Approximate XVA for European claims

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Abstract

We consider the problem of computing the Value Adjustment of European contingent claims when default of either party is considered, possibly including also funding and collateralization requirements.

As shown in Brigo et al. ([12], [13]), this leads to a more articulate variety of Value Adjustments (XVA) that introduce some nonlinear features. When exploiting a reduced-form approach for the default times, the adjusted price can be characterized as the solution to a possibly nonlinear Backward Stochastic Differential Equation (BSDE). The expectation representing the solution of the BSDE is usually quite hard to compute even in a Markovian setting, and one might resort either to the discretization of the Partial Differential Equation characterizing it or to Monte Carlo Simulations. Both choices are computationally very expensive and in this paper we suggest an alternative method based on an appropriate change of numeraire and on a Taylor's polynomial expansion when intensities are represented by means of affine processes correlated with the asset's price. The numerical discussion at the end of this work shows that, at least in the case of the CIR intensity model, even the simple first-order approximation has a remarkable computational efficiency.

Keywords: Credit Value Adjustment, Defaultable Claims, Counterparty Credit Risk, Wrong Way Risk, XVA, Affine Processes

1. Introduction

Many financial institutions trade contracts in over-the-counter (OTC) markets, their counterparties being other financial institutions or corporate clients. However, many of those contracts are subject to counterparty risk, or in other words, they are subject to some default event that might happen during their lifetime, concerning the solvency of either one of the parties. These contracts are called defaultable. Defaultable European options, originally named vulnerable, were the first to be studied, where the credit risk was coming only from the seller's potential default. Over the years, two approaches emerged to evaluate them: the structural approach and the reduced form approach.

Historically, the structural approach came first introduced by Johnson and Stulz in [29] when they considered the option as the sole liability of the counterparty. In the same framework, in [31] Klein discussed more general liability structures, in [32] he included interest rate risk, and in [33] he considered a (stochastic) default barrier depending on the value of the option. More recently, [36] extended this approach to jump-diffusion models, [27] considered multiple correlations, [18] treated it by using copulas.

Then researchers developed the alternative reduced-form approach. For a comprehensive presentation of the topic, we refer the reader to [34]. In [19], and the references therein, one can find a general

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overview of the approach for defaultable bonds. Later, the approach's mathematical framework was carefully formalized in [5] and [6], and recently [17] and [21] extended it to defaultable claims in Levy market models.

In the last decade, after the financial crisis of 2008-09, the interest in Counterparty Credit Risk increased remarkably, and attention focused on building a general framework to define and evaluate the premium to compensate the risk connected to defaultable products (in particular of Interest Rate Swaps). This premium took the name of Credit Value Adjustment (CVA) in the seminal paper by Zhu and Pykhtin [38], and it defines the appropriate reduction of the default-free value of a portfolio, to compensate for the default risk. This discount became the crucial quantity to take into account when trading derivatives in OTC markets, spurring much research in the field: see, for instance, [4], [10], [25].

Over the years, other value adjustments were introduced in the contract's evaluation, leading to the acronym (X)VA. Here, X stands for D= debt, L= liquidity, F=funding, to include also the risks due to the default of both parties, funding investment strategies, lack of liquidity. We refer the reader to [26] for a comprehensive exposition on the matter. In [24], one might find an updated overview of the recent research directions under investigation, where the authors explain the characterization of the adjusted value as the solution of a BSDE very well. In a Markovian setting, the connection between bilateral CVA and Partial Differential Equations (PDEs) is also thoroughly investigated in [15] and further developed in [16].

In this work, we treat a European claim, whose price is influenced by the default probabilities of either party as well by liquidity, financing, and collateralization risks when exploiting the intensity approach for the default times of both parties.

In a remarkable series of papers, ([12], [13], [14]), Brigo et al. describe in detail how introducing all the value adjustments implies the loss of an explicit expression for the adjusted value. Indeed the BSDE characterizing the contract's value is generally nonlinear and hence hardly solvable. It depends on the asset's price and many other, possibly correlated, factors such as default intensities, interest rate, stochastic volatility, so that even in a Markovian setting, also the associated PDE becomes extremely difficult to solve. Hence to provide a numerical approximation, one may resort either to the discretization of the PDE (see [30]) or to Monte Carlo simulations (as in [13]), which unfortunately result computationally very expensive for the average computing resources.

We are interested in devising an approximation procedure simple and computationally efficient even in the presence of many stochastic factors, provided we make some modeling choices. Indeed, we suggest to view the evaluation expectation as a smooth function of the correlation parameters and to approximate it by its Taylor polynomial expansion around the zero vector (the independent case), in the hope that the first or second-order are enough to provide an accurate approximation. We apply our method to estimate the price contribution that comes from considering stochastic default intensities correlated with the underlying's price. We remark, though, that we can straightforward extend the same technique to include further stochastic factors.

To evaluate Taylor polynomial's coefficients, we follow a two-step procedure to exploit, whenever possible, explicit formulae from option and bond's pricing theory. First, we condition the underlying's price with respect to the Brownian motions driving the intensities, retrieving a conditional explicit formula. Then, assuming the intensities to be described by affine models, we represent the single terms of the expansion using a change of Numeraire technique (similar to the one in [9]) to disentangle the correlations between the asset's price and the default intensities. The affinity of the processes makes it possible to use a "bond-like" expression for the default component.

To carry out the calculations in detail and to perform the numerical analysis of the method, we restrict to the Call option and we represent the intensities by two Cox Ingersoll Ross (CIR) processes. The final section shows the method's efficiency using Monte Carlo simulations as a benchmark.

A strong point of this approach is that it provides a relatively simple method that one can use with many correlated processes. Correlation often destroys any affine property the dynamical system might have, making the Riccati equations/Fourier transform framework inapplicable, and one can resort only to Monte Carlo or PDE's approximations. Since both are computationally expensive in several dimensions, the construction of an alternative approach with much shorter computational time and the same accuracy becomes very important.

Our method is particularly convenient when the correlation structure (as Monte Carlo simulations show for the CIR model) seems to follow a linear pattern. In this case, a first-order Taylor's polynomial is enough to produce an accurate approximation, providing a rather handy evaluation formula. We finally remark that the conditioning and change of numeraire techniques allow us to keep the coefficients' approximations to a minimum. The expansion's zeroth term corresponds to the independent case, and we need to have a semi-explicit formula to evaluate it. This fact forced us to restrict our model choices.

The paper is structured as follows. In the next section, we describe the general problem leading to the BSDE characterization under the reduced-form approach. We specify the model and the two-step evaluation procedure to compute Taylor's approximation in Section 3, while in Section 4, we specialize the calculations when the default intensities are CIR processes. Section 5 concerns the numerical analysis of our results.

2. XVA Evaluation of European claims under the intensity approach

We consider a finite time interval $[0, T]$ and a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, augmented with the $\mathbb{P}-$null sets and made right continuous. We assume that all processes have a cádlág version.

The market is described by the interest rate process r_t determining the money market account and by an adapted process X_t representing an asset log-price (we will specify its dynamics later), which may also depend on additional stochastic factors. We assume

- that the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is rich enough (and possibly more) to support all the stochastic processes that describe the market;
- to be in absence of arbitrage;
- that the given probability $\mathbb{P}$ is a risk-neutral measure, already selected by some criterion.

In this market model (as in [13]) we consider two parties (I = investor, C = counterparty) exchanging some European claim with default-free payoff $f(X_T)$, where f is a function (not necessarily nonnegative) as regular as needed. We take for granted that the market processes fulfill the necessary integrability hypotheses to guarantee a good definition of all the expectations we are going to write.

Both parties might default, due to some critical credit state, at respective random times τ^1 (Counterparty) and τ^2 (Investor), which are not stopping times with respect to the filtration $\mathcal{F}_t$. In this context we define the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}^1_t \vee \mathcal{H}^2_t$, where $\mathcal{H}^i_t = \sigma(\mathbf{1}_{\{\tau^i\leq s\}}, s \leq t)$, $i = 1, 2$, which is the smallest filtration extension that makes both random variables stopping times. Moreover, we assume there exists a unique extension of the risk-neutral probability to $\mathcal{G}_t$, that we keep denoting by $\mathbb{P}$.

In general, the following fundamental assumption, known as the H-hypothesis (see e.g. [23] and [22] and the references therein), ensures price coherence:

(H) Every $\mathcal{F}_t-$martingale remains a $\mathcal{G}_t-$ martingale.

By Lemma 7.3.5.1 in [28], (H) is automatically satisfied, under square integrability of the payoff, by the default-free price of any European contingent claim, whence we may affirm that

$$\mathrm{e}^{\int_0^t r_u du}\mathrm{e}^{X_t} = \mathbb{E}(\mathrm{e}^{\int_0^T r_u du}\mathrm{e}^{X_T}|\mathcal{F}_t) = \mathbb{E}(\mathrm{e}^{\int_0^T r_u du}\mathrm{e}^{X_T}|\mathcal{G}_t)$$

$$\mathrm{e}^{\int_0^t r_u du}c(t, T) := \mathbb{E}(\mathrm{e}^{\int_0^T r_u du}f(X_T)|\mathcal{F}_t) = \mathbb{E}(\mathrm{e}^{\int_0^T r_u du}f(X_T)|\mathcal{G}_t)$$

remain $\mathcal{G}_t-$ martingales under $\mathbb{P}$, for all $t \in [0,T]$.

In what follows, to stress the significance of the term "adjustment", we will point the corrections out step by step, with their signs determined by the fact that we are taking the investor's viewpoint.

We start assuming full knowledge that is we are in the $\mathcal{G}_t-$ filtration. The contract makes sense only if the default of either party has not occurred yet at the evaluation time t. Denoting by $\tau = \min(\tau^1, \tau^2)$, this fact is represented by the indicator function $\mathbf{1}_{\{\tau>t\}}$ to be placed in front of the price.

Either party may default, so a bilateral adjustment is needed. For the moment we assume nothing is recovered at default. Denoting by $\mathrm{CVA}^0(t,T)$ the Credit Value Adjustment due to the counterparty's default, this quantity has to act as a discount to the default-free price to balance the investor's risk assumption. On the other hand, the Debt Value Adjustment due to the investor's default, $\mathrm{DVA}^0(t,T)$, has to act as an accrual of the default-free price as it compensates the counterparty's risk assumption. So, for the $\mathcal{G}_t-$adapted adjusted value of the European claim $c^{\mathcal{G}}(t,T)$, we may write

$$\mathbf{1}_{\{\tau>t\}} c^{\mathcal{G}}(t,T) = \mathbf{1}_{\{\tau>t\}} \Big[c(t,T) - \mathrm{CVA}^0(t,T) + \mathrm{DVA}^0(t,T) \Big], \tag{1}$$

where $\mathrm{CVA}^0(t,T)$ and $\mathrm{DVA}^0(t,T) \geq 0$.

Now, let us admit the defaulting party might partially compensate for the loss due to his/her default. In this case, we have to include other two nonnegative terms, $\mathrm{CVA}^{rec}(t,T)$ and $\mathrm{DVA}^{rec}(t,T)$ (respectively for the counterparty and the investor), and we can rewrite the above as

$$\mathbf{1}_{\{\tau>t\}} c^{\mathcal{G}}(t,T) = \mathbf{1}_{\{\tau>t\}} \Big[c(t,T) - \mathrm{CVA}^0(t,T) + \mathrm{DVA}^0(t,T) + \mathrm{CVA}^{rec}(t,T) - \mathrm{DVA}^{rec}(t,T) \Big].$$

Moreover, as explained in [14], the two parties might be asked to collateralize their participation to the contract, they might need to borrow money to finance this participation and/or the risky asset(s) from a repo market to rea,lize their hedging strategies. All this leads to funding and liquidity risks that, again, have to be included for the correct contract's evaluation. Thus, we should write

$$\begin{aligned} \mathbf{1}_{\{\tau>t\}} c^{\mathcal{G}}(t,T) = & \mathbf{1}_{\{\tau>t\}} \Big[c(t,T) - \mathrm{CVA}^0(t,T) + \mathrm{DVA}^0(t,T) \\ & + \mathrm{CVA}^{rec}(t,T) - \mathrm{DVA}^{rec}(t,T) + \mathrm{FVA}(t,T) + \mathrm{LVA}(t,T) \Big], \end{aligned} \tag{2}$$

with $\mathrm{FVA}(t,T)$, $\mathrm{LVA}(t,T) \in \mathbb{R}$.The first represents the Funding Value Adjustment, the second the Liquidity Value Adjustment, and they are both determined by strategy financing and collateralization.

It is then necessary to model these terms to get to a manageable formula. The range of possible choices of mechanisms to include in the formation of prices is quite broad, and we refer the reader again to [12], [13] and [14] for a detailed discussion. Of course, there is an interplay among the different cash flows. For instance, collateralization changes the parties' exposures, the amount of cash borrowed at rate r^{ϕ} increases its value at a rate r_s.

Here we use the following set of assumptions.

1. The claim pays no dividends.
2. The adjustment processes all depend on a close-out value, ϵ_t, determined by a contractual agreement. It is natural to consider it $\mathcal{F}_t-$adapted since it is established on the basis of the information before default. Usually, it is taken as the default-free price or as the price of the defaultable claim itself.
3. We denote the collateralization process by C_s and it is a, possibly time-varying, percentage of the close-out value

$$C_s = \begin{cases} \alpha_s \epsilon_s^+, & \text{when due by the counterparty} \\ \alpha_s \epsilon_s^-, & \text{when due by the investor} \end{cases} \qquad 0 < \alpha_s < 1, \quad \forall s \in [0,T]. \tag{3}$$

Thus the net exposure is $(\epsilon_s - C_s)^+ = (1-\alpha_s)\epsilon_s^+$ for the investor and $(\epsilon_s - C_s)^- = (1-\alpha_s)\epsilon_s^-$ for the counterparty. Moreover, the rate for collateralization is r_s^c.

Symbol	Definition	Symbol	Definition
r_t	Risk-free rate	τ_1	Default time Counterparty
r_t^ϕ	Funding rate	τ_2	Default time Investor
r_t^c	Collateral rate	ϵ_t	Close-out value
h_t	Hedging rate	λ_t^i	Default intensities
α_t	collateralization level	$f(\cdot)$	Option payoff
$R_i(t)$	Recovery rates $i=1,2$	$\bar{v}_t$	$\int_t^T v_s ds$
$\tilde{r}_t$	$r_t^\phi - h_t$	$\hat{r}_t$	$r_t^\phi - r_t^c$

Table 1: Summary of notations.

4. $R_1(s)$ and $R_2(s)$ denote the close-out value recovery percentage in case of Counterparty's or Investor's default. Similarly, the Losses Given Default are $L_i(s) = (1 - R_i(s))$, $i = 1, 2$.
5. To construct investing strategies, the parties may invest in the riskless asset at a rate r^ϕ and in the risky asset(s) on a parallel repo market at a rate h_t. We denote by ϕ_u the quantity of riskless asset, and by H_t the value of the portion of the risky asset(s) that hedging the contract requires. Both can be either positive or negative. At the same time ϕ generates wealth at a rate r_s and H at a rate r^ϕ. This aspect will have to be taken into account, too.

As we said, the recovery and the collateral agreements are usually fractions of the close-out value hence they should be $\mathcal{F}_t-$adapted. On the contrary, the funding and hedging processes (ϕ, H) could be a priori $\mathcal{G}_t-$adapted, since they might incorporate a contribution from the default events.

Finally, the price must verify

$$c^{\mathcal{G}}(t,T) = \phi_t + H_t + C_t. \tag{4}$$

Following the crystal clear exposition in [13] (or in [12] and [14]), keeping in mind hypothesis (H) and (4), one can obtain the following BSDE in the $\mathcal{G}-$filtration

$$\begin{aligned}
\mathbf{1}_{\{\tau>t\}} c^{\mathcal{G}}(t,T) = \mathbf{1}_{\{\tau>t\}} \Bigg\{ & \mathbb{E}\left[\mathrm{e}^{-\int_t^T r_u du} f(X_T) \mathbf{1}_{\{\tau>T\}} \Big| \mathcal{G}_t \right] \\
+ & \mathbb{E}\left[\mathrm{e}^{-\int_t^\tau r_u du} \mathbf{1}_{\{\tau \le T\}} \Big(\epsilon_\tau - (1-\alpha_\tau) \big[L_1(\tau)\epsilon_\tau^+ \mathbf{1}_{\{\tau^1=\tau\}} - L_2(\tau)\epsilon_\tau^- \mathbf{1}_{\{\tau^2=\tau\}} \big] \Big) \Big| \mathcal{G}_t \right] \\
+ & \left[\int_t^{\tau\wedge T} \mathrm{e}^{-\int_t^s r_u du} \Big\{ [r_s - r_s^\phi] c^{\mathcal{G}}(s,T) ds + [r_s^\phi - r_s^c] C_s + [h_s - r_s] H_s \Big\} ds \Big| \mathcal{G}_t \right] \Bigg\}.
\end{aligned} \tag{5}$$

The random variables $\tau^i, i = 1, 2$, are not $\mathcal{F}_t-$stopping times. Indeed, traders can only observe whether the default events happened or not, conditioned to the available information. Thus, any risk-neutral evaluation that would naturally take place in the $\mathcal{G}-$filtration needs translating in terms of $\{\mathcal{F}_t\}$. For that, we have the following well known Key Lemma, to be found in [6] or [4], just to quote some references.

Lemma 2.1. *Given a $\mathcal{G}_t-$stopping time τ, for any integrable $\mathcal{G}_T-$measurable r.v. Y the following equality holds*

$$\mathbb{E}\Big[\mathbf{1}_{\{\tau>t\}} Y | \mathcal{G}_t \Big] = \mathbf{1}_{\{\tau>t\}} \frac{\mathbb{E}\Big[\mathbf{1}_{\{\tau>t\}} Y | \mathcal{F}_t \Big]}{\mathbb{P}(\tau > t | \mathcal{F}_t)}. \tag{6}$$

This Lemma calls for the conditional distributions of the default times that we are going to treat within the (Cox) reduced-form framework. We denote the random times' conditional distributions by

$$F_t^i = \mathbb{P}(\tau^i \le t | \mathcal{F}_t), \qquad i = 1, 2 \qquad \forall\, t \ge 0, \tag{7}$$

and we assume that they both verify $F_t^i < 1$. Hence we can define the corresponding $\mathcal{F}$- hazard processes of the τ^i's as

$$\Gamma_t^i := -\ln(1 - F_t^i) \quad \Rightarrow \quad F_t^i = 1 - \mathrm{e}^{-\Gamma_t^i} \quad \forall\, t > 0, \qquad \Gamma_0 = 0. \tag{8}$$

We assume them to be differentiable, implying the existence of $\mathcal{F}_t-$adapted intensity processes λ^i such that

$$\Gamma_t^i = \int_0^t \lambda_u^i du \quad \Rightarrow \quad F_t^i = 1 - \mathrm{e}^{-\int_0^t \lambda_u^i du}.$$

As in the classical framework of [20], we assume the default times are conditionally independent

$$\mathbb{P}(\tau^1 > t_1, \tau^2 > t_2|\mathcal{F}_t) = \mathbb{P}(\tau^1 > t_1|\mathcal{F}_t)\mathbb{P}(\tau^2 > t_2|\mathcal{F}_t), \quad \forall t > 0,\, t_1, t_2 \in [0, t].$$

Thus we may conclude that $\lambda_t := \lambda_t^1 + \lambda_t^2$ is the intensity process of $\tau = \inf\{\tau^1, \tau^2\}$.

Remark 2.2. *It is worth noting that the independence assumption certainly simplifies computations, but it does not take into consideration default contagion effects. Within the intensity framework, more realistic models allowing default dependence were recently proposed (see [7], [8] and the references therein), and we remark that we could extend our method to the correlated case, provided we introduce an additional parameter.*

Exploiting the Key Lemma and the intensity processes as in [3], the above equation gets projected on the smaller filtration, obtaining

$$\begin{aligned}
&\mathbf{1}_{\{\tau>t\}} c^{\mathcal{G}}(t,T) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}\Big[\mathrm{e}^{-\int_t^T (r_u+\lambda_u)du} f(X_T) \\
&+ \int_t^T \mathrm{e}^{-\int_t^s (r_u+\lambda_u)du} \big[\lambda_s \epsilon_s - (1-\alpha_s)\big(\lambda_s^1 L_1(s)\epsilon_s^+ - \lambda_s^2 L_2(s)\epsilon_s^-\big)\big] ds \\
&+ \int_t^T \mathrm{e}^{-\int_t^s (r_u+\lambda_u)du} \big[\big(r_s - r_s^\phi\big) c^{\mathcal{G}}(s,T) + \big(r_s^\phi - r_s^c\big)\alpha_s \epsilon_s + (h_s - r_s) H_s\big] ds \Big| \mathcal{F}_t\Big].
\end{aligned} \tag{9}$$

Applying the Key Lemma and its extension (Lemma 2 in [13]) to (9), we may conclude that there exists an $\mathcal{F}_t-$adapted adjusted price of the European claim, $c^a(t,T)$ and an adapted hedging strategy (the part hedging the default-free risks) $\tilde{H}$ such that

$$c^a(t,T)\mathbf{1}_{\{\tau>t\}} = c^{\mathcal{G}}(t,T)\mathbf{1}_{\{\tau>t\}}, \quad \tilde{H}_t \mathbf{1}_{\{\tau>t\}} = H_t \mathbf{1}_{\{\tau>t\}}.$$

Hence on $\{\tau > t\}$ we have

$$\begin{aligned}
&\mathbf{1}_{\{\tau>t\}} c^a(t,T) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}\Big[\mathrm{e}^{-\int_t^T (r_u+\lambda_u)du} f(X_T) \\
&+ \int_t^T \mathrm{e}^{-\int_t^s (r_u+\lambda_u)du} \big[\lambda_s \epsilon_s - (1-\alpha_s)\big(\lambda_s^1 L_1(s)\epsilon_s^+ - \lambda_s^2 L_2(s)\epsilon_s^-\big)\big] ds \\
&+ \int_t^T \mathrm{e}^{-\int_t^s (r_u+\lambda_u)du} \big[\big(r_s - r_s^\phi\big) c^a(s,T) + \Big(r_s^\phi - r_s^c\Big)\alpha_s \epsilon_s + (h_s - r_s) \tilde{H}_s\big] ds \big| \mathcal{F}_t\Big]
\end{aligned} \tag{10}$$

Remark 2.3. *Following [14], a few issues about the above BSDE need addressing.*

1. *We remark that the above equation has a unique strong solution, if the intensities and the close-out value are square-integrable and the processes r, r^c, r^ϕ, h are bounded. This is going to be our standing set of assumptions.*

2. *The process $\tilde{H}_t$ is linked to the solution of the BSDE. If we restrict to a diffusion setting with deterministic coefficients, the theory of BSDEs gives an explicit representation for this process. To deal with it, we extend the observation made in [14] when they assume deterministic intensities. More precisely, we assume that the stock price, $S_u = \mathrm{e}^{X_u}$, and the intensities processes, under the given risk-neutral probability, verify*

$$\begin{aligned} dS_u =& r_u S_u du + \sigma(t, S_u) dY_u, \quad \text{and} \\ d\lambda^i_u =& a_i(u, \lambda^i_u) du + b_i(u, \lambda^i_u) dB^i_u, \quad i = 1, 2, \end{aligned}$$

for correlated Brownian motions Y, B^1, B^2 and deterministic coefficients $\sigma(u,x), a_i(u,\lambda), b_i(u,\lambda)$ chosen to ensure the existence and uniqueness of strong solutions. Then (10) can be equivalently written on $\{\tau > t\}$ as

$$\begin{aligned} &\mathrm{e}^{-\int_0^t (r_u+\lambda_u)du} c^a(t,T) = c^a(0,T) + \int_0^t Z_s dY_s + M_t \\ &- \int_0^t \mathrm{e}^{-\int_0^s (r_u+\lambda_u)du} \left[\lambda_s \epsilon_s - (1-\alpha_s)\Big(\lambda^1_s L_1(s)\epsilon^+_s - \lambda^2_s L_2(s)\epsilon^-_s\Big)\right] ds \\ &- \int_0^t \mathrm{e}^{-\int_0^s (r_u+\lambda_u)du} \Big[\big(r_s - r^\phi_s\big) c^a(s,T) + \big(r^\phi_s - r^c_s\big)\alpha_s \epsilon_s + (h_s - r_s)\tilde{H}_s\Big] ds, \end{aligned} \tag{11}$$

where Z is the component of the solution of the BSDE coming from the martingale representation theorem, while M is a martingale depending on the intensities and possibly on some other stochastic factors (again represented by diffusions). In this context, $c^a(t,T)$ is a deterministic function of the state variables, and if it is regular enough, $\tilde{H}$ represents the $\delta-$hedging of the contract given by

$$\tilde{H}_u = \frac{\partial c^a(u,T)}{\partial S} S_u.$$

On the other hand, Itô's formula implies that Z is also given by

$$Z_u = \sigma(u, S_u)\frac{\partial c^a(u,T)}{\partial S} \quad \text{whence} \quad \tilde{H}_u = \frac{S_u}{\sigma(u,S_u)} Z_u,$$

provided that $\sigma(u,x) > 0$ for all u, x.

In addition to the hypotheses stated in remark 1., we assume that

$$0 < \sigma_0 x \leq \sigma(u,x) \leq \sigma_1 x, \qquad \forall\, u, x$$

for some constants σ_0 and σ_1.

As in [13] or [14], this implies, that we may apply Girsanov's theorem to change the Brownian motion driving the above BSDE to include the term $\tilde{H}$. Indeed,

$$B_t = Y_t + \int_0^t (r_u - h_u)\frac{S_u}{\sigma(u,S_u)} du$$

is a new Brownian motion with respect to the probability defined by the Radon-Nykodim derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathrm{e}^{-\int_0^T (r_u - h_u)\frac{S_u}{\sigma(u,S_u)} dY_u + \frac{1}{2}\int_0^T (r_u-h_u)^2 \frac{S_u^2}{\sigma^2(u,S_u)} du}$$

which verifies the Novikov condition. Consequently, under $\mathbb{Q}$ the asset price equation and (11) become

$$\begin{aligned} &dS_t = S_t h_t dt + \sigma(t,S_t)dB_t \\ &\mathrm{e}^{-\int_0^t (r_u+\lambda_u)du} c^a(t,T) = c^a(0,T) + \int_0^t Z_s dB_s + M_t \\ &- \int_0^t \mathrm{e}^{-\int_0^s (r_u+\lambda_u)du} \big[\lambda_s \epsilon_s - (1-\alpha_s)\big(\lambda_s^1 L_1(s)\epsilon_s^+ - \lambda_s^2 L_2(s)\epsilon_s^-\big)\big] ds \\ &- \int_0^t \mathrm{e}^{-\int_0^s (r_u+\lambda_u)du} \big[\big(r_s - r_s^\phi\big)c^a(s,T) + \Big(r_s^\phi - r_s^c\Big)\alpha_s \epsilon_s\big] ds. \end{aligned} \tag{12}$$

Passing again to the conditional expectation and multiplying both sides by $\mathrm{e}^{\int_0^t (r_u+\lambda_u)du}$, we obtain

$$\begin{aligned} &\mathbf{1}_{\{\tau>t\}} c^a(t,T) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{Q}}\Big[\mathrm{e}^{-\int_t^T (r_u+\lambda_u)du} f(X_T) \\ &+ \int_t^T \mathrm{e}^{-\int_t^s (r_u+\lambda_u)du} \big[\lambda_s \epsilon_s - (1-\alpha_s)\big(\lambda_s^1 L_1(s)\epsilon_s^+ - \lambda_s^2 L_2(s)\epsilon_s^-\big)\big] ds \\ &+ \int_t^T \mathrm{e}^{-\int_t^s (r_u+\lambda_u)du} \big[\big(r_s - r_s^\phi\big)c^a(s,T) + \Big(r_s^\phi - r_s^c\Big)\alpha_s \epsilon_s\big] ds \big| \mathcal{F}_t\Big]. \end{aligned} \tag{13}$$

The latter equation is linear, or nonlinear depending on the choice of ϵ_s. In the literature, there are fundamentally two possibilities: either $\epsilon_s = c(s,T)$ (the default-free value of the claim) or $\epsilon_s = c^a(s,T)$.

The first choice will always give a solvable linear BSDE. With the second choice, we might obtain a solvable linear BSDE if the adjusted value has constant sign otherwise, the negative and positive parts generate a nonlinear, not explicitly solvable, BSDE.

Here, we always choose $\epsilon_s = c(s,T)$ (that corresponds to asking collateralization proportional to the default-free price rather than to the current price) to guarantee the solvability of the BSDE. This choice will help us exploit some explicit formulas from the classical literature.

Consequently, (13) becomes

$$\mathbf{1}_{\{\tau>t\}} c^a(t,T) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{Q}}\Big[\mathrm{e}^{-\int_t^T (r_u+\lambda_u)du} f(X_T) + \int_t^T \mathrm{e}^{-\int_t^s (r_u+\lambda_u)du} \big[\Psi_s + (r_s - r_s^\phi) c^a(s,T)\big] ds \big| \mathcal{F}_t\Big],$$

where

$$\Psi_s = \big[\lambda_s + (r_s^\phi - r_s^c)\alpha_s\big] c(s,T) - (1-\alpha)\big[\lambda_s^1 L_1(s) c(s,T)^+ - \lambda_s^2 L_2(s) c(s,T)^-\big].$$

The equation is linear and it can be solved, obtaining

$$\mathbf{1}_{\{\tau>t\}} c^a(t,T) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{Q}}\Big[\mathrm{e}^{-\int_t^T (r_u^\phi+\lambda_u)du} f(X_T) + \int_t^T \mathrm{e}^{-\int_t^s (r_u^\phi+\lambda_u)du} \Psi_s ds \big| \mathcal{F}_t\Big]. \tag{14}$$

One can push the above equation to further generality, by considering distinct collateral rates, recovery processes, and close-out values for the two parties. These generalizations, though, do not introduce any additional mathematical complexity. Indeed, the nonlinearity of the equation is crucially determined by the recovery terms, when the close-out value is chosen equal to the adjusted price.

Remark 2.4. *The contribution to the price coming from funding, collateralization, rehypothecation is not irrelevant. Indeed, if those features are present, even in absence of default (which implies that default-free and close-out value coincide), (13) has the explicit solution*

$$c^a(t,T) = \mathbb{E}_{\mathbb{Q}}\Big[\mathrm{e}^{-\int_t^T [(1-\alpha_u) r_u^\phi + \alpha_u r_u^c] du} f(X_T) \big| \mathcal{F}_t\Big],$$

which reduces to the usual Black & Scholes formula only if all the rates coincide with the risk-free one.

From now on, we always refer to (14) and, to lighten notation, we omit the probability $\mathbb{Q}$ as subindex of the expectation. In the next section, we introduce the market model, and in the next ones, we outline our evaluation procedure by steps.

3. The evaluation procedure

In what follows, we specify the market model, where the asset price is a stochastic exponential, and the default intensities are assumed to be affine processes. Then, we illustrate a conditioning procedure that helps us exploit explicit expressions for the default-free price, as in the case of European Vanilla Options or Futures. Finally, we apply a change of Numeraire that allows using the well-known expression for Zero-Coupon Bonds for affine models. This last step helps to disentangle the contribution due to the intensities and the one coming from the derivative.

In section 4, we specialize this procedure when the intensities are CIR processes. In that setting, we will be able to derive semi-explicit formulas that we approximate by Taylor's expansion with respect to the correlation parameters.

3.1. The model

We keep denoting by $t \in [0,T]$ the initial time, and we make the following simplifying hypotheses:

1. all the rates, r, r^c, r^ϕ, h are deterministic;
2. for $i = 1, 2$, $(1-\alpha)L_i$ are constant and we keep denoting them simply by L_i.

So in (14) we have $\Psi_s = \big[\lambda_s + (r^\phi_s - r^c_s)\alpha\big]c(s,T) - \big[\lambda^1_s L_1 c(s,T)^+ - \lambda^2_s L_2 c(s,T)^-\big]$. For fixed initial conditions $(t, x, \lambda_1, \lambda_2) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, $\forall s \in [t,T]$, our fundamental processes verify

$$X_s = x + \int_t^s (h_u - \frac{\sigma^2}{2})du + \sigma(B_s - B_t) \qquad x \in \mathbb{R} \tag{15}$$

$$\lambda^i_s = \lambda_i + \int_t^s [\gamma^i_u \lambda^i_u + \beta^i_u]du + \int_t^s [\eta^i_u \lambda^i_u + \delta^i_u]^{\frac{1}{2}} dB^i_u,\ \lambda_i > 0,\ i = 1, 2. \tag{16}$$

We take $\sigma > 0$, $r, \gamma^i, \beta^i, \eta^i, \delta^i$, $i = 1, 2$ bounded deterministic functions of time, while (B^1, B^2, B^3) is a 3-dimensional Brownian motion, with

$$B_s = \rho_1 B^1_s + \rho_2 B^2_s + \sqrt{1 - \rho_1^2 - \rho_2^2} B^3_s, \qquad \rho_1^2 + \rho_2^2 \leq 1.$$

The processes $X_s, \lambda^1_s, \lambda^2_s$ are jointly Markovian, therefore $c(s,T)$ and $c^a(s,T)$ are deterministic functions respectively of the state variables X and $(X, \lambda^1, \lambda^2)$, which also depend on the correlation parameters $\boldsymbol{\rho} = (\rho_1, \rho_2)$.

For any $t \leq s \leq T$, we define the processes

$$N_i(u,s) := \mathbb{E}(\mathrm{e}^{-\int_t^s \lambda^i_v dv} | \mathcal{F}_u), \quad i = 1, 2, \tag{17}$$

which are martingales for $t \leq u \leq s$. Having chosen the intensities as affine processes, we have the explicit expressions

$$N_i(t,s) = \mathrm{e}^{A_i(t,s)\lambda_i + B_i(t,s)} \Rightarrow N_i(u,s) = \mathrm{e}^{A_i(u,s)\lambda_i + B_i(u,s) - \int_t^u \lambda^i_v dv}, \tag{18}$$

where λ_i is the initial condition of the intensity and A_i and B_i are deterministic functions verifying a set of Riccati equations. We remark that by independence of the intensities we also have

$$N(u,s) := \mathbb{E}(\mathrm{e}^{-\int_t^s \lambda_v dv} | \mathcal{F}_u) = \mathbb{E}(\mathrm{e}^{-\int_t^s (\lambda^1_v + \lambda^2_v) dv} | \mathcal{F}_u) = N_1(u,s) N_2(u,s),$$

which is still a martingale as product of independent martingales. By applying Itô's formula, the dynamics of these martingales are given by

$$\begin{aligned} dN_i(u,s) &= N_i(u,s)A_i(u,s)(\eta^i_u\lambda^i_u + \delta^i_u)^{\frac{1}{2}}dB^i_u \\ dN(u,s) &= N(u,s)\Big[A_1(u,s)(\eta^1_u\lambda^1_u + \delta^1_u)^{\frac{1}{2}}dB^1_u + A_2(u,T)(\eta^2_u\lambda^2_u + \delta^2_u)^{\frac{1}{2}}dB^2_u\Big]. \end{aligned} \tag{19}$$

In some classical specifications of the affine modeling framework:

- $\gamma^i_u = -\gamma_i,\ \beta^i(\lambda) = \gamma_i\theta_i,\ \ \delta^i_u = \delta^2_i,\ \eta^i_u = 0$ (Vasicek)

- $\gamma^i_u = -\gamma_i,\ \beta_i(\lambda) = \gamma_i\theta_i,\ \delta^i_u = 0,\ \eta^i_u = \eta^2_i$ (CIR),

for $\gamma_i, \theta_i, i = 1, 2$ positive constants, it is possible to compute $A_i(t, s)$ and $B_i(t, s)$ in closed form.

3.2. Conditioning

In this subsection, we express an alternative formulation for the expectations in (14), useful to exploit (conditionally) when possible, the explicit formula for the default-free price. To simplify notation, from now on we denote by $\mathbb{E}_t$ the conditional expectation with respect to $\mathcal{F}_t$.

Since the interest rate r^ϕ is deterministic, we rewrite (14) as

$$\mathbf{1}_{\{\tau>t\}}c^a(t,T) = \mathbf{1}_{\{\tau>t\}}\Big\{\mathrm{e}^{-\int_t^T r^\phi_u du}\mathbb{E}_t\Big(\mathrm{e}^{-\int_t^T \lambda_u du}f(X_T)\Big) + \int_t^T \mathrm{e}^{-\int_t^s r^\phi_u du}\mathbb{E}_t\Big(\mathrm{e}^{-\int_t^s \lambda_u du}\Psi_s\Big)ds\Big\} \tag{20}$$

and we focus on the inner expectations.

Proposition 3.1. *Let*

$$\mathcal{A}^t_s = \mathcal{F}^{B^1,B^2}_s \vee \mathcal{F}_t = \sigma(\{B^1_u, B^2_u, u \le s\}) \vee \mathcal{F}_t, \quad t \le s \le T.$$

Then

$$\mathbb{E}_t\Big[\mathrm{e}^{-\int_t^T \lambda_u du}f(X_T)\Big] = \mathrm{e}^{\int_t^T h_u du}\mathbb{E}_t\Big[\mathrm{e}^{-\int_t^T \lambda_u du}\mathbb{E}\Big(\mathrm{e}^{-\int_t^T h_u du}f(X_T)\Big|\mathcal{A}^t_T\Big)\Big],$$

where $X_T\Big|\mathcal{A}^t_T \sim \mathcal{N}\Big(\zeta_T(\boldsymbol{\rho}) + \int_t^T \big(h_u du - \frac{\Sigma^2(\boldsymbol{\rho})}{2}\big)du; \Sigma^2(\boldsymbol{\rho})(T-t)\Big)$ *and*

$$\zeta_T(\boldsymbol{\rho}) = x + \sigma(B^1_T - B^1_t)\rho_1 + \sigma(B^2_T - B^2_t)\rho_2 - \frac{\sigma^2|\boldsymbol{\rho}|^2}{2}(T-t), \quad \Sigma(\boldsymbol{\rho}) = \sigma\sqrt{1-|\boldsymbol{\rho}|^2}.$$

Proof: From (15) the log-price at time T is

$$X_T = \zeta_T(\boldsymbol{\rho}) + \int_t^T h_u du + \Sigma(\boldsymbol{\rho})(B^3_T - B^3_t) - \frac{\Sigma^2(\boldsymbol{\rho})}{2}(T-t),$$

and a simple application of the conditional expectation's tower-property gives

$$\begin{aligned} \mathbb{E}_t\Big[\mathrm{e}^{-\int_t^T \lambda_u du}f(X_T)\Big] &= \mathbb{E}_t\Big[\mathbb{E}\Big(\mathrm{e}^{-\int_t^T \lambda_u du}f(X_T)\Big|\mathcal{A}^t_T\Big)\Big] = \mathbb{E}_t\Big[\mathrm{e}^{-\int_t^T \lambda_u du}\mathbb{E}\Big(f(X_T)\Big|\mathcal{A}^t_T\Big)\Big] \\ &= \mathrm{e}^{\int_t^T h_u du}\mathbb{E}_t\Big[\mathrm{e}^{-\int_t^T \lambda_u du}\mathbb{E}\Big(\mathrm{e}^{-\int_t^T h_u du}f(X_T)\Big|\mathcal{A}^t_T\Big)\Big]. \end{aligned}$$

□

3.3. Changing Numeraires

As a final step to evaluate the expectations E_t in the previous expression, we apply the following family of changes of probability

$$\frac{d\mathbb{Q}^s}{d\mathbb{Q}}\Big|_{\mathcal{F}_s} = \frac{N(s,s)}{N(t,s)}, \tag{21}$$

defining the $s-$forward measures, for any $t \le s \le T$. Recalling (19), by Girsanov's theorem under $\mathbb{Q}^s$,

$$W_v^i = B_v^i - \int_t^v A_i(u,s)(\eta_u^i \lambda_u^i + \delta_u^i)^{\frac{1}{2}} du, \quad i = 1,2,\, t \le v \le s$$

define independent Brownian motions. The market dynamics, for $t \le v \le s \le T$, become

$$X_v = x + \int_t^v \Big(h_u - \frac{\sigma^2}{2} + \sigma \sum_{i=1,2} \rho_i A_i(u,s)(\eta_u^i \lambda_u^i + \delta_u^i)^{\frac{1}{2}}\Big) du + \sigma(W_s - W_t) \tag{22}$$

$$\lambda_v^i = \lambda_i + \int_t^v \Big[(\gamma_u^i + A_i(u,s)\eta_u^i)\lambda_u^i + (\beta_u^i + A_i(u,s)\delta_u^i)\Big] du + \int_t^v [\eta_u^i \lambda_u^i + \delta_u^i]^{\frac{1}{2}} dW_u^i, \tag{23}$$

where (W^1, W^2, B^3) is a 3-dimensional Brownian motion, on $[t,s]$ with

$$W_v = \rho_1 W_v^1 + \rho_2 W_v^2 + \sqrt{1 - \rho_1^2 - \rho_2^2} B_v^3, \qquad \rho_1^2 + \rho_2^2 \le 1$$

and we may conclude that the affine structure of the model is preserved.

We remark that for each fixed s, different Brownian motions are generated. We keep denoting them in the same manner, as they all have the same distributional properties.

In conclusion, for any $t \le s \le T$ and any $\mathcal{F}_s-$measurable random variable Y, we have

$$\mathbb{E}_t\Big(\mathrm{e}^{\int_t^s \lambda_u du} Y\Big) = N(t,s)\mathbb{E}_t^s\big(Y\big), \tag{24}$$

where $\mathbb{E}_t^s$ denotes the expectation under $\mathbb{Q}^s$.

4. Semiexplicit formulae

In this section, we restrict our analysis to a European call with strike price e^κ, $\kappa \in \mathbb{R}$, and maturity T, for which we will be able to exploit conditionally the Black & Scholes formula. We remark that by exploiting the put-call parity, it is possible to extend the evaluation to forward contracts.

Both intensities are described by CIR processes verifying Feller's condition in order to guarantee their positivity. By contrast, we do not consider the Vasicek model appropriate to represent intensities, even though it has been previously employed in credit risk modeling (see for instance [21]) as it allows writing very computable explicit formulas.

4.1. The CIR specification

In this case, the dynamics of the market, for any $t \le s \le T$, are given by

$$X_s = x + \int_t^s \Big(h_u - \frac{\sigma^2}{2}\Big) du + \sigma(B_s - B_t) \tag{25}$$

$$\lambda_s^i = \lambda_i + \int_t^s \gamma_i(\theta_i - \lambda_u^i) du + \eta_i \int_t^s \sqrt{\lambda_u^i} dB_u^i, \quad i = 1,2. \tag{26}$$

We denote by $\tilde{r}_u = r^\phi_u - h_u$ and , $\hat{r}_u = r^\phi_u - r^c_u$, and we have to compute

$$\begin{aligned}\mathbf{1}_{\{\tau>t\}} c^a(t,T) = \mathbf{1}_{\{\tau>t\}} \Big\{ & \mathrm{e}^{-\int_t^T \tilde{r}_u du} \mathbb{E}_t\Big[\mathrm{e}^{-\int_t^T \lambda_u du} \mathrm{e}^{-\int_t^T h_u du} f(X_T)\Big] \\ & + \int_t^T \mathrm{e}^{-\int_t^s r^\phi_u du} \mathbb{E}_t\Big[\mathrm{e}^{-\int_t^s \lambda_u du} \Lambda_s c(s,T)\Big] ds \Big\},\end{aligned} \tag{27}$$

where $\Lambda_s = \lambda_s + \alpha \hat{r}_s - L_1 \lambda^1_s$.

Proposition 4.1. *Let* $f(x) = (\mathrm{e}^x - \mathrm{e}^\kappa)^+$ *and*

$$\begin{aligned} c(s,T) &\equiv c(s,T)^+ = c_{BS}(X_s, s, \bar{v}_s, \sigma) \\ c_{BS}(x, s, \bar{v}_s, \sigma) &= \mathrm{e}^x \mathcal{N}\big(d_1(x,s,\bar{v}_s,\sigma)\big) - \mathrm{e}^{\kappa - \bar{v}_s} \mathcal{N}\big(d_2(x,s,\bar{v}_s,\sigma)\big) \\ d_{1,2}(x,s,\bar{v}_s,\sigma) &= \frac{x - \kappa + \bar{v}_s \pm \frac{\sigma^2}{2}(T-s)}{\sigma\sqrt{(T-s)}}, \end{aligned}$$

where we denoted by $\bar{v}_s = \displaystyle\int_s^T v_u du$, *for any* $v : [0,T] \longrightarrow \mathbb{R}$. *Then we have*

$$\begin{aligned}\mathbf{1}_{\{\tau>t\}} c^a(t,T;\boldsymbol{\rho}) = \mathbf{1}_{\{\tau>t\}} \Big\{ & \mathrm{e}^{-\int_t^T \tilde{r}_u du} \mathbb{E}_t\Big[\mathrm{e}^{-\int_t^T \lambda_u du} c_{BS}\big(\zeta_T(\boldsymbol{\rho}), t, \bar{h}_t, \Sigma(\boldsymbol{\rho})\big)\Big] \\ & + \int_t^T \mathrm{e}^{-\int_t^s r^\phi_u du} \mathbb{E}_t\Big[\mathrm{e}^{-\int_t^s \lambda_u du} \Lambda_s c_{BS}\big(X_s(\boldsymbol{\rho}), s, \bar{r}_s, \sigma\big)\Big] ds \Big\}.\end{aligned} \tag{28}$$

Proof: Applying inside the first expectation the conditioning with respect to $\mathcal{A}^t_T$, we obtain

$$\begin{aligned} &\mathbb{E}_t\Big[\mathrm{e}^{-\int_t^T \lambda_u du} \mathrm{e}^{-\int_t^T h_u du} f(X_T)\Big] = \mathbb{E}_t\Big[\mathbb{E}_t\Big(\mathrm{e}^{-\int_t^T \lambda_u du} \mathrm{e}^{-\int_t^T h_u du} f(X_T) | \mathcal{A}^t_T\Big)\Big] \\ =&\mathbb{E}_t\Big[\mathrm{e}^{-\int_t^T \lambda_u du} \mathbb{E}_t\Big(\mathrm{e}^{-\int_t^T h_u du} f(X_T)\Big| \mathcal{A}^t_T\Big)\Big] = \mathbb{E}_t\Big[\mathrm{e}^{-\int_t^T \lambda_u du} c_{BS}\big(\zeta^t_T(\boldsymbol{\rho}), t, \bar{h}_t, \Sigma(\boldsymbol{\rho})\big)\Big]. \end{aligned}$$

We may view the second expectation in (27) as $\mathbb{E}_t\Big[\mathrm{e}^{-\int_t^s \lambda_u du} \Lambda_s c_{BS}(X_s(\boldsymbol{\rho}), s, \bar{r}_s, \sigma)\Big]$ where, setting $M^i_s = B^i_s - B^i_t$, for $i = 1,2$ and $t \le s \le T$, we have

$$X_s(\boldsymbol{\rho}) = x + \int_t^s \Big(h_u - \frac{\sigma^2}{2}\Big) du + \sigma\big(M^1_s \rho_1 + M^2_s \rho_2 + M^3_s \sqrt{1 - |\boldsymbol{\rho}|^2}\big).$$

Consequently, we have

$$d_{1,2}(\zeta_T(\boldsymbol{\rho}), t, \bar{h}_t, \Sigma(\boldsymbol{\rho})) = \left\{ \begin{array}{l} \Big[d_1(x,t,\bar{h}_t,\sigma) + \frac{M^1_T}{\sigma\sqrt{T-t}}\rho_1 + \frac{M^2_T}{\sigma\sqrt{T-t}}\rho_2 - \sigma\sqrt{T-t}|\boldsymbol{\rho}|^2\Big] \frac{1}{\sqrt{1-|\boldsymbol{\rho}|^2}} \\ \Big[d_2(x,s,\bar{h}_t,\sigma) + \frac{M^1_T}{\sigma\sqrt{T-t}}\rho_1 + \frac{M^2_T}{\sigma\sqrt{T-t}}\rho_2\Big] \frac{1}{\sqrt{1-|\boldsymbol{\rho}|^2}}. \end{array} \right.$$

Pointing out the dependence on $\boldsymbol{\rho}$ of $c^a(t,T)$, we get (28). □

We want to approximate (28) by Taylor's expansion with respect to the correlation parameters $\boldsymbol{\rho} = (\rho_1, \rho_2)$ around $\mathbf{0} = (0,0)$ on $\{\tau > t\}$.

Remark 4.2. *For the sake of exposition, we restrict our discussion to the first-order expansion. This approximation might be accurate if the model has a roughly linear dependence upon the correlation parameters. Monte Carlo simulations showed that was the case in the CIR intensity setting (section 5), and indeed the accuracy of our method turned out to be very good. If the dependence on the correlation parameters is more markedly nonlinear, one may develop Taylor's polynomial to a higher order to capture this behavior. We computed fully also a second-order formula: it is computationally longer, but it does not present any additional theoretical complexity. We did not report it here to keep the exposition light.*

The first-order approximation then is

$$c^a(t,T;\boldsymbol{\rho}) \approx c^a(t,T;\mathbf{0}) + \frac{\partial c^a(t,T;\mathbf{0})}{\partial \rho_1}\rho_1 + \frac{\partial c^a(t,T;\mathbf{0})}{\partial \rho_2}\rho_2.$$

Since the integrability conditions are satisfied, the derivatives pass under the integral and expectation signs, and the problem reduces to computing the derivatives with respect to the correlation parameters of $c_{BS}\big(\zeta_T(\boldsymbol{\rho}), t, T, \Sigma(\boldsymbol{\rho})\big)$ and $c_{BS}\big(X_s(\boldsymbol{\rho}), s, T, \sigma\big)$ and evaluating them at $\mathbf{0}$. After some calculations, one arrives at the following expressions

$$\begin{aligned}
c_{BS}\big(\zeta_T(\boldsymbol{\rho}), t, \bar{h}_t, \Sigma(\boldsymbol{\rho})\big) &\approx c_{BS}\big(x, t, \bar{h}_t, \sigma\big) + \sigma e^x \mathcal{N}\big(d_1(x,t,\bar{h}_t,\sigma)\big)\big[M_T^1\rho_1 + M_T^2\rho_2\big]\,,\\
c_{BS}\big(X_s(\boldsymbol{\rho}), s, \bar{r}_s, \sigma\big) &\approx c_{BS}\big(X_s(\mathbf{0}), s, \bar{r}_s, \sigma\big) + \sigma e^{X_s(\mathbf{0})}\mathcal{N}\big(d_1(X_s(\mathbf{0}), s, \bar{r}_s, \sigma)\big)\big[M_s^1\rho_1 + M_s^2\rho_2\big]\,,
\end{aligned}$$

to be plugged into (27), and each term is going to be computed following the procedure outlined in the previous section. Exploiting the independence between $X_s(\mathbf{0})$ and B^1, B^2, we have

$$\begin{aligned}
c^a(t,T;\boldsymbol{\rho}) \approx & e^{-\int_t^T \tilde{r}_u du}\Big\{N(t,T)c_{BS}\big(x,t,\bar{h}_t,\sigma\big) + \sigma e^x \mathcal{N}\big(d_1(x,t,\bar{h}_t,\sigma)\big)\mathbb{E}_t\Big[e^{-\int_t^T \lambda_u du}(M_T^1\rho_1 + M_T^2\rho_2)\Big]\Big\}\\
& + \int_t^T e^{-\int_t^s r_u^\phi du}\Big\{\mathbb{E}_t\Big[e^{\int_t^s \lambda_u du}\Lambda_s\Big]\mathbb{E}_t\Big[c_{BS}\big(X_s(\mathbf{0}), s, \bar{r}_s, \sigma\big)\Big]\\
& \qquad + \sigma\mathbb{E}_t\Big[e^{X_s(\mathbf{0})}\mathcal{N}\big(d_1(X_s(\mathbf{0}), s, \bar{r}_s, \sigma)\big)\Big]\sum_{i=1}^{2}\mathbb{E}_t\Big(e^{-\int_t^s \lambda_u du}\Lambda_s M_s^i\Big)\rho_i\Big\}ds
\end{aligned}$$

and we now have to compute each single expectation. We proceed by steps, showing that we may restrict to some basic cases.

1. Noticing that

$$\begin{aligned}
& M_s^3 \sim N(0;\sigma^2(s-t))\\
& X_s(\mathbf{0}) = x + \int_t^s (h_u - \frac{\sigma^2}{2})du + M_s^3 \sim N\Big(x + \int_t^s (h_u - \frac{\sigma^2}{2})du; \sigma^2(s-t)\Big),\\
& d_i(X_s(\mathbf{0}), s, \bar{r}_s, \sigma) = \frac{X_s(\mathbf{0}) - k + \bar{r}_s \pm \frac{\sigma^2}{2}(T-s)}{\sigma\sqrt{T-s}}\\
& \qquad = \frac{M_s^3}{\sqrt{T-s}} + d_i(x, s, \bar{r}_s, \sigma) + \frac{1}{\sigma\sqrt{T-s}}\int_t^s (h_u - \frac{\sigma^2}{2})du\\
& \qquad \sim N\Big(d_i(x, s, \bar{r}_s, \sigma) + \frac{1}{\sigma\sqrt{T-s}}\int_t^s (h_u - \frac{\sigma^2}{2})du, \frac{s-t}{T-s}\Big), \quad i = 1, 2\\
& \mathbb{E}_t\Big[c_{BS}\big(X_s(\mathbf{0}), s, \bar{r}_s, \sigma\big)\Big] = \mathbb{E}_t\Big[e^{X_s(\mathbf{0})}\mathcal{N}\big(d_1(X_s(\mathbf{0}), s, \bar{r}_s, \sigma)\big)\Big] - e^{\kappa - \bar{r}_s}\mathbb{E}_t\Big[\mathcal{N}\big(d_2(X_s(\mathbf{0}), s, \bar{r}_s, \sigma)\big)\Big],
\end{aligned}$$

   we have that the Gaussian integrals can be computed explicitly

$$\begin{aligned}
\mathbb{E}_t\Big[e^{X_s(\mathbf{0})}\mathcal{N}\big(d_1(X_s(\mathbf{0}), s, \bar{r}_s, \sigma)\big)\Big] &= e^{x + \int_t^s h_u du}\mathcal{N}\Big(d_1(x + (\bar{r}_s - \bar{h}_s), t, \bar{h}_t, \sigma)\Big)\\
\mathbb{E}_t\Big[\mathcal{N}\big(d_2(X_s(\mathbf{0}), s, \bar{r}_s, \sigma)\big)\Big] &= \mathcal{N}\Big(d_2(x + (\bar{r}_s - \bar{h}_s), t, \bar{h}_t, \sigma)\Big),
\end{aligned}$$

   by applying the following

Lemma 4.3. *Let $p \in \mathbb{R}$ and $X \sim N(\mu, \nu^2)$, then*

$$\mathbb{E}(\mathrm{e}^{pX}\mathcal{N}(X)) = \mathrm{e}^{p\mu + \frac{(p\nu)^2}{2}}\mathcal{N}\left(\frac{\mu + p\nu^2}{\sqrt{1+\nu^2}}\right)$$

where by $\mathcal{N}$ we denote the standard Normal distribution function.

Proof: When $p = 0$ we refer the reader to [37], while the general case follows by a "completing the squares" argument. $\square$

Therefore, we may conclude that

$$\mathbb{E}_t\Big[c_{BS}\big(X_s(\mathbf{0}), s, \bar{r}_s, \sigma\big)\Big] = \mathrm{e}^{-(\bar{r}_s - \bar{h}_s) + \int_t^s h_u du} c_{BS}\big(x + (\bar{r}_s - \bar{h}_s), t, \bar{h}_t, \sigma\big) \tag{29}$$

2. It remains to evaluate the expectations

$$\mathbb{E}_t\Big(\mathrm{e}^{-\int_t^s \lambda_u du}\Lambda_s\Big), \quad \mathbb{E}_t\Big(\mathrm{e}^{-\int_t^s \lambda_u du}(B_s^i - B_t^i)\Big), \quad \mathbb{E}_t\Big(\mathrm{e}^{-\int_t^s \lambda_u du}\Lambda_s\ (B_s^i - B_t^i)\Big) \quad i = 1, 2.$$

Recalling that $\Lambda_s = \lambda_s + \alpha\hat{r}_s - L_1\lambda_s^1$, the above expressions reduce to computing

$$\mathbb{E}_t\Big[\mathrm{e}^{-\int_t^s \lambda_u du}(\lambda_s^i)^\alpha(B_s^j - B_t^j)^k\Big]$$

for $i, j = 1, 2$, and $\alpha, k = 0, 1$. To do so, we apply the change of Numeraire described in subsection 3.3, obtaining

$$\mathbb{E}_t\Big[\mathrm{e}^{-\int_t^s \lambda_u du}(\lambda_s^i)^\alpha(B_s^j - B_t^j)^k\Big] = N(t,s)\mathbb{E}_t^s\Big[(\lambda_s^i)^\alpha\big[(W_s^j - W_t^j) + \eta_j\int_t^s A_j(u,s)\sqrt{\lambda_u^j}du\big]^k\Big]. \tag{30}$$

For $i \neq j$, we can exploit the independence of W^1 and W^2, so the above expectation becomes

$$\eta_j\mathbb{E}_t^s\Big[(\lambda_s^i)^\alpha\Big]\left[\int_t^s A_j(u,s)\mathbb{E}_t^s\Big(\sqrt{\lambda_u^j}\Big)du\right]^k,$$

where for $t \leq u \leq s$

$$\lambda_u^i = \lambda_i + \int_t^u \Big[\gamma_i\theta_i - \Big(\gamma_i - \eta_i^2 A_i(v,s)\Big)\lambda_u^i\Big]\, dv + \eta_i\int_t^u \sqrt{\lambda_v^i}dW_v^i.$$

When $i = j$, if $k = 0$, in (30) clearly we have only the first expectation, if $\alpha = 0$ only the second, and for $\alpha = k = 1$, we end up with

$$\mathbb{E}_t^s\Big[\lambda_s^i(W_s^i - W_t^i)\Big] + \eta_j\int_t^s A_i(u,s)\mathbb{E}_t^s\Big[\lambda_s^i\sqrt{\lambda_u^i}\Big]du.$$

3. Thus we reduced the problem to considering the expectations, for $u \leq s$,

$$\mathbb{E}_t^s\big(\lambda_s^i\big), \quad \mathbb{E}_t^s\big(\sqrt{\lambda_u^i}\big), \quad \mathbb{E}_t^s\big(\lambda_s^i\sqrt{\lambda_u^i}\big), \tag{31}$$
$$\mathbb{E}_t^s\big(\lambda_s^i(W_s^i - W_t^i)\big), \tag{32}$$

The third of (31), again by the independence of the increments, can be written as

$$\mathbb{E}_t^s\big(\lambda_s^i\sqrt{\lambda_u^i}\big) = \mathbb{E}_t^s\big((\lambda_s^i - \lambda_u^i)\sqrt{\lambda_u^i}\big) + \mathbb{E}_t^s\big((\lambda_u^i)^{\frac{3}{2}}\big) = \mathbb{E}_t^s\big(\lambda_s^i - \lambda_u^i\big)\mathbb{E}_t^s\big(\sqrt{\lambda_u^i}\big) + \mathbb{E}_t^s\big((\lambda_u^i)^{\frac{3}{2}}\big).$$

To compute the expectations of the intensities' powers, we apply Itô's formula, and for $t \le u \le s \le T$, we obtain

$$\mathbb{E}_t^s\big(\lambda_u^i\big) = \mathrm{e}^{-\int_t^u[\gamma_i - \eta_i^2 A_i(\xi,s)]d\xi}\left\{\lambda_i + \gamma_i\theta_i\int_t^u \mathrm{e}^{\int_t^v[\gamma_i-\eta_i^2A_i(\xi,s)]d\xi}dv\right\},$$

$$\mathbb{E}_t^s\Big[\sqrt{\lambda_u^i}\Big] = \mathrm{e}^{-\frac{1}{2}\int_t^u[\gamma_i-\eta_i^2A_i(\xi,s)]d\xi}\left[\sqrt{\lambda_i}+\frac{1}{2}\Big[\gamma_i\theta_i-\frac{\eta_i^2}{4}\Big]\int_t^u \mathrm{e}^{\frac{1}{2}\int_t^v[\gamma_i-\eta_i^2A_i(\xi,s)]d\xi}\mathbb{E}_t^s\Big[\frac{1}{\sqrt{\lambda_v^i}}\Big]dv\right],$$

$$\mathbb{E}_t^s\Big[(\lambda_u^i)^{\frac{3}{2}}\Big] = \mathrm{e}^{-\frac{3}{2}\int_t^u[\gamma_i-\eta_i^2A_i(\xi,s)]d\xi}\left[(\lambda_i)^{\frac{3}{2}}+\frac{3}{2}\Big[\gamma_i\theta_i+\frac{\eta_i^2}{4}\Big]\int_t^u \mathrm{e}^{\frac{3}{2}\int_t^v[\gamma_i-\eta_i^2A_i(\xi,s)]d\xi}\mathbb{E}_t^s\Big[\sqrt{\lambda_v^i}\Big]dv\right],$$

From the above expressions, it is clear that the key point is computing $\mathbb{E}_t^s\Big[\frac{1}{\sqrt{\lambda_v^i}}\Big]$. We decide to approximate this quantity by either $\frac{1}{\sqrt{\lambda_i}}$ or $\frac{1}{\sqrt{\theta_i}}$, freezing the process either at the initial condition or at the mean reversion parameter. This choice usually provides simple and numerically quite accurate approximations of the (fractional) moments of a CIR process.
Finally, applying the integration by parts, we can compute expectation (32), obtaining

$$\mathbb{E}_t^s\big(\lambda_s^i(W_s^i - W_t^i)\big) = \eta_i\int_t^s \mathrm{e}^{-\int_u^s[\gamma_i-\eta_i^2A_i(\xi,s)]d\xi}\mathbb{E}_t^s\Big[\sqrt{\lambda_u^i}\Big]du.$$

In conclusion, all the pieces appearing in (4.1) can be computed explicitly, provided we freeze the process $(\lambda_u^i)^{-\frac{1}{2}}$.

Summarizing, we can write

$$c^a(t,T;\boldsymbol{\rho}) \approx g_0(t,T;\mathbf{0}) + g_1(t,T;\mathbf{0})\rho_1 + g_2(t,T;\mathbf{0})\rho_2, \tag{33}$$

where the zeroth term is (with $R_1 = 1 - L_1$)

$$\begin{aligned} g_0(t,T;\mathbf{0}) &= \mathrm{e}^{-\int_t^T\tilde r_u du}N(t,T)c_{BS}\big(x,t,\bar h_t,\sigma\big)\\ &+\int_t^T \mathrm{e}^{-\int_t^s\tilde r_u du-(\bar r_s-\bar h_s)}N(t,s)\big[R_1\mathbb{E}_t^s(\lambda_s^1)+\mathbb{E}_t^s(\lambda_s^2)+\alpha\hat r_s\big]c_{BS}\big(x+(\bar r_s-\bar h_s),t,\bar h_t,\sigma\big)ds, \end{aligned}\tag{34}$$

and the first-order coefficients are

$$\begin{aligned} g_1(t,T;\mathbf{0}) &= \sigma\Bigg\{\eta_1\mathrm{e}^{x-\int_t^T\tilde r_u du}N(t,T)\mathcal{N}\big(d_1(x,t,\bar h_t,\sigma)\big)\int_t^T A_1(s,T)\mathbb{E}_t^T\big(\sqrt{\lambda_s^1}\big)ds\\ &+\int_t^T \mathrm{e}^{x-\int_t^s\tilde r_u du}N(t,s)\mathcal{N}\Big(d_1(x+(\bar r_s-\bar h_s),t,\bar h_t,\sigma)\Big)\Big[R_1\mathbb{E}_t^s\big(\lambda_s^1(W_s^1-W_t^1)\big)\\ &+\eta_1\int_t^s A_1(u,s)\big[\mathbb{E}_t^s(\lambda_s^1-\lambda_u^1)\mathbb{E}_t^s(\sqrt{\lambda_u^1})+\mathbb{E}_t^s\big((\lambda_u^1)^{\frac{3}{2}}\big)+\big(\mathbb{E}_t^s(\lambda_s^2)+\alpha\hat r_s\big)\mathbb{E}_t^s(\sqrt{\lambda_u^1})\big]du\Big]ds\Bigg\}, \end{aligned}\tag{35}$$

$$\begin{aligned} g_2(t,T;\mathbf{0}) &= \sigma\Bigg\{\eta_2\mathrm{e}^{x-\int_t^T\tilde r_u du}N(t,T)\mathcal{N}\big(d_1(x,t,\bar h_t,\sigma)\big)\int_t^T A_2(s,T)\mathbb{E}_t^T\big(\sqrt{\lambda_s^2}\big)ds\\ &+\int_t^T \mathrm{e}^{x-\int_t^s\tilde r_u du}N(t,s)\mathcal{N}\Big(d_1(x+(\bar r_s-\bar h_s),t,\bar h_t,\sigma)\Big)\Big[\mathbb{E}_t^s\big(\lambda_s^2(W_s^2-W_t^2)\big)\\ &+\eta_2\int_t^s A_2(u,s)\big[\big(R_1\mathbb{E}_t^s(\lambda_s^1)+\alpha\hat r_s\big)\mathbb{E}_t^s(\sqrt{\lambda_u^2})+\mathbb{E}_t^s(\lambda_s^2-\lambda_u^2)\mathbb{E}_t^s(\sqrt{\lambda_u^2})+\mathbb{E}_t^s\big((\lambda_u^2)^{\frac{3}{2}}\big)\big]du\Big]ds\Bigg\}, \end{aligned}\tag{36}$$

and we recall that
$$N_i(t,s) = \mathrm{e}^{A_i(t,s)\lambda_i + B_i(t,s)}, \quad N(t,s) = N_1(t,s)N_2(t,s),$$
for $t \le s \le T$ and $i = 1, 2$, with
$$h_i = \sqrt{\gamma_i^2 + 2\eta_i^2}, \qquad A_i(t,T) = -\frac{2(\mathrm{e}^{h_i(T-t)} - 1)}{h_i - \gamma_i + (h_i + \gamma_i)\mathrm{e}^{h_i(T-t)}}$$
$$B_i(t,T) = \frac{2\gamma_i\theta_i}{\eta_i^2} \ln\left(\frac{2h_i \mathrm{e}^{\gamma_i + h_i(T-t)}}{h_i - \gamma_i + (h_i + \gamma_i)\mathrm{e}^{h_i(T-t)}}\right).$$

5. Numerical results

In this section, we present some numerical results of our approximation method for the call price. As a first step, we assess the performance of the first-order approximation (33) by using the Monte Carlo evaluations with control variates as a benchmark. We employ the default-free price as control: in the considered cases, this reduces the length of the confidence interval by at least one order of magnitude. For the simulations, we generated $M = 10^6$ sample paths with a time step equal to 10^{-3} for any considered maturity. The benchmark Monte Carlo method was implemented to approximate the call price (14) by using Euler's discretization scheme with full truncation for the intensity processes λ_t^1 and λ_t^2 (see [35]) and with an exact simulation of the Brownian motion for the underlying X_t. The running integrals appearing in the expectations were evaluated by a trapezoidal routine. All the algorithms were implemented in MatLab (R2019b).

The evaluation of the zeroth and first-order terms of our approximation ((34), (35), (36)) requires the computation of nested one-dimensional integrals of well-behaved functions once for each set of chosen parameters and this step was implemented through the vectorized global adaptive quadrature MatLab algorithm.

The parameters of the intensity processes were chosen as in [11] and [3] (see Table 2), and they agree with calibrated default intensities. The strike price was fixed to $K = \mathrm{e}^{\kappa} = 100$ and we considered two maturities, $T = 0.5$ and $T = 2$. Lastly, without loss of generality, we took $t = 0$, the log-asset's initial value was set to 4.6052, and its volatility to $\sigma = 40\%$. The remaining parameters were chosen as $r = h = 0.001$, $r^{\phi} = 0.005$, $r^c = 0.002$ and $\alpha = 0.5$.

The accuracy of the first-order approximation is summarized in Tables (3), (4), listing the errors with respect to the benchmark MC prices (see also figure (1)) at the two maturities, using two distinct sets of the default parameters for the Investor and the Counterparty. It is apparent how the approximation is highly satisfactory for short term maturity while it tends to deteriorate a little when the horizon increases.

In Table (5) we report the separate contributions of the zeroth and first-order terms in (33). These are quite sizeable, highlighting that the correlations between the underlying and the intensities affect significantly the price. This fact supports the choice of stochastic processes versus deterministic functions to represent the intensities. We notice that g_1 is more relevant than g_2, which instead appears to be always rather small. This is to be expected, since we are considering a call option and its price should be affected more by the Counterparty's default than by the Investor's default. As natural, we also observe g_1 to decrease as the collateralization tends to one.

The contribution coming from the stochastic nature of the intensities can be better appreciated by looking at the results of the further set of numerical experiments reported in Table (6). In order to compare with the results in [14], we considered the rates $r = 0.001$, $h = 0.005$, $r^{\phi} = 0.005$, $r^c = 0.002$, $\lambda_0^1 = 0.04$, $\lambda_0^2 = 0.02$, and the other parameters as in (2). The losses given default were set to $L_1 = L_2 = 60\%$ and we took $T = 0.5$. The correction that we obtain with respect to the prices in [14] (constant intensities) is of order 10^{-2}, which can become very relevant as the volume of the transaction grows.

	λ_0	γ	θ	η	6-months surv. prob.	2-years surv. prob.
τ_1 (counterparty)	0.03	0.02	0.161	0.08	0.9848	0.9371
τ_2 (investor)	0.035	0.35	0.45	0.15	0.9660	0.7399

Table 2: Parameter sets for the CIR default intensities.

$\rho_2 \backslash \rho_1$	-0.6	-0.4	-0.2	0	0.2	0.4	0.6
-0.6	-7.478e-04	-5.850e-04	-3.852e-04	-1.951e-04	-4.362e-05	7.122e-05	1.881e-04
-0.4	-5.338e-04	-3.423e-04	-1.508e-04	5.306e-05	1.955e-04	3.118e-04	3.636e-04
-0.2	-3.104e-04	-9.415e-05	8.240e-05	2.456e-04	3.640e-04	4.693e-04	5.321e-04
0	-1.194e-04	8.440e-05	2.489e-04	4.203e-04	5.234e-04	6.252e-04	7.105e-04
0.2	5.723e-05	2.527e-04	4.217e-04	5.816e-04	7.102e-04	8.091e-04	9.161e-04
0.4	2.584e-04	4.708e-04	6.296e-04	7.458e-04	8.760e-04	9.736e-04	1.079e-03
0.6	4.854e-04	6.768e-04	8.431e-04	9.614e-04	1.074e-03	1.167e-03	1.241e-03

Table 3: Approximation errors, Set 1 for τ_1, Set 2 for τ_2, $T = 0.5$. The average length of the 95% confidence interval for the MC estimates is $5.3939e-04$.

As a final remark, we write explicitly our evaluation formula when constant intensities $\lambda_t^i \equiv \lambda^i$ are taken. It is immediately seen by using (29) that the price (27) becomes

$$\begin{aligned} c^a(t,T) &= \mathrm{e}^{(\lambda^1+\lambda^2-(r^\phi-h))(T-t)} c_{BS}(x,t,\bar{h},\sigma) + (\lambda^1+\lambda^2+(r^\phi-r^c)\alpha-\lambda^1 L_1)\times \\ &\int_t^T \mathrm{e}^{-(\lambda^1+\lambda^2+(r^\phi-h))(s-t)}\mathrm{e}^{-(r-h)(T-s)} c_{BS}(x+(r-h)(T-s),t,\bar{h},\sigma)ds \end{aligned} \tag{37}$$

which, as noticed in [13] and [14], shows that the interplay among all the rates changes significantly the classical default-free price.

Last but not least, we would like to point out that our approximation implies a very big reduction of the computational time compared with Monte Carlo Simulations, providing a fast and flexible alternative approximation method.

$\rho_2\backslash\rho_1$	-0.6	-0.4	-0.2	0	0.2	0.4	0.6
-0.6	-6.619e-02	-5.728e-02	-4.887e-02	-3.983e-02	-3.114e-02	-2.303e-02	-1.427e-02
-0.4	-5.191e-02	-4.320e-02	-3.409e-02	-2.552e-02	-1.726e-02	-9.017e-03	-7.615e-04
-0.2	-3.706e-02	-2.828e-02	-1.938e-02	-1.138e-02	-3.327e-03	4.780e-03	1.299e-02
0	-2.246e-02	-1.338e-02	-5.165e-03	2.822e-03	1.095e-02	1.877e-02	2.686e-02
0.2	-7.224e-03	1.505e-03	9.585e-03	1.776e-02	2.559e-02	3.352e-02	4.164e-02
0.4	8.800e-03	1.771e-02	2.568e-02	3.327e-02	4.091e-02	4.864e-02	5.639e-02
0.6	2.543e-02	3.414e-02	4.206e-02	4.961e-02	5.704e-02	6.453e-02	7.191e-02

Table 4: Approximation errors, Set 1 for τ_1, Set 2 for τ_2, $T = 2$. The average length of the 95% confidence interval for the MC estimates is 0.0086.

T	g_0	g_1	g_2
0.5	11.3300	-0.0071	0.0003
2	22.4224	-0.0435	0.0370

Table 5: Contribution of zero-th and first order terms in the expansion approximation with Set 1 for τ_1 and Set 2 for τ_2. The corresponding default-free prices according to the B&S formula are $c_{BS}(X_0, 0, \bar{r}_s, \sigma) = 11.2685$ ($T = 0.5$) and $c_{BS}(X_0, 0, \bar{r}_s, \sigma) = 22.3480$ ($T = 2$).

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	$K=90$			$K=100$			$K=110$		
	$\alpha=0$	$\alpha=0.5$	$\alpha=1$	$\alpha=0$	$\alpha=0.5$	$\alpha=1$	$\alpha=0$	$\alpha=0.5$	$\alpha=1$
g_0	16.3455	16.4559	16.5663	11.2208	11.2965	11.3723	7.4639	7.5142	7.5646
g_1	−0.0317	−0.0155	0.0007	−0.0254	−0.0124	0.0006	−0.0193	−0.0094	0.0004
g_2	0.0004	0.0004	0.0003	0.0004	0.0003	0.0002	0.0003	0.0002	0.0001

Table 6: Values of the zero-th and first order terms of the expansion approximation with different strikes K and levels of collateralization α.

m	−0.2	−0.1	0	0.1	0.2
			$\alpha=0$		
c^{const}	5.5458	8.3127	11.9943	16.7047	22.5212
g_0	5.2034	7.7995	11.2539	15.6736	21.1312
			$\alpha=0.5$		
c^{const}	5.5728	8.3532	12.0527	16.7862	22.6310
g_0	5.2307	7.8405	11.3130	15.7561	21.2423
			$\alpha=1$		
c^{const}	5.5998	8.3937	12.1112	16.8676	22.7408
g_0	5.2581	7.8815	11.3722	15.8385	21.3535

Table 7: Values of the zero-th and first order terms of the expansion approximation with different moneyness m and levels of collateralization α. The prices c^{const} are obtained from (37).

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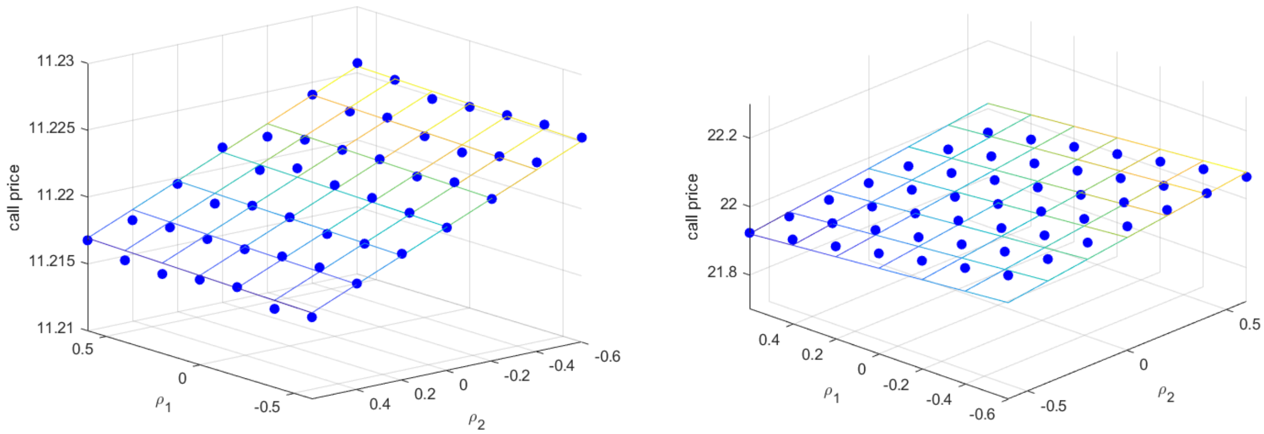

Figure 1: *MC prices (dot) vs approximated prices (lines).On the left $T = 0.5$, on the right $T = 2$.*

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