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Aharonov–Bohm superselection sectors

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Abstract

We show that the Aharonov–Bohm effect finds a natural description in the setting of QFT on curved spacetimes in terms of superselection sectors of local observables. The extension of the analysis of superselection sectors from Minkowski spacetime to an arbitrary globally hyperbolic spacetime unveils the presence of a new quantum number labelling charged superselection sectors. In the present paper, we show that this “topological” quantum number amounts to the presence of a background flat potential which rules the behaviour of charges when transported along paths as in the Aharonov–Bohm effect. To confirm these abstract results, we quantize the Dirac field in the presence of a background flat potential and show that the Aharonov–Bohm phase gives an irreducible representation of the fundamental group of the spacetime labelling the charged sectors of the Dirac field. We also show that non-Abelian generalizations of this effect are possible only on spacetimes with a non-Abelian fundamental group.

Keywords Quantum field theory on curved space · Superselection sectors · Aharonov–Bohm effect

Mathematics Subject Classification 81T05 · 81T20

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1 Introduction

The analysis of superselection sectors is one of the central results of algebraic quantum field theory. It allows to derive from first principles particle statistics, the particle–antiparticle correspondence, quantum charges and, as a consequence, the appearance of global gauge symmetries, independently of the model under consideration [12].

In the original formulation, superselection sectors were defined by Doplicher, Haag and Roberts in terms of localized and transportable endomorphisms $\rho(o) : \mathcal{A} \rightarrow \mathcal{A}$ of the global observable C^* -algebra \mathcal{A} , where o is a double cone in Minkowski spacetime. In this way, a new representation (sector)

$$\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_0), \quad \pi := \pi_0 \circ \rho(o),$$

is defined starting from a vacuum representation π_0 . The term *localized* indicates that $\rho(o) \upharpoonright \mathcal{A}_e$ is the identity on any local algebra $\mathcal{A}_e \subset \mathcal{A}$ generated by observables localized in a double cone e causally disjoint from o , while *transportable* means that, for any double cone a , $\rho(o)$ is unitary equivalent to an endomorphism localized in a . This yields the DHR selection criterion [20].

It was later recognized by Roberts that an equivalent formulation of superselection sectors could be achieved by considering *charge transporters*, that is, families z of unitaries $z(a, o) \in \mathcal{A}_a$, $o \subset a$, fulfilling a cocycle relation. Each z defines a family $\{\rho^z(o)\}_o$ of localized endomorphisms, such that

$$z(a, o) \rho^z(o)(A) = \rho^z(a)(A) z(a, o), \quad A \in \mathcal{A}.$$

In Minkowski spacetime, this approach turns out to be equivalent to the original one [29,30]. Yet things change in the context of curved spacetimes M , mainly due to topological obstructions that arise when the family of *diamonds*, the analogues of double cones, is not upward directed under inclusion. A first point is that the notion of an algebra of global observables is not well-defined and must be replaced by the *universal algebra*, indicated with $\tilde{\mathcal{A}}$ [16], having the property of lifting any family of representations

$$\pi_o : \mathcal{A}_o \rightarrow \mathcal{B}(\mathcal{H}) : \pi_a \upharpoonright \mathcal{A}_o = \pi_o, \quad o \subset a.$$

A second point is that localized endomorphisms alone cannot encode the whole physical content of sectors [6,32]. Therefore, the use of charge transporters becomes necessary in the more general scenario of a curved spacetime M .

A condition that must be imposed on z in order to get a well-defined representation π of $\tilde{\mathcal{A}}$ is

$$z_p := z(a, o_{n-1}) \cdots z(o_2, o_1) z(o_1, a) \equiv \mathbb{I},$$

where $p := \{a, o_1, \dots, o_{n-1}\}$ is a suitable covering (*path-approximation*, Sect. 2.2) of an arbitrary loop (closed curve) $\gamma : [0, 1] \rightarrow M$. This condition is called *topological triviality*, and the justification of this terminology is given by the fact that z_p only

depends on the homotopy class of γ . The question of what happens if topological triviality is not imposed then naturally arises. An answer is given in a series of papers, in which the following facts were established:

1. Statistics, charge content and particle–antiparticle correspondence are well defined on generic charge transporters. Thus, they have all the properties that allow to assign a physical interpretation to topologically trivial transporters. Moreover, each z defines a representation

$$\sigma : \pi_1(M) \rightarrow \mathbb{U}(n), \quad (1)$$

which affects the transport of localized endomorphisms along loops. The integer $n \in \mathbb{N}$ is called the *topological dimension* [6].

2. Generic charge transporters do not define representations of $\vec{\mathcal{A}}$. Rather, they define representations of the net $\mathcal{A}_{KM} := \{\mathcal{A}_o\}_o$ on flat Hilbert bundles over M , which correspond to covariant representations of a universal C^* -dynamical system $\alpha : \pi_1(M) \rightarrow \mathbf{aut} \mathcal{A}_*$ encoding parallel transport along loops [33–35]. It turns out that $\vec{\mathcal{A}}$ is a quotient of \mathcal{A}_* , corresponding to the set of representations of \mathcal{A}_* with trivial action of $\pi_1(M)$ on the Hilbert space.

The physical interpretation of these results was proposed by Brunetti and the second named author in the scenario of the Aharonov–Bohm effect. There, superposition of wavefunctions of charged particles is affected by phases (parallel transports) of type

$$\sigma(\gamma) = \exp \oint_{\gamma} A, \quad (2)$$

where A is a potential with vanishing electromagnetic field, and it is natural to note the similarity with (1). The suitable model in which such an interpretation can be tested at the level of quantum field theory is the Dirac field, describing charged quantum particles in a curved spacetime M . A step in this direction has been made in [38], where it is proven that introducing a background potential represented by a closed 1-form yields a representation of the observable net \mathcal{A}_{KM} of the free Dirac field over a flat Hilbert bundle over M .

In the present paper, we complete the above partial results and perform an analysis of the superselection structure of \mathcal{A}_{KM} . Our main results show that:

- Superselection sectors z define background flat potentials A^z , that are, at the mathematical level, flat connections on suitable flat Hermitean vector bundles $\mathcal{L}^z \rightarrow M$ whose rank coincides with the topological dimension of z (Theorem 2.4). We shall be interested, in particular, in those sectors with topological dimension 1, so that \mathcal{L}^z is a line bundle.
- Superselection sectors with topological dimension 1 and “charge 1”, in a sense that shall be clarified later, are in one-to-one correspondence with *twisted field nets*. These are representations of the field net of the Dirac field on flat Hilbert bundles over M with monodromy given by the “second quantization” of the monodromy of \mathcal{L}^z (Theorem 4.4);

- When no torsion appears in the first homology group of the spacetime, \mathcal{L}^z is topologically trivial. In this case, we are able to construct a twisted field solving the Dirac equation with interaction term A^z , and the representation (1), with $n = 1$, takes the form (2) (Theorem 4.3). This occurs in physically relevant scenarios such as de Sitter and anti-de Sitter spacetimes, as well as the spacetime complementary to the ideally infinite solenoid of the Aharonov-Bohm apparatus.

Both the twisted field net and the twisted Dirac field are indistinguishable from the analogous untwisted objects in simply connected regions of M . This is the reason why \mathcal{A}_{KM} is able to detect them. The non-trivial parallel transport carried by the family of localized endomorphisms defined by a sector is interpreted in terms of the background flat potential, shedding some light on the role that the corresponding interaction plays on quantum charges.

The paper is organized as follows.

In Sect. 2, we recall results from [6, 32] describing the superselection structure, say $Z^1(\mathcal{A}_{KM})$, of a generic observable net. Moreover, by applying a result by Barrett [3], for any sector z we exhibit a flat $u(n)$ -connection such that (1) is the associated parallel transport. This yields a Chern character defined on $Z^1(\mathcal{A}_{KM})$ with values in the odd cohomology $H^{odd}(M, \mathbb{R}/\mathbb{Q})$ of the spacetime (Remark 2.3). The first component of the Chern character will be shown to correspond to Aharonov–Bohm phases in the subsequent sections.

In Sect. 3, we focus on the case in which \mathcal{A}_{KM} is the observable net of the free Dirac field. By applying results in [4], we show that Haag duality holds in the GNS representation π_ω of any pure quasi-free state ω of the Dirac-CAR algebra defined on the spinor space of a globally hyperbolic spacetime (Theorem 3.2). This, and the *III*-property fulfilled by the local Von Neumann algebras living in π_ω , [8], yields two key properties for the analysis of sectors of \mathcal{A}_{KM} . Finally, we construct a family of topologically trivial sectors of \mathcal{A}_{KM} labelling the electron–positron charge (Remark 3.7).

In Sect. 4, we construct Dirac fields interacting with background potentials A , which are closed 1-forms ($dA = 0$). These fields are “twisted”, as in [24], namely they are defined on the space of sections of the Dirac bundle tensor a flat line bundle with connection A . We show that these fields are in one-to-one correspondence with sectors in $Z^1(\mathcal{A}_{KM})$ such that \mathcal{L}^z is topologically trivial (Theorem 4.3). The parallel transport defined by A appears, from one side as a byproduct of the interaction term in the Dirac equation, and on the other side as the parallel transport of DHR-charges (localized endomorphisms) of \mathcal{A}_{KM} .

In Sect. 5, we draw our conclusions and we illustrate further developments, in particular for $\pi_1(M)$ non-Abelian and generalized Dirac fields with non-Abelian gauge group.

In “Appendix A”, we prove a technical result, allowing to adopt the usual Haag duality instead of the punctured one in the analysis of sectors.

Some of the results of the present paper have been presented in [39] in a simplified form, in particular by avoiding to discuss Haag duality and without going in details on the structure of twisted Dirac fields.

2 Aharonov–Bohm effects in terms of sectors

Aim of the present section is to describe the Aharonov–Bohm effect in terms of superselection sectors of the net of local observables on a 4-d globally hyperbolic spacetime M . From the intrinsic viewpoint of the electromagnetic field and of the vector potential, this effect has been studied instead in [5,37].

We start by recalling some basic facts on globally hyperbolic spacetimes, and we introduce the net of local observables defined in a reference representation. We then discuss DHR-charges of the observable net in terms of charge transporters. These are cocycles depending on a suitable family of regions of the spacetime, diamonds, taking values in the unitary group of the observable net. The key observation is that diamonds encode the fundamental group $\pi_1(M)$ of the spacetime M and cocycles define representations of $\pi_1(M)$. The Aharonov–Bohm effect is manifested in cocycles which define nontrivial representations of the fundamental group of the spacetime. We shall prove that these cocycles are nothing but the holonomy of smooth flat connections acting on a DHR charge. The form of the action is ruled by the charge quantum numbers.

The material appearing in the present section is essentially a convenient summary of [6,32]. Exceptions are Lemma 2.2 and Theorem 2.4, where the interpretation in terms of flat connections is given.

2.1 Local observables in a globally hyperbolic spacetime

Let M denote a four-dimensional, connected, globally hyperbolic spacetime. *Causal disjointness* relation is a symmetric binary relation \perp defined on subsets of M as follows:

$$o \perp \tilde{o} \iff o \subseteq M \setminus J(\tilde{o}), \quad (3)$$

where J denotes the causal set of o . The causal complement of a set o is the open set $o^\perp := M \setminus \text{cl}(J(o))$. An open set o is causally complete whenever $o = o^{\perp\perp}$.

A distinguished class of subsets of M is given by the set KM of *diamonds* $o \subset M$ [6, §3.1]. More precisely, a diamond $o \subset M$ is an open subset such that there exists a spacelike Cauchy surface Σ , a chart (U, φ) of Σ , and an open ball $\mathcal{B} \subset \mathbb{R}^3$ such that

$$o = D(\varphi^{-1}(\mathcal{B})), \quad \text{cl}(\mathcal{B}) \subset \varphi(U) \subset \mathbb{R}^3,$$

where $D(\varphi^{-1}(\mathcal{B}))$ is the domain of dependence of $\varphi^{-1}(\mathcal{B})$, while $\text{cl}(o)$, the closure of o , is compact. Observe that diamonds are open, relatively compact, connected and simply connected, causally complete, and the causal complement o^\perp is connected. Moreover, diamonds form a base of neighbourhoods for the topology of M .

In the present paper, we are interested in describing the observable net in a *reference* representation playing the same rôle as the vacuum representation in Minkowski spacetime. This is defined in terms of a net \mathcal{A}_{KM} given by the assignment of a von Neumann algebra $\mathcal{A}_o \subset \mathcal{B}(\mathcal{H}_0)$ for any $a \in KM$. We assume the following standard properties:

- (*Isotony*) $\mathcal{A}_o \subseteq \mathcal{A}_a$ for $o \subseteq a$,

- (*Causality*) $\mathcal{A}_o \subseteq \mathcal{A}'_a$ for $o \perp a$, where \mathcal{A}'_a stands for the commutant of \mathcal{A}_a ,
- (*The Borchers property*) for any inclusion $cl(o) \subset \tilde{o}$, if E is a non-vanishing orthogonal projection of \mathcal{A}_o then there is an isometry $V \in \mathcal{A}_{\tilde{o}}$ such that $VV^* = E$;
- (*Haag duality*) $\mathcal{A}_o = \cap \{\mathcal{A}'_a : a \perp o\}$, for any $o \in KM$.

In addition, \mathcal{A}_{KM} is assumed to be *irreducible* $(\cup_o \mathcal{A}_o)' = \mathbb{C}1$, and *outer regular* $\mathcal{A}_o = \cap_{cl(o) \subset a} \mathcal{A}_a$.

Meaningful examples satisfying the above properties are the observable net of the free scalar field [40] and, as we shall see, and that of the free Dirac field, in representations induced by pure quasi-free Hadamard states. Anyway, the above properties are expected to hold for any Wightman field over M .

2.2 1-cocycles of the observable net and topology

We introduce the mathematical structure underlying the charges studied in [6]. The main fact is that the theory is encoded by charge transporters which define representations of the observable net by means of Haag duality. A crucial mathematical property is that charge transporters satisfy a cocycle equation with respect to the partially ordered set (poset) KM and, as a consequence, give a representation of the fundamental group of the spacetime. In the present paper, we give an equivalent but simplified exposition of cocycles which do not rely on simplicial sets.

We start by recalling some elements of the “geometry” of KM , the poset of diamonds ordered under inclusion \subseteq . A pair $a, o \in KM$ is said to be *comparable* whenever either $o \subseteq a$ or $a \subseteq o$, and in this case we write $a \geq o$. Any comparable pair $o \geq a$ defines two oriented “elementary” paths: $(a, o) : o \rightarrow a$, starting from o and ending to a , and its *opposite* $\overline{(a, o)} := (o, a) : a \rightarrow o$. We call the *support* of (a, o) the set $|(a, o)| := a \cup o$. Arbitrary paths are obtained by composing elementary paths: for instance, given (a, o) and (o, \tilde{a}) we can form the path

$$p := (a, o) * (o, \tilde{a}) : \tilde{a} \rightarrow a$$

and the opposite $\overline{p} : a \rightarrow \tilde{a}$ of p as the path

$$\overline{p} := \overline{(a, o) * (o, \tilde{a})} := \overline{(o, \tilde{a})} * \overline{(a, o)} = (\tilde{a}, o) * (o, a).$$

The composition and the operation of taking the opposite extend to arbitrary paths in an obvious way: given $p : a \rightarrow o$ and $q : o \rightarrow \tilde{o}$ then

$$q * p : a \rightarrow \tilde{o}, \quad \overline{q * p} = \overline{p} * \overline{q} : \tilde{o} \rightarrow a.$$

A path of the type $p : o \rightarrow o$ is called a *loop* over o . It is verified that M is connected if, and only if, KM is *pathwise connected*, meaning that any pair of diamonds a, o can be joined by a path $p : a \rightarrow o$ [32]. There is a homotopy equivalence relation \sim on the set of paths: the quotient by \sim of the set of loops over a fixed $a \in KM$ gives the homotopy group $\pi_1(KM, a)$ which does not depend, up to isomorphism, on a , because of pathwise connectedness. The isomorphism class is called the *fundamental*

group of KM , written $\pi_1(KM)$. A key result is that, given $o \in KM$ and $x_o \in o$, there is an isomorphism

$$\pi_1(M, x_o) \ni [\gamma] \mapsto [p_\gamma] \in \pi_1(K, o), \quad (4)$$

where $\pi_1(M, x_o)$ is the homotopy group of M based at $x_o \in o$ [32]. In the previous expression, p_γ is a *path-approximation* of the closed curve $\gamma : [0, 1] \rightarrow M$: that is, a path

$$p_\gamma = (o_n, o_{n-1}) * \dots * (o_2, o_1) * (o_1, o_0)$$

such that there is a partition

$$t_0 = 0 < t_1 < \dots < t_{n-1} < t_n = 1 \quad : \quad \gamma(t_i) \in o_i \text{ and } \gamma([t_{i-1}, t_i]) \subset o_{i-1} \wedge o_i, \quad \forall i.$$

Thus, closed curves γ and β are homotopic, i.e. $[\gamma] = [\beta]$, if and only if $p_\gamma \sim p_\beta$, i.e. $[p_\gamma] = [p_\beta]$.

Remark 2.1 We highlight a few consequences of the previous result which are of relevance to this paper. If KM is directed under inclusion, then $\pi_1(KM)$ is trivial [32]. So, if $\pi_1(M)$ is not trivial, then KM is not upward directed. This is actually the physical situation we are interested in, since Aharonov–Bohm type effects appear when the spacetime has a non-trivial fundamental group.

We now introduce the set $Z^1(\mathcal{A}_{KM})$ of 1-cocycles of the observable net, i.e. the charge transporters. A *1-cocycle of the net* \mathcal{A}_{KM} is a map assigning to any *comparable* pair $o \geq a \in KM$ a unitary operator $z(a, o) \in \mathcal{A}_{|a, o|}$, fulfilling the property $z(a, o) = z(o, a)^*$ and the *cocycle equation*

$$z(\hat{o}, o) = z(\hat{o}, a) z(a, o), \quad o \subseteq a \subseteq \hat{o}. \quad (5)$$

The previous definition implies $z(o, o) = 1$ for any $o \in KM$, since $z(o, o) = z(o, o)z(o, o)$ and $z(o, o)$ are unitary. 1-cocycles are extended to paths by defining

$$z(p) := z(o, o_n) \cdots z(o_2, o_1) z(o_1, a), \quad p = (o, o_n) * \dots * (o_2, o_1) * (o_1, a).$$

Note that $z(\bar{p}) = z(p)^*$. Any $z \in Z^1(\mathcal{A}_{KM})$ preserves the homotopy equivalence relation, so that the mapping $[p] \mapsto z(p)$, $[p] \in \pi_1(KM, o)$, is well-defined. By (4), we get the representation

$$\sigma_z : \pi_1(M, x_o) \rightarrow \mathcal{U}(\mathcal{H}_0), \quad \sigma_z([\gamma]) := z(p_\gamma), \quad [\gamma] \in \pi_1(M, x_o), \quad (6)$$

where $p_\gamma : o \rightarrow o$ is a path-approximation of γ . A 1-cocycle z is said to be *topologically trivial* whenever σ_z is the trivial representation.

An *intertwiner* between two 1-cocycles z, \tilde{z} is a map $t_a \in \mathcal{A}_a$, $a \in KM$, fulfilling the relations $t_a z(p) = \tilde{z}(p) t_o$ for all $p : o \rightarrow a$. The set of *intertwiners* between z, \tilde{z} is denoted by (z, \tilde{z}) . The 1-cocycles z, \tilde{z} are said to be *unitarily equivalent* if there exists $t \in (z, \tilde{z})$ such that t_a is unitary for any $a \in KM$. The 1-cocycle z is said to

be *irreducible* whenever $(z, z) = \mathbb{C}\mathbb{I}$, where $\mathbb{I} \in \mathcal{B}(\mathcal{H}_0)$ is the identity, and *trivial* if it is unitarily equivalent to the trivial 1-cocycle $\iota(a, o) \equiv \mathbb{I}$. It is easily seen that cocycles form the set of objects for a category $Z^1(\mathcal{A}_{KM})$, with the corresponding sets of intertwiners as arrows. We denote the full subcategory of $Z^1(\mathcal{A}_{KM})$ whose objects are topologically trivial 1-cocycles by $Z^1_t(\mathcal{A}_{KM})$.

2.3 Charge quantum numbers

The physical content of 1-cocycles is encoded in the charge quantum numbers associated with their equivalence classes. In this section, we shall recall how these quantum numbers arise and explain the nature of these charges. As already mentioned, these results have been proved in [6, 32]. There is, however, an important difference: in the present paper, we assume Haag duality, whilst in the above references a stronger form of duality is used, the *punctured* Haag duality. In the appendix, we show succinctly that Haag duality suffices.

Using the defining properties of the observable net, one can introduce a tensor product \times (charge composition) and a permutation symmetry ε (statistics), making $Z^1(\mathcal{A})$ and $Z^1_t(\mathcal{A})$ symmetric, tensor C^* -categories. In particular, one can identify a subset of 1-cocycles called *objects with finite statistics* which turn out to be a finite direct sum of irreducible 1-cocycles. The unitary equivalence class of any such irreducible object z is classified by the following charge quantum numbers:

- the *statistical phase* $\kappa(z) \in \{-1, 1\}$ distinguishing between Fermi and Bose statistics;
- the *statistical dimension* $d(z) \in \mathbb{N}$ giving the order of the (para)statistics;
- the *charge conjugation* assigning to z a conjugated irreducible 1-cocycle \bar{z} which has the same statistical phase and dimension as z .

We stress that objects with statistical dimension 1 obey the Bose/Fermi statistics according to their statistical phase. We denote the subcategory of *1-cocycles with finite statistics* by $Z^1_{AB}(\mathcal{A})$ and its subcategory of *topologically trivial 1-cocycles* by $Z^1_{DHR}(\mathcal{A})$.

To complete the physical interpretation, we need to observe that any 1-cocycle $z \in Z^1_{AB}(\mathcal{A})$ splits in two parts: the charged and the topological component. To see this, we need a *path frame* P_e consisting of a collection of arbitrary paths $p_{ae} : e \rightarrow a$ joining a fixed diamond e , the pole, with any a , and such that p_{ee} is the trivial path (e, e) . Once a path frame P_e is given, the *charged component* of z is a topologically trivial 1-cocycle z_c in $Z^1_{DHR}(\mathcal{A}_{KM})$ defined as

$$z_c(\tilde{a}, a) := z(p_{\tilde{a}e} * p_{ea}), \quad a \geq \tilde{a}, \quad (7)$$

where $p_{e\tilde{a}}$ is the opposite $\overline{p_{\tilde{a}e}}$ of $p_{\tilde{a}e}$. This cocycle encodes the charge content of z as the maps $z \mapsto z_c$ and $(z, \tilde{z}) \ni s \rightarrow s \in (z_c, \tilde{z}_c)$ define a functor

$$Z^1_{AB}(\mathcal{A}_{KM}) \rightarrow Z^1_{DHR}(\mathcal{A}_{KM}), \quad (8)$$

preserving the tensor product, the permutation symmetry and, hence, the conjugation.¹ On the other end, setting

$$u_z(\tilde{a}, a) := z(p_{e\tilde{a}} * (\tilde{a}, a) * p_{ae}), \quad \tilde{a} \gtrsim a, \quad (9)$$

one gets a 1-cocycle u_z , the *topological component* of z , taking values in the algebra of the chosen pole e , i.e. $u_z(\tilde{a}, a) \in \mathcal{A}_e$ for all $\tilde{a} \gtrsim a$. It encodes the topological content of z since it defines the same representation (6). Finally, the elements of $Z_{AB}^1(\mathcal{A}_{KM})$ are completely characterized by their topological and the charged component since

$$z(\tilde{a}, a) = (u_z \bowtie z_c)(\tilde{a}, a) = \alpha_{\tilde{a}e}(u_z(\tilde{a}, a)) \cdot z_c(\tilde{a}, a), \quad \tilde{a} \gtrsim a, \quad (10)$$

where $\alpha_{\tilde{a}e}(A) := z_c(p_{\tilde{a}e}) A z_c(p_{\tilde{a}e})^*$. The composition \bowtie is called the *join*. These constructions depend neither on the choice of the pole e nor on that of the path frame P_e : different choices lead to equivalent, in the corresponding categories, charged and topological components.

Now, any 1-cocycle z defines representations of the observable net which are sharp excitations of the reference one. To be precise, given a diamond o define

$$\rho^z(o)_a(A) := z(p) A z(\bar{p}), \quad a \in KM, A \in \mathcal{A}_a, \quad (11)$$

where $p : e \rightarrow o$ is an arbitrary path such that $e \subset a^\perp$. By (70), given in appendix, this definition is independent of p , and $\rho^z(o)_a : \mathcal{A}_a \rightarrow \mathcal{B}(\mathcal{H}_0)$ is a unital morphism satisfying the relation $\rho^z(o)_a = \rho^z(o)_{\tilde{a}} \upharpoonright \mathcal{A}_a$ for any inclusion $a \subseteq \tilde{a}$. This amounts to saying that the family $\rho^z(o) := \{\rho^z(o)_a\}_{a \in KM}$ defines a Hilbert space representation

$$\rho^z(o) : \mathcal{A}_{KM} \rightarrow \mathcal{B}(\mathcal{H}_0) \quad (12)$$

in the sense of [33]. Furthermore, always using the property (70), we find that $\rho^z(o)$ is *localized* in o , that is,

$$\rho^z(o)_a = \text{id}_{\mathcal{A}_a}, \quad a \perp o, \quad (13)$$

and *transportable*, i.e.

$$z(p)\rho^z(o) = \rho^z(a)z(p), \quad p : o \rightarrow a, \quad (14)$$

where the above equation should be read as

$$z(p) \cdot \rho^z(o)_e(A) = \rho^z(a)_e(A) \cdot z(p), \quad e \in KM, A \in \mathcal{A}_e. \quad (15)$$

Equation (13) says that $\rho^z(o)$ describes a charge localized in o , since it equals the reference representation in the causal complement of o , whereas (14) illustrates the role of z as a charge transporter along paths. The important point is that the charges described by AB and DHR cocycles are the same since $\rho^z(o) = \rho^{z_c}(o)$ for any $o \in KM$. This is an immediate consequence of the path independence of the definition

¹ The functor is not full, as we may have $(z, \tilde{z}) \subsetneq (z_c, \tilde{z}_c)$ [6].

of $\rho^z(o)$ and of the definition of the charged component z_c . However, an important difference arises when the charge is transported from o to a along to different paths $p, q : o \rightarrow a$. If z is of DHR type, then the transport is path-independent,

$$z(p)\rho^z(o)z(\bar{q}) = \rho^z(a),$$

in fact $z(p * \bar{q}) = 1$; whilst if z is of AB type, then

$$z(p)\rho^z(o)z(\bar{q}) = z(p * \bar{q})\rho^z(a) = u_z(p * \bar{q})\rho^z(a). \quad (16)$$

Hence, $z(p * \bar{q})\rho^z(a)$ depends on the homotopy equivalence class of the loop $p * \bar{q}$. In particular, $z(p * \bar{q}) \neq 1$ if $p * \bar{q} \approx 1$. We conclude that *the Aharonov–Bohm effect manifests itself in sectors of AB type and, as we shall prove in the next sections, it is due to a presence of a background flat potential.*

2.4 DHR charges and WWW superselection rules

To briefly illustrate how the notion of DHR charge fits the traditional approach of Wick, Wigner and Wightman (WWW), we assume that our observable net is obtained by taking the gauge-invariant part of a field net (see [19] for details).

The scenario is that of a Hilbert space \mathcal{H}_{phys} on which a compact gauge group acts by means of a unitary representation $U : G \rightarrow \mathcal{U}(\mathcal{H}_{phys})$. It is given a net \mathcal{F}_{KM} of von Neumann algebras $\mathcal{F}_o \subseteq \mathcal{B}(\mathcal{H})$, $o \in KM$, that are typically generated by smeared field operators $\psi(f)$, where ψ is a Wightman field and f are test functions with $\text{supp}(f) \subseteq o$. We define the gauge-invariant net \mathcal{F}_{KM}^0 by setting $\mathcal{F}_o^0 := \mathcal{F}_o \cap U(G)'$, where $U(G)'$ is the commutant of $U(G)$. As well-known, \mathcal{H}_{phys} splits into a direct sum labelled by the irreducible representations κ of G ,

$$\mathcal{H}_{phys} \simeq \oplus_{\kappa} \mathcal{H}_{\kappa}.$$

The spectral projections $E_{\kappa} \in \mathcal{B}(\mathcal{H}_{phys})$, $E_{\kappa}\mathcal{H}_{phys} = \mathcal{H}_{\kappa}$, commute with \mathcal{F}_o^0 for all $o \in KM$, so \mathcal{F}_{KM}^0 is reduced on each \mathcal{H}_{κ} by defining $\pi_{\kappa}(F) := E_{\kappa}F$, $F \in \mathcal{F}_o^0$. The maps π_{κ} define representations of \mathcal{F}_{KM}^0 and yield the superselection sectors according to the standard WWW approach. In particular, we have the observable net \mathcal{A}_{KM} , $\mathcal{A}_o := E_0\mathcal{F}_o^0 \subset \mathcal{B}(\mathcal{H}_0)$, where E_0 is the projection onto the subspace of G -invariant vectors \mathcal{H}_0 .

By standard assumptions for field nets, for any κ and o there is a multiplet $V_{\kappa,o,i} \in \mathcal{F}_o$, $i = 1, \dots, \dim_{\kappa}$, of mutually orthogonal isometries with total support the identity, transforming according κ and defining the unitaries

$$z(a, o) := E_0 \sum_i V_{\kappa,a,i} V_{\kappa,o,i}^* \in \mathcal{A}_a, \quad o \subset a.$$

These unitaries are charge transporters and define localized endomorphisms $\rho^z(o)$ such that $\rho^z(o) \circ \pi_0$ and π_{κ} are unitarily equivalent representations of \mathcal{F}_{KM}^0 . Thus, π_{κ}

and $\rho^z(o)$ carry the same physical information, yet $\rho^z(o)$ has the advantage of being intrinsically defined on the observable net \mathcal{A}_{KM} .

2.5 Background flat potentials and the AB effect

In this section, we complete the picture drawn in the preceding sections aimed at explaining the Aharonov–Bohm effect in terms of superselection sectors of local observables. In particular, we shall prove that the topological component of 1-cocycles is nothing but the holonomy of a flat connection. To this end, we shall use a result that has been rediscovered by several authors.² We refer in particular to a paper by Barrett [3].

Before proceeding, it is convenient to specify the notion of flat connection. Let G be a compact Lie group with Lie algebra \mathfrak{g} and $\mathcal{P} \rightarrow M$ denote a smooth G -principal bundle. We assume that \mathcal{P} is trivialized on any diamond, and that it is *locally constant* in the sense that it admits a set of *locally constant* transition maps

$$g_{oa} : o \cap a \rightarrow G, \quad o, a \in KM.$$

Thus, $dg_{oa} \equiv 0$, and following [25, §I.1-2] we define a flat connection on \mathcal{P} as a collection A of flat \mathfrak{g} -valued 1-forms A_o defined on $o \subset M$, fulfilling the relations

$$A_a \upharpoonright_{o \cap a} = g_{oa}^* A_o g_{oa} \upharpoonright_{o \cap a}, \quad o \cap a \neq \emptyset$$

(see [25, Eq.1.16]). When $G = \mathbb{U}(1)$, each g_{oa} is a phase and $A_a \upharpoonright_{o \cap a} = A_o \upharpoonright_{o \cap a}$; thus, the forms A_o can be glued and give a $\mathfrak{u}(1)$ -valued 1-form that we denote again by A . Because of flatness, and since $\mathfrak{u}(1) \simeq \mathbb{R}$, calling $Z_{dR}^1(M)$ the vector space of closed real 1-forms on M , it holds

$$A \in Z_{dR}^1(M), \quad dA = 0. \quad (17)$$

Now, following [6], if $z \in Z_{AB}^1(\mathcal{A}_{KM})$ is irreducible with statistical dimension $d(z)$, the von Neumann algebra

$$\mathcal{A}(M, o) := \{z(p) = u_z(p) \mid p : o \rightarrow o\}'' \subseteq \mathcal{A}_o \quad (18)$$

is a factor of type I_n with $n \leq d(z)$, where $d(z)$ is the statistical dimension of z . This amounts to saying that $\mathcal{A}(M, o) \cap \mathcal{A}(M, o)' = \mathbb{C}1$ and

$$U\mathcal{A}(M, o)U^* = 1_{\mathcal{H}_0} \otimes \mathbb{M}_n(\mathbb{C}), \quad (19)$$

for some unitary operator $U : \mathcal{H}_0 \rightarrow \mathcal{H}_0 \otimes \mathbb{C}^n$. The integer n is an invariant of the equivalence class of z which is called the *topological dimension* of z and it is denoted by $\tau(z)$. Letting P_o be a path frame and calling $\text{hol}_{A^z}(\ell_{(\tilde{a}, a)}) \in \mathbb{U}(n)$ the holonomy

² See, for example, [22] and references therein.

over any closed curve $\ell_{(\tilde{a}, a)} : x_o \rightarrow x_o$, $x_o \in o$, approximated by $p_{o\tilde{a}} * (\tilde{a}, a) * p_{ao}$, the following result holds true

Lemma 2.2 *Let z be an irreducible 1-cocycle of $Z_{\text{AB}}^1(\mathcal{A}_{KM})$ with topological dimension $n := \tau(z)$ and let P_o be a path frame. Then, there exists a smooth $\mathbb{U}(n)$ -principal bundle $\mathcal{P}^z \rightarrow M$ with a flat connection A^z such that*

$$u_z(\tilde{a}, a) = U^* \left(1 \otimes \text{hol}_{A^z}(\ell_{(\tilde{a}, a)}) \right) U \in \mathcal{U}(\mathcal{A}_o), \quad \tilde{a} \geq a, \quad (20)$$

where U is the unitary realizing (19).

Proof As observed in the previous section, z and its topological component define the same representation σ_z (6) of the fundamental group of M . As the algebra $\mathcal{A}(M, o)$ is a type I_n factor, there exists an irreducible n -dimensional representation σ of the fundamental group of M such that

$$\sigma_z([\ell]) = u_z(p_\ell) = U^*(1_{\mathcal{H}_0} \otimes \sigma([\ell]))U, \quad [\ell] \in \pi_1(M, x_o), \quad (21)$$

where U is the unitary realizing (19). The key observation is that the mapping $\ell \mapsto \sigma([\ell])$ assigning to any closed curve ℓ over x_o the unitary $\sigma([\ell]) \in \mathbb{U}(n)$ is a holonomy map in the sense of Barrett [3, cfr. Section 2.4]. By Barrett's reconstruction theorem [3], $\ell \mapsto \sigma([\ell])$ is the parallel transport of a flat connection A^z on a smooth $\mathbb{U}(n)$ -principal bundle \mathcal{P}^z , that is, $\sigma([\ell]) = \text{hol}_{A^z}(\ell)$ for any closed curve $\ell : x_e \rightarrow x_e$. Explicitly,

$$\mathcal{P}^z := \hat{M} \times_{\sigma} \mathbb{U}(n), \quad (22)$$

where \hat{M} is the universal covering of M carrying the well-known right $\pi_1(M)$ -action, whilst $\hat{M} \times_{\sigma} \mathbb{U}(n)$ is the quotient of $\hat{M} \times \mathbb{U}(n)$ by the equivalence relation $(y\ell, v) \simeq (y, \sigma(\ell)v)$, $y \in \hat{M}$, $\ell \in \pi_1(M)$, $v \in \mathbb{U}(n)$. Hence by (6) and (21) we have that

$$u_z(p_\ell) = U \left(\mathbb{I} \otimes \text{hol}_{A^z}(\ell) \right) U^*, \quad (23)$$

for any loop p_ℓ and any closed curve ℓ such that p_ℓ is a path approximation of ℓ . Given a comparable pair $\tilde{a} \geq a$, if we take the loop $p_{o\tilde{a}} * (\tilde{a}, a) * p_{ao}$ and a closed curve $\ell_{(\tilde{a}, a)} : x_o \rightarrow x_o$ such that $p_{o\tilde{a}} * (\tilde{a}, a) * p_{ao}$ is a path-approximation of $\ell_{(\tilde{a}, a)}$, the proof follows from (21), (23) and from the definition of the topological component. \square

It is easily seen that a change of the path frame P_o used in the previous proof yields a representation $\sigma' : \pi_1(M, x_{o'}) \rightarrow \mathbb{U}(n)$ unitarily equivalent to σ . This implies that we find a flat connection A'^z providing, up to conjugation, the same parallel transport as A^z .

Remark 2.3 (The Chern character on $Z_{\text{AB}}^1(\mathcal{A}_{KM})$) Let z denote an irreducible cocycle and let $\sigma : \pi_1(M) \rightarrow \mathbb{U}(n)$, $n := \tau(z)$, denote the associated representation of the fundamental group (21). By standard results in differential geometry, σ defines a flat Hermitean vector bundle $E^z \rightarrow M$ with rank n ,

$$E^z := \hat{M} \times_{\sigma} \mathbb{C}^n, \quad (24)$$

defined analogously to (22). Using the results in [7], we can define the Chern character

$$ccs : Z_{AB}^1(\mathcal{A}_{KM}) \rightarrow H^{odd}(M, \mathbb{R}/\mathbb{Q}), \quad ccs(z) := n + \sum_k^n \frac{(-1)^{k-1}}{(k-1)!} c_k(E^z)_{\text{mod } \mathbb{Q}}, \quad (25)$$

defined via the Chern–Cheeger–Simons classes $c_k(E^z) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$ [7, §4] (here, for convenience $H^{odd}(M, \mathbb{R}/\mathbb{Q})$ includes by definition \mathbb{Z} as the first direct summand, so $n \in \mathbb{Z}$ is regarded as an element of $H^{odd}(M, \mathbb{R}/\mathbb{Q})$). The interpretation of $ccs(z)$ is that of a generalized Aharonov–Bohm phase, as we shall see in the next sections.

From now on, to emphasize the geometric aspects, we shall write

$$\text{Hol}_{A^z}(\tilde{a}, a) := u_z(\tilde{a}, a), \quad \tilde{a} \gtrsim a,$$

instead of the expression appearing in (20). We can state the main result of this section.

Theorem 2.4 *Let z be an irreducible 1-cocycle of $Z_{AB}^1(\mathcal{A}_{KM})$ with $d(z) = \tau(z)$. Then, the charged component z_c is a multiplet of the type $z_c = \vec{\zeta}$, where $\vec{\zeta} := \zeta \oplus \dots \oplus \zeta$ is a $\tau(z)$ -fold direct sum of a topologically trivial 1-cocycle ζ such that $d(\zeta) = 1$ and $\kappa(\zeta) = \kappa(z)$. Hence,*

$$z(\tilde{a}, a) = \left\{ \text{Hol}_{A^z} \rtimes \vec{\zeta} \right\}(\tilde{a}, a), \quad \tilde{a} \gtrsim a,$$

where A^z is the flat connection of the $\mathbb{U}(n)$ -principal bundle \mathcal{P}^z defined by z .

Proof In general, since $\sum_{k=1}^m d(\zeta_k) = d(\tau)$, the charged component can be written as a direct sum of m irreducible and topologically trivial 1-cocycles $z_c = \bigoplus_{k=1}^m \zeta_k$, with $m \leq d(z)$ [6]. We prove that $m = d(z)$. First of all observe that

$$\mathcal{A}(M, e) \subseteq (z, z)_e := \{t_e, t \in (z, z)\},$$

where the r.h.s. is a C^* -algebra formed by the component e of the intertwiners of (z, z) . If $m < d(z)$, then the dimension of $(z, z)_e$ would be smaller than $d(z)^2$ because any ζ_k is irreducible. This leads to a contradiction because of the above inclusion and because the dimension of $\mathcal{A}(M, e)$ is $\tau(z)^2$ and $\tau(z) = d(z)$ by assumption. This implies that all ζ_k have statistical dimension 1 and, on account of the same dimensional argument just used, that all ζ_k are equivalent. Thus, the proof follows by applying the previous theorem to (10). \square

We shall refer to A^z as the *background flat potential* associated with z .

Hence, an irreducible 1-cocycle z with statistical dimension $d(z) = \tau(z)$ is the transporter of charge of Fermionic/Bosonic type in the presence of a background field A^z . If, in particular, $d(z) > 1$ then we see that the charge z_c is a multiplet, and the background field acting upon this multiplet, realizing the AB effect, is non-Abelian. The background field quantifies how the charge depends on the homotopy equivalence

of paths when transported along paths. Precisely, referring to (16), if $p : o \rightarrow a$ and $q : o \rightarrow a$ then

$$z(p) \rho^z(o) z(\bar{q}) = \text{Hol}_{A^z}(p * \bar{q}) \rho^z(a),$$

so the final representation $\rho^z(o)$ differs from $\rho^z(a)$ by the factor $\text{Hol}_{A^z}(p * \bar{q})$ (see (15) for the notation).

The next result proves that non-Abelian AB effects can appear only in spacetimes having a non-Abelian fundamental group.

Corollary 2.5 *Let $z \in Z_{\text{AB}}^1(\mathcal{A}_{KM})$ be irreducible with topological dimension $\tau(z) = 1$. Then, Hol_{A^z} takes values in $\mathbb{U}(1)$ and*

$$z(\tilde{a}, a) = \text{hol}_{A^z}(\ell_{(\tilde{a}, a)}) \cdot z_c(\tilde{a}, a), \quad \forall \tilde{a} \gtrsim a. \quad (26)$$

When $\pi_1(M)$ is Abelian, this holds for any irreducible 1-cocycle.

Proof It is enough to observe that if $\tau(z) = 1$ the topological component takes values in $\mathbb{U}(1)$ and the joining reduces to the multiplication by the so-defined phase. Moreover, if $\pi_1(M)$ is Abelian, then any irreducible representation of $\pi_1(M)$ is a phase. \square

Now, a principal $\mathbb{U}(1)$ -bundle \mathcal{P} with a flat connection A is trivial if, and only if,

$$\text{hol}_A([\ell]) = \exp\left(i \oint_{\ell} A\right), \quad \forall \ell, \quad (27)$$

where here ℓ stands for any closed path on M . That such a \mathcal{P} is trivial shall be illustrated for convenience in the following Sect. 4.1; on the converse, it is well-known that any trivial bundle has holonomy of the type (27). On the other hand, a flat principal $\mathbb{U}(1)$ -bundle either is trivial or has non-trivial first Chern class in the torsion subgroup of the singular cohomology $H^2(M)$ [17, Sec.3]. Thus, applying Corollary 2.5 we conclude:

Corollary 2.6 *Let $z \in Z_{\text{AB}}^1(\mathcal{A}_{KM})$ be irreducible with $\tau(z) = 1$, and $\sigma : \pi_1(M) \rightarrow \mathbb{U}(1)$ denote the morphism (21). If \mathcal{P}^z is trivial, then*

$$z(\tilde{a}, a) = \exp\left(i \oint_{\ell_{(\tilde{a}, a)}} A^z\right) \cdot z_c(\tilde{a}, a), \quad \tilde{a} \gtrsim a. \quad (28)$$

When $H^2(M)$ has no torsion \mathcal{P}^z is always trivial, and (28) holds for all irreducible z with $\tau(z) = 1$.

By well-known results, the above condition of $H^2(M)$ being torsion free is equivalent to the first homology $H_1(M)$ being torsion free (see [17, Sec.3] and references cited therein). This covers the cases of physical interest, such as anti-de Sitter spaces ($H_1(M) = \mathbb{Z}$), de Sitter spaces ($H_1(M) = 0$), and the “Aharonov-Bohm spacetime” $M := (\mathbb{R}^3 \setminus S) \times \mathbb{R}$, where $S \sim \mathbb{R}$ stands for the ideally infinite solenoid ($H_1(M) = \mathbb{Z}$).

3 On the observable net of the free Dirac field

In the previous section, we have shown that the nontrivial topology of the spacetime induces on the observable net superselection sectors affected by the topology of the spacetime. These sectors factorize in a part which describes a quantum charge (DHR charges) and in a topological part ruling the behaviour of the quantum charge when moved along a path in the same way as it occurs in the Aharonov–Bohm effect.

Our final aim is to show that this is not merely an analogy but the real physical Aharonov–Bohm effect: we want to show that the quantization of the free Dirac field in the presence of a background flat potential gives rise to the same type of superselection sectors as those described in the abstract analysis of the previous section.

As a preliminary step, in this section we analyze the free Dirac field in a 4-d globally hyperbolic spacetime *without* the presence of a background flat potential. We show that the net of local observables in the representation defined by a pure quasi-free Hadamard state satisfies all the properties assumed in Sect. 2.1, in particular Haag duality and (a strengthening) of the Borchers property. Then, we provide an explicit construction of the superselection sectors of the Dirac field and we show, as expected, that these are DHR cocycles obeying the Fermi statistics.

3.1 The Dirac operator

It is convenient to recall some notions on spin structures. The existence of any such structure is guaranteed if and only the second Stiefel–Whitney class of the underlying manifold $w_2(M) \in H^2(M, \mathbb{Z}_2)$ vanishes. This is always the case if we consider four-dimensional globally hyperbolic spacetimes and, in addition the corresponding Dirac bundle is trivial, namely $DM \simeq M \times \mathbb{C}^4$, cf. [23] and references therein. We introduce also the space $\mathcal{E}(DM)$ of smooth sections of DM , the space $\mathcal{S}(DM)$ of *compactly supported* smooth sections of DM (*spinors*), and the subspaces $\mathcal{S}_o(DM)$ of those sections $f \in \mathcal{S}(DM)$ whose support is contained in a diamond o , $\text{supp}(f) \subset o$. Similar notions hold for the dual Dirac bundle $D^*M \cong M \times (\mathbb{C}^4)^*$. We denote the space of smooth sections by $\mathcal{E}(D^*M)$, that of compactly supported smooth sections (co-spinors) by $\mathcal{S}(D^*M)$ and the subset of smooth sections of $\mathcal{S}(D^*M)$ whose support is contained in a diamond o denoted by $\mathcal{S}_o^*(DM)$.

On top of these structures, we define the Dirac operator $\mathcal{D} := i\gamma^\mu \nabla_\mu - m$ acting on spinors, where $m > 0$ is the mass parameter, the γ -matrices are fiberwise linear maps of DM obeying $\{\gamma_a, \gamma_b\} = 2g_{ab}$, while ∇ is the spin-connection. Similarly, we define the dual Dirac operator $\mathcal{D}_* = -i\gamma^\mu \nabla_\mu - m$ acting on co-spinors. Here, duality refers to the metric induced pairing $\langle, \rangle : \mathcal{E}(D^*M) \times \mathcal{S}(DM) \rightarrow \mathbb{C}$

$$\langle h, f \rangle := \int_M h \cdot f \, d\mu_g, \quad h \in \mathcal{E}(D^*M), \, f \in \mathcal{S}(DM).$$

Since the Dirac operator is Green hyperbolic, it possesses unique advanced and retarded fundamental solutions $S^\pm : \mathcal{S}(DM) \rightarrow \mathcal{E}(DM)$, such that $\mathcal{D}S^\pm = S^\pm \mathcal{D} = id$ on $\mathcal{S}(DM)$ while $\text{supp}(S^\pm(f)) \subseteq J^\pm(\text{supp}(f))$ for all $f \in \mathcal{S}(DM)$. The same

conclusion can be drawn for \mathcal{D}_* and it turns out that the associated advanced and retarded fundamental solutions $S_*^\pm : \mathcal{S}(D^*M) \rightarrow \mathcal{E}(D^*M)$ obey the structurally defining property $\langle h, S^\pm(f) \rangle = \langle S_*^\mp(h), f \rangle$ for any $h \in \mathcal{S}(D^*M)$ and $f \in \mathcal{S}(DM)$. Finally, the corresponding propagators $S := S^+ - S^-$ and $S_* := S_*^+ - S_*^-$ enjoy

$$\langle S_*(h), f \rangle = -\langle h, S(f) \rangle, \quad h \in \mathcal{S}(D^*M), \quad f \in \mathcal{S}(DM).$$

The vector spaces whose elements are spinors and co-spinors are isomorphic via the Dirac adjoint defined in terms of the gamma matrices as

$$\begin{aligned} \mathcal{E}(DM) \ni f &\mapsto f^\dagger := f^* \gamma_0 \in \mathcal{E}(D^*M); \\ \mathcal{E}(D^*M) \ni h &\mapsto h^\dagger := \gamma_0^{-1} h^* \in \mathcal{E}(DM). \end{aligned}$$

Here, we are taking the gamma matrices in the standard representation: $\gamma_0^* = \gamma_0$ and $\gamma_k^* = -\gamma_k$ for $k = 1, 2, 3$.

3.2 The (CAR) algebra of the Dirac field

There is a fairly vast literature on the CAR algebra and the Dirac quantum field in a 4-dimensional globally hyperbolic spacetime. In the present section, we give a brief description of these topics and refer the reader to the references [9,11,36] for details.

A convenient approach to quantize the Dirac field and to study the corresponding representations is the self-dual approach [1]. We consider the following Whitney sum of bundles

$$\tilde{DM} := DM \oplus D^*M,$$

as well as the space of compactly supported smooth sections $\mathcal{S}(\tilde{DM}) \cong \mathcal{S}(DM) \oplus \mathcal{S}(D^*M)$. In addition, we extend the Dirac operators and the corresponding propagators on this space by setting

$$\tilde{D}(f \oplus h) := Df \oplus D_*h, \quad f \oplus h \in \mathcal{S}(\tilde{DM}).$$

Observe that $\mathcal{S}(\tilde{DM})$ is equipped with a positive semi-definite sesquilinear form

$$(f_1 \oplus h_1, f_2 \oplus h_2) := (f_1, f_2)_s + (h_1, h_2)_c, \quad (29)$$

with

$$(f_1, f_2)_s := i \int_M f_1^\dagger S(f_2); \quad (h_1, h_2)_c := -i \int_M S_*(h_2) h_1^\dagger, \quad (30)$$

for any $f_i \oplus h_i \in \mathcal{S}(\tilde{DM})$, $i = 1, 2$ (both $(\cdot, \cdot)_s$ and $(\cdot, \cdot)_c$ are positive semi-definite sesquilinear forms on $\mathcal{S}(DM)$ and $\mathcal{S}(D^*M)$, respectively). This form (29) annihilates on the image of the extended Dirac operator \tilde{D} . Taking the quotient we get, up to a suitable closure, a Hilbert space $(\mathbf{h}, (\cdot, \cdot))$. Finally, setting

$$\Gamma(f \oplus h) := h^\dagger \oplus f^\dagger, \quad f \oplus h \in \mathcal{S}(\tilde{DM})$$

one observes that Γ extends to an anti-unitary operators on \mathbf{h} :

$$(\Gamma[f_1 \oplus h_1], \Gamma[f_2 \oplus h_2]) = ([f_2 \oplus h_2], [f_1 \oplus h_1]), \quad [f_1 \oplus h_1], [f_2 \oplus h_2] \in \mathbf{h}.$$

We now are ready to give the definition of the CAR algebra.

Definition 3.1 The **algebra of the Dirac field** is the C^* -algebra $\mathcal{C}(M)$ generated by the unit $\mathbb{1}$ and $\Psi([f \oplus h])$ as $[f \oplus h]$ varies on the Hilbert space \mathbf{h} and satisfying

- (i) The map $\mathbf{h} \ni [f \oplus h] \mapsto \Psi([f \oplus h]) \in \mathcal{C}(M)$ is linear;
- (ii) Self-duality: $\Psi(\Gamma([f \oplus h])) = \Psi([f \oplus h])^*$ for any $[f \oplus h] \in \mathbf{h}$;
- (iii) Canonical Anticommutation Relations (CARs):

$$\{\Psi([f_1 \oplus h_1]), \Psi([f_2 \oplus h_2])\} = (\Gamma[f_1 \oplus h_1], [f_2 \oplus h_2])$$

for any $[f_1 \oplus h_1], [f_2 \oplus h_2] \in \mathbf{h}$.

We draw on the consequences of this definition.

(1) The generators of the CAR algebra encode the Dirac field ψ and the conjugated Dirac field $\bar{\psi}$:

$$\psi(f) := \Psi[f \oplus 0], \quad f \in \mathcal{S}(DM); \quad \bar{\psi}(h) := \Psi[0 \oplus h], \quad h \in \mathcal{S}(D^*M).$$

Notice in fact that, since $\Psi \circ \tilde{D} = 0$, the fields ψ and $\bar{\psi}$ satisfy the Dirac equation

$$\psi \circ (i\nabla - m) = 0, \quad \bar{\psi} \circ (i\nabla + m) = 0.$$

Furthermore, we have

$$\bar{\psi}(h) = \psi(h^\dagger)^*, \quad h \in \mathcal{S}(D^*M), \quad (31)$$

and thus $\Psi[f \oplus h] = \psi(f) + \psi(h^\dagger)^* = \bar{\psi}(f^\dagger)^* + \bar{\psi}(h)$. This implies in particular that the algebra $\mathcal{C}(M)$ can be generated by taking only either the Dirac field or the conjugated one.

(2) The Dirac field satisfies the usual Canonical Anticommutation Relations:

$$\{\psi(f_1)^*, \psi(f_2)\} = (f_1, f_2)_s, \quad \{\psi(f_1), \psi(f_2)\} = 0. \quad (32)$$

From these and (31), it follows that $\{\bar{\psi}(h), \psi(f)\} = (h^\dagger, f)_s$ and $\{\bar{\psi}(h_1), \bar{\psi}(h_2)\} = 0$.

(3) The algebra $\mathcal{C}(M)$ is acted upon by $\mathbb{U}(1)$ as a gauge group. The mapping $[f \oplus h] \mapsto [\zeta f \oplus \bar{\zeta} h]$, which leaves invariant the scalar product and commutes with the conjugation Γ for any $\zeta \in \mathbb{U}(1)$, induces an action $\mathbb{U}(1) \ni \zeta \mapsto \eta_\zeta \in \text{Aut}(\mathcal{C}(M))$ where

$$\eta_\zeta(\Psi([f \oplus h])) = \Psi(\zeta f \oplus \bar{\zeta} h). \quad (33)$$

In particular, $\eta_\zeta(\psi(f)) = \zeta \psi(f)$ and $\eta_\zeta(\bar{\psi}(h)) = \bar{\zeta} \bar{\psi}(h)$. We denote by η_- the automorphism induced by $\zeta = -1$.

(4) Finally, we note that ψ is a C^* -valued distribution when $S(DM)$ is equipped with the topology of test-functions (the same holds true for $\bar{\psi}$). In fact by the CARs (32), we have $\psi(f)^*\psi(f) = -\psi(f)\psi(f)^* + (f, f)_s \mathbb{1}$. So, $\psi(f)^*\psi(f)\psi(f)^*\psi(f) = \psi(f)^*\psi(f)(f, f)_s \mathbb{1}$ because $\psi(f)\psi(f) = 0$. Using the C^* property of the norm, we get

$$\begin{aligned}\|\psi(f)^*\psi(f)\|^2 &= \|\psi(f)^*\psi(f)\psi(f)^*\psi(f)\| = \|\psi(f)^*\psi(f)\|(f, f)_s \\ &\iff (f, f)_s = \|\psi(f)^*\psi(f)\| = \|\psi(f)\|^2,\end{aligned}$$

and the proof follows by the continuity of $(\cdot, \cdot)_s$ with respect to the test function topology.

3.3 Quasi-free Hadamard states

We describe the net of the Dirac field in the representation associated with a pure and gauge invariant quasi-free Hadamard state. We show that the net satisfies twisted Haag duality and the local algebras are type III factors. As a consequence, the resulting net of local observables in the vacuum (zero charge) representation verifies all the properties assumed in Sect. 2.1.

To begin with, we consider a *pure quasi-free state* ω of the algebra $\mathcal{C}(M)$ and the corresponding GNS triple $(\mathcal{H}, \pi, \Omega)$ (see [1]). Here, \mathcal{H} is the antisymmetric Fock space associated with a closed subspace of $P\mathbf{h}$ defined by an orthogonal projection P , the *base projection* enjoying the relation $\Gamma P = (1 - P)\Gamma$; π is an irreducible representation of the CAR algebra on the Hilbert space \mathcal{H} and Ω is a unit norm vector \mathcal{H} such that $\omega(\cdot) = (\Omega, \pi(\cdot)\Omega)$. To ease notation, we keep symbols Ψ , ψ and $\bar{\psi}$ to denote the generator of the CAR algebra, the Dirac field and its conjugate, respectively, in the Fock representation. Finally, if ω is also *gauge invariant* $\omega \circ \eta = \omega$, then there exists a unitary representation $U : \mathbb{U}(1) \rightarrow \mathcal{U}(\mathcal{H})$ such that for any $\zeta \in \mathbb{U}(1)$, it holds $U(\zeta)\Omega = \Omega$

$$U(\zeta)\Psi([f \oplus h])U(\zeta)^* = \Psi([\zeta f \oplus \bar{\zeta} h]). \quad (34)$$

In addition, $U(\zeta)\psi(f)U(\zeta)^* = \zeta\psi(f)$ and $U(\zeta)\bar{\psi}(h)U(\zeta)^* = \bar{\zeta}\bar{\psi}(h)$.

So given a pure and gauge invariant quasi-free state of the CAR algebra and $\mathcal{H}, \pi, \omega, U$ as above, the *net of local Dirac fields* is the correspondence $\mathcal{F} : KM \ni o \rightarrow \mathcal{F}_o \subseteq \mathcal{B}(\mathcal{H})$ associating the von Neumann algebra

$$\mathcal{F}_o := \{\Psi([f \oplus h]) : [f \oplus h] \in \tilde{\mathcal{S}}_o(DM)\}'' ,$$

to any diamond o . The net \mathcal{F}_{KM} satisfies isotony, $\mathcal{F}_o \subseteq \mathcal{F}_{\tilde{o}}$ for any $o \subseteq \tilde{o}$, but *not* causality because of the CARs. This is replaced by *twisted causality*

$$\mathcal{F}_o^t \subseteq \mathcal{F}_a' , \quad a \perp o .$$

where \mathcal{F}_o^t is the von Neumann algebra defined as $\mathcal{F}_o^t := \tilde{U}_- \mathcal{F}_o \tilde{U}_-^*$ in terms of the *twist operator* \tilde{U}_- :

$$\tilde{U}_- := \frac{1}{1+i} (1 + iU_-) \quad \text{such that} \quad \tilde{U}_- \Psi(\cdot) \tilde{U}_-^* = iU_- \Psi(\cdot).$$

Theorem 3.2 *The net of the Dirac field \mathcal{F}_{KM} in a Fock representation induced by a pure and gauge invariant quasi-free state satisfies **twisted Haag duality***

$$\mathcal{F}_o^t = \cap \{ \mathcal{F}_a^t : a \perp o \}, \quad o \in KM. \quad (35)$$

Proof The proof relies on abstract twisted duality and on a density result. We recall that for the CAR algebra in a Fock representation, abstract twisted duality holds for any closed subspace which is invariant under the conjugation defining the abstract CAR algebra [4, Theorem 7.1]. Observing that the closure \mathbf{h}_o of $\tilde{S}_o(DM)$ in \mathbf{h} is Γ invariant for any o , abstract twisted duality in our case reads as

$$\mathcal{F}_o^t = \mathcal{F}((\mathbf{h}_o)^c)',$$

where $(\mathbf{h}_o)^c$ is the orthogonal complement of \mathbf{h}_o in the Hilbert space of the Cauchy data \mathbf{h} (cfr Theorem 7.1 of [4]). So, it is enough to prove $\cup \{ \tilde{S}_a(DM) : a \subset o^\perp \}$ is dense in $(\mathbf{h}_o)^c$ where o^\perp is the causal complement of the diamond o .

We now need two observations. First, any diamond o is of the form $D(G)$ with $G \subset \Sigma$ where Σ is a spacelike Cauchy surface and G is an open relatively compact subset of G diffeomorphic to an open 3-ball with $cl(G) \subsetneq \Sigma$. So, it is easily seen that $o^\perp = I(\Sigma \setminus cl(G))$ where I stands for the chronological set. Secondly, the Hilbert space \mathbf{h} is isomorphic to the Hilbert space $L^2(\Sigma, \mathbb{C}^4) \oplus L^2(\Sigma, (\mathbb{C}^4)^*)$ with the scalar product $(u_1 \oplus v_1, u_2 \oplus v_2) := \int_\Sigma (u_1^\dagger \not{n} u_2 + v_2 \not{n} v_1^\dagger) d\Sigma$ where n^μ is the timelike future pointing normal vector to the Cauchy surface. In particular, the isomorphism relies on the following identity

$$([f_1 \oplus h_1], [f_2 \oplus h_2]) = \int_\Sigma \left((Sf_1)^\dagger \not{n} (Sf_2) + (S_* h_2) \not{n} (S_* h_1)^\dagger \right) d\Sigma.$$

On these grounds, it is enough to prove that the space $C_0^\infty(G, \mathbb{C}^4) \oplus C_0^\infty(\Sigma \setminus cl(G), \mathbb{C}^4)$ is dense in $L^2(\Sigma, \mathbb{C}^4)$ and similarly for the dual. This follows from a partition of unity argument, similarly to [21, Prop. III.1] for Majorana fields, because G is a relatively compact open subset of Σ with a smooth boundary. \square

A class of quasi-free states for quantum fields on curved spacetimes having several physically meaningful applications is that of Hadamard states [26] (see also [18]). The existence of pure quasi-free and gauge invariant Hadamard states has been shown in [15]. The local algebras of the Dirac field in a representation defined by such a state are type III factors [21].³ This property is stronger than the Borchers property since

³ Actually the authors give a stronger characterization of the local algebras showing that these are III_1 factors.

for any $o \in KM$ any projection $E \in \mathcal{F}_o$ is equivalent in \mathcal{F}_o to the identity, i.e. there exists $V \in \mathcal{F}_o$ s.t. $V^*V = E$ and $VV^* = 1$ (V is an isometry).

Definition 3.3 We take as a **reference representation** for the Dirac field, the representation defined by a **pure quasi-free and gauge invariant Hadamard state**.

As shown above, the net \mathcal{F}_{KM} is isotonomous, satisfies twisted Haag duality and the local algebras \mathcal{F}_o are type III factors.

We define the observable net. To this end, we consider the *spectral subspaces*

$$\mathcal{F}_o^n := \{T \in \mathcal{F}_o : U(\zeta)TU(\zeta)^* = \zeta^n T, \forall \zeta \in \mathbb{U}(1)\}'', \quad n \in \mathbb{Z}. \quad (36)$$

For the grade $n = 0$, any \mathcal{F}_o^0 is a von Neumann algebra which is *gauge invariant* and such that $[T, S] = 0$ for $T \in \mathcal{F}_o^0$ and $S \in \mathcal{F}_a^0$ with $a \perp o$. This yields the *net of local observables* $\mathcal{F}_{KM}^0 : KM \ni o \rightarrow \mathcal{F}_o^0 \subseteq \mathcal{B}(\mathcal{H})$ which satisfies isotony and causality. The local observable algebras \mathcal{F}_o^0 inherit from \mathcal{F}_o the properties of being type III factors, see for instance [21].

Finally, we have to take into the game the zero-charge “vacuum” representation of the net of local observables. The *vacuum Hilbert space* \mathcal{H}_0 is the subspace

$$\mathcal{H}_0 := \text{closure} \{A\Omega : A \in \mathcal{F}_o^0, o \in KM\} \subseteq \mathcal{H}, \quad (37)$$

and the *vacuum representation* $\pi_0 : \mathcal{F}_{KM}^0 \rightarrow \mathcal{B}(\mathcal{H}_0)$ is defined as

$$\pi_0(A) := AE_0, \quad A \in \mathcal{F}_o^0, o \in KM, \quad (38)$$

defined in terms of the projection E_0 onto \mathcal{H}_0 . Since E_0 is an element of $(\mathcal{F}_o^0)'$ for any diamond o and since the local algebras \mathcal{F}_o^0 are type III factors, π_0 is faithful and the von Neumann algebras

$$\mathcal{A}_o := \pi_0(\mathcal{F}_o^0) \subseteq \mathcal{B}(\mathcal{H}_0), \quad o \in KM. \quad (39)$$

turn out to be type III factors. We call the correspondence $\mathcal{A}_{KM} : KM \ni o \rightarrow \mathcal{A}_o \subseteq \mathcal{B}(\mathcal{H}_0)$ the *net of local observables in the vacuum representation*. So, \mathcal{A}_{KM} is an irreducible net satisfying isotony, causality and whose local algebras \mathcal{A}_o are type III factors. Furthermore, it is a well known fact [10] that as the net of fields \mathcal{F}_{KM} satisfies twisted Haag duality, the net \mathcal{A}_{KM} satisfies *Haag duality*.

Remark 3.4 The net of local observables $o \mapsto \mathcal{F}_o^0$ encodes all the physical content of the theory. Very briefly, gauge-invariant Wick polynomials of the field, like the *Dirac current*, when evaluated on test functions supported in a region o yield unbounded self-adjoint operators affiliated with the von Neumann algebra \mathcal{F}_o^0 i.e., the corresponding spectral projections belong, by twisted Haag duality, to the algebra \mathcal{F}_o^0 , see [13,14,27] and references therein for technical details.

3.4 Charge transporters of the free Dirac field

Having established that the observable net \mathcal{A}_{KM} of the Dirac field in the vacuum representation satisfies the same properties as those assumed in Sect. 2.1, now we construct the charge transporters of the Dirac field, which provide the DHR-sectors for \mathcal{A}_{KM} .

We start by taking for any $o \in KM$ a normalized spinor f_o of $\mathcal{S}_o(DM)$, that is $(f_o, f_o)_s = 1$. With this choice, any $\psi(f_o)$ turns out to be a partial isometry of \mathcal{F}_o since, as it is easily verified by (32), the operators $\psi(f_o)\psi(f_o)^*$ and $\psi(f_o)^*\psi(f_o)$ are projections. Notice, in particular, that $\psi(f_o)$ is a partial isometry of the spectral subspace \mathcal{F}_o^1 , see (36). We can now make $\psi(f_o)$ equivalent to a unitary of the same spectral subspace. To this end, we note that both the projections $\psi(f_o)\psi(f_o)^*$ and $\psi(f_o)^*\psi(f_o)$, since gauge invariant (34), belong to the algebra \mathcal{F}_o^0 which is a type III factor. Hence, there exist $W_o, V_o \in \mathcal{F}_o^0$ s.t.

$$V_o^*V_o = 1, \quad V_oV_o^* = \psi(f_o)\psi(f_o)^*, \quad W_o^*W_o = 1, \quad W_oW_o^* = \psi(f_o)^*\psi(f_o).$$

The operator

$$\varphi_o := W_o^*\psi(f_o)V_o, \quad o \in KM \quad (40)$$

is a unitary in \mathcal{F}_o and an element of \mathcal{F}_o^1 because by construction $\eta_\zeta(\varphi_o) = \zeta\varphi_o$, $\zeta \in \mathbb{U}(1)$. The operators φ_o do not fulfill the CARs (32). Nevertheless, if $a \perp o$, $A \in \mathcal{F}_o^0$ and $F \in \mathcal{F}_a$, then $[A, F] = 0$, so that

$$\begin{aligned} \varphi_o^*\varphi_a &= V_o^*\psi(f_o)^*W_oW_a^*\psi(f_a)V_a = V_o^*\psi(f_o)^*W_a^*W_o\psi(f_a)V_a \\ &= V_o^*W_a^*\psi(f_o)^*\psi(f_a)W_oV_a = -V_o^*W_a^*\psi(f_a)\psi(f_o)^*W_oV_a \\ &= -\varphi_a\varphi_o^* \end{aligned}$$

and similarly for the other CARs. In conclusion, for all $o \perp a$

$$\varphi_o^*\varphi_a = -\varphi_a\varphi_o^*, \quad \varphi_o\varphi_a = \varphi_a\varphi_o, \quad \varphi_o \in (\mathcal{F}_a^0)'. \quad (41)$$

Now, we define the charge transporter z associated with the Dirac field as

$$z(a, o) := \pi_0(\varphi_a^*\varphi_o) = E_0\varphi_a^*\varphi_o, \quad o \subseteq a, \quad (42)$$

where π_0 is the vacuum representation. Clearly, z is localized and $z(a, o) \in \mathcal{A}_a$ for any $o \subseteq a$, satisfies the 1-cocycle identity

$$z(\tilde{o}, a)z(a, o) = \pi_0(\varphi_{\tilde{o}}^*\varphi_a\varphi_o^*) = z(\tilde{o}, a), \quad a \subseteq o \subseteq \tilde{o},$$

and

$$z(p) = z(o, o_n) \cdots z(o_2, o_1)z(o_1, a) = \pi_0(\varphi_o^*\varphi_{o_n} \cdots \varphi_{o_1}^*\varphi_a) = \pi_0(\varphi_o^*\varphi_a) \quad (43)$$

for any path $p : a \rightarrow o$, and for $o = a$ we find $z(p) = 1$. Thus, z is topologically trivial, $z \in Z_t^1(\mathcal{A}_{KM})$.

Lemma 3.5 *The following assertions hold true:*

- (i) *the definition of the 1-cocycle z is independent from the choice of the normalized spinors;*
- (ii) *$z \in Z_{\text{DHR}}^1(\mathcal{A}_{KM})$ and obeys the Fermi statistics, i.e. it has statistical dimension $d(z) = 1$ and statistical phase $\kappa(z) = -1$.*

Proof (i) Let $g_o \in S_o(DM)$, $o \in KM$, be another choice of normalized spinors. As above, denote with S_o and T_o the isometries of \mathcal{F}_o^0 making the operator $\chi_o := T_o^* \psi(g_o) S_o$ a unitary of \mathcal{F}_o and of \mathcal{F}_o^1 . Then, $z'(a, o) := \pi_0(\chi_a^* \chi_o)$ is a 1-cocycle of $Z_t^1(\mathcal{A}_{KM})$ which is unitary equivalent to z . In fact, $t_o := \pi_0(\chi_o \phi_o^*)$ is a unitary of \mathcal{A}_o for any o satisfying $t_a z(a, o) = \pi_0(\chi_a^* \phi_a) \pi_0(\phi_a^* \phi_o) = \pi_0(\chi_a^* \phi_o) = z'(a, o) t_o$, i.e. $t : KM \rightarrow o \rightarrow t_o \in \mathcal{A}_o$ is a unitary intertwiner of (z, z') .

(ii) It is enough to prove that the symmetry intertwiner $\varepsilon \in (z \times z, z \times z)$ is such that $\varepsilon_a = -1$ for all a , where

$$\varepsilon_a := z(q)^* \times z(p)^* \cdot z(p) \times z(q),$$

$p : a \rightarrow \tilde{o}$ and $q : a \rightarrow o$ are paths with $o \perp \tilde{o}$, and \times is the tensor product of $Z^1(\mathcal{A}_{KM})$ [32, Theorem 4.9]. The latter is defined by the expressions

$$z(q)^* \times z(p)^* := z(q)^* z(p_1) z(p)^* z(p_1)^*, \quad z(p) \times z(q) := z(p) z(p_1) z(q) z(p_1)^*,$$

with $p_1 : o_1 \rightarrow a$, $o_1 \perp |q|$. According to these relations, we may take $o \perp a$, $\tilde{o} = a$ and $p : a \rightarrow a$ the trivial path. So, $z(p) = 1$. Moreover, by (43) we have $z(q) = \pi_0(\varphi_o^* \varphi_a)$. Finally, we may take o_1 causally disjoint from o and a and $z(p_1) = \pi_0(\varphi_a^* \varphi_{o_1})$. In this way,

$$\begin{aligned} z(p)^* \times z(q)^* \cdot z(p) \times z(q) &= \pi_0(\varphi_a^* \varphi_o \varphi_a^* \varphi_{o_1} \varphi_{o_1}^* \varphi_a \cdot \varphi_a^* \varphi_{o_1} \varphi_o^* \varphi_a \varphi_{o_1}^* \varphi_a) \\ &= \pi_0(\varphi_a^* \varphi_o \varphi_a^* \varphi_{o_1} \varphi_o^* \varphi_a \varphi_{o_1}^* \varphi_a) \\ &= -\pi_0(\varphi_a^* \varphi_o \varphi_a^* \varphi_o^* \varphi_{o_1} \varphi_a \varphi_{o_1}^* \varphi_a) \\ &= \pi_0(\varphi_a^* \varphi_o \varphi_a^* \varphi_o^* \varphi_a \varphi_{o_1} \varphi_{o_1}^* \varphi_a) \\ &= -\pi_0(\varphi_a^* \varphi_o \varphi_o^* \varphi_a^* \varphi_a \varphi_a) = -1, \end{aligned}$$

having used repeatedly (41) for the relations $o_1 \perp o, a$ and $o \perp a$. We conclude that $\varepsilon_a = -1$, and this completes the proof. \square

Thus, the superselection sector associated with z has the same charge quantum number as an electron. We now define the charge transporter associated with the conjugated Dirac field. In analogy to what has been done for ψ , we choose a co-spinor $h_o \in \mathcal{S}(D^*M)$ with $(h_o, h_o)_c = 1$ for any $o \in KM$ and set $\bar{\varphi}_o := \bar{W}_o^* \bar{\psi}(h_o) \bar{V}_o$, where \bar{W}_o^* , \bar{V}_o are the isometries of \mathcal{F}_o^0 making $\bar{\varphi}_o$ a unitary of \mathcal{F}_o and an element of the spectral subspace \mathcal{F}_o^{-1} . Then, we define

$$\bar{z}(a, o) := \pi_0(\bar{\varphi}_a^* \bar{\varphi}_o), \quad o \subseteq a. \quad (44)$$

In complete analogy to what has been shown for z , one can prove that \bar{z} is independent from the choice of the normalized co-spinor and it is a topologically trivial 1-cocycle obeying the Fermi statistics: $d(\bar{z}) = 1$ and $\kappa(\bar{z}) = -1$.

We now prove that \bar{z} is the conjugate of z in the sense of the theory of superselection sectors.

Lemma 3.6 *\bar{z} is the conjugate of z in the category $Z_{\text{DHR}}^1(\mathcal{A}_{KM})$.*

Proof This amounts to verifying triviality of the tensor product $(z \times \bar{z})$ defined as

$$(z \times \bar{z})(o, a) := z(o, a) z(p_1) \bar{z}(o, a) z(p_1)^*, \quad a \subseteq o,$$

where $p_1 : o_1 \rightarrow a$, with $o_1 \perp o, a$, is an arbitrary path as in the previous proof [6, 32]. As z and \bar{z} do not depend on the choice of the normalized spinors and co-spinors, respectively, it is enough to prove the above triviality for a suitable choice of the normalized co-spinors. So if f_o, W_o and V_o are the normalized spinor and isometries defining φ_o , we take $h_o := f_o^\dagger$ and notice that $(h_o, h_o)_c = (f_o^\dagger, f_o^\dagger)_c = (f_o, f_o)_s = 1$. Furthermore, as $\bar{\psi}(h_o) = \psi(h_o^\dagger)^* = \psi(f_o)^*$, we may take, for the definition of $\bar{\varphi}$, $\bar{W}_o := V_o$ and $\bar{V}_o := W_o$. Then, $\bar{\varphi}_o := V_o^* \psi(f_o)^* W_o = \varphi_o^*$ and

$$\begin{aligned} (z \times \bar{z})(o, a) &= \pi_0(\varphi_o^* \varphi_a \varphi_a^* \varphi_{o_1} \bar{\varphi}_o^* \bar{\varphi}_a \varphi_{o_1}^* \varphi_a) = \pi_0(\varphi_o^* \varphi_{o_1} \varphi_o \varphi_a^* \varphi_{o_1}^* \varphi_a) \\ &= \pi_0(\varphi_o^* \varphi_{o_1} \varphi_{o_1}^* \varphi_o \varphi_a^* \varphi_a) = 1 \end{aligned}$$

where we have used the fact that o_1 is causally disjoint from a and o and that $\varphi_o \varphi_a^*$ is an observable. \square

Remark 3.7 Performing tensor powers $z^{\times n}$ and $\bar{z}^{\times n}$, $n \in \mathbb{N}$, we obtain a structure $Z_{ec}^1(\mathcal{A}_{KM})$ of DHR-sectors labelled by \mathbb{Z} , where positive integers are associated with $z^{\times n}$ and negative integers to $\bar{z}^{\times n}$ (we define z^0 as the reference representation). Using the techniques explained in the present section, it is easily seen that $z^{\times n}$ may be defined starting from products of field operators $\psi(f_1) \cdots \psi(f_n)$, and similarly $\bar{z}^{\times n}$ with products of conjugate field operators. Thus, it is natural to interpret $Z_{ec}^1(\mathcal{A}_{KM})$ as the charge superselection rule.

4 Dirac fields in background flat potentials

The Aharonov–Bohm effect concerns the behaviour of electrically charged quantum particles in a space M where a classical background potential A is defined, with the property of having vanishing electromagnetic tensor $F := dA$. This amounts to saying that A is a closed de Rham 1-form, $dA = 0$, and we say that A is a flat background potential. Existence of non-trivial flat background potentials is equivalent to the property of having a non-trivial de Rham cohomology, $H_{dR}^1(M) \neq 0$. By standard results, this implies that $\pi_1(M) \neq 0$.

Passing to a relativistic setting, the scenario we adopt for describing quantum interactions with the flat background potential A is given a spinor field on a globally

hyperbolic spacetime M , fulfilling the Dirac equation with interaction A . In the present section, we construct such interacting Dirac fields and show that they can be interpreted as superselection sectors of a given reference free Dirac field.

4.1 Preliminaries on flat potentials

We start by collecting some properties of flat potentials, closed de Rham forms $A \in Z_{dR}^1(M)$, in particular their relations with flat line bundles, and we establish a mapping $A \mapsto \hat{A}$ with \hat{A} a real 1-cocycle in the cohomology of KM . This will be useful in the discussion of AB-sectors of the observable net of the free Dirac field.

As a first step, we note that, since diamonds $o \in KM$ are simply connected, there are C^∞ local primitives $\phi : o \rightarrow \mathbb{R}$ such that

$$d\phi_o(x) = A(x), \quad x \in o \in KM.$$

If $a, o \in KM$ and $a \cap o \neq \emptyset$, then $\{d\phi_o - d\phi_a\}(x) = A(x) - A(x) = 0$, thus the function $\phi_o - \phi_a$ is constant on each connected component of $o \cap a$ (that is, $\phi_o - \phi_a : o \cap a \rightarrow \mathbb{R}$ is *locally constant*). On these grounds, defining

$$g_{oa}(x) := e^{i(\phi_o - \phi_a)(x)} \in \mathbb{U}(1), \quad x \in a \cap o, \quad (45)$$

we get a family of locally constant functions fulfilling the cocycle relations $g_{oa}g_{ac} = g_{oc}$, as it can be trivially verified on each overlap $o \cap a \cap c \neq \emptyset$. The next result is standard in geometry: for a proof, we refer the reader to [39]:

Lemma 4.1 *Let $A \in Z_{dR}^1(M)$. Then, $g := \{g_{oa}\}$, defined as in (45), is a set of transition maps for the flat line bundle $\mathcal{L}_A := \hat{M} \times_\sigma \mathbb{C}$ defined as in (24), where*

$$\sigma([\ell]) := \exp i \oint_\ell A, \quad [\ell] \in \pi_1(M).$$

Let now

$$\pi_o : \mathcal{L}_A \upharpoonright_o \rightarrow M \times \mathbb{C}, \quad o \in KM,$$

denote local charts such that $g_{oa}\pi_a = \pi_o$ on $o \cap a$. The previous Lemma implies that the relation $e^{-i\phi_o}\pi_o = e^{-i\phi_a}\pi_a$ holds for any $a \cap o \neq \emptyset$. Thus, the maps $e^{-i\phi_o}\pi_o : \mathcal{L}_A \upharpoonright_o \rightarrow M \times \mathbb{C}$, $o \in KM$, glue in the correct way and they induce the bundle isomorphism $\vartheta : \mathcal{L}_A \rightarrow M \times \mathbb{C}$ defined as

$$\vartheta \upharpoonright_o := e^{-i\phi_o}\pi_o, \quad o \in KM. \quad (46)$$

Note that ϑ locally looks like the multiplication by $e^{-i\phi_o}$. Let now $a \subseteq o$; then $\phi_o - \phi_a$ is a constant real function on a and we define

$$\hat{A}_{oa} := \phi_o - \phi_a \in \mathbb{R}, \quad \hat{A}_{ao} := -\hat{A}_{oa}, \quad a \subseteq o. \quad (47)$$

The relations $\hat{A}_{co} + \hat{A}_{oa} = \hat{A}_{ca}$ are clearly fulfilled for all $a \subset o \subset c \in KM$, that is, \hat{A} is a real cocycle in the cohomology of KM . Note that by definition (45), we have

$$g_{oa} = e^{i\hat{A}_{oa}}, \quad a \subseteq o. \quad (48)$$

The cocycle \hat{A} can be used to express the holonomy of A . In view of [38, Eq.4.7], given a loop $\gamma : [0, 1] \rightarrow M$ and $p_\gamma = (o_n, o_{n-1}) * \cdots * (o_1, o_0)$ a poset approximation of γ , we have

$$\oint_\gamma A = \hat{A}_{p_\gamma} := \sum_{i=1}^n \hat{A}_{o_i o_{i-1}}. \quad (49)$$

4.2 The Dirac field interacting with a background flat potential

We construct a quantum Dirac field interacting with a background flat electromagnetic potential, and we describe it in terms of a family of Dirac fields locally defined on diamonds. These fields yield a “local coordinate” description of the interacting field and will play an important role at the level of the associated nets of von Neumann algebras.

The interacting field shares with twisted fields in the sense of Isham [2,23] the property of being defined on the space of sections of a twisted bundle. Yet it is not exactly a field of this type as we shall see in the following lines.

Let $A = (A_\mu) \in Z_{dR}^1(M)$. Our task is the construction of a Dirac field ψ_A such that

$$\psi_A \circ (i\nabla + \not{A} - m) = 0,$$

where \not{A} is defined in local frames as $\gamma_o^\mu A_\mu$, and since the local Dirac matrices γ_o^μ can be arranged to form a section of $T^*M \otimes \mathbf{end}DM$, we have $\not{A} \in \mathcal{S}(\mathbf{end}DM)$. We note, that if ψ_A is given, then for any $o \in KM$ and $f \in \mathcal{S}_o(DM)$ it turns out

$$\begin{aligned} 0 &= \psi_A((i\nabla + \not{A} - m)(e^{i\phi_o} f)) \\ &= \psi_A(-\gamma_o^\mu (\partial_\mu \phi_o) e^{i\phi_o} f + i e^{i\phi_o} \nabla f + (\not{A} - m)(e^{i\phi_o} f)) \\ &= \psi_A(e^{i\phi_o} (-\not{A} f + i\nabla f + (\not{A} - m)f)) \\ &= \psi_A(e^{i\phi_o} ((i\nabla - m)f)). \end{aligned}$$

Thus, applying the local gauge transformation $\psi_A \rightarrow \psi_{A,o} := \psi_A \circ e^{i\phi_o}$, we find that $\psi_{A,o}$, as a field evaluated on the test space $\mathcal{S}_o(DM)$, solves the free Dirac equation.

The idea of the previous computation is that A appears as an exact 1-form on each (simply connected) diamond $o \in KM$, thus it can be represented locally as a local gauge transformation making ψ_A a free Dirac field. Thus, to construct our field ψ_A , it comes natural to reverse the above argument and to start with a free Dirac field $\psi : \mathcal{S}(DM) \rightarrow \mathcal{B}(\mathcal{H})$ and then to define

$$\psi_o : \mathcal{S}_o(DM) \rightarrow \mathcal{B}(\mathcal{H}), \quad \psi_o(f) := \psi(e^{-i\phi_o} f), \quad o \in KM.$$

This implies

$$\begin{aligned} 0 &= \psi((i\mathbb{V} - m)(e^{-i\phi_o} f)) = \psi(\gamma_o^\mu (\partial_\mu \phi_o) e^{i\phi_o} f + i e^{i\phi_o} \mathbb{V} f - m(e^{i\phi_o} f)) \\ &= \psi(e^{i\phi_o} ((\hat{A} + i\mathbb{V} - m)f)) = \psi_o((i\mathbb{V} + \hat{A} - m)f), \end{aligned} \quad (50)$$

for $f \in \mathcal{S}_o(DM)$. Thus, each ψ_o is a local solution of the Dirac equation with interaction A . Yet for $a \subseteq o$ and $f \in \mathcal{S}_a(DM)$ we find $\psi_o(f) = \psi(e^{-i\phi_o} f) = \psi(e^{-i\hat{A}_{oa}} e^{-i\phi_a} f)$, and we conclude

$$\psi_o(f) = e^{-i\hat{A}_{oa}} \psi_a(f), \quad f \in \mathcal{S}_a(DM), \quad a \subseteq o. \quad (51)$$

The above relations show that there is a topological obstruction to gluing the local fields ψ_o , encoded by the cocycle \hat{A} . To get a globally defined and interacting Dirac field, we introduce the twisted Dirac bundle

$$D_A M := DM \otimes \mathcal{L}_A \rightarrow M,$$

which, by Lemma 4.1, is endowed with “local charts” $\pi_o : (D_A M)|_o \rightarrow DM|_o$, $o \in KM$. Here to be concise, we make an abuse of notation by identifying π_o and $id_o \otimes \pi_o$, where id_o is the identity of $DM|_o$. Note that $\mathbf{end}(D_A M) \simeq \mathbf{end} DM$.

In the following lines, we collect some simple properties of $D_A M$. First, we note that sections $\varsigma \in \mathcal{S}(D_A M)$ are in one-to-one correspondence with families $\{\varsigma_o\}$, where each $\varsigma_o := \pi_o \circ \varsigma|_o$ is a section of $DM|_o$ and the relations $\varsigma_o = g_{oa} \varsigma_a$ hold for $o \cap a \neq \emptyset$ with g_{oa} defined in (45). Secondly, we note that tensoring (46) by the identity of DM induces the isomorphism

$$\vartheta : \mathcal{S}(D_A M) \rightarrow \mathcal{S}(DM) \quad (52)$$

such that $\vartheta(\varsigma) = e^{-i\phi_o} \varsigma_o$ for all $o \in KM$ and $\varsigma \in \mathcal{S}(D_A M)$ with $\text{supp}(\varsigma) \subseteq o$. In particular, by definition of \hat{A} , we have

$$e^{i\hat{A}_{oa}} \cdot id_a = \pi_o \circ \pi_a^{-1} = g_{oa}, \quad a \subseteq o. \quad (53)$$

Thus, if $\varsigma \in \mathcal{S}_a(D_A M)$, then

$$\varsigma_o = e^{i\hat{A}_{oa}} \varsigma_a, \quad a \subseteq o. \quad (54)$$

Defining

$$\psi_{A,a}(\varsigma) := \psi_a(\varsigma_a) = \psi(e^{-i\phi_a} \varsigma_a), \quad \varsigma \in \mathcal{S}_a(D_A M), \quad (55)$$

we have that if $a \subseteq o$ then, combining (51) and (54),

$$\begin{aligned} \psi_{A,o}(\varsigma) &= \psi_o(\varsigma_o) = e^{i\hat{A}_{oa}} \psi_o(\varsigma_a) = e^{i\hat{A}_{oa}} e^{-i\hat{A}_{oa}} \psi_a(\varsigma_a) \\ &= \psi_{A,a}(\varsigma), \quad \varsigma \in \mathcal{S}_a(D_A M). \end{aligned}$$

The previous computation shows that the operator $\psi_{A,a}(\varsigma)$ is independent of the choice of $a \in KM$ such that $\text{supp}(\varsigma) \subseteq a$. By (52) we find $e^{-i\phi_a}\varsigma_a = \vartheta(\varsigma)$, implying that each $\psi_{A,a}$ extends to the well-defined field

$$\psi_A : \mathcal{S}(D_A M) \rightarrow \mathcal{B}(\mathcal{H}), \quad \psi_A := \psi \circ \vartheta.$$

Remark 4.2 In the previous expression, ψ_A appears as a field defined on $D_A M$ rather than DM . This kind of field was introduced and studied by Isham [2,23]. Note that ψ_A fulfills the CARs

$$\{\psi_A(\varsigma_1)^*, \psi_A(\varsigma_2)\} = i \int_M \vartheta(\varsigma_1)^\dagger S(\vartheta(\varsigma_2)), \quad \varsigma_1, \varsigma_2 \in \mathcal{S}(D_A M).$$

To evaluate the Dirac equation on ψ_A , it is convenient to give a description of ∇ as an operator on sections of $\mathcal{S}(D_A M)$ explicitly presented as tensors $\varsigma = f \otimes s$, $f \in \mathcal{S}(DM)$, $s \in \mathcal{S}(\mathcal{L}_A)$. To this end, we define $\mathbf{d} := \vartheta^{-1}d\vartheta$, where ϑ is defined in (46) and $d : C^\infty(M, \mathbb{C}) \rightarrow \mathcal{S}(T^*M \otimes \mathbb{C})$ is the complex exterior derivative. In this way, we obtain a connection \mathbf{d} on \mathcal{L}_A compatible with the inner product of $\mathcal{S}(\mathcal{L}_A)$ in the sense that $d(\langle s, s' \rangle) = \langle \mathbf{d}s, s' \rangle + \langle s, \mathbf{d}s' \rangle$, $s, s' \in \mathcal{S}(\mathcal{L}_A)$. The Leibniz rule for the Dirac connection leads to the following extension:

$$\nabla(f \otimes s) := \nabla f \otimes s + f \otimes \mathbf{d}s \in \mathcal{S}(D_A M \otimes T^*M). \quad (56)$$

Before constructing the associated Dirac operator, we introduce some useful objects. We start with the normalized section $e^{i\phi} \in \mathcal{S}(\mathcal{L}_A)$ obtained by applying the inverse of (46) to the constant section 1 of $M \times \mathbb{C}$. Note that by definition $e^{i\phi_o} = \pi_o \circ e^{i\phi}|_o$ for all $o \in KM$ (and this justifies our notation). Hence, we can write

$$\vartheta(s) = \langle e^{i\phi}, s \rangle, \quad s \in \mathcal{S}(\mathcal{L}_A), \quad (57)$$

having regarded the complex function $\langle e^{i\phi}, s \rangle$ as a section of $M \times \mathbb{C}$. We can write explicitly the complex 1-form $X_s := d(\langle e^{i\phi}, s \rangle) \in \mathcal{S}(T^*M \otimes \mathbb{C})$,

$$X_s = -iA \cdot \langle e^{i\phi}, s \rangle + \langle e^{i\phi}, \mathbf{d}s \rangle, \quad (58)$$

having used $i\partial_\mu\phi_o = iA_\mu$ and the fact that $\langle \cdot, \cdot \rangle$ is conjugate linear in the first variable. Also note that $\mathbf{d}s \in \mathcal{S}(\mathcal{L}_A \otimes T^*M)$, so the pairing $\langle e^{i\phi}, \mathbf{d}s \rangle$ yields a complex 1-form. Applying a contraction with local Dirac matrices γ_o^μ , we get an operator $\mathbb{X}_s \in \mathcal{S}(\text{end}DM)$,

$$\mathbb{X}_s(f) = -iA f \cdot \langle e^{i\phi}, s \rangle + \langle e^{i\phi}, f \otimes \mathbf{d}s \rangle, \quad f \in \mathcal{S}(DM),$$

where, in local frames over $o \in KM$,

$$f \otimes \mathbf{d}s := \gamma_o^\mu f \otimes \partial_\mu s \in \mathcal{S}(D_A M). \quad (59)$$

Here, ∂_μ has lower indices, thus $\partial_\mu s$ lives in $\mathcal{L}_A \otimes TM$ and contraction by γ_o^μ drops components in T^*M and TM . Moreover, pairing by $e^{i\phi}$ yields $\langle e^{i\phi}, f \otimes ds \rangle \in \mathcal{S}(DM)$. With the notation (59), pairing (56) with the Dirac matrices yields

$$\nabla(f \otimes s) = \nabla f \otimes s + f \otimes ds. \quad (60)$$

In conclusion,

$$\begin{aligned} \nabla \circ \vartheta(f \otimes s) &= \nabla(f \cdot \langle e^{i\phi}, s \rangle) = (\nabla f) \cdot \langle e^{i\phi}, s \rangle + \mathbb{X}_s(f) \\ &= \nabla f \cdot \langle e^{i\phi}, s \rangle - i\mathbb{A}f \cdot \langle e^{i\phi}, s \rangle + \langle e^{i\phi}, f \otimes ds \rangle \\ &= \vartheta((\nabla - i\mathbb{A})f \otimes s + f \otimes ds) \\ &= \vartheta((\nabla - i\mathbb{A})(f \otimes s)), \end{aligned}$$

having written $\mathbb{A}(f \otimes s) := (\mathbb{A}f) \otimes s$ and used (57), (58), (60). This implies

$$(i\nabla + \mathbb{A} - m)\psi_A(f \otimes s) = \psi(\vartheta((i\nabla + \mathbb{A} - m)(f \otimes s))) = \psi((i\nabla - m)(\vartheta(f \otimes s))) = 0, \quad (61)$$

as expected. Note that if one starts with the field ψ_A , then defining for all $o \in KM$

$$\psi_o : \mathcal{S}_o(DM) \rightarrow \mathcal{B}(\mathcal{H}), \quad \psi_o(f) := \psi_A(\pi_o^{-1}(f))$$

we get a family of fields fulfilling (51) (see (53)). Moreover, the chain of identities (50) shows that each field $\psi_o \circ e^{i\phi_o}$ fulfills the free Dirac equation and, again by (53), the family $\{\psi_o \circ e^{i\phi_o}\}$ glue in the correct way to define a free Dirac field $\psi := \psi_A \circ \vartheta^{-1}$. In conclusion:

- We accomplished our task of constructing a Dirac field ψ_A interacting with A , by paying the price of switching from DM to the twisted Dirac bundle $D_A M$;
- ψ_A is equivalently described by the family $\{\psi_o\}$ fulfilling (51) and such that each ψ_o is gauge-equivalent to a free field ψ .

4.3 Charge transporters and Aharonov–Bohm effect

In the present section, we compare the net defined by ψ and the ones defined by the interacting fields ψ_A , verifying that they all define the same observable net \mathcal{A}_{KM} . We show that any interacting field defines an irreducible 1-cocycle in $Z_{AB}^1(\mathcal{A}_{KM})$ obeying the Fermi statistics and having topological dimension one.

As a first step, recalling (55), we define

$$\mathcal{F}_{A,o} := \{\psi_o(f), f \in \mathcal{S}_o(DM)\}'' , \quad o \in KM.$$

Clearly, $\mathcal{F}_{A,o} = \mathcal{F}_o$, where \mathcal{F}_o is the von Neumann algebra generated by the free field ψ . The additional information carried by A is, instead, encoded by (51), which imposes to define the *-monomorphisms $J_{oa} : \mathcal{F}_{A,a} \rightarrow \mathcal{F}_{A,o}$,

$$J_{oa}(T) := \eta(e^{-i\hat{A}_{oa}})(T) = U(e^{-i\hat{A}_{oa}}) T U(e^{i\hat{A}_{oa}}), \quad a \subseteq o, \quad (62)$$

where η is the gauge action (33) and U is the corresponding unitary action on the Fock space (34). This definition allows to recover the relation between the locally defined fields ψ_o ,

$$J_{oa}(\psi_a(f)) = e^{-i\hat{A}_{oa}}\psi_a(f) = \psi_o(f), \quad f \in \mathcal{S}_a(DM),$$

and by applying (47) and subsequent remarks, we find

$$J_{oa} \circ J_{ae} = J_{oe}, \quad e \subseteq a \subseteq o. \quad (63)$$

The pair (\mathcal{F}_A, J) is called the *field net twisted by A* , and, at the mathematical level, it defines a *precosheaf* [38]. These objects are more general than nets since, instead of the inclusion maps, we have the non-trivial $*$ -monomorphisms J_{oa} . As a consequence, if $T_o \in \mathcal{F}_{A,o}$ and $T_a \in \mathcal{F}_{A,a}$, then the correct way to perform their product is

$$T_o J_{oa}(T_a) \in \mathcal{F}_{A,o}. \quad (64)$$

We define the gauge-invariant von Neumann algebras

$$\mathcal{F}_{A,o}^0 := \{S \in \mathcal{F}_{A,o} : U(\zeta)S = SU(\zeta), \forall \zeta \in \mathbb{U}(1)\}, \quad o \in KM.$$

This implies $J_{oa}(S) = S$ for all $S \in \mathcal{F}_{A,a}^0$, thus \mathcal{F}_A^0 is a *net*, coinciding with the gauge-invariant subnet \mathcal{F}^0 of the free Dirac field since $\mathcal{F}_{A,o}^0 = \mathcal{F}_o^0$ for all $o \in KM$.

We now pass to the construction of charge transporters starting from the twisted field net (\mathcal{F}_A, J) . To this end, we note that the operators φ_o , $o \in KM$, defined in (40), can be regarded as unitaries φ_o of $\mathcal{F}_{A,o}$ carrying charge 1. To construct the associated charge transporters, we invoke (64) and define

$$z(o, a) := \pi^0(\varphi_o^* J_{oa}(\varphi_a)) = \pi^0(e^{-i\hat{A}_{oa}}\varphi_o^*\varphi_a), \quad a \subseteq o. \quad (65)$$

Here, $z(a, o)$ is a unitary of \mathcal{A}_o and by (63) it fulfills the cocycle relations,

$$z(o, a)z(a, e) = z(a, e), \quad e \subseteq a \subseteq o,$$

thus we get a 1-cocycle $z \in Z^1(\mathcal{A}_{KM})$.

Recalling the definitions of charged and topological component of a 1-cocycle given in Sect. 2.2, we can now prove the following theorem which yields a converse for Corollary 2.6.

Theorem 4.3 *Let \mathcal{A}_{KM} denote the observable net of the free Dirac field and let $A \in Z_{dR}^1(M)$. Then, the cocycle z defined by (65) is an element of the category $Z_{AB}^1(\mathcal{A}_{KM})$ satisfying*

$$z(o, a) = \exp\left(-i \oint_{\ell(o,a)} A\right) \cdot z_c(o, a), \quad a \subseteq o. \quad (66)$$

The charged component z_c obeys the Fermi statistics, and the holonomy of z is given by

$$z(p_\ell) = \exp\left(-i \oint_\ell A\right) \mathbb{I}, \quad [\ell] \in \pi_1(M). \quad (67)$$

Proof The holonomy of z is readily computed by using (49), (43) and (65):

$$z(p_\ell) := z(o_n, o_{n-1}) \cdots z(o_1, o_0) = e^{-i \hat{A}_{p_\ell}} \mathbb{I} = \exp\left\{-i \oint_\ell A\right\} \mathbb{I},$$

where $p_\ell = (a, o_{n-1}) * \cdots * (o_1, a)$ is a path-approximation of the loop $\ell : x \rightarrow x$, $x \in a \in KM$. Concerning the first part of the theorem, given a path frame P_e and recalling the definitions of topological and charged component given in Sect. 2.3 we have

$$z_c(o, a) = z(p_{oe} * p_{ea}) = \pi^0(e^{i(\hat{A}_{p_{oe}} - \hat{A}_{p_{ae}})} \varphi_o^* \varphi_a) = e^{i(\hat{A}_{p_{oe}} - \hat{A}_{p_{ae}})} \pi^0(\varphi_o^* \varphi_a).$$

Defining the unitary $t_a := e^{-i \hat{A}_{p_{ae}}} \mathbb{I}$ for any $a \in KM$, we find $t_o z_c(o, a) = \pi^0(\varphi_o^* \varphi_a) t_a$ for any inclusion $a \subseteq o$. So z_c is unitary equivalent to the DHR cocycle (42) obeying the Fermi statistics. Finally, from the splitting formula (10) and from the above relation we get

$$\begin{aligned} z(\tilde{a}, a) &= \alpha_{\tilde{a}e}(u_z(\tilde{a}, a)) z_c(\tilde{a}a) = \alpha_{\tilde{a}e}(z(p_{e\tilde{a}} * (\tilde{a}, a) * p_{ae})) z_c(\tilde{a}a) \\ &= \alpha_{\tilde{a}e}\left(\exp\left(-i \oint_{\ell_{oa}} A\right) \mathbb{I}\right) z_c(\tilde{a}, a) = \exp\left(-i \oint_{\ell_{oa}} A\right) z_c(\tilde{a}, a). \end{aligned}$$

□

We stress that the connection 1-form A^z defined by z in the sense of Corollary 2.5 is gauge equivalent to A up to a singular cocycle, that is, A^z stands in the same de Rham class of A modulo a cohomology class $\xi^z \in H^1(M)$. In fact, by iterating (26) over the path approximation p_ℓ , we obtain

$$z(p_\ell) = \exp\left(-i \sum_i \oint_{\ell(o_i o_{i-1})} A^z\right) \cdot z_c(p_\ell) = \exp(-i \oint_\ell A^z) \cdot \mathbb{I},$$

having used topological triviality of z_c , that is, $z_c(p_\ell) = \mathbb{I}$. Thus, by applying (67), we arrive to

$$\exp\left(-i \oint_\ell A\right) = \exp\left(-i \oint_\ell A^z\right) \quad (68)$$

for all loops ℓ . Since the exponential map has kernel \mathbb{Z} , and since any singular 1-cycle can be interpreted as a loop, we find $(2\pi i)^{-1} \oint_c A \bmod \mathbb{Z} = (2\pi i)^{-1} \oint_c A^z \bmod \mathbb{Z}$, for all 1-cycles c . The previous equality says that A and A^z define the same differential character in the sense of Cheeger and Simons, thus by the third exact sequence of [7, Theorem 1.1] we conclude that the de Rham cohomology class of $A - A^z$ defines by integration the desired class $\xi^z \in H^1(M)$.

4.4 Twisted nets and cocycles

Indeed, the notion of twisted net can be given for arbitrary families $\sigma_{oa} \in \mathbb{U}(1)$, $a \subseteq o \in KM$, fulfilling the cocycle relations $\sigma_{oa}\sigma_{ae} = \sigma_{oe}$. By the results in [31], this is equivalent to giving the morphism

$$\sigma : \pi_1(M) \rightarrow \mathbb{U}(1), \quad \sigma([\ell]) := \sigma(p_\ell) := \sigma_{ao_{n-1}} \cdots \sigma_{o_1 a},$$

where $p = (a, o_{n-1}) * \dots * (o_1, a)$ is a path-approximation of the loop $\ell : [0, 1] \rightarrow M$. We call σ a *topological twist* of \mathcal{F}_{KM} . Defining $J_{oa} := \eta(\sigma_{oa}) \upharpoonright \mathcal{F}_a$, we get the desired twisted field net, that we denote by (\mathcal{F}_σ, J) , $\mathcal{F}_{\sigma,o} := \mathcal{F}_o$, $\forall o \in KM$. By the results in [34, 35], a twisted field net is nothing but a representation of \mathcal{F}_{KM} on the net of Hilbert spaces $U_{oa} : \mathcal{H}_a \rightarrow \mathcal{H}_o$, $a \subseteq o$, with $\mathcal{H}_o \equiv \mathcal{H}$, $o \in KM$, and $U_{oa} := U(\sigma_{oa})$, $o \subseteq a$, or, equivalently, a representation of \mathcal{F}_{KM} over the flat Hilbert bundle $H \rightarrow M$ with fibre \mathcal{H} and monodromy $U \circ \sigma : \pi_1(M) \rightarrow \mathcal{U}(\mathcal{H})$.

Let us now return on the observable net \mathcal{A}_{KM} and consider the charge 1 DHR-sector (42), that here we denote by $z_1(o, a) := \pi_0(\varphi_o^* \varphi_a)$, $a \subseteq o$.

Theorem 4.4 *Let \mathcal{A}_{KM} denote the observable net of the free Dirac field. Then, there is a one-to-one correspondence between sectors $z \in Z_{AB}^1(\mathcal{A}_{KM})$ with charge component z_1 and topological dimension 1, and twisted field nets (\mathcal{F}_σ, J) .*

Proof Let $\sigma_{oa} \in \mathbb{U}(1)$, $a \subseteq o$, denote a cocycle and $\sigma : \pi_1(M) \rightarrow \mathbb{U}(1)$ denote the associated topological twist of (\mathcal{F}_σ, J) . Then, in accord with (65) we define

$$z(o, a) := \pi^0(\varphi_o^* J_{oa}(\varphi_a)) = \sigma_{oa} z_1(o, a), \quad a \subseteq o.$$

A straightforward verification then shows that $z \in Z_{AB}^1(\mathcal{A}_{KM})$ has the desired properties. Conversely, let $z \in Z_{AB}^1(\mathcal{A}_{KM})$ as in the statement of the Theorem. For all $[\ell] \in \pi_1(M)$, we set $\sigma([\ell]) := \text{hol}_{A^z}(\ell) \in \mathbb{U}(1)$ and we define the associated twisted field net (\mathcal{F}_σ, J) . The cocycle z' defined by σ as in the previous part of the previous theorem is then given by

$$z'(o, a) = \text{Hol}_{D^z}(\ell_{(o,a)}) \cdot z_1(o, a),$$

and Corollary 2.5 implies $z = z'$. \square

Considering in particular topological twists arising from closed 1-forms, we obtain the following Theorem, which resumes the results of the last two sections:

Theorem 4.5 *Let \mathcal{A}_{KM} denote the observable net of the free Dirac field. Then, for any closed de Rham form $A \in Z_{dR}^1(M)$ there are:*

1. A twisted Dirac field $\psi_A : \mathcal{S}(D_A M) \rightarrow \mathcal{B}(\mathcal{H})$ solving the Dirac equation with interaction A ;
2. A twisted field net (\mathcal{F}_A, J) with topological twisting $\sigma := \exp -i \oint A$;

3. A cocycle $z_A \in Z_{AB}^1(\mathcal{A}_{KM})$ with charge component z_1 and holonomy

$$z_A(p_\ell) = \exp \left\{ -i \oint_\ell A \right\} \mathbb{I}, \quad [\ell] \in \pi_1(M), \quad (69)$$

where p_ℓ is a path approximation of ℓ .

If $H_1(M)$ has no torsion elements, then any irreducible $z \in Z_{AB}^1(\mathcal{A}_{KM})$ with charge component z_1 and topological dimension 1 is equivalent to a cocycle of the type z_A .

Proof The twisted Dirac field ψ_A has been constructed in Sect. 4. Starting from ψ_A , we constructed in (62) the twisted field net (\mathcal{F}_A, j) , having the required twisting. The cocycle z_A is obtained by the previous theorem and has the form

$$z_A(o, a) = \exp \left\{ -i \oint_{\ell(o,a)} A \right\} \cdot z_c(o, a), \quad a \subseteq o,$$

from which (69) follows. Finally, if $H_1(M)$ has no torsion, then by the considerations after (27) we conclude that any holonomy $\sigma : \pi_1(M) \rightarrow \mathbb{U}(1)$ is of the desired type $\exp -i \oint A$. \square

5 Conclusion and outlooks

In the present paper, we studied the superselection sectors defined on curved spacetimes [6] by analyzing them in the specific model of the observable net of the free Dirac field. We provided the expected physical interpretation of the sectors, showing that they correspond to background flat potentials interacting with Dirac fields. The correspondence is essentially one-to-one, because potentials having the same Aharonov–Bohm phase define the same sector, and the phase is the actual observable quantity. The picture is then completed in terms of twisted field nets, whose inclusion morphisms are designed to reproduce the relative phases that appear when one tries to describe the interacting Dirac field in terms of “local charts” (51). Twisted field nets are, on turns, in one-to-one correspondence with superselection sectors.

For the physical interpretation of our results, a crucial point is that the background flat potentials can be reconstructed having merely as input the localized loop observables (6) defined by the sectors. Thus, the potential is a byproduct of the sector, which, as well-known, corresponds to a state of the observable net, in general affected by the spacetime topology. We then conclude that the potential is codified in the preparation of the state.

In particular, in the case of the classical Aharonov–Bohm effect, we argue that:

- When the experimenter shields the solenoid, the space where the charged particles are confined acquires a non-trivial topology, with fundamental group \mathbb{Z} .
- Switching on the magnetic field \vec{B} inside the solenoid makes the system fall into a superselection sector of AB type, labelled by the Aharonov–Bohm phase $\gamma \mapsto \exp -i \oint_\gamma A$. This fits the abstract discussion by Morchio and Strocchi [28], as well

as old results in path-integral quantization (see [22] and references cited therein). Thus, the presence of the non-trivial potential A , $d\vec{A} = \vec{B}$, is regarded as part of the preparation of the state: if the experimenter switches off the magnetic field, $\vec{B} = 0$, then we have a topologically trivial sector.

The lesson that we learn is that A is interpreted as a generalized DHR-charge of Aharonov–Bohm type and can be reconstructed using exclusively localized observables.

Finally, we remark that for the construction of our interacting field ψ_A it is essential that the twisting bundle \mathcal{L}_A is topologically trivial. Yet in general the twisting bundle

$$\mathcal{L}_z := \hat{M} \times_{\sigma} \mathbb{C}^n,$$

where $\sigma : \pi_1(M) \rightarrow \mathbb{U}(n)$ is as usual the holonomy representation defined by $z \in Z_{AB}^1(\mathcal{A}_{KM})$, is non-trivial. This may occur when:

1. $n = 1$ and the homology $H_1(M)$ has torsion;⁴
2. $n > 1$, a case that appears for $\pi_1(M)$ non-Abelian or for Dirac fields with non-Abelian gauge group. Under this hypothesis, the argument for proving triviality of \mathcal{L}_A fails. We note that this is the case of explicit physical interest: for example, two parallel solenoids in the Aharonov–Bohm apparatus yield $\pi_1(M)$ isomorphic to the free group with two generators.

Even if there exists no problem in defining the twisted field net $(\mathcal{F}_{\sigma}, j)$ in these cases, we believe that it is desirable from the point of view of physical interpretation to construct the interacting field in correspondence of the sector, thus this point is object of a work in progress. One possible approach may be to embed \mathcal{L}_z into some trivial bundle $M \times \mathbb{C}^m$, which always exists for m great enough. As an alternative, we may directly take twisted Dirac bundles of the type $D_z M := DM \otimes \mathcal{L}_z$ and construct a twisted Dirac field $\psi_z : \mathcal{S}(D_z M) \rightarrow \mathcal{B}(\mathcal{H})$ in sense of Isham.

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⁴ For examples, see [17]. EV thanks M. Benini and A. Schenkel for remarking this fact.

A Results on topological sectors

This appendix is devoted to proving that one can avoid using punctured Haag duality in the analysis of superselection sectors; only Haag duality is enough.

We consider the observable net \mathcal{A}_{KM} defined in a representation satisfying Haag duality. The key property of 1-cocycles that we need in this appendix is the following: for any pair $a, a_1 \in KM$ in the causal complement of $o \in KM$ one has that [6, Corollary 1]

$$z(q) \in \mathcal{A}'_o, \quad q : a \rightarrow a_1. \quad (70)$$

which is a consequence of homotopy invariance of 1-cocycles and of pathwise connectedness of the causal complement of a diamond.

Lemma A.1 *Let \mathcal{A}_{KM} be a causal net satisfying Haag duality. Given an inclusion of diamonds $o \subseteq \tilde{o}$, let $a \perp \tilde{o}$ and $p : a \rightarrow o$. Then, the following properties hold:*

- (i) *The adjoint action $z(p)\mathcal{A}_{\tilde{o}}z(\bar{p})$ is independent of the choice of p and a .*
- (ii) $z(p)\mathcal{A}_{\tilde{o}}z(\bar{p}) \subseteq \mathcal{A}_{\tilde{o}}$.

Proof (i) If $q : a_1 \rightarrow o$ with $a_1 \perp \tilde{o}$ observe that

$$z(p)Az(\bar{p}) = z(q)z(\bar{q} * p)Az(\bar{p} * q)z(\bar{q}) = z(q)Az(\bar{q}), \quad A \in \mathcal{A}_o$$

where we have used (70) as $\bar{q} * p : a \rightarrow a_1$ and a, a_1 are in the causal complement of \tilde{o} . (ii) Here, we use Haag duality. Take $o_1 \perp \tilde{o}$ and $B \in \mathcal{A}_{o_1}$. Since $o_1 \perp \tilde{o}$, we can find a_1 with $a_1 \perp \tilde{o}$, o_1 and a path $q : a_1 \rightarrow o$. By the just proved result, we have that for any $A \in \mathcal{A}_{\tilde{o}}$

$$\begin{aligned} z(p)Az(\bar{p})B &= z(q)Az(\bar{q})B = z(q)ABz(\bar{q}) = z(q)BAz(\bar{q}) \\ &= Bz(q)Az(\bar{q}) = Bz(p)Az(\bar{p}) \end{aligned}$$

where, again, we have used (70) as $q : a_1 \rightarrow o$ and a_1, o are in the causal complement of o_1 . \square

We can now prove [32, Lemma 4.5(i)].

Proposition A.2 *Let \mathcal{A}_{KM} be a causal net satisfying Haag duality. Given $o, a \in KM$ with $o \perp a$. Then, for any inclusion $\tilde{a} \subseteq a$ and any path $p : \tilde{a} \rightarrow o$ there holds*

$$z(p)\mathcal{A}'_a z(\bar{p}) \subseteq \mathcal{A}'_a.$$

Proof Take $B \in \mathcal{A}_a$, $A \in \mathcal{A}'_a$ and p as in the statement. By assumption, we have that $\bar{p} : o \rightarrow \tilde{a}$ and that $o \perp \tilde{a}$. By the previous Lemma, we have that $z(\bar{p})Bz(p) \in \mathcal{A}_a$; hence,

$$z(p)Az(\bar{p})B = z(p)Az(\bar{p})Bz(p)z(\bar{p}) = z(p)z(\bar{p})Bz(p)Az(\bar{p}) = Bz(p)Az(\bar{p}),$$

completing the proof. \square

This result implies that (11) gives an endomorphism $\rho^z(o)_{\tilde{a}} : \mathcal{A}'_a \rightarrow \mathcal{A}'_a$ for any $\tilde{a} \subseteq a$ and $o \perp a$ and that we can reply the analysis of superselection sectors without making use of punctured Haag duality.

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