the Jones projection for $M_N(\mathbb{C})^{\otimes^{k-1}} \subseteq B$.

Then we perform the same construction as above for $B \subseteq A$ (instead of $M_N(\mathbb{C}) \subseteq M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$). We get a subfactor inclusion $N^\lambda(Q) \subseteq M^\lambda(Q)$. The iterated Jones' basic construction steps (up to $k-1$) are then obtained by adding consecutively the projections $e_1, e_2, \ldots, e_{k-1}$. As we will see below by reducing by a minimal projection in B it follows that these subfactors are isomorphic to the original ones. In particular it follows that the factor and subfactor are always isomorphic to the algebra corresponding to D and D^{-1}.

Lemma 1.5. *Let ϕ be a state on $M_N(\mathbb{C})$ with $\phi = tr(D\cdot)$, where D is diagonal with eigenvalues list $\lambda_1, \ldots, \lambda_N$. Let (e_{ij}) be a matrix unit for $M_N(\mathbb{C})$, such that e_{ii} are the projections onto the eigenvectors of D. Let Q be a type II_1 factor and consider the type III factor $Q * M_N(\mathbb{C})$ (see, e.g., [4], [8], [32], [35]) where the free product is with respect to the state ϕ on $M_N(\mathbb{C})$ and the trace on Q. Let*

$$\Phi(\alpha) = \sum_{ij} \lambda_j^{-1} e_{ij} \alpha e_{ji}, \qquad \alpha \in Q * M_N(\mathbb{C}),$$

and let $\mathcal{M}^\lambda(Q)$ be the minimal subalgebra of $Q * M_N(\mathbb{C})$ that contains Q and is closed under Φ. Let $\mathcal{N}^\lambda(Q)$ be the image of M through Φ. Then

$$[\mathcal{M}^\lambda(Q) : \mathcal{N}^\lambda(Q)] = \sum_i \frac{1}{\lambda_i}; \quad \mathcal{M}^\lambda(Q) \cap \mathcal{N}^\lambda(Q)' = \mathbb{C}1.$$

Moreover $\mathcal{M}^\lambda(Q)$ has non-trivial fundamental group and the type of the subfactor inclusion is A_∞. The algebra of the subfactor is isomorphic with the factor $M^{(\lambda_i^{-1})}(Q)$.

Proof. We apply Popa's construction for the inclusion $M_N(\mathbb{C}) \subseteq M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$ described in Lemma 1.2. Since this is the basic construction for $\mathbb{C} \subseteq M_N(\mathbb{C})$ we can apply Lemma 1.1 and one obtains an equivalent description of the subfactor $\mathcal{M}^\lambda(Q)$ sitting in $Q * M_N(\mathbb{C})$. Since $\lambda_j^{-1} e_{ij}$ is a Pimsner-Popa basis for the inclusion $\mathbb{C} \subseteq M_N(\mathbb{C})$, the result follows. That the subfactor inclusion is of type A_∞ is proved in a more general context in [33]. $\square$

By analogy with the case of traces, we have the following result which shows that the algebra obtained by Popa's construction from the inclusion of finite-dimensional algebras (with states), $\mathbb{C} \subseteq M_N(\mathbb{C})$ is a minimal fixed point algebra.

Theorem 1.6. *Let A be the function algebra of Woronowicz quantum group G and let V_π be finite-dimensional unitary representation of dimension $N \geqslant 2$. Assume that the operator Q_π associated to the representation as in [43], [2] has eigenvalues $\lambda_1^{-1}, \ldots, \lambda_N^{-1}$. Let $Q*B(V_\pi)$ and consider the Ueda's ([37], [38]) type of free product coaction on $Q * B(V_\pi)$, that acts trivially on Q and by conjugation on $B(V_\pi)$. Let $\mathcal{M}$ be the Popa's type factor constructed in Lemma 1.5.*

Then

$$\mathcal{M} \subseteq (Q * B(V_\pi))^G.$$

If A is the function algebra of $SU_q(2)$, and π is the fundamental representation then we have equality.

Proof. Let (e_{st}) be a matrix unit for $B(V_\pi)$ that diagonalizes Q_π. Assume that the unitary implementing the representation is represented as

$$U = \sum_{s,t} e_{st} \otimes u_{st}.$$

Note that we use another indexing for the entries of the unitary than the one used in [43], Section 4. Since as proved in [2],

$$Q_\pi^{1/2} \overline{U} (Q_\pi)^{-1/2}$$

is the unitary corresponding to the conjugate representation it follows that the matrix

$$\alpha_{ij} = \lambda_i^{-1/2} \lambda_j^{1/2} u_{ji}$$

is a unitary too. Thus we have the following unitarity conditions:

$$\sum_j u_{sj} u_{tj}^* = \delta_{st}; \qquad \sum_j u_{js}^* u_{jt} = \delta_{st}, \qquad s, t = 1, \ldots, N, \qquad (1.1)$$

$$\sum_j \lambda_j^{-1} u_{sj}^* u_{tj} = \lambda_s^{-1} \delta_{st}; \qquad \sum_j \lambda_j u_{js} u_{jt}^* = \lambda_s \delta_{st}; \qquad s, t = 1, \ldots, N. \qquad (1.2)$$

The corepresentation for the algebra $B(V_\pi)$ obtained by conjugation by the unitary U is given by

$$U(e_{ij} \otimes 1)U^* = \sum_{rs} e_{rs} \otimes u_{ri} u_{sj}^*. \qquad (1.3)$$

The elements in the algebra $\mathcal{M}$ have an open and closing parenthesis structure ([24], [7]). This proves that these elements are fixed points under the coaction, by recursively using (first equality in each of) the relations (1.1), (1.2). For example the coaction on an element of the form

$$x = \sum_{\alpha, \beta} \lambda_\beta^{-1} q_1 e_{\alpha,\beta} q_2 e_{\beta,\alpha} q_3$$

gives

$$x = \sum_{r_1, r_2, s_1, s_2} q_1 e_{r_1, s_1} q_2 e_{r_2, s_2} q_3 \left(\sum_{\alpha, \beta} \lambda_\alpha^{-1} u_{r_1 \alpha} u_{s_1 \beta}^* u_{r_2 \beta} u_{s_2 \alpha}^* \right). \qquad (1.4)$$

By applying first equation 1.1 and then 1.2 it follows that

$$\sum_{\alpha, \beta} \lambda_\alpha^{-1} u_{r_1 \alpha} u_{s_1 \beta}^* u_{r_2 \beta} u_{s_2 \alpha}^* = \lambda_{s_1}^{-1} \delta_{r_1 s_2} \delta_{s_1 r_2}. \qquad (1.5)$$

This proves that the coaction on x in (1.4) gives $x \otimes 1$. The only other fixed point are obtained by recurrence. Note that the algebra Q is in the fixed point algebra of the coaction.

More precisely if x in the fixed point algebra and h is the Haar measure, we need to show that

$$(Id \otimes h) \left(\sum_{\alpha\beta} \lambda_\beta^{-1} (U(e_{\alpha, \beta} \otimes 1) U^*)(x \otimes 1)(U(e_{\beta, \alpha} \otimes 1) U^*) \right) = \sum_{\alpha\beta} \lambda_\beta^{-1} e_{\alpha, \beta} x e_{\beta, \alpha}.$$

This follows by using the modular properties of the Haar measure and using the relations (1.4). Once we have shown this and since Q is obviously in the fixed point algebra the result follows from the definition of the algebra $M^\lambda(Q)$ as the minimal algebra containing Q closed to the operation in the statement of Lemma 1.5.

Let Q_0 be the subspace of Q consisting of elements of zero trace. The fixed point algebra and Popa's algebra admit a filtration given by the subspaces $B(V_\pi) Q_0 B(V_\pi) \cdots Q_0 B(V_\pi)$ corresponding to the number of occurrences of copies of $B(V_\pi)$. Moreover the averaging argument (with respect to the Haar measure) used in [18] or [14] to establish that the fixed point algebra in the Powers factor is the algebra generated by the Jones projections, allows to reduce the determination of the the fixed point algebra, to the determination of the intersection of the fixed point algebra with the subspaces in the filtration.

Therefore, to show the equality, when G is $SU_q(2)$, of the algebra in Popa's construction with the fixed point algebra, it is therefore sufficient to check that the intersections of these two algebras with the space

$$B(V_\pi) Q_0 B(V_\pi) \cdots Q_0 B(V_\pi)$$

coincide. (This comes also to to the determine the fixed point algebra in spaces of the form $B(V_\pi)^{\otimes^n}$, by formally replacing elements in Q_0 with tensor product sign. Counting dimensions of fixed point algebra could probably give an alternative for the argument below.)

Since we already have shown the reverse equality, it is sufficient to check that any element in the fixed point algebra intersected with one of the above finite-dimensional spaces $B(V_\pi) Q_0 B(V_\pi) \cdots Q_0 B(V_\pi)$ is one of the elements that respects the open and closing paranthesis structure of the recursive construction of the Popa's algebra.

So assume that for given scalars $\mu_{\alpha_0,\ldots,\beta_n}$ we have an fixed point element of the form

$$\left(\sum_{\alpha_0,\ldots,\beta_n} \mu_{\alpha_0,\ldots,\beta_n} e_{\alpha_0\beta_0} q_1 e_{\alpha_1\beta_1} q_2 \cdots q_n e_{\alpha_n\beta_n} \right) \otimes 1$$

$$= \sum_{r_0,\ldots,s_n} e_{r_0 s_0} q_1 e_{r_1 s_1} \cdots q_n e_{r_n s_n} \tag{1.6}$$

$$\otimes \left(\sum_{\alpha_0,\ldots,\beta_n} \mu_{\alpha_0,\ldots,\beta_n} u_{r_0\alpha_0} u_{s_0\beta_0}^* u_{r_1\alpha_1} u_{s_1\beta_1}^* \cdots u_{r_n\alpha_n} u_{s_n\beta_n}^* \right).$$

Then the fixed point condition comes to the following condition that has to hold for all $r_0,\ldots,s_n$:

$$\sum_{\alpha_0,\ldots,\beta_n} \mu_{\alpha_0,\ldots,\beta_n} u_{r_0\alpha_0} u_{s_0\beta_0}^* u_{r_1\alpha_1} u_{s_1\beta_1}^* \cdots u_{r_n\alpha_n} u_{s_n\beta_n}^* = \mu_{r_0,\ldots,s_n}. \tag{1.7}$$

By denoting Θ the matrix $(\mu_{\alpha_0,ldots,\beta_n})_{\alpha_0,\ldots,\beta_n}$, and by U the matrix $(u_{ij})_{ij}$ and by $U^\#$ the matrix $(u_{ji})_{ij}$, the relation (1.7) comes to the following equation:

$$(U \otimes U^\# \cdots \otimes U \otimes U^\#)\Theta = \Theta. \tag{1.8}$$

Let $I_{(ik)(rs)}$ and $E_{(ik)(rs)}$ be the matrices defined by $I_{(ik)(rs)} = \delta_{rs}$ and $E_{(ik)(rs)} = \lambda_s^{-1}\delta_r s$. Then the equations (1.1), (1.2) correspond to

$$(U \otimes U^\#)I = I; \qquad\qquad I(U^\# \otimes U) = I, \tag{1.9}$$

$$(U \otimes U^\#)E = E; \qquad\qquad E(U^\# \otimes U) = E, \tag{1.10}$$

where $(U \otimes U^\#)$ and $(U^\# \otimes U)$ are considered as the matrices indexed as follows: $(U \otimes U^\#)_{((st)(ij))} = u_{si}u_{tj}^*$ and $(U^\# \otimes U)_{((st)(ij))} = u_{si}u_{tj}^*$. As in [22], [44], the relations (1.9), (1.10) are the only relations defining $SU_q(2)$. Thus if a matrix Θ verifies the relation (1.8), then Θ is obtained by recurrence (and linear span) from consecutive applications of the relations (1.9), (1.10).

To have a reduction we therefore have to use the relation $\sum_j \lambda_j^{-1} u_{sj}^* u_{tj} = \lambda_s^{-1}\delta_{st}$, or $\sum_j u_{sj}u_{tj}^* = \delta_{st}$ (the other two equations in (1.1), (1.2) involve a summation index which is not appropriate for the corresponding sum). This means that the only possibility to have a simplification in (1.7), by using the defining relations of $SU_q(2)$, is if somewhere in the sum of (1.7), we have that $\mu_{\alpha_0\cdots\beta_n}$ splits as a sum of elements the form

$$\lambda_{\beta_i}^{-1}\delta_{\beta_i a_{i+1}} \times \mu'_{\alpha_0,\ldots,\hat{\beta}_i,\hat{\alpha}_{i+1},\ldots,\beta_n}, \tag{1.11}$$

or in the form

$$\delta_{\alpha_i b_i} \times \mu''_{\alpha_0,\ldots,\hat{\alpha}_i,\hat{\beta}_i,\ldots,\beta_n}. \tag{1.12}$$

The symbol $\hat{\ }$ corresponding to omission of the corresponding symbol. Moreover after applying such a simplification as in (1.11) and respectively (1.12) in (1.6) we obtain a sum in $r_0,\ldots,s_n$ involving a corresponding $\lambda_{s_i}\delta_{s_i r_{i+1}}$ or respectively $\delta_{r_i s_i}$.

Repeated application of the above two procedures outlined in the equations (1.11) and (1.12) to the sum in (1.6), will give, once we have finished the simplification procedure to the sum reducing (1.7) to scalars, it follows that the fixed point element has exactly the open and closing paranthesis structure in the indices $\alpha_0, \ldots, \beta_n$ that corresponds to an element in the algebra derived from Popa's construction.   □

We can also use the model described in the paper ([32]), where the structure of the free product algebra $Q * M_2(\mathbb{C})$ is described in terms of the "infinite" free semicircular element in [30] (here $M_2(\mathbb{C})$ is endowed with a state). We obtain an explicit random matrix ([40]) model for the structure of the subfactor $\mathcal{N}^\lambda(Q) \subseteq \mathcal{M}^\lambda(Q)$, when Q is a free group factor or $Q = L^\infty([0,1])$.
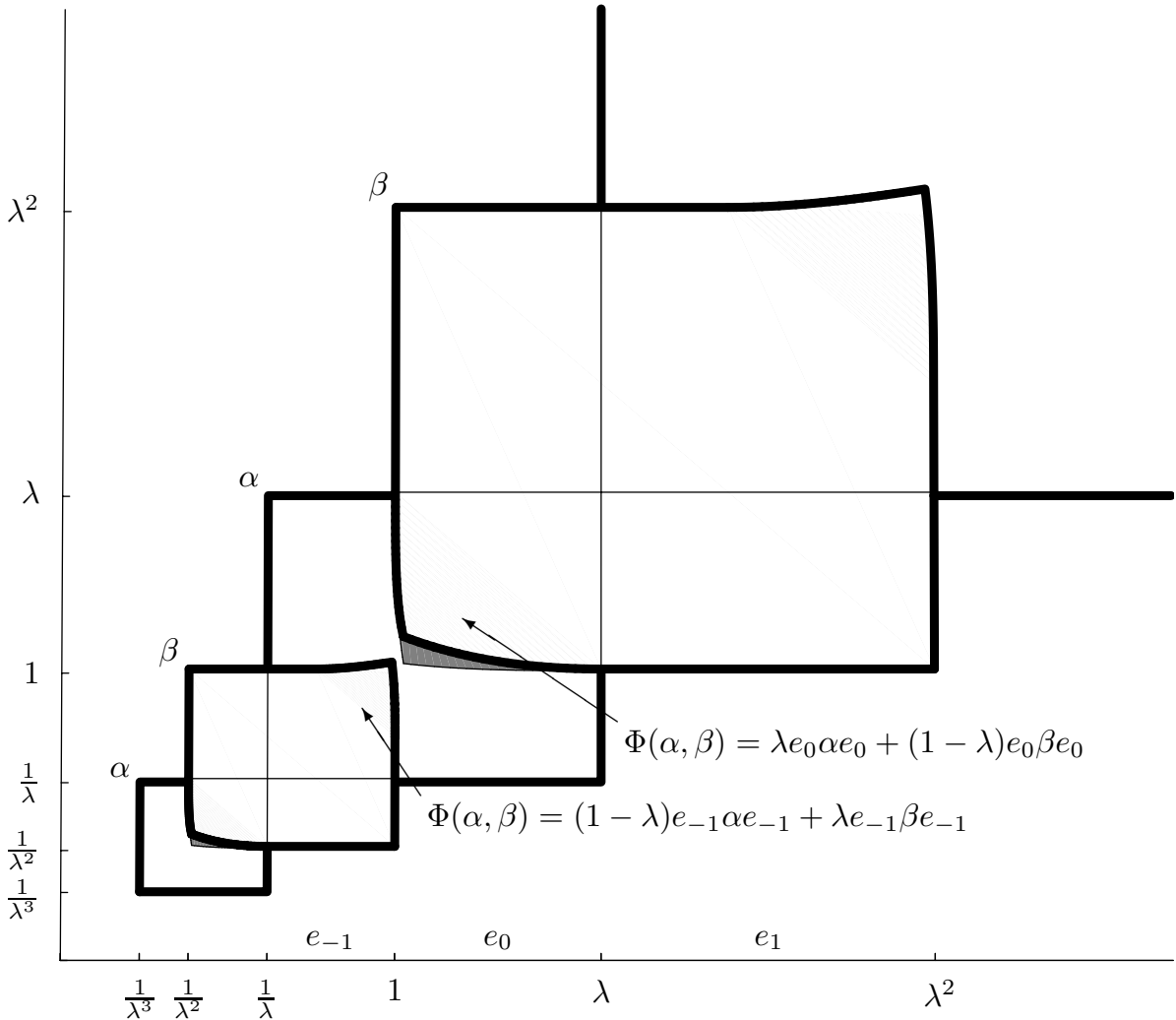
We describe this model below:

FIGURE 1. Action of Φ

Theorem 1.7. *Let λ be a number in $(0,1)$ and let D be the algebra generated by the characteristic functions of the intervals $e_n = [\lambda^n, \lambda^{n+1}]$, $n \in \mathbb{Z}$. Let X, Y be two "infinite" free semicircular elements as considered in [30] (see also [9]). Let f_i be the projections $e_{2i} + e_{2i+1}$ and let $g_i = e_{2i+1} + e_{2i+2}$. Let F, G be the algebras respectively generated by this projections. Let X_0, Y_0 be the (bounded) free semicircular family obtained by diagonally carving X and respectively Y by the projections f_i and respectively g_i. In [32] it was proved that the algebra $\mathcal{A}$ generated by X_0, Y_0 and D is isomorphic to $M_2(\mathbb{C}) * L^\infty([0,1]$, where on $M_2(\mathbb{C})$ we consider a state with weights λ and $1 - \lambda$.*

Note that the automorphism of homothety by λ (which is a restriction of the one parameter group introduced in [30]), scales trace by λ.

Let $\mathcal{B}_0, \mathcal{B}_1$ be the algebras $F' \cap \mathcal{A}$ and respectively $G' \cap \mathcal{A}$. Let E_0, E_1 be the sum of the even, respectively odd, indexed projections in (e_i). On the algebra $\mathcal{B}_0 \oplus \mathcal{B}_1$ we consider the following linear map

$$\Phi(\alpha, \beta) = \lambda E_0 \alpha E_0 + (1 - \lambda) E_0 \beta E_0 + (1 - \lambda) E_1 \alpha E_1 + \lambda E_1 \beta E_1, \quad (\alpha, \beta) \in \mathcal{B}_0 \oplus \mathcal{B}_1.$$

Let $\mathcal{M}$ be the minimal subalgebra of $\mathcal{B}_0 \oplus \mathcal{B}_1$ containing (X_0, Y_0) and closed under Ψ, and let $\mathcal{N}$ be the image of Φ.

Then $\mathcal{N} \subseteq \mathcal{M}$ is an irreducible subfactor of index $\frac{1}{\lambda} + \frac{1}{1-\lambda}$.

Proof. Indeed, reducing $\mathcal{M}$ by the projection $e_0 + e$, we get the model described in 1.5 (for $N = 2$). $\qquad \square$

2. Determination of the cross product algebra

In this section we determine the structure of the crossed product algebra $(Q * B(V_\pi)) \rtimes G$. This, by Takesaki Duality, is in turn used to determine, the structure of the fixed point algebra.

The fixed point algebra could be a type III factor or a type II_1 factor. In the case of the factor in the previous section the fixed point algebra is a type II_1 factor which makes it easier to determine the structure of this algebra. This is similar to the results in [20], [14], [15], where the fixed point algebra of an infinite tensor product of $B(V_\pi)$ under a natural action of $SU_q(N)$ is determined to be the Temperly-Lieb algebra.

To establish that the cross product is stably isomorphic to the fixed point algebra we will need to use instead a quotient group "$SU_q(2)/\mathbb{Z}_2 = SO_q(3)$" (see [21]). Indeed, in the statement of Theorem 1.6 the action of $SU_q(N)$ on $B(V_\pi)$ is not faithful, in the sense that its tensor products do not contain all other representations of the group.

We will use an idea from Wasserman's paper ([41]) that in the case of an infinite tensor product action of a compact group, by a faithful, selfadjoint, unitary irreducible action of a compact group, the Takesaki's duality gives a stable isomorphism between the fixed point algebra and the cross product. We are also indebted

to Y. Ueda for pointing out to us that part of the statement in Theorem 5.6 in [20] does not hold, as also mentioned in the abstract of the paper by M. Izumi [10].

Proposition 2.1. *Let V be the Hilbert space of the fundamental representation of $SU_q(2)$. The adjoint coaction of $SU_q(2)$ on $B(V)$ corresponds to a coaction α of $SO_q(3)$ on $B(V)$. Moreover the fixed point algebras for any of the two (free product) coactions on $Q * B(V)$ coincide. Thus*

$$(Q * B(V))^{SU_q(2)} = (Q * B(V))^{SO_q(3)}$$

Proof. Let as in [21] d_0, $d_{1/2}$, d_1, ... be the representations of $SU_q(2)$, with $d_{1/2}$ the fundamental representation. Then the fundamental unitary for $SO_q(3)$ is given by the unitary representing d_1. As $V \oplus \overline{V}$ is $d_0 \oplus d_1$ it follows that the coaction of $SU_q(2)$ gives the coaction α of $SO_q(3)$ on $B(V)$, (for classical groups that means that the representation comes from a representation of the quotient). Since the unitary implementing both coactions is the same, it also follows that we have the same fixed point algebra, (in terms of groups representations if we have a representation that factors through the quotient, then we have the same fixed point algebra). □

In the next proposition we prove that for any finite-dimensional Hilbert space V, if the tensor product of a selfadjoint coaction α of a Woronowicz quantum group G on $B(V)$ contains any other finite-dimensional unitary representation of the quantum group, then the coaction on $Q * B(V)$ is semidual in the sense of [19] and [41]. Thus the fixed point algebra is Morita equivalent to the cross product by the action. We will use Takesaki Duality, in the sense described in [20].

Proposition 2.2. *Let A be the function algebra of a Woronowicz compact quantum group G, with faithful Haar state. Let $L^2(A)$ be the Hilbert space associated to the Haar measure on A. Assume that α is a faithful selfadjoint corepresentation of G (i.e., a corepresentation such that its tensor product contains any other unitary, finite-dimensional irreducible representation of G). Let Q be a II_1 factor (or a diffuse abelian algebra). Then*

$$(Q \star B(V)) \rtimes A \cong (Q \star B(V))^G \otimes B(L^2(A)).$$

Proof. Let $\mathcal{A} = Q \star B(V)$. By the argument in the proof of Theorem 5.6 in [20] (and using also Lemma 20 in [6], as pointed out in [36]) it is sufficient to prove that for any irreducible unitary coaction $\hat{\alpha}$ of G, the corresponding spectral subspace ([6], [41], [13]) $\mathcal{A}_{\hat{\alpha}}$ is non-trivial. But, since $B(V)$ appears in a tensor product situation in $Q \star B(V)$ this follows from the fact that the tensor product of the representation α with itself contains any other representation (see [41], [19]). □

The following was proved in [37]. It expresses the natural fact that the cross product distributes when we have an action on a free product of two algebras, such that the coaction is trivial on one of the two factors in the free product.

Note that one has to specify a state on the algebra over which we amalgamate in order to get a von Neumann algebra.

Proposition 2.3. *([37]) Let A be the function algebra of a Woronowicz quantum group G, let α be a corepresentation of G on the bounded linear operators $B(V)$ acting on a finite-dimensional Hilbert space V. Let ϕ_α be an α invariant faithful state on $B(V)$, that is such that $(\phi_\alpha \otimes Id)(\alpha(x)) = \phi_\alpha(x)1$, for all x in $B(V)$. Let $\hat{A}$ be the dual algebra. Then we have the following isomorphism*

$$(Q * B(V)) \rtimes A \cong (Q \otimes \hat{A}) *_{\hat{A}} (B(V) \rtimes A).$$

The amalgamated free product is with respect to the canonical conditional expectation of $B(V) \rtimes A$ onto $\hat{A}$ (i.e., the restriction of $\phi_\alpha \otimes Id$). Moreover the free amalgamated von Neumann algebra on the right-hand side of the above equality is determined by endowing the algebra $\hat{A}$ with a state (or weight) that is the restriction of a faithful normal state on $B(L^2(A))$.

We note in the case of $SO_q(3)$ the hypothesis are satisfied. Indeed we have the following:

Lemma 2.4. *We use the notations from the above lemma. For the action α in 2.1, if we consider the state $\phi_\alpha = \tau = tr(Q_\pi \cdot)$, where Q_π is the operator associated in [2] to the fundamental representation π of $SU_q(2)$, then ϕ_α is $SO_q(3)$ invariant.*

Proof. Indeed the unitary implementing the adjoint corepresentation $Ad\,\pi$ is the same as the unitary implementing α. Since τ has this property, the same will hold for ϕ_α. $\qquad\square$

The algebras $\hat{A} \subseteq B(V) \rtimes A$ are discrete and the structure of the corresponding inclusion matrix can be easily described in the case when the representation ring of A is the same as the representation ring of a classical compact group. We are indebted to V. Toledano, D. Bisch, M. Pimsner and G. Nagy for pointing us the more precise description of this inclusion matrix, which essentially appears in [41]. We will not use the explicit description of this matrix.

Lemma 2.5. *Let A be the function algebra of a Woronowicz quantum group G. Let $\hat{A}$ be the dual algebra and let π be a finite-dimensional unitary corepresentation of A on a Hilbert space V_π. Then $B(V_\pi) \rtimes A$ is isomorphic to $B(V_\pi) \otimes \hat{A}$. Moreover $\hat{A} \subseteq B(V_\pi) \rtimes A$ has a Bratelli inclusion matrix with finite multiplicities.*

Proof. By definition $B(V_\pi) \rtimes A$ is isomorphic to the von Neumann algebra generated in $B(V_\pi) \otimes B(L^2(A))$ by $u^*(B(V(\pi) \otimes 1)u$ and $\hat{A}$. But $u(1 \otimes \hat{A})u^*$ is contained in $B(V_\pi) \otimes \hat{A}$, since $Ad\,u$ acting on $B(L^2(A))$ maps the matrix coefficients (viewed as elements of $\hat{A}$) corresponding to a finite-dimensional unitary representation u^α of A into a linear combination of the matrix coefficients (viewed as elements in $\hat{A}$) of the representation $u \otimes u^\alpha$.

We describe this map in classical setting and then explain the modifications needed for a quantum compact group. Indeed in a classical setting, if G is a compact group, let V be the fundamental unitary viewed as an element in $L^\infty(G) \otimes B(L^2(G))$ defined by $V(g) = \lambda_g$, $g \in G$, where $\lambda_g, g \in G$ is the left regular representation of G into the unitary group on $L^2(G)$.

Assume $\pi(g) = (u_g) \in \mathcal{U}(B(V_\pi))$, $g \in G$, is a finite-dimensional unitary representation of G on a Hilbert space V_π. Let U be the unitary in $L^\infty(G) \otimes B(V_\pi)$ given by u. Then in $B(L^2(G)) \otimes B(V_\pi)$ the following holds (the absorption principle for the left regular representation):

$$U^*(\lambda_g \otimes 1)U = \lambda_g \otimes u_g.$$

By using the fundamental unitary $V = V_{12}$ defined above, in $L^\infty(G) \otimes B(L^2(G)) \otimes B(V_\pi)$, the last equality gives (by using Woronowicz's notations)

$$U_{23}^* V_{12} U_{23} = V_{12} U_{13}. \tag{2.1}$$

Let e_{ij} be a matrix unit for $B(V_\pi)$ and let $u_{ij}(g)$ be the matrix coefficients for u_g in this matrix unit. For convolution operators in $L(G)$ the equality (2.1) corresponds, for f in $L^1(G)$ to the following equality in $B(L^2(G)) \otimes B(V_\pi)$.

$$U^* \left(\int f(g)\lambda_g dg \right) U = \int \sum_{ij} (f(g)u_{ij}(g))(1 \otimes e_{ij})\lambda_g dg. \tag{2.2}$$

This gives, using the decomposition of tensor products of irreducible unitary representations of G, a complete description of the inclusion $L(G) \subseteq B(V_\pi) \rtimes G$ (since we have described the inclusion $U(L(G) \otimes 1)U^*$ into $B(V_\pi) \otimes L(G)$ in terms of fusion rules of tensor by the representation π (on matrix coefficients).

If we want to generalize the above statement for arbitrary quantum groups, we will use the corresponding absorption principle for arbitrary quantum groups described in formula (5.11) in [45] which replaces (2.1). This formula now holds in $A \otimes B(L^2(A) \otimes B(V_\pi)$ (where π is the representation in our statement), V is now one of the fundamental unitary in $A \otimes B(L^2(A)$ replacing the V used for the classical case. In this context a convolution operator $\int f(g)\lambda_g dg$ is replaced (in $A \otimes B(L^2(A))$) by

$$(h(f\cdot) \otimes Id)(V) \in \hat{A}, \quad f \in A.$$

This is a generic element in a weakly dense subspace of $\hat{A}$ (here we use V_{12} which is the flip of what is usually the fundamental unitary). The formula (2.2) now reads in $A \otimes B(L^2(A))$ as follows

$$U_{23}[(h(f\cdot) \otimes Id)(V_{12})]U_{23}^* \otimes Id_{B(V_\pi)} = (h(f\cdot) \otimes Id \otimes Id_{B(V_\pi)})(U_{23}V_{12}U_{23}^*).$$

By using the formula (5.11) in the paper [45] we get that this is further equal to (and using a matrix unit e_{ij} in $B(V_\pi)$ with respect to which $u \in A \otimes B(V_\pi)$ has components u_{ij})

$$(h(f\cdot) \otimes Id \otimes Id_{B(V_\pi)})(V_{12}U_{13})$$

which is thus the following element in $B(V_\pi) \otimes \hat{A}$,

$$\sum_{ij} (h(fu_{ij}\cdot) \otimes Id)(V) \otimes e_{ij}. \tag{2.3}$$

This shows that in $A \otimes B(L^2(A))$, the transformation $Ad\, u$ maps $B(V_\pi) \rtimes A$ onto $B(V_\pi) \otimes \hat{A}$. $\qquad\square$

Remark 2.6. If $G = SU_q(2)$, and π is the fundamental representation, since the representation ring of G is the same as the classical one, it follows that the inclusion matrix of $\hat{A}$ into $B(V_\pi) \rtimes A$, is the same as in the classical one, which is a matrix of type A_∞.

We need to apply the previous lemma to the case of the coaction α of $SO_q(3)$ on $B(V)$ which was described in 2.1.

Lemma 2.7. *Let $\hat{B}$ be the dual Woronowicz algebra for $SO_q(3)$. With the notation in Proposition 2.1 we have that $B(V) \rtimes SO_q(3)$ is isomorphic to $B(V) \otimes \hat{B}$. Moreover the inclusion matrix $\hat{B} \subseteq B(V) \rtimes SO_q(3)$ has a Bratelli inclusion matrix with finite multiplicities.*

Proof. Denote $B = SO_q(3)$ and $A = SU_q(2)$ and let $\hat{B}, \hat{A}$ be the dual algebras. The representation $\alpha : B(V) \to B(V) \otimes B$ is the restriction of the adjoint corepresentation induced by π. This holds because $\alpha(x) = u^*(x \otimes 1)u$ belongs to $B(V) \otimes B$, since B is generated by the matrix coefficients of d_1. But then if P is the projection (in $B(L^2(A))$) onto $L^2(B)$, then

$$B(V) \rtimes B = P(B(V) \rtimes A)P,$$

since $P\hat{A}P = \hat{B}$ and since B acts on $L^2(A)$ as the Haar measure on B is the restriction of the Haar measure on A ([21]). $\square$

The following lemma is a direct consequence of the method used in [29]. There it was proved that amalgamated free products of the type $(L(F_N) \otimes D) *_D C$, where $D \subseteq C$ is an inclusion of discrete von Neumann algebras, C with a faithful trace, are isomorphic to a free group factors.

Lemma 2.8. *Let $D \subseteq C$ be von Neumann algebras that are infinite sums of algebras of finite-dimensional matrices, and such that the Bratelli inclusion matrix has finite multiplicities and $\mathcal{Z}(C) \cap \mathcal{Z}(D) = \mathbb{C}1$. Let Q be a free group factor or a diffuse abelian von Neumann algebra. Assume that D is endowed with a (semi)finite faithful trace tr and that the amalgamation is performed ([40], [24]) with respect to a normal faithful conditional expectation from C onto D, which then gives a trace state on the amalgamated free product, (which by GNS construction gives the amalgamated free product von Neumann algebra).*

Assume that the amalgamated free product von Neumann algebra is finite or semifinite.

Then $tr \circ E$ is a faithful trace on C and $\mathcal{L}(F_N) \otimes D \star_D C$ is Morita equivalent to a free group factor.

Proof. Indeed if $\tau \circ E$ is not a trace then the modular group of $\tau \circ E$ will be nontrivial on off diagonal elements. If the trace is finite this is exactly the content of Theorem 5.1 in [29] (see also [34] for a different, more recent proof). If the trace is infinite, the arguments in the proof of the theorem mentioned above can obviously be modified to handle this case (by changing the principle of counting the blocks as in the construction of a one parameter group of automorphisms in [30]). $\square$

Remark 2.9. With the notations in Theorem 1.6, one can prove directly that factor $(Q * B(V_\pi)) \rtimes SU_q(2)$ is a finite type II von Neumann algebra. Indeed because of Lemma 2.3, it is sufficient to that the inclusion $\hat{A} \subseteq B(V_\pi) \rtimes A$, and the corresponding conditional expectation $tr(\phi_\alpha \cdot) \otimes Id$ (see also Lemma 2.4) from $B(V_\pi) \rtimes A$ onto A, has the property that the composition of the trace on $\hat{A}$ with the conditional expectation is again a trace. The trace on $\hat{A}$ that we are considering here is the restriction to $\hat{A} \subseteq B(L^2(A))$ of the canonical trace on $B(L^2(A))$.

Let p_π be the projection from $\hat{A}$ onto $B(V_\pi)$ (by viewing $\hat{A}$ as the direct sum over all bounded linear operators on the Hilbert spaces of an enumeration of the irreducible finite-dimensional unitary representations of A, ([16], [23])).

Let $\hat{\phi} : \hat{A} \to \hat{A} \otimes \hat{A}$ be dual comultiplication map. Then the inclusion $\hat{A} \subseteq B(V_\pi) \rtimes A$, (in the identification $B(V_\pi) \rtimes A \cong B(V_\pi) \otimes \hat{A}$, described in Proposition 2.5) is, (because of the description in [23]) exactly the map

$$(p_\pi \otimes Id)\hat{\phi} : \hat{A} \to B(V_\pi) \otimes \hat{A}.$$

Moreover if $\hat{\gamma}$ is the density matrix in $\hat{A}$ of the dual Haar measure $\hat{h}$ on $\hat{A}$, then $\hat{\phi}(\hat{\gamma}) = \hat{\gamma} \otimes \hat{\gamma}$ ([46]). But this amounts exactly to the fact that the composition of the conditional expectation $tr(\phi_\alpha \cdot) \otimes Id$ with $\hat{h}$ gives the state $tr(\phi_\alpha \cdot) \otimes \hat{h}$ on $B(V_\pi) \rtimes A \cong B(V_\pi) \otimes \hat{A}$. Hence, the composition of the trace on $\hat{A}$ (coming from $B(L^2(A))$) with the conditional expectation is a trace.

Corollary 2.10. *The factor $\mathcal{M}^\lambda(\mathcal{L}(F_k))$ in Theorem 1.6 (for $N = 2$) is isomorphic to $\mathcal{L}(F_\infty)$.*

Proof. From 2.8, 2.3 it follows that $(Q * B(V_\pi)) \rtimes SO_q(3)$ is isomorphic to $\mathcal{L}(F_\infty) \otimes B(H)$, where H is an infinite-dimensional space. The result follows now from 1.6, 2.1, since d_1 is a faithful representation of $SO_q(3)$ (no adjoint is required here). $\square$

In this way we reobtain, by a different method, the result that was recently proved by D. Shlyakhtenko and Y. Ueda in [36]. We do not know if the subfactors obtained by using the method in this paper, which are derived from Popa's construction of irreducible subfactors from the Temperly-Lieb algebra coincide with those constructed in the paper [36]. Both of the two subfactors have A_∞ invariants and both are the fixed point algebra of a coaction of a quantum group, although in one case the fixed point algebra is a type III factor while in the present case this is a type II_1 factor.

Corollary 2.11. *In particular $\mathcal{L}(F_\infty)$ has irreducible subfactors of index λ (and type A_∞) for all index values bigger than 4.*

The previous result, since we are using a non-tracial version of [24] is strong evidence to Popa's conjecture that the subfactors constructed in the breakthrough papers [24] (or [25], [26]) are isomorphic to free group factors. The only case in which Popa's subfactors ([24], [25]) are known to be free group factors is for index values less than 4 ([29]).

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