Braided Actions of DHR Categories
and Reconstruction of Chiral Conformal Field Theories

vorgelegt von
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'Nel quartiere siracusano dell'antica Neapolis, *sul declivio del colle Temenite, così chiamato perché in esso sorgevano dei santuari (Témenoi), sorge il teatro greco, il più famoso dei teatri greci di Occidente. Sulla sua data di fondazione i pareri sono discordi: c'è chi la pone nel V secolo a.C. e chi la abbassa fino all'inizio dell'età ellenistica. Il primo studio scientifico di epoca moderna è Il teatro greco di Siracusa di Giulio Emanuele Rizzo, saggio scritto nel 1916 ma pubblicato solo dopo la parentesi bellica, nel 1923. Secondo questo studioso, la prima fase di costruzione è da collocare agli inizi del V secolo a.C. Tale tesi è confortata da varie fonti che ne attestano l'esistenza fin da quel tempo, tra cui Eustazio che riferisce la notizia secondo la quale Sofrone, fiorito negli ultimi anni del V secolo, avrebbe citato il nome del costruttore, l'architetto Domocopo, soprannominato Mirilla (da myroi = unguenti) perché ultimata la costruzione del teatro, distribuì a coloro che erano intervenuti all'inaugurazione degli unguenti profumati.'*

'*Non bisogna comunque mai dimenticare il fascino del Teatro, inteso sia come elemento ambientale ma anche come memoria di un passato che è presente: Eschilo era fisicamente qui nel V secolo a.C. ed è qui ora con il suo genio a parlare con noi di cose che sono del nostro essere umani.*'

Pietro Giorgetti. *Siracusa 2012: Il Prometeo e le Baccanti.*

Tesi di Laurea in Drammaturgia del Mondo Greco e Romano. Alma Mater Studiorum Università di Bologna, Facoltà di Conservazione dei Beni Culturali, 2012.

Introduction

Or, why that much maths in a thesis on QFT?

Quantum Field Theory (QFT) is our modern understanding of particles and matter at small scales, where quantum behaviour replaces macroscopical phenomena, which are much closer to intuition, and dynamics is rather driven by the more fundamental interactions between fields. Using quantum fields one can describe particle production, annihilation and scattering processes and they can all together be cast into the Standard Model of particle physics. The latter gives a recipe to predict cross sections of high-energy collisions which fit remarkably well with experimental data. On the other hand our mathematical understanding of the framework, and how to replace diverging series, ad hoc renormalized or truncated to get finite numbers, is still a deep open question.

Since the early days it was clear that quantum fields, even when they arise from the classical picture of Lagrangian functionals and actions, are more singular objects than those employed in classical physics. Their values in points of spacetime, i.e., their point-like dependence as operator-valued functions, is easily seen to clash with their realizability as operators on an Hilbert space on one hand, on the other hand it is neither dictated by physics. The structure of spacetime itself, at very small scales, is by now out of our experimental reach.

In order to overcome the previous difficulties the notion of field can be relaxed to that of an (unbounded) operator-valued distribution (Wightman axiomatization), elevating the smearing with test functions to an essential feature of a local quantum theory. This generalization introduces more difficult mathematical objects (distributions, compared to functions) but which can be rigorously (without ambiguities) treated, and which are suitable enough to obtain a complete scattering theory, once a Wightman QFT is assigned.

In the same spirit, but with different mathematics, QFTs can be dealt

with using techniques from the theory of operator algebras. The first main characteristics of the algebraic approach (AQFT) is that one describes *local measurements* or *observable fields* and regards them as the primary objects of interest to study matter, particles and fundamental interactions, relegating the non-observable quantities to theoretical tools. Secondly, one treats them by means of *bounded* operators on Hilbert space (e.g., by considering bounded functions of the fields), advantageous at least for the analysis of the framework. More in details, physically relevant quantities such as observables (and states) of a QFT are described in terms of abstract operator algebras associated to open bounded regions of spacetime ("local algebras"). By abstract we mean independent of any specific Hilbert space realization, and then we regard the choice of different representations of the local algebras as the choice of different states (mathematically speaking via the GNS construction). In particular these objects encode both quantum behaviour, in their intrinsic non-commutativity, and Einstein's causality principle, in the triviality of commutation relations between local algebras sitting at space-like distances. This second approach is what this thesis is devoted to.

The relation between these two formalisms is not completely understood, from distributions to local algebras one has to take care of spectral commutation relations on suitable domains, vice versa one should control the scaling limits of the local algebras in order to exploit the distributional "point-like" generators. In both cases, and (theoretically) in any other mathematically sound description of QFT, consequences become proofs, and different features of models or more general model-independent principles (particle content, covariance, local commutation relations) can be separated and analysed. Moreover, beyond the needs of rigorous description of models, the "axiomatic" approaches to QFT have the advantage of being more independent from classical analogies, like field equations and Lagrangians. Fields themselves are not an essential input to model local measurements obeying the constraints of Einstein's special relativity and quantum theory.

AQFT can be thought of as being divided into two lines, the first aims to the construction of models (either in low or high dimensions, both starting from physical counterparts or using the theory of operator algebras), the second is devoted to the analysis of the assumptions and of the possibly new mathematical structures arising from them. The work presented in this thesis has been developed and expresses its contribution in the second line of research. Our aim is to introduce new invariants for local quantum field

theories, more specifically to complete a well established construction (the DHR construction) which associates a certain category of representations (collection of superselection sectors together with their fusion rules, exchange symmetry, statistics) to any local quantum field theory, once the latter is formulated as a local net of algebras.

Overview

This thesis is organized as follows.

In Chapter 1 we introduce the main character of our work, namely chiral conformal nets of von Neumann algebras. They are the AQFT counterpart of chiral CFTs studied in the usual physical literature. They are special because they provide rigorous proofs of almost all expected properties of their higher-dimensional analogues (motivated on physical grounds) and because they are sharp enough to realize almost all desirable models, together with their properties, e.g., superselection structure, modular invariant partition functions. We review the most important properties of such chiral conformal nets, their algebraic structure, their dependence on the vacuum sector and how conformal covariance can be generated from algebraic data. We choose to work on the one-dimensional line instead of its conformal compactification, the circle, which is the usual setting adopted in the literature on conformal nets. This choice is motivated by representation-theoretical needs: on the line the representation of the nets can be unambiguously described by endomorphisms of a global object, moreover we have a left/right distinction and a unique definition of the braiding symmetry of such representations, which are the main ingredients of our research. Along the way we give a full detailed proof on the extension procedure from the line to the circle (Lemma 1.2.1) and review the notion of isomorphism of nets which makes the circle and the line pictures equivalent. We also give a proof of a fact concerning split property of local algebras and mutual geometric positions of intervals which is well known to experts, but which we could not find in the literature (Proposition 1.6.12).

In Chapter 2 we introduce category theory as a useful and clear language to deal with representations (superselection sectors) of local quantum theories. We start from the very basics and we review the useful graphical tensor calculus which is intuitive and of essential importance in computations. Priority is given to tensor (monoidal) categories, i.e., to categories with an abstract operation

which mimics the usual tensor product of spaces. We focus on the case of braided (and modular) categories which arise from (low-dimensional) QFTs in the algebraic setting through the DHR construction (to be introduced in Chapter 3). Remarkably these categories and the numerical invariants which can be computed out of them, encode already many of the features of CFTs as they are discovered using point-like quantum fields and imposing commutation relations. (Notice that none of these two ingredients is required in an AQFT local conformal net.) We review modern known results on modular categories, state two interesting open questions about their classification by means of modular data, and elaborate a proof due to K.-H. Rehren about another numerical invariant of modular categories, namely the trace of the self-braiding operators in an arbitrary channel (Proposition 2.7.7).

In Chapter 3 we review the main object studied in this thesis, the DHR category of a (chiral conformal) net, after Doplicher, Haag and Roberts. These type of categories describe irreducible positive energy representations ("particles") of a CFT, their fusion rules, their braiding symmetry and statistics. On one hand they fall into the abstract setting introduced in Chapter 3 of (modular) braided tensor categories, on the other hand they describe superselection sectors and they "explain" numerical relations conjectured in the usual physical literature of CFT. The objects of the DHR category of a local net describe excitations of the vacuum which are localizable in bounded regions (for us, intervals of the line) and can be equivalently treated as endomorphisms (algebra preserving maps) of the net which are sensitive to its local substructure via their characteristic "localizability" properties. In the case of completely rational nets, the DHR categories are, remarkably, examples of modular categories (which are very rigid and difficult to construct in mathematics) in a natural way. We also review the connection of the numerical invariants (modular data), introduced in Chapter 2 for abstract categories, with more concrete analytic properties of subfactors and spectral properties arising from endomorphisms, namely the index-dimension theorem and the spin-statistics theorem.

In Chapter 4, which is the main original part of this work, we focus on the concrete realization of DHR categories as braided categories of endomorphisms. This is motivated on one hand to understand better their structure, e.g., to find new invariants related to QFT, on the other hand to decide in which case two different CFTs can give rise to the same DHR category. A simple observation, namely the trivialization property of the DHR braiding of endomorphisms

which sit space-like far apart (on the line in our case), shows a difference between abstract braided tensor categories and those generated by a QFT. We exploit this difference by studying the action of the DHR endomorphisms on local algebras (Definition 4.2.2) and the realization of the DHR braiding. The latter is a combination of the action itself and suitable commutation relations between unitary operators living inside the local algebras (responsible for transporting localizable charges between different regions of spacetime). It is defined via left/right separation of algebras hence contains information about the local structure of the net, and, remarkably, it obeys the naturality and tensoriality constraints axiomatized in Chapter 2 and dictated by conceptual cleanness and tractability of mathematical objects only. We introduce "duality relations" which put local subalgebras and local subcategories on the same footing, and we show two useful facts concerning relative commutants of local algebras and fixed points of DHR endomorphisms, valid in the completely rational conformal setting (Proposition 4.1.4 and 4.3.7). The second gives a positive answer to a conjecture of S. Doplicher, formulated in the physical high-dimensional setting, in our present case of chiral Rational CFTs.

Our main new tool is the notion of abstract point (Definition 4.5.1), which exploits the characteristic trivialization condition of braiding operators in QFT, and which we use to characterize a suitable subfamily of CFTs, which we call "prime" (Definition 4.7.5), for which the DHR braiding contains sufficiently sharp information about locality in order to uniquely determine the local algebras. On one hand, we give different examples of abstract points which have nothing to do with geometric points of the line (unless one changes the field theory living over its bounded intervals), on the other hand we show a "total ordering" result for abstract points which "belong to the same field theory" (Proposition 4.8.5). The latter result is based on the notion of "prime" modular category, which we adopt as an essential ingredient. On the QFT side this assumption is fulfilled by many examples and excludes nets which are obtained by tensor products of other nets (in this sense they are not prime nets). We also show a way of deciding whether the DHR category and its braided action on the net can be turned into a complete invariant on prime CFTs. Here we must rely on two assumptions which still have to be controlled in models, or sharpened, and on the "continuum" structure of the real line expressed by Dedekind's completeness axiom (Proposition 4.9.1). Chapter 4 is joint work with K.-H. Rehren and is submitted online as [GR15] (to appear in Comm. Math. Phys. (2016)).

Contents

Chapter 1

Conformal nets on the line

Conformal field theories (CFTs) in one spacetime dimension can be described, in the language developed by Haag and Kastler, see [Haa55], [Haa59], [HK64] by means of *nets of local algebras*. They answer to the need of giving to relativistic Quantum Field Theories (QFTs) a mathematically rigorous formulation and to understand it in a model-independent way. Basic features ("first principles") that one would expect from a reasonable QFT are singled out and formulated as simple mathematical statements ("axioms").

As already outlined in the introduction, the *local algebras* prescribe local measurements (local observables) of a QFT to each bounded region of spacetime where the respective measurements can be performed, and then constrain them by imposing locality, spacetime covariance and spectral properties of the energy-momentum operators. See the book of [Haa96] for explanations and deeper physical motivations.

Due to its generality and conceptual cleanness, the algebraic framework provided by AQFT is very flexible and one can inspect very different situations (different spacetimes with different dimensions together with their spacetime symmetries, different classes of states, extensions and restrictions of theories, cosets, gauge fixed points) with similar ideas and instruments. In this work we deal with CFTs in one dimension for several reasons: the related physical literature is very abundant, the mathematical structure is on one side well understood (properties expected to hold in general for QFTs are theorems) and on the other side exotic and unexplored (especially, for us, from a representation-theoretical point of view and concerning the connection between local algebras and their representations), and many nontrivial models have been rigorously constructed hence sit in the intersection of mathematics

and physics. Moreover, several statements in the following sections admit natural generalizations in the direction of higher dimensions or smaller spacetime symmetry groups (richer particle content).

1.1   General structure

In this section we introduce the general (algebraic) structure describing a CFT in one spacetime dimension, i.e., on the real line $\mathbb{R}$. We can think of it as a theory of left/right movers (on a one-dimensional light-ray) of a (1+1)-dimensional CFT. Using standard terminology we will also refer to the one-dimensional case as *chiral* field theory.

Fix for convenience a (complex, separable) Hilbert space $\mathcal{H}$ and consider the family $\mathcal{I}$ of non-empty open *bounded* intervals $I \subset \mathbb{R}$. The first serves to allow quantum behaviours, the second to set up a local field theory. The CFT is then described by assigning to every $I \in \mathcal{I}$ an algebra of operators $\mathcal{A}(I) \subset \mathcal{B}(\mathcal{H})$ and the correspondence

$$I \mapsto \mathcal{A}(I)$$

is a *net of algebras*, because the family $\mathcal{I}$ is partially ordered by inclusion and directed. Here $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$ and $\mathcal{A}(I)$, $I \in \mathcal{I}$, are assumed to be unital *-subalgebras of $\mathcal{B}(\mathcal{H})$ closed in the weak operator topology, i.e., $\mathcal{A}(I) = \mathcal{A}(I)''$ by von Neumann's bicommutant theorem, see [Ped89]. Here $\mathcal{N}'$ denotes the commutant of $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$. We can (but need not) think of them as the von Neumann algebras generated in the vacuum sector by bounded functions of local gauge currents or stress-energy tensor of the CFT, smeared with test functions supported inside I.

More generally, we could have considered $\mathcal{A}(I)$, $I \in \mathcal{I}$, as abstract C^*-algebras, see [Ped79], and regard $\mathcal{H}$ as *some* representation space $\mathcal{H}_\pi$ on which the algebras act via a unital *-homomorphism π. One can consider then different representations $(\pi, \mathcal{H}_\pi)$ of the net, some of which might have a physical interpretation and describe e.g. different superselection sectors. The previous special case corresponds to the choice of the defining representation $\pi_0 = \text{id}$ on $\mathcal{H}_{\pi_0} = \mathcal{H}$, for which we have $\mathcal{A}(I) = \pi_0(\mathcal{A}(I))''$ for all $I \in \mathcal{I}$.

Definition 1.1.1. A net of von Neumann algebras $\{I \in \mathcal{I} \mapsto \mathcal{A}(I)\}$ on $\mathcal{H}$ as above is called a **local net on the line** if it fulfills **isotony** and **locality**, i.e., respectively

- if $I \subset J$, $I, J \in \mathcal{I}$ then $\mathcal{A}(I) \subset \mathcal{A}(J)$,
- if $I \cap J = \emptyset$, $I, J \in \mathcal{I}$ then $\mathcal{A}(I)$ and $\mathcal{A}(J)$ elementwise commute.

We denote the net by $\{\mathcal{A}\}$ and call its elements $\mathcal{A}(I)$, $I \in \mathcal{I}$, **local algebras**.

Remark 1.1.2. In general, the first condition is motivated by coherence of local measurements performed in bigger spacetime regions, the second condition by Einstein's causality principle, i.e., independence of measurements performed in space-like separated regions. In the case of chiral observables on a light-ray notice that space-like separation reduces to disjointness.

So far we have not mentioned neither implementation of spacetime symmetries nor conformal covariance. This is to stress the idea that *all* the physically relevant information about the theory, hence its model realization, is encoded in the collection of local algebras. Thanks to [Wei11], [CW05] this idea is a theorem in chiral CFT, under some natural assumptions, as we shall see in details in Section 1.6.
Furthermore, under the same assumptions one can show that the local algebras $\mathcal{A}(I)$ are, individually, canonical objects, see [BGL93], [Haa87] and Chapter 4 for explanation. Hence they cannot distinguish different models, and the whole information about the CFT is encoded in the inclusions, intersections and commutation relations among different local algebras, i.e., in the local net structure.
Mathematically speaking, this makes the algebraic formulation of chiral CFT rigid enough to be susceptible to classification questions. See e.g. [KL04] for the complete classification of chiral CFTs with Virasoro stress-energy tensor and central charge $c < 1$.

Definition 1.1.3. A net of von Neumann algebras $\{I \in \mathcal{I} \mapsto \mathcal{A}(I)\}$ on $\mathcal{H}$ is called a **local conformal net on the line** if it is local and there is a strongly continuous unitary representation of $\mathsf{M\ddot{o}b} = PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm \mathbb{1}\}$, the **Möbius group** (see Remark 1.1.4) on $\mathcal{H}$ denoted by $g \mapsto U(g)$ such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI) \tag{1.1}$$

whenever $I \in \mathcal{I}$ and $gI \in \mathcal{I}$ (open bounded intervals of $\mathbb{R}$), we ask nothing otherwise. We also assume to have a positive generator H (**conformal Hamiltonian**) of the rotations subgroup of $\mathsf{M\ddot{o}b}$ and a unique, up to scalar multiples, Möbius invariant vector $\Omega \in \mathcal{H}$ (**vacuum vector**) which is cyclic for $\{\mathcal{A}(I), U(g) : I \in \mathcal{I}, g \in \mathsf{M\ddot{o}b}\}''$, cf. [LR04], [FJ96].

We call $\mathcal{H}$ the **vacuum space** and refer to the quadruple $(\{\mathcal{A}\}, U, \Omega, \mathcal{H})$ as a local conformal net on the line (in the vacuum sector).

The conditions of Definition 1.1.3 describe "conformal" transformations of the net, and they can be naturally generalized to diffeomorphism covariance, see Section 1.6 for more comments. Positivity of the conformal Hamiltonian is a one-dimensional version of the physically motivated *spectral condition* imposed on infinitesimal generators of spacetime symmetry groups in higher dimensions.

Moreover, we can invoke general statements on unitary representation theory of the Möbius group (a connected simple Lie group) to get several interesting applications to chiral CFT.

Remark 1.1.4. The Möbius group $\mathsf{M\ddot{o}b} = PSL(2, \mathbb{R})$ acts naturally on $\mathbb{R}$ as fractional linear transformations, namely if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ then

$$g . x := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \frac{ax + b}{cx + d}$$

for every $x \in \mathbb{R}$ or in its "conformal closure" $\overline{\mathbb{R}}$, which is isomorphic to the unit circle $\mathbb{S}^1$. The latter can be seen as the one-point compactification of the real line $\overline{\mathbb{R}} \cong \mathbb{S}^1$ via the *Cayley map*, a suitably parametrized stereographic projection, namely $x \in \mathbb{R} \mapsto (x + i)(x - i)^{-1} \in \mathbb{S}^1 \smallsetminus \{1\}$ with inverse $z \in \mathbb{S}^1 \smallsetminus \{1\} \mapsto i(z + 1)(z - 1)^{-1} \in \mathbb{R}$ mapping 1 to ∞. The Möbius group consist of three types of transformations: dilations, translations and rotations, in the sense that every $g \in \mathsf{M\ddot{o}b}$ can be uniquely decomposed (Iawasawa decomposition) as a product of such transformations, [Lon08a, Prop. 1.2.1]. The first two are geometrically better understood on the line, the third on the circle. It is also useful to keep in mind the following characterization: dilations, translations and rotations are those elements of $\mathsf{M\ddot{o}b}$ respectively with two, one or no fixed points. Moreover $\mathsf{M\ddot{o}b}$ acts transitively on (single, pairs of, triples of) points, and faithfully on ordered triples of those. We refer to [GF93, App. I] and [Lon08a, Sec. 1.1] for a more detailed treatment of these facts.

Concerning the positive energy condition we have

Proposition 1.1.5. [GL96], [Lon08a]. *Let $g \mapsto U(g)$ be a nontrivial strongly continuous unitary representation of $\mathsf{M\ddot{o}b}$ on Hilbert space $\mathcal{H}$. Then positivity*

of the generator of rotations H is equivalent to positivity of the generator of translations. In this case the spectrum of the latter coincides with $[0, +\infty)$.

Remark 1.1.6. Notice that the generator of dilations in a nontrivial representation has always spectrum equal to $\mathbb{R}$, while the conformal Hamiltonian has discrete spectrum because the rotations subgroup is topologically isomorphic to the circle group, hence compact.

Concerning Möbius invariance and uniqueness of the vacuum vector we have

Proposition 1.1.7. [GL96]. *Let $g \mapsto U(g)$ be a strongly continuous unitary representation of* Möb *on Hilbert space $\mathcal{H}$ and let $\Omega \in \mathcal{H}$. The following are equivalent*

- *$\mathbb{C}\Omega$ are the only Möbius invariant vectors.*
- *$\mathbb{C}\Omega$ are the only translation invariant vectors.*
- *$\mathbb{C}\Omega$ are the only dilation invariant vectors.*

If in addition the rotations have positive generator H, then the previous are also equivalent to

- *$\mathbb{C}\Omega$ are the only rotation invariant vectors.*

Moreover, uniqueness of the vacuum vector in chiral CFT turns out to be equivalent to algebraic conditions on $\{\mathcal{A}\}$. We refer to those algebraic conditions as **irreducibility** of the net.

Proposition 1.1.8. [GL96], [Lon08b]. *Let $\{\mathcal{A}\}$ be a local conformal net on the line as in Definition 1.1.3 and drop uniqueness of the vacuum vector $\Omega \in \mathcal{H}$. Then the following are equivalent*

- *$\mathbb{C}\Omega$ are the only Möbius invariant vectors.*
- *The local algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$ are factors. In this case, they are either type III_1 factors or trivial, i.e., $\mathcal{A}(I) = \mathbb{C}$, for all $I \in \mathcal{I}$. In the second case we also have $\mathcal{H} = \mathbb{C}$.*
- *Let $\{I_\alpha\} \subset \mathcal{I}$ be a family of intervals intersecting at a point $\{p\} = \bigcap_\alpha I_\alpha$, then $\bigcap_\alpha \mathcal{A}(I_\alpha) = \mathbb{C}$.*
- *Let $\{I_\alpha\} \subset \mathcal{I}$ be a family of intervals covering $\mathbb{R} = \bigcup_\alpha I_\alpha$, then $\bigvee_\alpha \mathcal{A}(I_\alpha) = \mathcal{B}(\mathcal{H})$, where $\bigvee$ denotes the von Neumann algebra generated in $\mathcal{B}(\mathcal{H})$.*

Proof. The original proof of this statement [GL96, Prop. 1.2] is given for nets on the circle $\mathbb{S}^1$ (see later) and covariant under the universal covering group $\widetilde{\mathsf{Möb}}$ of the Möbius group. The latter can be replaced by $\mathsf{Möb}$ because by the vacuum spin-statistics theorem [GL95] the rotation of angle 2π is always represented by $\mathbb{1}$, see also [GL96, Prop. 1.1 (d)], [Lon08b, Prop. 6.2.9]. We only observe that $\mathbb{S}^1$ can be replaced by $\mathbb{R}$ (punctured circle) because the Reeh-Schlieder theorem (see below) holds under the assumptions 1.1.3, see [FJ96], and by Möbius covariance the choice of the point at infinity is irrelevant. $\qquad\qquad\square$

For the physical interpretation of the (very special) type of local algebras see [Yng05], we just mention that the factor property is a nontrivial issue in algebraic QFT, see [Bor00, Sec. 5.2], and that its proof (together with the type) in the conformal case is due to [Lon79], [Dri77]. Moreover, the previous alternative $\mathbb{C}$ or type *III* factor is mathematically rather natural, cf. [Kaw15, Sec. 2.2]. A von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called a *factor* if it has trivial center $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}$. It is of type *III* if every orthogonal projection e in $\mathcal{M}$, i.e., $e = e^*e \in \mathcal{M}$, is either 0 or equivalent to $\mathbb{1}$ in $\mathcal{M}$ (in the sense of Murray and von Neumann). If the last case is only trivially satisfied (by the identity projection $\mathbb{1}$) we obtain the trivial factor $\mathcal{M} = \mathbb{C}$, which is better regarded as a trivial type *I* factor (1×1 matrix algebra), hence we have to exclude it in the definition of type *III*.

The observation that local algebras associated to intervals intersecting at a point must have trivial intersection (scalar operators) was first made in [Wig64]. It says that the concept of *bounded* point-like quantum field is not consistent with the other assumptions we made to describe a QFT.

The previous proposition relies on the following technical result, the Reeh-Schlieder theorem, which is of paramount importance in any theory of local observables. The proof in our context is due to [FJ96].

Theorem 1.1.9. (**Reeh-Schlieder**). *Let $\{\mathcal{A}\}$ be a local conformal net on the line as in Definition 1.1.3, then the vacuum vector Ω is cyclic and separating for each local algebra $\mathcal{A}(I)$, $I \in \mathcal{I}$.*

This means that, for every interval $I \in \mathcal{I}$, the linear sub-space $\mathcal{A}(I)\Omega$ is norm dense in $\mathcal{H}$ (cyclic) and that $a \in \mathcal{A}(I)$, $a\Omega = 0$ implies $a = 0$ (separating). Moreover, it is easy to see that a vector is cyclic (resp. separating) for a

von Neumann algebra if and only if it is separating (resp. cyclic) for its commutant.

The previous proposition can be interpreted by saying that it is in principle possible to approximate arbitrarily well every state $\Psi \in \mathcal{H}$ by acting on the vacuum with local operators. Macroscopically speaking, we can (almost) construct cathedrals on the moon by working in our particle physics laboratory for a finite amount of time. To make this statement more acceptable, we specify that Reeh-Schlieder says nothing about the amount of energy (spectral properties) we should spend in the process.

1.2 Extension to the circle

Before seeing the consequences of the Reeh-Schlieder theorem 1.1.9, we show how local conformal nets on the line as in Definition 1.1.3 extend uniquely to the unit circle $\mathbb{S}^1$. Despite this fact is well known, cf. [BGL93], [FJ96], [LR04], we give here an argument in order to stress the different geometric situation and to make explicit contact with the literature on conformal nets, which is more abundant in the latter setting, see e.g. [BMT88], [BSM90], [FRS92], [GF93], [GL96], [Was98], [GLW98], [Xu00b], [KL04], [Car04], [LX04], [CCHW13].

Lemma 1.2.1. *Let $\{\mathcal{A}\}$ be a local conformal net on the line as in Definition 1.1.3, then it extends uniquely to a local conformal net on the circle $\{\tilde{\mathcal{A}}\}$ and the correspondence is one-to-one up to unitary equivalence.*

Proof. Let $\tilde{\mathcal{I}}$ be the family of open non-empty non-dense intervals $J \subset \mathbb{S}^1$ and observe that $\mathcal{I} \subset \tilde{\mathcal{I}}$ via Cayley map. The Möbius group acts naturally on the circle and transitively on $\tilde{\mathcal{I}}$, moreover every $J \in \tilde{\mathcal{I}}$ can be written as $J = gI$ for some $I \in \mathcal{I}$, $g \in \mathsf{M\ddot{o}b}$ thanks to rotations. Hence by setting $\tilde{\mathcal{A}}(gI) := U(g)\mathcal{A}(I)U(g)^*$ for arbitrary $I \in \mathcal{I}$, $g \in \mathsf{M\ddot{o}b}$ we obtain a net (or better a precosheaf) of von Neumann algebras indexed by $\tilde{\mathcal{I}}$. The definition is well posed by covariance assumption on bounded intervals of $\mathbb{R}$ and the extended net is manifestly Möbius covariant, i.e., $U(g)\tilde{\mathcal{A}}(J)U(g)^* = \tilde{\mathcal{A}}(gJ)$ for all $g \in \mathsf{M\ddot{o}b}$, $J \in \tilde{\mathcal{I}}$. Moreover $\{\tilde{\mathcal{A}}\}$ extends $\{\mathcal{A}\}$ in the sense that $\tilde{\mathcal{A}}(J) = \mathcal{A}(J)$ whenever $J \in \mathcal{I}$. Also vacuum space, spectral conditions and vacuum vector are left untouched. The extended net is trivially isotonous, while showing locality requires some work. Any pair of intervals $I, J \in \tilde{\mathcal{I}}$ such that $I \cap J = \emptyset$ and $J \neq I'$, up to a suitable rotation, does not contain

the infinity point $\mathbb{S}^1 \smallsetminus \mathbb{R} = \{\infty\}$, where we denoted $I' := \mathbb{S}^1 \smallsetminus \overline{I} \in \tilde{\mathcal{I}}$. In this case I and J can be seen, after rotation, as bounded disjoint intervals of $\mathbb{R}$ and $\tilde{\mathcal{A}}(I)$ commutes with $\tilde{\mathcal{A}}(J)$ by locality of $\{\mathcal{A}\}$ and conformal covariance. The case $J = I'$ which corresponds to considering two half-lines on $\mathbb{R}$ with opposite orientation and common origin is more subtle.

Assuming additivity on $\{\mathcal{A}\}$ (see Proposition 1.4.3) would allow us to conclude, see [BGL93, Prop. 1.10]. But given that we want to derive additivity out of Definition 1.1.3, we first have to cover a small but instructive gap in the proof of the Reeh-Schlieder theorem contained in [FJ96, Sec. 3]. Namely, we can assume $I = \mathbb{R}_+$ and follow the argument of [FJ96, Sec. 3] until it shows that the vacuum vector Ω is cyclic for $\tilde{\mathcal{A}}(I)$. Then take $K \in \tilde{\mathcal{I}}$ such that $K \cap I = \emptyset$ and $K \neq I'$, hence $\tilde{\mathcal{A}}(K) \subset \tilde{\mathcal{A}}(I)'$ by the above argument and Ω is separating for $\tilde{\mathcal{A}}(K)$. Now, reparametrizing the geometric transformations involved in the proof of cyclicity for $\mathbb{R}_+$ we can conclude that Ω is also cyclic for $\tilde{\mathcal{A}}(K)$ for all $K \in \tilde{\mathcal{I}}$.

In order to show additivity we can observe that the modular group of $(\tilde{\mathcal{A}}(I), \Omega)$, see [BR87, Sec. 2.5], acts as the reparametrized dilations with respect to I (modular covariance, see later) irrespectively of locality assumptions on the net, see [DLR01, Thm. 2.1 (ii)]. Hence we can follow again the arguments of [FJ96, Sec. 3] and get additivity, i.e., $\tilde{\mathcal{A}}(I) = \bigvee_{J \Subset I} \tilde{\mathcal{A}}(J)$ where $I \in \tilde{\mathcal{I}}$ and all $J \in \tilde{\mathcal{I}}$, from which $\tilde{\mathcal{A}}(I) \subset \tilde{\mathcal{A}}(I')'$ follows easily by the same argument as above, hence we have locality of $\{\tilde{\mathcal{A}}\}$.

Alternatively, the relation $\tilde{\mathcal{A}}(I) \subset \tilde{\mathcal{A}}(I')'$ can be shown directly by taking an *increasing* sequence of proper intervals I_n which covers I. Consider the family of $g \in \text{Möb}$ such that $\overline{g(I)} \subset I$, hence each $g(I) \subset I_n$ for some n and for every $a \in \tilde{\mathcal{A}}(I)$ we have $U(g)aU(g)^* \in \bigvee_n \tilde{\mathcal{A}}(I_n)$. By strong continuity of U, if we let g go to the identity inside the previous family, we get $a \in \bigvee_n \tilde{\mathcal{A}}(I_n)$ and obtain $\tilde{\mathcal{A}}(I) = \bigvee_n \tilde{\mathcal{A}}(I_n)$. This weaker form of additivity, cf. [Lon08a, Cor. 3.3.3], is sufficient to conclude locality of $\{\tilde{\mathcal{A}}\}$ as before. We thank Y. Tanimoto for explaining us this second more natural argument.

The uniqueness of the construction $\{\mathcal{A}\} \mapsto \{\tilde{\mathcal{A}}\}$ is ensured by the observation that two local conformal nets on the circle coincide if they share one algebra, the unitary representation of Möb and the vacuum vector.

Vice versa, let $\{\tilde{\mathcal{A}}\}$ be a local conformal net on the circle, choose an infinity point, cut the circle and restrict the net to bounded intervals of $\mathcal{I}$. The restricted net depends on the choice of the point only up to a rotation, which is implemented by a vacuum preserving unitary, hence we have the

statement.   □

Despite the previous correspondence, we take the real line setting as more natural from the point of view of DHR representation theory, action of DHR endomorphisms and definition of DHR braiding which involves a left/right choice, see Chapter 3 and 4, and cf. [KLM01]. If not specified, we always consider local conformal nets on the line.

Anyway the main part of this work is concerned with the action of the *local* DHR categories on *local* algebras, for which the distinction between real line picture or circle picture is irrelevant.

1.3   Modular theory and Bisognano-Wichmann property

Let $\{\mathcal{A}\}$ be a local conformal net on the line with $\Omega \in \mathcal{H}$ the vacuum vector and fix some interval $I \in \mathcal{I}$. Thanks to the Reeh-Schlieder theorem one can compute the modular objects $(\Delta_{I,\Omega}, J_{I,\Omega})$ associated to the pair $(\mathcal{A}(I), \Omega)$, or better to the vacuum state $\omega := (\Omega| \cdot \Omega)$ restricted to $\mathcal{A}(I)$, see [BR87, Sec. 2.5], [Str81], [BDNR76].

In general a pair $(\mathcal{M}, \Omega)$ where $\mathcal{M}$ is a von Neumann algebra in $\mathcal{B}(\mathcal{H})$ and $\Omega \in \mathcal{H}$ a cyclic and separating vector for $\mathcal{M}$ is called a von Neumann algebra in *standard form*, after [Haa75]. There always exist such *standard vectors* Ω for $\mathcal{M}$, e.g., whenever $\mathcal{H}$ is separable. One can consider then the maps $S : a\Omega \mapsto a^*\Omega$, $a \in \mathcal{M}$, which is a densely defined (typically unbounded) antilinear operator on $\mathcal{H}$, and similarly $S' : b\Omega \mapsto b^*\Omega$, $b \in \mathcal{M}'$, replacing $\mathcal{M}$ with $\mathcal{M}'$. Both these operators turn out to be closable and their polar decompositions define uniquely the pair $(\Delta_{\mathcal{M},\Omega}, J_{\mathcal{M},\Omega})$ where $\Delta_{\mathcal{M},\Omega}$ is the modulus squared (positive self-adjoint) of S and $J_{\mathcal{M},\Omega}$ is its phase (antiunitary). Similarly one gets another pair $(\Delta_{\mathcal{M}',\Omega}, J_{\mathcal{M}',\Omega})$ for S' and it turns out that the two pairs are related as follows, see [BR87, Prop. 2.5.11],

$$J_{\mathcal{M},\Omega} = J_{\mathcal{M}',\Omega}, \quad \Delta_{\mathcal{M}',\Omega} = \Delta_{\mathcal{M},\Omega}^{-1}$$

and

$$J_{\mathcal{M},\Omega}^2 = \mathbb{1}, \quad J_{\mathcal{M},\Omega}\Delta_{\mathcal{M},\Omega}^{1/2}J_{\mathcal{M},\Omega} = \Delta_{\mathcal{M},\Omega}^{-1/2}.$$

The fundamental result of Tomita and Takesaki [BR87, Thm. 2.5.14] shows that the adjoint action of $J := J_{\mathcal{M},\Omega}$ (*modular conjugation*) turns $\mathcal{M}$ onto

$\mathcal{M}'$ and vice versa, while $\Delta := \Delta_{\mathcal{M},\Omega}$ (*modular operator*) gives rise to a group of automorphisms of $\mathcal{M}$ (*modular group*).

Theorem 1.3.1. (**Tomita-Takesaki**). *Let* $(\mathcal{M},\Omega)$ *a von Neumann algebra in standard form, in the previous notation it holds*

$$J\mathcal{M}J = \mathcal{M}', \quad \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$$

for all $t \in \mathbb{R}$, *where* $t \mapsto \Delta^{it}$ *is the strongly continuous one-parameter group of unitaries generated by* $\ln(\Delta)$ *and* $\sigma_t^{\mathcal{M},\Omega}(\cdot) := \Delta^{it}\cdot\Delta^{-it}$, $t \in \mathbb{R}$ *is the modular group of* $(\mathcal{M},\Omega)$.

Remarkably, in the case of a local algebra together with the vacuum state of a CFT, the above relations turn out to have a geometrical explanation. Namely, we have the following theorem due to [BGL93], [GF93]

Theorem 1.3.2. (**Bisognano-Wichmann property**). *Let* $\{\mathcal{A}\}$ *be a local conformal net on the line, let* $\{\tilde{\mathcal{A}}\}$ *be its extension to the circle as in Lemma 1.2.1 and consider for simplicity the positive half-line* $\mathbb{R}_+$ *and the local algebra* $\tilde{\mathcal{A}}(\mathbb{R}_+)$ *of the extended net. Then the modular objects* $(\Delta_{\mathbb{R}_+,\Omega}, J_{\mathbb{R}_+,\Omega})$ *associated to* $\tilde{\mathcal{A}}(\mathbb{R}_+)$ *and the vacuum vector* Ω *are given by*

$$\Delta_{\mathbb{R}_+,\Omega}^{it} = U(\Lambda_{\mathbb{R}_+}(-2\pi t))$$

for all $t \in \mathbb{R}$, *and*

$$J_{\mathbb{R}_+,\Omega} = U(r_{\mathbb{R}_+})$$

where $s \mapsto \Lambda_{\mathbb{R}_+}(s)$ *denotes the one-parameter dilations subgroup of* Möb *which preservers* $\mathbb{R}_+$, *namely* $\Lambda_{\mathbb{R}_+}(s) : t \mapsto e^s t$, *and* $r_{\mathbb{R}_+}$ *is the reflection with respect to the origin of* $\mathbb{R}$ *(and to* ∞ *on* $\mathbb{S}^1$*) mapping* $\mathbb{R}_+$ *onto* $\mathbb{R}_-$ *and vice versa, namely* $r_{\mathbb{R}_+} : t \mapsto -t$.

Remark 1.3.3. To be precise the reflection $r_{\mathbb{R}_+}$ does not belong to Möb $= PSL(2,\mathbb{R})$ (it is orientation reversing) so the previous statement is concerned with an extension of the representation U. Consider the group generated by Möb and $r_{\mathbb{R}_+}$ and call it the *improper Möbius group*, denoted by Möb$_2$. The adjoint action of $r_{\mathbb{R}_+}$ defines a group automorphism of Möb, moreover $r_{\mathbb{R}_+}^2 = \mathbb{1}$, hence Möb$_2$ can be written as a semidirect product Möb$_2 \cong$ Möb $\rtimes \mathbb{Z}_2$. In matrix form one also sees that Möb$_2 \cong PSL(2,\mathbb{R})_\pm$ where $+$ stands for determinant equal to 1 (orientation preserving) and $-$ stands for determinant equal to -1 (orientation reversing). Möb sits in Möb$_2$ as a normal subgroup of order two, see [GF93, App. I].

The representation U in Theorem 1.3.2 is indeed an *(anti)unitary* extension to $\mathsf{M\ddot{o}b}_2$ of the original unitary representation given by Möbius covariance. Namely it is a strongly continuous representation $g \mapsto U(g)$ of $\mathsf{M\ddot{o}b}_2$ on $\mathcal{H}$ where $U(g)$ is unitary if g is orientation preserving and antiunitary if g is orientation reversing. The correspondence between representations of $\mathsf{M\ddot{o}b}$ and representation of $\mathsf{M\ddot{o}b}_2$ is clarified by the following

Proposition 1.3.4. [Lon08a, Thm. 1.6.3]. *Every (strongly continuous) unitary positive energy representation of* $\mathsf{M\ddot{o}b}$ *on Hilbert space* $\mathcal{H}$ *extends to an (anti)unitary representation of* $\mathsf{M\ddot{o}b}_2$ *on the same space* $\mathcal{H}$ *and every (anti)unitary representation of* $\mathsf{M\ddot{o}b}_2$ *arises in this way. Moreover, the correspondence preserves the unitary equivalence classes of representations.*

In the case of Bisognano-Wichmann the extended representation is given by $U(r_{\mathbb{R}_+}) := J_{\mathbb{R}_+,\Omega}$ because the relations $J_{\mathbb{R}_+,\Omega}U(g)J_{\mathbb{R}_+,\Omega} = U(r_{\mathbb{R}_+}gr_{\mathbb{R}_+})$ hold for all $g \in \mathsf{M\ddot{o}b}$. Moreover, the net $\{\tilde{\mathcal{A}}\}$ transforms covariantly under the extended representation of $\mathsf{M\ddot{o}b}_2$ because $J_{\mathbb{R}_+,\Omega}\tilde{\mathcal{A}}(I)J_{\mathbb{R}_+,\Omega} = \tilde{\mathcal{A}}(r_{\mathbb{R}_+}(I))$ can be shown for all $I \in \tilde{\mathcal{I}}$, see [GF93, Thm. 2.19 (i),(ii)], [GL96, Prop. 1.1 (b)].

Remark 1.3.5. Thanks to conformal covariance, it is easy to see that the modular objects associated to $(\tilde{\mathcal{A}}(I), \Omega)$ for each $I \in \tilde{\mathcal{I}}$ have a similar geometrical origin. Namely, each $\tilde{\mathcal{A}}(I) = U(g)\tilde{\mathcal{A}}(\mathbb{R}_+)U(g)^*$ for some $g \in \mathsf{M\ddot{o}b}$ hence the operator $a \in \Omega \mapsto a^*\Omega$, $a \in \tilde{\mathcal{A}}(I)$ reads $U(g)b\Omega \mapsto U(g)b^*\Omega$, $b \in \tilde{\mathcal{A}}(\mathbb{R}_+)$ which is the adjoint action of $U(g)$ on the operator $b\Omega \mapsto b^*\Omega$, $b \in \tilde{\mathcal{A}}(\mathbb{R}_+)$. Then by uniqueness of the polar decomposition we have

$$J_{I,\Omega} = U(g)J_{\mathbb{R}_+,\Omega}U(g)^*, \quad \Delta_{I,\Omega} = U(g)\Delta_{\mathbb{R}_+,\Omega}U(g)^*.$$

Thanks to the last remark and observing that $\mathsf{M\ddot{o}b}$ is generated by dilations with respect to arbitrary intervals of the circle we have

Corollary 1.3.6. [BGL93], [GF93]. *Let* $\{\mathcal{A}\}$ *be a local conformal net on the line, let* $\{\tilde{\mathcal{A}}\}$ *be its extension to the circle as in Lemma 1.2.1. Then the representation* U *of* $\mathsf{M\ddot{o}b}$ *which defines covariance is completely determined by the modular structure of the (extended) net in the vacuum sector, that is by local algebras* $\tilde{\mathcal{A}}(I)$, $I \in \tilde{\mathcal{I}}$ *and the vacuum vector* Ω, *through the formula*

$$\Delta_{I,\Omega}^{it} = U(\Lambda_I(-2\pi t))$$

for all $t \in \mathbb{R}$ *and arbitrary* $I \in \tilde{\mathcal{I}}$.

We denoted by $s \mapsto \Lambda_I(s)$ the one parameter dilations subgroup of Möb which preserves the interval I, i.e., fixes its boundary points. It is given, e.g., by $\Lambda_I = g\Lambda_{\mathbb{R}_+}g^{-1}$ where $g \in$ Möb is such that $I = g(\mathbb{R}_+)$. Notice that the definition does not depend on the choice of g because the only Möbius transformations k such that $k(\mathbb{R}_+) \subset \mathbb{R}_+$ and $k^{-1}(\mathbb{R}_+) \subset \mathbb{R}_+$ are the dilations with respect to $\mathbb{R}_+$ and they form an Abelian subgroup. Similarly for the reflection with respect to I one can define $r_I = gr_{\mathbb{R}_+}g^{-1}$. This is again independent of the choice of g such that $I = g(\mathbb{R}_+)$ because reflections and dilations with respect to $\mathbb{R}_+$ mutually commute, as can be easily seen from their matrix representation.

We conclude with a few remarks. First the Bisognano-Wichmann property owes its name to [BW75] who proved it in the Wightman setting for four-dimensional QFTs on wedge regions. It has been also generalized to double cones in the free massless case by [HL82] thanks to conformal covariance. In the algebraic setting it holds for general conformal nets [BGL93] and for well behaved massive theories [Mun01]. Second the proof of Bisognano-Wichmann in the algebraic setting relies on a deep result of Borchers [Bor92] which shows that the commutation relations between dilations (or Lorentz boosts) and translations are somehow intrinsic in the Tomita-Takesaki modular theory. Third the property of *modular covariance*, namely that the modular group $\sigma_t^{I,\Omega}$ implements the dilation automorphisms if $\mathcal{A}(I)$ is possibly weaker than the sharp Bisognano-Wichmann equality of the one-parameter unitary groups e.g. in a priori non-local theories, see [DLR01, Thm. 2.1, Prop. 2.3].
On the other hand, in four-dimensional theories, the modular conjugation is the antiunitary representation of the CPT operator up to a rotation of π.

1.4   Essential duality, additivity and modular extension to the circle

In this section we stress once more the real line picture for chiral CFT as more natural for the purposes of this work. In the end we will see another procedure to extend nets of algebras from the line to the circle, starting from more natural real line covariance assumptions but making use of modular theory. Now, thanks to the Bisognano-Wichmann property many useful properties of local algebras can be shown. The following property is usually

referred to as "Haag duality", but we prefer to specify "Haag duality on $\mathbb{S}^1$" for local conformal nets, which is in turn **essential duality** on $\mathbb{R}$ by conformal covariance [1]. Namely

Proposition 1.4.1. [BGL93], [GF93]. *Let $\{\mathcal{A}\}$ be a local conformal net on the line, let $\{\tilde{\mathcal{A}}\}$ be its extension to the circle as in Lemma 1.2.1 and consider the positive and negative half-lines $\mathbb{R}_+$ and $\mathbb{R}_-$. Then the associated local algebras of the extended net are each other's commutant in $\mathcal{B}(\mathcal{H})$, i.e.,*

$$\tilde{\mathcal{A}}(\mathbb{R}_+)' = \tilde{\mathcal{A}}(\mathbb{R}_-).$$

Remark 1.4.2. By conformal covariance it follows that $\tilde{\mathcal{A}}(I)' = \tilde{\mathcal{A}}(I')$ for all $I \in \tilde{\mathcal{I}}$ where both I and $I' := \mathbb{S}^1 \smallsetminus \overline{I}$ are by definition open non-empty non-dense intervals of $\mathbb{S}^1$. Translated on the line, i.e., when $I \in \mathcal{I}$, this does *not* mean however that $\mathcal{A}(I)' = \mathcal{A}(I')$, not even in Haag dual theories (see next section) where taking commutants in $\mathcal{B}(\mathcal{H})$ one gets $\mathcal{A}(I)'' = \mathcal{A}(I')' = \mathcal{A}(I)$ for each *bounded* interval I (i.e. not containing infinity in its closure). For the sake of DHR representation theory (see later) the algebra $\mathcal{A}(I')$ associated to $I' := \mathbb{R} \smallsetminus \overline{I}$, which still has to be defined on the line, will be considered as a *-algebra or equivalently C^*-algebra but not necessarily weakly closed. The inclusion $\mathcal{A}(I') \subset \mathcal{A}(I)'$, which is in many cases proper, contains a great part of the information on the DHR superselection structure of the chiral theory and can be regarded as a *real line version* of the *two-interval subfactor* in the terminology of [KLM01]. In chiral CFT it is also determined by the DHR superselection category itself, see Corollary 4.3.9, at least in the completely rational case (see Chapter 4).

Modular covariance also implies the following continuity property of local algebras called **additivity**, see e.g. [Bis12a, Cor. 3.2.4]. Namely

Proposition 1.4.3. [FJ96]. *Let $\{\mathcal{A}\}$ be a local conformal net on the line, then for every interval $I \in \mathcal{I}$ it holds*

$$\mathcal{A}(I) = \bigvee_{K \Subset I} \mathcal{A}(K)$$

where all $K \in \mathcal{I}$, and $K \Subset I$ means $\overline{K} \subset I$. The symbol $\bigvee$ denotes the von Neumann algebra generated in $\mathcal{B}(\mathcal{H})$.

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[1]by "Haag duality" we will always mean "Haag duality on $\mathbb{R}$" which is in general a condition stronger than essential duality on $\mathbb{R}$.

Remark 1.4.4. By additivity the boundary points of the interval I are irrelevant at the level of associated local algebras (of any chiral CFT), i.e., $\mathcal{A}(I) = \mathcal{A}(\overline{I})$ for any reasonable definition of $\mathcal{A}(\overline{I})$.

Consider now the subgroup of Möbius transformations which preserve the real line, namely those which preserve the infinity point. Is is easy to see that it is generated by dilations with respect to $\mathbb{R}_+$ (hence to $\mathbb{R}_-$) and translations with respect to $\mathbb{R}$. Call it the **translation-dilation group** and denote it by $\mathsf{Möb}_{\mathbb{R}}$. The next proposition shows that $\mathsf{Möb}$ covariance on the circle (conformal closure of the line) is equivalent to $\mathsf{Möb}_{\mathbb{R}}$ covariance on the line, provided the Bisognano-Wichmann property holds on half-lines.

Proposition 1.4.5. [GLW98]. *Let $\{\mathcal{A}\}$ be a local net on the line as in Definition 1.1.1, assume it is translation-dilation covariant via a strongly continuous unitary representation $g \mapsto U(g)$ of $\mathsf{Möb}_{\mathbb{R}}$ on $\mathcal{H}$ and assume there is a unique, up to scalar multiples, translation-dilation invariant vector $\Omega \in \mathcal{H}$ which is cyclic and separating for each $\mathcal{A}(I)$, $I \in \mathcal{I}$. Then the following are equivalent*

- *$\{\mathcal{A}\}$ extends to a local conformal net on the circle.*

- *The Bisognano-Wichmann property holds on half-lines, namely*

$$\Delta_{\mathbb{R}_+,\Omega}^{it} = U(\Lambda_{\mathbb{R}_+}(-2\pi t))$$

  for all $t \in \mathbb{R}$, where $\Delta_{\mathbb{R}_+,\Omega}$ is the modular operator associated to the von Neumann algebra $\bigvee_{I \in \mathbb{R}_+} \mathcal{A}(I)$ and to the vacuum vector Ω.

As remarked in [Lon08b, p. 96] positivity of the generator of translations is not assumed in the second point of the previous proposition. Indeed, it is a consequence of the Wiesbrock theorem [Wie93], [AZ05]. We also thank R. Longo for pointing out the result [GLW98, Thm. 1.4] reviewed in the previous proposition.

1.5 Modular reconstruction of nets

In the previous sections we have seen how modular theory enters in chiral CFT and allows to show that the implementation of covariance is intrinsic in the local algebras (local net structure) and the vacuum vector. In this

section we go further, we see how the net itself can be reconstructed from "modular prescriptions", i.e., from two or three local algebras only and the vacuum vector. Those modular prescriptions, which can be made abstract, correspond to particular "geometric positions" of the local algebras in the vacuum sector. All this section is due to the work of Borchers [Bor92] and to the intuitions of [Wie92], [Wie93], [GLW98]. We first introduce some terminology.

Let $\mathcal{M}, \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be two von Neumann factors, then a *subfactor* is an inclusion $\mathcal{N} \subset \mathcal{M}$. Assume there is a vector $\Omega \in \mathcal{H}$ cyclic and separating for $\mathcal{M}$ and $\mathcal{N}$, then in particular $(\mathcal{M}, \Omega)$ is in standard form and we can compute the associated modular group $\sigma_t^{\mathcal{M},\Omega}$, $t \in \mathbb{R}$. The pair $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is called a *+half-sided modular subfactor*, for short +hsm subfactor, if the modular group of $\mathcal{M}$ compresses $\mathcal{N}$ for positive parameter, in symbols if $\sigma_t^{\mathcal{M},\Omega}(\mathcal{N}) \subset \mathcal{N}$ for $t \geq 0$. Similar definition for $-$hsm subfactor if the same holds for $t \leq 0$. Let $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 \subset \mathcal{B}(\mathcal{H})$ be three pairwise commuting factors and $\Omega \in \mathcal{H}$ a cyclic and separating vector for all of them. The quadruple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \Omega)$ is called a *+half-sided modular factorization* if each $(\mathcal{N}_i \subset \mathcal{N}_{i+1}', \Omega)$ for all $i \in \mathbb{Z}_3$ is a +hsm subfactor. Similar definition for $-$hsm factorization. Notice that we are stressing the factor case because we want uniqueness of the vacuum vector, i.e., irreducibility of the net.

A +hsm factorization is the abstract modular-theoretical version of three local algebras (of a local conformal net in the vacuum sector) associated to the three intervals arising as connected components of $\mathbb{S}^1 \setminus \{p_1, p_2, p_3\}$ where the p_i's are distinct points of $\mathbb{S}^1$, or equivalently by conformal covariance from $\mathbb{R} \setminus \{p_1, p_2, \infty\}$ where the p_i's are distinct points of $\mathbb{R}$ [2]. Namely

Proposition 1.5.1. [GLW98]. *Let $\{\mathcal{A}\}$ be a local conformal net on the line, let $\{\tilde{\mathcal{A}}\}$ be its extension to the circle as in Lemma 1.2.1. Consider three intervals $I_1, I_2, I_3 \in \tilde{\mathcal{I}}$ forming a partition of $\mathbb{S}^1$ in counter-clockwise order and the vacuum vector Ω, then $(\tilde{\mathcal{A}}(I_1), \tilde{\mathcal{A}}(I_2), \tilde{\mathcal{A}}(I_3), \Omega)$ is a +hsm factorization. Vice versa, every +hsm factorization $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \Omega)$ in $\mathcal{B}(\mathcal{H})$ arises in this way, i.e., there is a (unique) local conformal net on the circle such that $\tilde{\mathcal{A}}(I_i) = \mathcal{N}_i$ for all $i \in \mathbb{Z}_3$ and having Ω as the vacuum vector. The positive energy unitary implementation of Möbius covariance is determined by the formula $\Delta_{I_i,\Omega}^{it} = U(\Lambda_{I_i}(-2\pi t))$ for all $t \in \mathbb{R}$ and $i \in \mathbb{Z}_3$.*

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[2]notice that Möb acts transitively on ordered triples of points of $\mathbb{S}^1$.

In the previous proposition the information about locality is contained in the mutual position of the three algebras. So let $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \Omega)$ be a +hsm factorization in $\mathcal{B}(\mathcal{H})$ and observe that e.g. $\mathcal{N}_1 \vee \mathcal{N}_2 \subset \mathcal{N}_3'$. We can strengthen this condition to $\mathcal{N}_1 \vee \mathcal{N}_2 = \mathcal{N}_3'$ and get a class of +hsm factorizations which correspond to a particular class of local conformal nets, those for which locality assumption is maximally satisfied, namely nets for which Haag duality holds.

Definition 1.5.2. Let $\{\mathcal{A}\}$ be a local net on the line and take a (bounded) interval $I \in \mathcal{I}$. Denote by $I' := \mathbb{R} \smallsetminus \overline{I}$ and define $\mathcal{A}(I')$ as the C^*-algebra generated by all local algebras $\mathcal{A}(J)$, $J \in \mathcal{I}$ such that $J \subset I'$. Then $\{\mathcal{A}\}$ is said to fulfill **Haag duality** on $\mathbb{R}$ if $\mathcal{A}(I')' = \mathcal{A}(I)$ holds for all $I \in \mathcal{I}$.

Remark 1.5.3. Notice that for the sake of defining Haag dual nets on the line we could have taken the set theoretic union of the $\mathcal{A}(J)$'s which is a unital *-closed subset of $\mathcal{B}(\mathcal{H})$, but a priori not even an algebra because I' is not connected and we cannot rely on isotony.

Geometrically speaking we can say that essential duality (see previous section) can be checked on two points, e.g. $\{0, \infty\}$, while Haag duality requires three, e.g. $\{0, 1, \infty\}$.

Moreover, Haag duality on $\mathbb{R}$ does *not* hold in general for CFTs. The first examples of non Haag dual nets (on $\mathbb{R}$) are due to [BSM90], see also [Lon08a, Prop. 6.5.5] for more examples coming from free (second quantized) chiral CFTs (the n-derivatives of the $U(1)$-current algebra net). Observe that in all these examples essential duality on $\mathbb{R}$, i.e., "Haag duality on $\mathbb{S}^1$", does hold by conformal covariance, see Proposition 1.4.1.

It is an instructive exercise to check that

Lemma 1.5.4. [GLW98]. *Let $\{\mathcal{A}\}$ be a local conformal net on the line, let $\{\tilde{\mathcal{A}}\}$ be its extension to the circle as in Lemma 1.2.1, then the following are equivalent*

- $\mathcal{A}(I')' = \mathcal{A}(I)$ *for every $I \in \mathcal{I}$ and $I' = \mathbb{R} \smallsetminus \overline{I}$. (Haag duality on $\mathbb{R}$).*

- $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ *for every $I \in \mathcal{I}$ and $I_1, I_2 \in \mathcal{I}$ arising as connected components of $I \smallsetminus \{p\}$, $p \in I$. (Strong additivity).*

- $\mathcal{A}(I_1)' \cap \mathcal{A}(I) = \mathcal{A}(I_2)$ *(and $\mathcal{A}(I_2)' \cap \mathcal{A}(I) = \mathcal{A}(I_1)$) for every $I \in \mathcal{I}$ and $I_1, I_2 \in \mathcal{I}$ as in the previous point.*

The second equivalent condition is usually referred to as **strong additivity** of the net. Moreover, the lemma shows that Haag duality on $\mathbb{R}$ is a local

condition, i.e., can be checked inside one single bounded interval $I \in \mathcal{I} \subset \tilde{\mathcal{I}}$ for which $\mathcal{A}(I) = \tilde{\mathcal{A}}(I)$. Anyway conformal covariance enters in the proof, together with additivity and essential duality $\tilde{\mathcal{A}}(I') = \tilde{\mathcal{A}}(I)'$, where $I \in \tilde{\mathcal{I}}$ and $I' = \mathbb{S}^1 \setminus \bar{I}$, $I' \in \tilde{\mathcal{I}}$, which are more natural in the circle picture.

Going back to modular theory of conformal nets, we have that in the *Haag dual* case the whole information about the CFT can be encoded in two algebras, instead of three, together with the vacuum vector. In general, a pair $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is called a *standard subfactor* if $\mathcal{N} \subset \mathcal{M}$ is a subfactor in $\mathcal{B}(\mathcal{H})$ and $\Omega \in \mathcal{H}$ is a cyclic and separating vector for $\mathcal{N}, \mathcal{M}$ *and* for the relative commutant $\mathcal{N}^c := \mathcal{N}' \cap \mathcal{M}$. Notice that if $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \Omega)$ is a $+$hsm factorization in $\mathcal{B}(\mathcal{H})$ such that $\mathcal{N}_1 \vee \mathcal{N}_2 = \mathcal{N}_3'$, then $(\mathcal{N}_1 \subset \mathcal{N}_1 \vee \mathcal{N}_2, \Omega)$ is a standard $-$hsm subfactor and $(\mathcal{N}_2 \subset \mathcal{N}_1 \vee \mathcal{N}_2, \Omega)$ a standard $+$hsm subfactor. This follows from definitions and from the general fact that $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is a standard $+$hsm subfactor if and only if $(\mathcal{M}' \subset \mathcal{N}', \Omega)$ is a standard $-$hsm subfactor, see [Wie93], [AZ05].

As before, we can say that a standard $+$hsm subfactor is the abstract modular-theoretical version of two local algebras (of an Haag dual local conformal net in the vacuum sector) associated to two intervals, one included in the other and sharing their left boundary point. As before the two intervals arise from the choice of three distinct points of $\mathbb{S}^1$, or equivalently of $\mathbb{R}$. In this case, however, the third algebra is algebraically determined by the first two algebras (see above), as the third interval is (always) geometrically determined by the first two intervals.

Proposition 1.5.5. [GLW98]. *Let $\{\mathcal{A}\}$ be a Haag dual local conformal net on the line, let $\{\tilde{\mathcal{A}}\}$ be its extension to the circle as in Lemma 1.2.1. Consider two intervals $I_1, I_2 \in \tilde{\mathcal{I}}$ such that $I_1 \subset I_2$ on $\mathbb{S}^1$ is conformally equivalent to $(0,1) \subset (0, \infty)$ on $\mathbb{R}$ and the vacuum vector Ω, then $(\tilde{\mathcal{A}}(I_1) \subset \tilde{\mathcal{A}}(I_2), \Omega)$ is a standard $+$hsm subfactor.*
Vice versa, every standard $+$hsm subfactor $(\mathcal{N} \subset \mathcal{M}, \Omega)$ in $\mathcal{B}(\mathcal{H})$ arises in this way, i.e., there is a (unique) Haag dual local conformal net on the circle such that $\tilde{\mathcal{A}}(I_1) = \mathcal{N}$, $\tilde{\mathcal{A}}(I_2) = \mathcal{M}$ and having Ω as the vacuum vector. The positive energy unitary implementation of Möbius covariance is determined by the Bisognano-Wichmann formula for the three intervals $I_1, I_2', I_1' \cap I_2$ and the three algebras $\mathcal{N}, \mathcal{M}', \mathcal{N}^c$.

Applying Proposition 1.1.8 or following the direct argument of [Wie93], it follows that a standard +hsm subfactor $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is either a proper inclusion of type III_1 factors or $\mathcal{N} = \mathcal{M} = \mathbb{C}$, in the second case we have trivially $\mathcal{H} = \mathbb{C}$.

Remark 1.5.6. Notice that every local conformal net $\{\tilde{\mathcal{A}}\}$, say on the circle, can be turned into an Haag dual local conformal net, the *dual net* $\{\tilde{\mathcal{A}}^d\}$, see [GLW98, Cor. 1.5] and remarks thereafter. See also [Rob11, Sec. 1.6].
The dual net $\{\tilde{\mathcal{A}}^d\}$ is uniquely determined up to unitary equivalence, its local algebras extend those of $\{\mathcal{A}\}$ on bounded intervals and vice versa on intervals containing the infinity point, while those associated to half-lines remain untouched. The unitary representations implementing Möbius covariance are in general different for $\{\tilde{\mathcal{A}}\}$ and $\{\tilde{\mathcal{A}}^d\}$ due to the Bisognano-Wichmann property.

See Section 3.2 for further discussion on Haag duality on $\mathbb{R}$ and notice that we will almost always assume it in this work because of its implications on DHR representation theory.

1.6 Complete invariants

By the results of the previous section we know that a local conformal net (in the vacuum sector), i.e., a quadruple $(\{\mathcal{A}\}, U, \Omega, \mathcal{H})$ as in Definition 1.1.3 is completely determined by either $(\{\mathcal{A}\}, \Omega, \mathcal{H})$ or $(\{\mathcal{A}\}, U, \mathcal{H})$.
In this section we will see that it is indeed determined, up to unitary equivalence, by the local algebras $\{\mathcal{A}\}$ only, thanks to a result of [Wei11]. In particular this justifies the terminology *the* vacuum sector for a chiral CFT. We first recall the useful notion of **isomorphism of nets**.

Definition 1.6.1. Two local conformal nets (in their vacuum sector) $\{\mathcal{A}\}$ and $\{\mathcal{B}\}$, or better $(\{\mathcal{A}\}, U_\mathcal{A}, \Omega_\mathcal{A}, \mathcal{H}_\mathcal{A})$ and $(\{\mathcal{B}\}, U_\mathcal{B}, \Omega_\mathcal{B}, \mathcal{H}_\mathcal{B})$, are **isomorphic**, or *unitarily equivalent*, if there exists a unitary operator $W : \mathcal{H}_\mathcal{A} \to \mathcal{H}_\mathcal{B}$ which intertwines all local algebras, i.e., $W\mathcal{A}(I)W^* = \mathcal{B}(I)$ for all $I \in \mathcal{I}$, the vacuum vectors $W\Omega_\mathcal{A} = \Omega_\mathcal{B}$ and the implementations of Möbius covariance $WU_\mathcal{A}(g)W^* = U_\mathcal{B}(g)$ for all $g \in$ Möb.
In this case we write $\{\mathcal{A}\} \cong \{\mathcal{B}\}$ and denote by $[\{\mathcal{A}\}]$ the **isomorphism class** of the net $\{\mathcal{A}\}$.

Remark 1.6.2. We know by [BGL93], [GF93, Sec. II.3] that the last two conditions are equivalent by irreducibility of the net and by the Bisognano-

Wichmann property.

Notice also that *every* unitary operator $W \in \mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ denotes the *unitary group* of $\mathcal{B}(\mathcal{H})$, is eligible as an isomorphism of nets. Indeed if $\{\mathcal{A}\}$, or better $(\{\mathcal{A}\}, U, \Omega, \mathcal{H})$, is a local conformal net, then $I \mapsto W\mathcal{A}(I)W^*$ is again local and covariant under $g \in \mathsf{M\ddot{o}b} \mapsto WU(g)W^*$. Positivity of the energy holds because the spectrum of unbounded operators is a unitary invariant. Hence $(\{W\mathcal{A}W^*\}, WUW^*, W\Omega, \mathcal{H})$ is isomorphic to $(\{\mathcal{A}\}, U, \Omega, \mathcal{H})$ via W.

Now we can formulate Proposition 1.5.1 and 1.5.5 in the language of **(complete) invariants**. Heuristically speaking, an *invariant* for local conformal nets is a property, quantity or structure that can be associated to any such net $\{\mathcal{A}\}$, in such a way it *does not vary* if we change the net inside its isomorphism class $[\{\mathcal{A}\}]$. An invariant is called *complete* if it distinguishes among different isomorphism classes of nets, In other words, if it gives rise to a bijection between the family (or a subfamily) of all isomorphism classes of nets and the "values" they take through the invariant, i.e., if it gives rise to a complete classification of all nets (or a subfamily) up to isomorphism.

Instead of trying to make this definition more formal, e.g. introducing functors and categorical equivalences (see Chapter 2), we give examples.

Example 1.6.3. Let $(\mathcal{N}_1^1, \mathcal{N}_2^1, \mathcal{N}_3^1, \Omega^1)$ and $(\mathcal{N}_1^2, \mathcal{N}_2^2, \mathcal{N}_3^2, \Omega^2)$ be two +hsm factorizations as in the previous section, respectively in $\mathcal{B}(\mathcal{H}^1)$ and in $\mathcal{B}(\mathcal{H}^2)$. We call them *isomorphic* if there exists a unitary operator $V : \mathcal{H}^1 \to \mathcal{H}^2$ which intertwines the factors $V\mathcal{N}_i^1 V^* = \mathcal{N}_i^2$, $i \in \mathbb{Z}_3$ and the standard vectors $V\Omega^1 = \Omega^2$. We denote by $[(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \Omega)]$ the **isomorphism class of the +hsm factorization** $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \Omega)$.

Let $\{\mathcal{A}\}$ be a local conformal net and fix the notation as in Proposition 1.5.1, then $[(\tilde{\mathcal{A}}(I_1), \tilde{\mathcal{A}}(I_2), \tilde{\mathcal{A}}(I_3), \Omega)]$ is trivially an invariant for local conformal nets. Proposition 1.5.1 says that the invariant is complete ("injectivity") and that the isomorphism classes of local conformal nets exhaust the isomorphism classes of +hsm factorizations through this invariant ("surjectivity" or "realizability").

Example 1.6.4. We can define analogously the **isomorphism class of the standard +hsm subfactor** $(\mathcal{N} \subset \mathcal{M}, \Omega)$ and denote it by $[(\mathcal{N} \subset \mathcal{M}, \Omega)]$. In the notation of Proposition 1.5.5, we have that $[(\tilde{\mathcal{A}}(I_1) \subset \tilde{\mathcal{A}}(I_2), \Omega)]$ is trivially an invariant for local conformal nets $\{\mathcal{A}\}$.

Now the difference is that this invariant is in general *not* complete. e.g., take a net $\{\mathcal{A}\}$ and its dual net $\{\mathcal{A}^d\}$ (defined up to isomorphism) and observe that

they have the same half-line algebras and vacuum vector up to isomorphism, but need not be isomorphic unless $\{\mathcal{A}\}$ is Haag dual on $\mathbb{R}$. The completeness of this second invariant, but for the subfamily of Haag dual local conformal nets, is part of the content of Proposition 1.5.5. As before, the proposition gives also a positive answer to the realizability problem of standard +hsm subfactors as inclusions of half-line algebras of some chiral CFT in the vacuum sector.

We have just seen two invariants which involve two or three suitably chosen local algebras and the vacuum vector. In order to drop the latter, making use of a deep result of [Wei11, Thm. 5.1], we need a further condition called *split property*. This amounts to strengthening the locality assumption, i.e., $\mathcal{A}(I)$, $\mathcal{A}(J)$ not only commute when $I \cap J = \emptyset$ but also "split" as two different (commuting) tensor factors acting on a factorization of the vacuum Hilbert space $\mathcal{H} \cong \mathcal{H}_1 \otimes \mathcal{H}_2$, at least when I and J have strictly positive distance. Namely $\mathcal{A}(I) \vee \mathcal{A}(J) \cong \mathcal{A}(I) \otimes \mathcal{A}(J)$ and $ab = ba$, $a \in \mathcal{A}(I)$, $b \in \mathcal{A}(J)$ reads $(a \otimes \mathbb{1})(\mathbb{1} \otimes b) = a \otimes b = (\mathbb{1} \otimes b)(a \otimes \mathbb{1})$. It is important to notice that the symbol $\cong$ denotes *spatial* isomorphism of von Neumann algebras, i.e., implemented by some unitary operator between the two Hilbert spaces via adjoint action. Also, $\otimes$ denotes the von Neumann tensor product in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Indeed, whenever $\mathcal{N}$ and $\mathcal{M}$ are commuting factors in $\mathcal{B}(\mathcal{H})$, a fundamental lemma of Murray and von Neumann [MVN36, p. 140] says that $a \otimes b \mapsto ab$ defines a *-algebraic isomorphism between the *-algebraic tensor product $\mathcal{N} \otimes_{\text{alg}} \mathcal{M}$ and the *-algebra generated by $\mathcal{N} \vee_{\text{alg}} \mathcal{M}$. Split property is requiring that this isomorphism is spatial, hence normally extends to the weak closures. *Normality* in the context of *-morphisms or states of von Neumann algebras means (is equivalent to) continuity in the ultraweak operator topology, see [Ped79, Sec. 3.6].
We recall now the following definition due to [DL83a], [DL84].

Definition 1.6.5. Let $\mathcal{N}, \mathcal{M}$ be a pair of *commuting* von Neumann factors (or von Neumann algebras) in $\mathcal{B}(\mathcal{H})$, i.e., $\mathcal{N} \subset \mathcal{M}'$ or equivalently $\mathcal{M} \subset \mathcal{N}'$. The pair $\mathcal{N}, \mathcal{M}$ fulfills the **split property**, i.e., $\mathcal{N} \subset \mathcal{M}'$ or equivalently $\mathcal{M} \subset \mathcal{N}'$ are *split subfactors* (or *split inclusions*), if there exists a type I factor $\mathcal{F}$ such that $\mathcal{N} \subset \mathcal{F} \subset \mathcal{M}'$ or equivalently $\mathcal{M} \subset \mathcal{F}' \subset \mathcal{N}'$.

Remark 1.6.6. Recall that there is only one type I factor, up to spatial isomorphism, namely $\mathcal{B}(\mathcal{H}_1) \otimes \mathbb{1}$ in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. In particular $\mathcal{F}$ is type I if

and only if $\mathcal{F}'$ is type I. Trivially the pair $\mathcal{N}, \mathcal{N}'$ fulfills the split property if and only if $\mathcal{N}$, hence $\mathcal{N}'$, is a type I factor.

The connection between the split property as stated in the definition and the preceding discussion is clarified in the next proposition. For simplicity we assume to deal with standard subfactors $(\mathcal{N} \subset \mathcal{M}', \Omega)$ and $(\mathcal{M} \subset \mathcal{N}', \Omega)$ as is usually the case if $\mathcal{N}$ and $\mathcal{M}$ are two commuting local algebras in chiral CFT and Ω is the vacuum vector. In this case, if the isomorphism $a \otimes b \mapsto ab$ normally extends to the weak closures then it is automatically spatial thanks to the Araki-Connes-Haagerup standard implementation, see [Haa75, Thm. 2.3].

Proposition 1.6.7. [DL83a], [DL84], [Lon08b]. *Let $(\mathcal{N} \subset \mathcal{M}', \Omega)$, or equivalently $(\mathcal{M} \subset \mathcal{N}', \Omega)$, be a standard subfactor in $\mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ separable and $\Omega \in \mathcal{H}$. Then the following are equivalent*

- $\mathcal{N} \vee \mathcal{M} \cong \mathcal{N} \otimes \mathcal{M}$ *spatially, i.e., the isomorphism is implemented by a unitary operator $U : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ via adjoint action.*

- *There exists a type I factor $\mathcal{F}$ such that $\mathcal{N} \subset \mathcal{F} \subset \mathcal{M}'$, or equivalently $\mathcal{M} \subset \mathcal{F}' \subset \mathcal{N}'$. (Split property for the pair $\mathcal{N}, \mathcal{M}$).*

- *For any given pair of normal (and faithful, resp.) states φ_1 on $\mathcal{N}$ and φ_2 on $\mathcal{M}$ there is a normal (and faithful, resp.) state φ on $\mathcal{N} \vee \mathcal{M}$ which extends both and in such a way there are no correlations between the two states, i.e., $\varphi(ab) = \varphi_1(a)\varphi_2(b)$ for every $a \in \mathcal{N}$, $b \in \mathcal{M}$.*

- *There exists a vector $\eta \in \mathcal{H}$ which is cyclic and separating for $\mathcal{N} \vee \mathcal{M}$ such that $(\eta|ab\eta) = (\Omega|a\Omega)(\Omega|b\Omega)$ for every $a \in \mathcal{N}$, $b \in \mathcal{M}$.*

Remark 1.6.8. Remarkably, given a standard split inclusion, it can be shown that there is a *canonical* choice of the intermediate type I factor $\mathcal{F}$. Moreover, in the standard split subfactor case, this choice can be characterized by means of modular theory (of the relative commutant of the inclusion), see [DL84, Thm. 2.1, 4.1].

The relevance of the split property in high-dimensional QFT has been first pointed out by Borchers. In the free field case, Buchholz proved [Buc74] that pairs of local algebras do fulfill the split property, in the above sense, whenever they are associated to pairs of space-like separated double cones with *strictly positive distance*. Indeed he shows the last equivalent condition of the previous proposition, which is, mathematically speaking, the most

fundamental, see the proof on [Haa96, Thm. 5.2.1]. More generally, one can interpret the second equivalent condition above as "statistical independence" of the local algebras associated to space-like well separated bounded regions of spacetime. Physically speaking, it says that (local) preparation of states in some bounded region $\mathcal{O}$ (the causal shadow of our laboratory) can be performed regardless of all other (local) preparation procedures taking place in regions space-like outside any proper neighborhood $\tilde{\mathcal{O}}$ of $\mathcal{O}$, at least for the sake of local measurements in $\mathcal{O}$. In this spirit, back to the chiral case, one can consider "split" local nets in the following sense

Definition 1.6.9. A local net on the line $\{\mathcal{A}\}$ fulfills the **split property** if *every* inclusion of local algebras $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$ is a split inclusion (in the sense of Definition 1.6.5) whenever $I, \tilde{I} \in \mathcal{I}$ are such that $I \Subset \tilde{I}$, i.e., $\bar{I} \subset \tilde{I}$.

Remark 1.6.10. The split property, stated in this way, is manifestly "local", i.e., can be checked inside bounded intervals. Moreover, if every inclusion $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$ as above is a split inclusion, then it is easy to see that every pair $\mathcal{A}(I), \mathcal{A}(J)$, where $I \cap J = \emptyset$ and I, J have positive distance, fulfills the split property (in the sense of Definition 1.6.5). *Vice versa* also holds, at least for local conformal nets.

It is known that the split property holds in the majority of explicitly constructed models of chiral CFTs, see e.g. [Xu00b], [Lon08b, Cor. 7.4.4], and that the existence of the *vacuum conformal character* $\beta \mapsto \mathrm{Tr}(e^{-\beta H})$ where $\beta > 0$ and H is the conformal Hamiltonian, i.e., $e^{-\beta H}$ is trace class in $\mathcal{B}(\mathcal{H})$ for all $\beta > 0$, is a sufficient condition for the split property to hold, see [Lon08b, Thm. 7.3.3].

On the other hand, the split property is not a consequence of the general assumptions. e.g., let $\{\mathcal{A}\}$ be a split local conformal net, then its infinite tensor powers $\{\otimes_{n=0}^{\infty}\mathcal{A}\}$ on $\otimes_{n=0}^{\infty}\mathcal{H}$ (driven by the sequence $\{\Omega, \Omega, \ldots\}$) give rise to non-split local conformal nets by [DL84, Thm. 9.2], see also [DL84, Sec.10].

We can now compare the notion of *standard split subfactor* with the notion of *standard $\pm hsm$ subfactor* seen in the previous section. In chiral CFT they correspond to two different types of inclusions between intervals $I \subset J$, in one case $I \Subset J$, in the other case I and J share one (left or right) boundary point. The case $I = J$ is excluded by assuming standard. Notice that these two classes of inclusions are stable under Möbius transformations. The next proposition, which rephrases a result of [Wie93, Lem. 13], says that this

difference somehow lifts to the local algebras as a sharp distinction between the two types of subfactors.

Proposition 1.6.11. [Wie93]. *Let* $(\mathcal{N} \subset \mathcal{M}, \Omega)$ *be a standard inclusion where* $\mathcal{N}$, $\mathcal{M}$ *and* $\mathcal{N}^c$ *are assumed to be type* III_1 *factors. Then* $(\mathcal{N} \subset \mathcal{M}, \Omega)$ *can be either* $+hsm$ *or* $-hsm$ *or split, but each case excludes the others.*

Now, let $\{A\}$ be a local conformal net on the line fulfilling the split property as in Definition 1.6.9. By Remark 1.6.10 the local algebras $\mathcal{A}(I)$, $\mathcal{A}(J)$ split commute whenever I and J are disjoint with strictly positive distance, but this need not be the case if I and J are disjoint and "touching" [3]. The next proposition shows that this is never the case in chiral CFT.

Proposition 1.6.12. *Let* $\{\mathcal{A}\}$ *be a local conformal net on the line, then pairs of local algebras* $\mathcal{A}(I), \mathcal{A}(J)$ *associated to disjoint touching intervals* $I, J \in \mathcal{I}$ *never fulfill the split property (in the sense of Definition 1.6.5), unless* $\{\mathcal{A}\} = \{\mathbb{C}\}$ *and* $\mathcal{H} = \mathbb{C}$.

Proof. We give two proofs of the statement, the first relies on the previous proposition. Take two local algebras $\mathcal{A}(I)$, $\mathcal{A}(J)$ as above and consider the extended net on circle $\{\tilde{\mathcal{A}}\}$ as in Lemma 1.2.1 for which $\mathcal{A}(I) = \tilde{\mathcal{A}}(I)$ and $\mathcal{A}(J) = \tilde{\mathcal{A}}(J)$. Assume that $\tilde{\mathcal{A}}(I), \tilde{\mathcal{A}}(J)$ split commute, i.e., $\tilde{\mathcal{A}}(I) \subset \tilde{\mathcal{A}}(J)'$ is a split subfactor, hence by essential duality $\tilde{\mathcal{A}}(I) \subset \tilde{\mathcal{A}}(J')$ is also a split subfactor where $J' = \mathbb{S}^1 \smallsetminus \overline{J} \in \tilde{\mathcal{I}}$. After a suitable rotation we can assume that I and J touch at $\infty \in \mathbb{S}^1$ and then substitute $\{\tilde{\mathcal{A}}\}$ with the dual net $\{\tilde{\mathcal{A}}^d\}$, see Remark 1.5.6, for which $\tilde{\mathcal{A}}(I) = \tilde{\mathcal{A}}^d(I)$ and $\tilde{\mathcal{A}}(J') = \tilde{\mathcal{A}}^d(J')$. Observe that the splitting of the inclusion $\tilde{\mathcal{A}}(I) \subset \tilde{\mathcal{A}}(J')$ is not affected by conjugation with the unitary implementing the rotation. Now the point is that the relative commutant of $\tilde{\mathcal{A}}^d(I) \subset \tilde{\mathcal{A}}^d(J')$ is a local algebra by Lemma 1.5.4, namely $(\tilde{\mathcal{A}}^d(I))^c = \tilde{\mathcal{A}}^d(K)$ where $K = J' \smallsetminus \overline{I} \in \tilde{I}$, in particular it is a type III_1 factor and we are in the case of Proposition 1.6.11. It is clear that $\tilde{\mathcal{A}}^d(I) \subset \tilde{\mathcal{A}}^d(J')$ is a $+/-$hsm standard subfactor, cf. Proposition 1.5.5, depending on the mutual left/right position of I and J, hence cannot be split and we have a contradiction.

The second proof is based on [Bis14] and uses the Bisognano-Wichmann property and the characterization of the canonical intermediate type I factor in terms of modular theory, as the proof of [Wie93, Lem. 13] does. First extend

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[3]two disjoint intervals I, J are called *touching* if the interior of $\overline{I \cup J}$ is again an interval.

the net to the circle and then assume, up to a Möbius transformation, that I and J are the lower right / upper right quadrant of the circle, respectively. In particular they touch at $\infty \in \mathbb{S}^1$, which we usually place at $1 \in \mathbb{C}$. As before we have $\mathcal{A}(I) = \tilde{\mathcal{A}}^d(I)$ and $\mathcal{A}(J) = \tilde{\mathcal{A}}^d(J)$. Now, assume that $\tilde{\mathcal{A}}^d(I) \subset \tilde{\mathcal{A}}^d(J')$ is a (standard) split subfactor then by [DL84, Thm. 4.1] the canonical intermediate type I factor can be expressed as

$$\mathcal{F} = \tilde{\mathcal{A}}^d(I) \vee J_{(\tilde{\mathcal{A}}^d(I))^c,\Omega}\, \tilde{\mathcal{A}}^d(I)\, J_{(\tilde{\mathcal{A}}^d(I))^c,\Omega}$$

where $J_{(\tilde{\mathcal{A}}^d(I))^c,\Omega}$ is the modular conjugation of the relative commutant with respect to the vacuum vector. Now by Lemma 1.5.4 the relative commutant is again a local algebra, namely $(\mathcal{A}^d(I))^c = \tilde{\mathcal{A}}^d(K)$ where $K = J' \smallsetminus \overline{I} \in \tilde{I}$ and in this case coincides with the left semicircle. By the Bisognano-Wichmann property of $\{\tilde{\mathcal{A}}^d\}$, see Theorem 1.3.2, the modular conjugation acts geometrically as the reflection r_K, hence by strong additivity, see Lemma 1.5.4, $\mathcal{F}$ coincides with the local algebra in $\{\tilde{\mathcal{A}}^d\}$ associated to the lower semicircle. In particular $\mathcal{F}$ is a type III_1 factor which is a contradiction. $\square$

Remark 1.6.13. By conformal covariance, the previous proposition says that the split property cannot hold for commuting pairs of half-line algebras in chiral CFT. This is a special case of the statement that conformal field theories in $n+1$ dimensions cannot satisfy the split property for commuting pairs of wedge algebras. This fact is well known to experts, cf. [Müg98] for theories on $\mathbb{R}^{1+1}$, but we could not find a proof in the literature. On one hand, the split property for wedges (SPW) is a strong assumption, indeed together with Haag duality (on double cones of $\mathbb{R}^{1+1}$) kills nontrivial DHR superselection sectors (and soliton sectors as well), see [Müg98, Thm. 3.1]. On the other hand, the SPW holds e.g. for the highly nontrivial models constructed by [Lec08] describing *massive interacting* particles on $\mathbb{R}^{1+1}$, and is expected to hold in more complicated models [CT15] (with bound states), [Ala14] (non-scalar). Summing up, we have at least three possibly different shades of split property: on pairs of unbounded regions (e.g. wedge regions), between bounded and unbounded regions (e.g. one double cone and one wedge), on pairs of bounded regions (e.g. two double cones). In conformal field theory the last two are equivalent and the first is forbidden. In free field theory the second holds thanks to [Buc74, Thm. 3.5], see also [Buc74, p. 292 (a)].

Going back to our motivation for introducing the split property in this section, we can now enunciate a theorem of M. Weiner [Wei11, Thm. 5.1]

which will play on our side in the analysis of completely rational models. We reformulate it in the language of invariants and in the chiral CFT case only, see [Wei11] for its general formulation in $n + 1$ dimensions. See also [Wei11, Conj. 1.1] for an interesting question on the isomorphism classes of standard $\pm$hsm subfactors.

We first need some terminology. Two families of factors $\{\mathcal{N}_\alpha^1, \alpha \in A\}$ and $\{\mathcal{N}_\alpha^2, \alpha \in A\}$, respectively in $\mathcal{B}(\mathcal{H}^1)$ and in $\mathcal{B}(\mathcal{H}^2)$, indexed by the same set of indices A, are called *isomorphic* if there exists a unitary operator $V : \mathcal{H}^1 \to \mathcal{H}^2$ such that $V\mathcal{N}_\alpha^1 V^* = \mathcal{N}_\alpha^2$ for all $\alpha \in A$. Similarly for two families of subfactors of the form $\{\mathcal{N}_\alpha^1 \subset \mathcal{M}^1, \alpha \in A\}$ and $\{\mathcal{N}_\alpha^2 \subset \mathcal{M}^2, \alpha \in A\}$ if there is such a unitary V which fulfills in addition $V\mathcal{M}^1 V^* = \mathcal{M}^2$.

Theorem 1.6.14. [Wei11] (**Algebraic version of Haag's theorem**). *Let $\{\mathcal{A}\}$ be a local conformal net on the line, let $\{\tilde{\mathcal{A}}\}$ be its extension to the circle as in Lemma 1.2.1. Suppose that the net fulfills the split property as in Definition 1.6.9, then the **isomorphism class of all local factors** $[\{\tilde{\mathcal{A}}(I), I \in \tilde{\mathcal{I}}\}]$ is a complete invariant for the family of split local conformal nets $\{\mathcal{A}\}$.*
*Moreover, fix arbitrarily a bounded interval $I_0 \in \mathcal{I}$, then the **isomorphism class of all local subfactors in** I_0 $[\{\mathcal{A}(I) \subset \mathcal{A}(I_0), I \in \mathcal{I}, I \subset I_0\}]$ is also a complete invariant for split local conformal nets $\{\mathcal{A}\}$.*

We thank M. Bischoff and R. Longo for drawing our attention to this theorem, crucial for our work.

Remark 1.6.15. Now, the second point says that in chiral CFT *all* the information is contained in the local algebras sitting in any arbitrarily small fixed interval I_0. Explicitly this means that if there exists a unitary V such that $V\mathcal{A}(I)V^* = \mathcal{B}(I)$ for all $I \subset I_0$, including $I = I_0$, then there is *another* unitary W which intertwines all local algebras and the vacuum vectors (hence the implementations of covariance), hence realizes the isomorphism $\{\mathcal{A}\} \cong \{\mathcal{B}\}$. Notice also that Haag duality on $\mathbb{R}$ is not assumed here.

Physically speaking the theorem says that the collection of all local algebras, or at least those sitting on a space-like infinite hyperplane in $n + 1$ dimensions, determine uniquely the dynamics of the QFT (the representation) hence completely fix the model up to isomorphism. Notice also that the two unitaries realizing the isomorphism of local subfactors and of local nets in

their vacuum sectors need not coincide, differently to what happens in the original Wightman QFT formulation of Haag's theorem. For better explanations see the comments of the author himself [Wei11] and references therein.

We conclude this section by mentioning that Theorem 1.6.14 holds also for **diffeomorphism covariant** nets [Wei11, Prop. 5.2], namely for those local conformal nets (in the sense of Definition 1.1.3) whose implementation of Möb covariance extends to a strongly continuous (projective) unitary representation of $\text{Diff}^+(\mathbb{S}^1)$. Here $\text{Diff}^+(\mathbb{S}^1)$ denotes the infinite-dimensional Lie group of orientation preserving diffeomorphism of $\mathbb{S}^1$. See [CW05, Sec. 2.2] for the precise definition.

On one hand it is known that such an extension, when it exists, is uniquely determined by the underlying Möbius covariant net, see [CW05, Thm. 5.5], [Wei05, Thm. 6.1.9], and is completely classified by a number, the *central charge* of the representation, which moreover can take values only in a discrete series, see [FQS85]. Also, diffeomorphism covariant models are the majority in the physical literature, see e.g. the Virasoro models generated by a stress-energy tensor in the vacuum sector [Car04], and they are the most interesting ones.

On the other hand it is also known that Möbius covariance need not extend to diffeomorphism covariance and that neither Haag duality on $\mathbb{R}$ [CW05, Sec. 6] nor the split property [Kös03, Ch. VI] are sufficient conditions for such an extension to exist. In [CW05] the authors argue that there might be some other "regularity" condition on local conformal nets under which the extension becomes automatic.

Chapter 2

Unitary Braided Tensor Categories

Category theory is a quite recent discipline of mathematics which was invented, at least to our knowledge, in order to introduce the notion of "functor" and "natural transformation". A natural transformation is a transition between two functors. A functor is a map between two categories which preserves some or all of their respective properties. A category is a third step in the following abstraction climax: one usually encounters first examples of structures, e.g. groups or vector spaces, then realizes that many different examples can be written using the same symbols (and that theorems can be shown at once for all of these examples, independently of the specific realization), as a further step one can consider the collection of all the different instances of some structure and call it a *category*, e.g., the category of groups or the category of vector spaces. It is particularly convenient and conceptually very clear to talk about morphisms (or arrows of some category) between different instances of the same structure (called objects of that category). e.g., group homomorphisms and linear maps are respectively arrows in the category of groups and in the category of vector spaces. This helps to make important distinctions in everyday work, e.g., when dealing with an algebra, which can be seen also as a group or vector space or ring or set at the same time, hence sits in at least five different categories and we can talk about five different types of morphisms depending on what is the (part of the) structure we want to single out.

For our taste and purposes, category theory is a metamathematical language very powerful for finding generalizations and connections. An example of

such a connection is provided by AQFT. As an historical remark we mention that category theory has officially entered mathematical physics in the work of Doplicher and Roberts [DR72], as reported by the MathSciNet reviewer. More specifically, they realized that superselection sectors arising from global gauge symmetries in QFT (DHR superselection sectors) have the structure of a *unitary braided tensor category* (symmetric in higher dimensions). This observation led the authors, after many years, to a (re)construction result [DR90] which allows to determine a unique Bose/Fermi field theory with usual (anti)commutation relations together with a compact gauge group acting on it, having as a *unique* input a net of local observables (measurements procedures) and in such a way the gauge invariant part of the fields is exactly the given observable net. This shows in a completely model-independent way how groups of gauge symmetries arise automatically in QFT once we accept Einstein's causality principle. The core of this (re)construction is a duality theorem [DR89] which characterizes abstractly all the categories of representations of some compact group (which then becomes our gauge group). This epoch-making theorem, and circle of ideas, are the main motivations for doing research in this "categorical" direction of QFT, especially in the low-dimensional case where the situation is quite different from the one exploited by Doplicher and Roberts, and more complicated. Despite this difference (in one case we have symmetric, or "commutative", tensor categories in the other case *modular*, or "factorial", ones) we want to mention that particles in the four-dimensional world and left/right movers on a light-ray still have many things in common, see e.g. Proposition 4.3.7 and Remark 4.3.8.

For further motivation, we also mention that the second quantization construction is functorial (Fock functor) and that the Gelfand equivalence between (the categories of) commutative C^*-algebras and locally compact topological spaces provides a rigorous example where commutativity means classical statistical physics, and non-commutativity becomes a rigorous fingerprint of quantum field theories.

We shall not talk about all the possible generalizations of category theory to higher category theory, and content ourselves, in the following chapters, to give some new inputs on the connection between chiral CFTs and *unitary modular tensor categories* (of DHR superselection sectors).

35

2.1 Basics of category theory

In the spirit of the title of S. Mac Lane's book [ML98] "Categories for the Working Mathematician" we begin with our as-more-complete-as-possible exposition of the definitions, terminology and results that will be used in the next two chapters. See [ML98], [Bae10], [nLa].

Definition 2.1.1. A **category** $\mathcal{C}$ is the assignment of a collection of "*objects*", denoted by $\mathcal{C}^{(0)}$, together with a collection of "*morphisms*" (or "*arrows*"), denoted by $\mathcal{C}^{(1)}$, fulfilling the following properties.

- Every arrow $t \in \mathcal{C}^{(1)}$ has a *source* object (or *domain*), say $a \in \mathcal{C}^{(0)}$, and a *target* object (or *codomain*), say $b \in \mathcal{C}^{(0)}$, and we can write $t : a \to b$.

- Every object $a \in \mathcal{C}^{(0)}$ has an *identity arrow* $1_a : a \to a$.

- For every ordered pair of arrows $s : a \to b$, $t : c \to d$, if $b = c$ then there is a *composite arrow* $t \circ s : a \to d$ and we say that s, t are *composable*.

- For every triple of arrows r, s, t, if r, s and s, t are composable then $t \circ (s \circ r) = (t \circ s) \circ r$ holds (*associative law*).

- For every object $a \in \mathcal{C}^{(0)}$ and for every pair of arrows $s : b \to a$, $t : a \to c$ then $1_a \circ s = s$ and $t \circ 1_a = t$ hold (*left and right unit law*).

Definition 2.1.2. Given two objects a, b in a category $\mathcal{C}$, the collection of all morphisms having a as source and b as target is denoted by

$$\text{Hom}_{\mathcal{C}}(a, b) := \{t \in \mathcal{C}^{(1)}, t : a \to b\}.$$

A morphism $t \in \text{Hom}_{\mathcal{C}}(a, b)$ is called *invertible* (or *isomorphism*) if there is a morphism $s \in \text{Hom}_{\mathcal{C}}(b, a)$ such that $s \circ t = 1_a$ and $t \circ s = 1_b$. In this case we say that the objects a and b are *isomorphic* in $\mathcal{C}$, and write $a \cong_{\mathcal{C}} b$ (or $a \cong b$). The **isomorphism class** of some object a in $\mathcal{C}$ is defined as

$$[a]_{\cong_{\mathcal{C}}} := \{b \in \mathcal{C}^{(0)}, b \cong_{\mathcal{C}} a\}.$$

Inverse arrows and identity arrows are easily seen to be unique.

Remark 2.1.3. Notice that two categories might have the same objects but different morphisms, hence different isomorphism classes, or the same isomorphisms classes but very different sizes in the sense of "cardinality" of the collections of objects.

Moreover, the objects of a category (e.g. vector spaces of the category $\mathbb{C}$-Vec = complex vector spaces, not necessarily finite-dimensional, with linear maps as morphisms) do not see the elements inside them (e.g. vectors), indeed such elements need not exist at at all. On the other hand, the whole category might still see some properties of such "elements inside some object", through its morphisms in the following sense. Take, e.g., the category Set whose objects are sets and morphisms are functions between them [1], then an object $S \in Set^{(0)}$ contains only one element (has cardinality one) if and only if for every other object $R \in Set^{(0)}$ there is exactly one morphisms $t : R \to S$, i.e., $\mathrm{Hom}_{Set}(R, S) = \{t\}$. One can also go the other way around in the abstraction process and find more concrete structures inside the language of categories, e.g., a monoid is a category with only one object and a group is a category with one object where all morphisms are invertible.

In the following we will sometimes drop the apices (0) and (1) and talk about objects $a, b, c, \ldots$ and morphisms $t, r, s, \ldots$ of some category $\mathcal{C}$ by writing improperly, e.g., $a \in \mathcal{C}$ and $t \in \mathcal{C}$. We can also write ts instead of $t \circ s$.

Definition 2.1.4. A **functor** F between two categories $\mathcal{C}$ and $\mathcal{D}$, denoted by $F : \mathcal{C} \to \mathcal{D}$, consists of two maps both denoted by $F : \mathcal{C}^{(0)} \to \mathcal{D}^{(0)}$ (on objects) and $F : \mathcal{C}^{(1)} \to \mathcal{D}^{(1)}$ (on arrows) fulfilling the following properties.

- If $t \in \mathcal{C}^{(1)}$, $t : a \to b$, then $F(t) : F(a) \to F(b)$.

- If $s, t \in \mathcal{C}^{(1)}$ are composable, then $F(t \circ s) = F(t) \circ F(s)$.

- For every $a \in \mathcal{C}^{(0)}$, it holds $F(1_a) = 1_{F(a)}$.

Remark 2.1.5. The last two conditions are equivalent to asking that F sends commutative diagrams of arrows in $\mathcal{C}$ to commutative diagrams of arrows in $\mathcal{D}$. The most famous examples of functors are homology $H_n : Top \to AbGr$ and de Rham cohomology $H_{dR}^n : Mfd \to \mathbb{R}$-$Vec$, for all $n \in \mathbb{N}$, and curious examples are the forgetful functors which simply forget some part of structure, e.g., $Top \to Set$ and $\mathbb{R}$-$Vec \to AbGr$.

---

[1] note that the collection of all sets is not a set, in the usual Zermelo-Fraenkel set theory, but a *proper class*. Hence Set is a *large* category (opposed to *small* category) and one needs care to handle it. We don't enter in these delicate set-theoretic issues here, just mention that almost all categories one encounters in everyday life, e.g. Top, $\mathbb{C}$-Vec, Gr, $AbGr$ are of this type, and that there is more than one way to avoid paradoxes, see e.g. [ML98, Sec. 6, 7].

Definition 2.1.6. A functor $F : \mathcal{C} \to \mathcal{D}$ is called **faithful** if for every pair of objects $a, b \in \mathcal{C}^{(0)}$ and arrows $t, s : a \to b$ in $\mathcal{C}^{(1)}$, the equality $F(t) = F(s)$ implies $t = s$. A functor F is called **full** if for every pair of objects $a, b \in \mathcal{C}^{(0)}$ and every arrow $r : F(a) \to F(b)$ in $\mathcal{D}^{(1)}$, there is an arrow $t : a \to b$ in $\mathcal{C}^{(1)}$ such that $F(t) = r$.

Equivalently, $F : \mathcal{C} \to \mathcal{D}$ is faithful (resp. full) if for every pair of objects $a, b \in \mathcal{C}^{(0)}$ the map

$$t \mapsto F(t) : \; \mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{D}}(F(a), F(b))$$

is injective (resp. surjective). Then F is full and faithful if and only if for every pair of objects of $\mathcal{C}$ we have such a bijection on arrows. Notice that a full and faithful functor need *not* be a bijection on objects.

A functor can be also thought as a morphism of categories (in the category of all (small) categories Cat, see [ML98, Sec. II.5]). Indeed there is a natural composition of functors, which is associative, and an identity functor $1_{\mathcal{C}}$ can be defined for every category $\mathcal{C}$. Hence two categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic* if there is an invertible functor $F : \mathcal{C} \to \mathcal{D}$, with inverse functor $G : \mathcal{D} \to \mathcal{C}$, such that $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{\mathcal{D}}$, and in this case we write $\mathcal{C} \cong \mathcal{D}$ (or $\mathcal{C} \cong_{Cat} \mathcal{D}$). This notion turns out to be too strong in practice, in particular two isomorphic categories must have the same amount of objects and of morphisms (bijective correspondences). The notion is relaxed by introducing a further degree of freedom, namely *natural transformations* between functors and consequently the notion of *equivalence of categories*. We denote the latter by $\mathcal{C} \simeq \mathcal{D}$, in order to make the distinction clear.

A natural transformation is a morphism of functors, or better a 2-morphism between morphisms of categories, indeed Cat has the structure (and is the prototype) of a 2-category. For this reason we use the double arrow notation.

Definition 2.1.7. A **natural transformation** τ between two functors $F, G : \mathcal{C} \to \mathcal{D}$, denoted by $\tau : F \Rightarrow G$, consists of a family of morphisms in $\mathcal{D}$, $\{\tau_a\}_{a \in \mathcal{C}^{(0)}}$, one for every object of $\mathcal{C}$, such that

- $\tau_a : F(a) \to G(a)$ for every object $a \in \mathcal{C}^{(0)}$,
- $\tau_b \circ F(t) = G(t) \circ \tau_a$ for every morphism $t \in \mathcal{C}^{(1)}$, $t : a \to b$ (*naturality*),

i.e., the following diagrams commute

$$
\begin{array}{ccc}
F(a) & \xrightarrow{F(t)} & F(b) \\
\downarrow{\scriptstyle \tau(a)} & & \downarrow{\scriptstyle \tau(b)} \\
G(a) & \xrightarrow{G(t)} & G(b)
\end{array}
$$

for every $t : a \to b$ in $\mathcal{C}$.

In words, τ intertwines F and G both on objects and on morphisms. A natural transformation $\tau : F \Rightarrow G$ is called a **natural isomorphism** between the two functors, or better a 2-isomorphism, if all the morphisms τ_a are isomorphisms. Denoted by $(\tau_a)^{-1}$ the inverse morphisms, the inverse natural transformation $\tau^{-1} : G \Rightarrow F$ is given by $(\tau^{-1})_a := (\tau_a)^{-1}$.
In this case we write $F \cong G$ (or $F \cong_{Cat} G$), and say that the two functors are naturally isomorphic, or better 2-isomorphic.

The following will be the archetypal example of the type of categories we deal with, despite they will have in general more "nontrivial objects" or more "nontrivial commutativity constraints".

Example 2.1.8. Let Vec $\subset \mathbb{C}$-Vec be the subcategory of $\mathbb{C}$-Vec, whose objects are complex vector spaces *with finite dimension* and morphisms are complex linear maps. Consider the *cartesian product category* which arises by taking ordered pairs of objects and morphisms, denote it by (Vec, Vec) (or Vec^2, or also Vec $\times$ Vec). Consider the tensor product functor $\otimes :$ (Vec, Vec) $\to$ Vec and the opposite tensor product functor $\otimes^{\mathrm{op}} :$ (Vec, Vec) $\to$ Vec which are given, on objects, respectively as $\otimes(V, W) = V \otimes W$ and $\otimes^{\mathrm{op}}(V, W) = W \otimes V$. Now $V \otimes W \cong W \otimes V$ for every $(V, W) \in$ (Vec, Vec)$^{(0)}$ via the *flip* map, defined on indecomposable tensors as $\sigma(v \otimes w) = w \otimes v$. The family of morphisms $\{\sigma_{(V,W)}\}_{(V,W)}$ provides our first very special example of natural transformation (isomorphism) $\sigma : \otimes \Rightarrow \otimes^{\mathrm{op}}$. Observe that $\sigma^2 = \mathrm{id}$.
Later on we will see more general *braidings*, i.e., natural isomorphisms on abstract monoidal products (also called again tensor products), which need not fulfill the analogue of $\sigma^2 = \mathrm{id}$, i.e., need not give presentations of the infinite symmetric group but of the Artin braid group.

Definition 2.1.9. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called an **equivalence of categories** if there is another functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \cong 1_{\mathcal{C}}$ and $F \circ G \cong 1_{\mathcal{D}}$ via natural isomorphisms $\tau : G \circ F \Rightarrow 1_{\mathcal{C}}$

and $\beta : F \circ G \Rightarrow 1_{\mathcal{D}}$.

In this case the two categories are called **equivalent**, and we write $\mathcal{C} \simeq \mathcal{D}$.

Remark 2.1.10. Two isomorphic categories are obviously equivalent, but in the latter case they are allowed to have very different amounts of objects. e.g., a *skeleton* of a category $\mathcal{C}$ is a (full) subcategory which contains exactly one representative for each isomorphism class of objects in $\mathcal{C}$. A category $\mathcal{C}$ can contain more than uncountably many disjoint skeletons, and be *equivalent* to each of them at the same time.

It is a useful exercise to check that whenever a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence then F is automatically full and faithful. The next theorem [ML98, Thm. 1 §IV.4] says which is the missing ingredient in order to obtain the converse implication, namely *essential surjectivity*.

Theorem 2.1.11. [ML98]. *A full and faithful functor $F : \mathcal{C} \to \mathcal{D}$ (in the sense of Definition 2.1.6) is an equivalence of categories if and only if it is essentially surjective, i.e., for every object $d \in \mathcal{D}^{(0)}$ there is some $c \in \mathcal{C}^{(0)}$ such that $F(c) \cong_{\mathcal{D}} d$.*

2.2 Tensor categories (TCs)

Having Example 2.1.8 in mind, we recall the definition of *tensor* (or *monoidal*) category, namely a category with an additional structure, a multiplication operation denoted by $\otimes$ (or better by $\times$), resembling the tensor product of vector spaces. Observe that this tensor multiplication is *not* strictly associative, namely $(U \otimes V) \otimes W \neq U \otimes (V \otimes W)$, however it is associative up to (canonical) isomorphisms $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$. This observation leads, on one side, to the notion of natural associativity constraints and pentagon diagrams, on the other side, to a special class of tensor categories called *strict*, namely those for which associativity holds strictly. An example of the latter is provided by categories of endomorphisms of von Neumann algebras, where the tensor multiplication is defined by composition of endomorphisms, which is manifestly strictly associative. Categories of this type are used in AQFT to describe the composition of charge quantum numbers (see Chapter 3). On the other hand, despite every tensor category is tensor equivalent to a strict one [ML98], as we shall mention later, one should always keep in mind the degree of freedom of non-strictness. e.g., two strict tensor categories might be

equivalent, but only via a non-strict tensor functor, i.e., a functor equipped with natural associator which fulfills nontrivially 2-cocylce-like identities, as we shall see later. See [BK01], [Müg10b] in addition to the references of the previous section.

Definition 2.2.1. A **strict tensor category** $\mathcal{C}$, or better $(\mathcal{C}, \times, \mathrm{id})$, is a category $\mathcal{C}$ together with a map $\times : (\mathcal{C}, \mathcal{C}) \to \mathcal{C}$ both on objects and morphisms, and a distinguished object id fulfilling the following properties.

- $\times : (\mathcal{C}, \mathcal{C}) \to \mathcal{C}$ is a *functor* (also called bifunctor in this case), i.e., for every quadruple of suitably composable morphisms t, s, r, u in $\mathcal{C}$, and every pair of objects a, b in $\mathcal{C}$ it holds

$$(t \circ s) \times (r \circ u) = (t \times r) \circ (s \times u) \,, \quad 1_a \times 1_b = 1_{a \times b}$$

  which are better understood in graphical notation [2], respectively

- $\times : (\mathcal{C}, \mathcal{C}) \to \mathcal{C}$ is *associative* both on objects $a, b, c, \ldots$ and on morphisms $t, s, r, \ldots$ in $\mathcal{C}$, i.e.

$$a \times (b \times c) = (a \times b) \times c \,, \quad t \times (s \times r) = (t \times s) \times r$$

  also written as

- $\times : (\mathcal{C}, \mathcal{C}) \to \mathcal{C}$ has id as a *tensor unit* both on objects and on morphisms in $\mathcal{C}$, i.e.

$$a \times \mathrm{id} = a = \mathrm{id} \times a \,, \quad t \times 1_{\mathrm{id}} = t = 1_{\mathrm{id}} \times t$$

---

[2]to be read as follows, composition of morphisms: vertical from top to bottom, tensor multiplication of morphisms: horizontal from left to right.

also written as

$$
\boxed{t} \;\vdots\; \text{id} \quad = \quad \boxed{t} \quad = \quad \text{id} \;\vdots\; \boxed{t} \quad .
$$

Remark 2.2.2. One can "slide up and down" boxes (morphisms) along strings (labeled by two objects) even in diagrams with more than one string (n-fold tensor products of morphisms with $n > 1$) and get a coherent graphical calculus for strict tensor categories. This is a particular instance of functoriality of $\times$, because for every $t : a \to b$, $r : c \to d$ it holds

$$
(t \times 1_d) \circ (1_a \times r) = t \times r = (1_b \times r) \circ (t \times 1_c)
$$

i.e.

$$
\begin{array}{cc} a & c \\ & \boxed{r} \\ \boxed{t} & \\ b & d \end{array} \quad = \quad \begin{array}{cc} a & c \\ \boxed{t} & \boxed{r} \\ b & d \end{array} \quad = \quad \begin{array}{cc} a & c \\ \boxed{t} & \\ & \boxed{r} \\ b & d \end{array} \quad . \tag{2.1}
$$

Notice that vertical composition of morphisms is always associative, and horizontal composition of diagrams (tensor multiplication of morphisms) is also associative in a *strict* tensor category. This is similar to what happens for vertical and horizontal composition of natural transformations (2-morphisms, or 2-cells) between functors (morphisms, or 1-cells) in the 2-category Cat, see [ML98, Sec. XII.3]. Indeed, a strict tensor category can be seen as a 2-category with only one object (or 0-cell).

As already outlined, the notion of strict tensor category can be *relaxed* by dropping strict associativity of the tensor multiplication. This is exactly as dropping strict associativity of the horizontal composition of 2-morphisms, hence leading from a 2-category to the more general notion of bicategory. Again, a (non-necessarily strict) tensor category, see below, can be thought as a bicategory with only one object.

Definition 2.2.3. A **tensor category** $\mathcal{C}$, or better $(\mathcal{C}, \times, \mathrm{id}, \alpha, \lambda, \rho)$, is a category $\mathcal{C}$ together with a map $\times : (\mathcal{C}, \mathcal{C}) \to \mathcal{C}$ both on objects and morphisms, a distinguished object id and natural isomorphisms α, λ, ρ, fulfilling the following properties.

- $\times : (\mathcal{C}, \mathcal{C}) \to \mathcal{C}$ is a *functor* (as before).

- $\times : (\mathcal{C}, \mathcal{C}) \to \mathcal{C}$ is *associative up to isomorphism*, namely up to the natural isomorphism $\alpha : \times \circ (\times, 1_{\mathcal{C}}) \Rightarrow \times \circ (1_{\mathcal{C}}, \times)$, which is called *associator* and runs between functors $(\mathcal{C}, \mathcal{C}, \mathcal{C}) \to \mathcal{C}$. Explicitly, the associator is given by isomorphisms

$$\alpha_{a,b,c} : (a \times b) \times c \cong a \times (b \times c)$$

  for every triple a, b, c in $\mathcal{C}$, which are natural on arrows and make the *pentagon diagrams* commute

$$
\begin{array}{ccc}
((a \times b) \times c) \times d \longrightarrow (a \times b) \times (c \times d) \longrightarrow a \times (\times(c \times d)) \\
\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\
(a \times (b \times c)) \times d \longrightarrow\qquad\qquad\qquad\qquad\qquad a \times ((b \times c) \times d)
\end{array}
$$

$$(2.2)$$

  for every a, b, c, d in $\mathcal{C}$.

- $\times : (\mathcal{C}, \mathcal{C}) \to \mathcal{C}$ has id as a *tensor unit up to isomorphism*, namely up to the natural isomorphisms $\lambda : L_{\mathrm{id}} \Rightarrow 1_{\mathcal{C}}$ (resp. $\rho : R_{\mathrm{id}} \Rightarrow 1_{\mathcal{C}}$), where L_{id} (resp. R_{id}) are the functors $\mathcal{C} \to \mathcal{C}$ given by left (resp. right) tensor multiplication by id on objects and 1_{id} on arrows, hence they are called respectively *left and right unitor*. Explicitly

$$\lambda_a : \mathrm{id} \times a \cong a \,, \quad \rho_a : a \times \mathrm{id} \cong a$$

  for every a in $\mathcal{C}$, they are natural on arrows, make the *triangle diagrams* commute

$$
\begin{array}{ccc}
(a \times \mathrm{id}) \times b \longrightarrow a \times (\mathrm{id} \times b) \\
\searrow \qquad\qquad \swarrow \\
a \times b
\end{array}
$$

$$(2.3)$$

  for every a, b in $\mathcal{C}$, and also

$$\lambda_{\mathrm{id}} = \rho_{\mathrm{id}} : \mathrm{id} \times \mathrm{id} \to \mathrm{id}\,. \tag{2.4}$$

A tensor category is **strict** (in the sense of Definition (2.2.1)) if and only if α, λ, ρ are all identities.

Remark 2.2.4. In the following we will be mainly interested in *strict* tensor categories, also in view of Theorem 2.2.11. Examples coming from QFT are of this type (composition of endomorphisms of operator algebras is strictly associative). For these reasons, we will sometimes omit to specify "strict", when clear from the context or not "strictly" necessary.

Remark 2.2.5. Equation (2.2) says that two possible transitions (isomorphisms) between $((a \times b) \times c) \times d$ and $a \times (b \times (c \times d))$ obtained by moving brackets (applying α) are the same. Equations (2.3) and (2.4) say something similar about the insertion of id in a tensor product (applying λ and ρ). Mac Lane's *coherence* theorem says that the same is true for all the possibly different transitions obtained by moving brackets and inserting tensor units in n-fold tensor products of objects. All these isomorphisms coincide, i.e., all such diagrams are commutative. In order to formulate it in a more compact way, it is convenient to introduce first the notions of *tensor functor* and *tensor natural transformation*.

Definition 2.2.6. A **tensor functor** F, or better (F, μ, η), between two tensor categories $(\mathcal{C}, \times_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}, \ldots)$ and $(\mathcal{D}, \times_{\mathcal{D}}, \mathrm{id}_{\mathcal{D}}, \ldots)$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$, a natural isomorphism μ and an isomorphism η [3], which respect tensor products in the following sense. We improperly use the symbols $\times$, id, $\ldots$ for both $\mathcal{C}$ and $\mathcal{D}$.

- The natural isomorphism $\mu : \times \circ (F, F) \Rightarrow F \circ \times$, called *multiplier*, is explicitly given by

$$\mu_{a,b} : F(a) \times F(b) \cong F(a \times b)$$

for every a, b in $\mathcal{C}$, and makes the following diagrams commute

$$
\begin{array}{ccccc}
(F(a) \times F(b)) \times F(c) & \longrightarrow & F(a \times b) \times F(c) & \longrightarrow & F((a \times b) \times c) \\
\downarrow & & & & \downarrow \\
F(a) \times (F(b) \times F(c)) & \longrightarrow & F(a) \times F(b \times c) & \longrightarrow & F(a \times (b \times c))
\end{array}
$$

for every a, b, c in $\mathcal{C}$. We call them *2-cocycle diagrams*.

---

[3] these tensor functors are called *strong* in [ML98, Sec. XI.2] in order to distinguish with non-strong ones where μ and η are not required to be invertible. We consider only strong tensor functors in this work.

- The isomorphism

$$\eta : \mathrm{id} \cong F(\mathrm{id})$$

makes the following diagrams commute

$$
\begin{array}{ccc}
\mathrm{id} \times F(a) & \longrightarrow & F(a) \\
\downarrow & & \downarrow \\
F(\mathrm{id}) \times F(a) & \longrightarrow & F(\mathrm{id} \times a)
\end{array}
\quad , \quad
\begin{array}{ccc}
F(a) \times \mathrm{id} & \longrightarrow & F(a) \\
\downarrow & & \downarrow \\
F(a) \times F(\mathrm{id}) & \longrightarrow & F(a \times \mathrm{id})
\end{array}
$$

for every a in $\mathcal{C}$.

A tensor functor is called **strict** if μ and η are all identities.

Remark 2.2.7. Notice that a tensor functor between strict tensor categories need not be strict. Also, a strict tensor functor can run between two non-strict tensor categories. Moreover, in order to introduce *tensor equivalences* we must keep the degree of freedom of considering *non-strict* tensor functors.

Definition 2.2.8. A **tensor natural transformation** τ between two tensor functors (F, μ_F, η_F) and (G, μ_G, η_G) running between two tensor categories $\mathcal{C}$ and $\mathcal{D}$, is just a natural transformation $\tau : F \Rightarrow G$ which makes the following diagrams commute

$$
\begin{array}{ccc}
F(a) \times F(b) & \longrightarrow & F(a \times b) \\
\downarrow & & \downarrow \\
G(a) \times G(b) & \longrightarrow & G(a \times b)
\end{array}
\quad , \quad
\begin{array}{c}
\mathrm{id}
\end{array}
\begin{array}{c}
F(\mathrm{id}) \\
\nearrow \quad \downarrow \\
\searrow \\
G(\mathrm{id})
\end{array}
$$

for every a, b in $\mathcal{C}$. Similarly, a **tensor natural isomorphism** is a natural isomorphism which makes the same diagrams commute, and in this case we write $F \cong G$, or better $F \cong_{TC} G$.

Definition 2.2.9. Let $\mathcal{C}$ and $\mathcal{D}$ be two tensor categories. A tensor functor $F : \mathcal{C} \to \mathcal{D}$ is called an **equivalence of tensor categories** if there is another tensor functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \cong 1_{\mathcal{C}}$ and $F \circ G \cong 1_{\mathcal{D}}$ via tensor natural isomorphisms $\tau : G \circ F \Rightarrow 1_{\mathcal{C}}$ and $\beta : F \circ G \Rightarrow 1_{\mathcal{D}}$.

In this case the two categories are called **tensor equivalent**, and we write $\mathcal{C} \simeq \mathcal{D}$, or better $\mathcal{C} \simeq_{TC} \mathcal{D}$.

Remark 2.2.10. It might be surprising (and useful) to know that the characterization of equivalences seen in Theorem 2.1.11 holds verbatim for tensor equivalences, see [Müg10b, p. 101], [EGNO15, Def. 2.4.1, Rmk. 2.4.10]. In other words, a tensor functor is an equivalence of tensor categories if and only if it is an equivalence of categories.

Now we can state Mac Lane's coherence theorem [ML98, Thm. 1 §XI.3] for tensor categories, in one of its equivalent formulations.

Theorem 2.2.11. (**Mac Lane**). *Every tensor category is tensor equivalent to a strict one.*

Anyway notice that a tensor category *need not* be tensor *isomorphic* to a strict one [EGNO15, Rmk. 2.8.6], as shown by taking categories where objects are labeled by elements of some group G, tensor products are given by the group operation and strict associativity is perturbed with a 3-cocycle on G with nontrivial 3-cohomology class. Hence Mac Lane's theorem, despite its generality, is very sharp.

2.3 Braided tensor categories (BTCs)

In this section, and later on, we restrict ourselves to *strict* tensor categories and go back to the (non-strict) Example 2.1.8 of vector spaces with the usual tensor product. We recall the notion of *braiding* on a strict tensor category. It generalizes the flip isomorphism $\sigma_{V,W} : V \otimes W \cong W \otimes V$ to more general commutativity constraints $\mathcal{E}_{a,b} : a \times b \cong b \times a$, which are still compatible with the tensor structure but not necessarily "square to the identity", i.e., $\mathcal{E}_{b,a} \circ \mathcal{E}_{a,b}$ need not be $1_{a \times b}$ for all a, b. In other words we allow non-symmetric braidings, i.e., pure braid group symmetries on our tensor structures. A very natural class of examples of braided tensor categories, those coming from completely rational chiral CFTs, are indeed maximally far away from being symmetric, namely they are *modular*. In addition to the previous references, see [Reh90b], [FRS92], [Müg03], [Müg12], [DGNO10], [EGNO15].

Definition 2.3.1. Let $(\mathcal{C}, \times, \mathrm{id})$ be a strict tensor category as in Definition 2.2.3 and denote by $\times^{\mathrm{op}}$ the *opposite tensor multiplication* functor, namely $\times^{\mathrm{op}}(a, b) := b \times a$. Then $\mathcal{C}$, or better $(\mathcal{C}, \times, \mathrm{id}, \mathcal{E})$, is a **braided tensor category** if there is natural isomorphism $\mathcal{E} : \times \Rightarrow \times^{\mathrm{op}}$, called **braiding**, together with its inverse $\mathcal{E}^{-1} : \times^{\mathrm{op}} \Rightarrow \times$, such that both respect the strict

tensor structure. Namely, the braiding $\mathcal{E}$ and its inverse $\mathcal{E}^{-1}$ are given by a family of isomorphisms

$$\mathcal{E}_{a,b} : a \times b \cong b \times a, \quad \mathcal{E}_{a,b}^{-1} : b \times a \cong a \times b$$

for every a, b in $\mathcal{C}$, which are natural, i.e.

$$\mathcal{E}_{c,d} \circ (t \times s) = (s \times t) \circ \mathcal{E}_{a,b} \tag{2.5}$$

for every $t : a \to c$, $s : b \to d$ in $\mathcal{C}$, and they fulfill the following properties.

- $\mathcal{E}$ and $\mathcal{E}^{-1}$ are *multiplicative*, i.e.

$$\mathcal{E}_{a \times b, c} = (\mathcal{E}_{a,c} \times 1_b) \circ (1_a \times \mathcal{E}_{b,c}), \quad \mathcal{E}_{a,b \times c}^{-1} = (\mathcal{E}_{a,b}^{-1} \times 1_c) \circ (1_b \times \mathcal{E}_{a,c}^{-1}) \tag{2.6}$$

  or equivalently

$$\mathcal{E}_{a \times b, c} = (\mathcal{E}_{a,c} \times 1_b) \circ (1_a \times \mathcal{E}_{b,c}), \quad \mathcal{E}_{a,b \times c} = (1_b \times \mathcal{E}_{a,c}) \circ (\mathcal{E}_{a,b} \times 1_c) \tag{2.7}$$

  for every a, b, c in $\mathcal{C}$.

- $\mathcal{E}$ and $\mathcal{E}^{-1}$ are *unital*, i.e.

$$\mathcal{E}_{a,\mathrm{id}} = 1_a, \quad \mathcal{E}_{\mathrm{id},a} = 1_a. \tag{2.8}$$

Remark 2.3.2. Indeed, the two conditions expressed in (2.8) are equivalent because $\mathcal{E}_{a,\mathrm{id}} = \mathcal{E}_{\mathrm{id},a}^{-1}$, and more, they are consequences of the previous conditions (2.5), (2.7), see [EGNO15, Ex. 8.1.6]. Analogous statements hold also in the non-strict case.

Remark 2.3.3. In a strict tensor category, the previous conditions are better understood by means of string diagrams instead of commutative diagrams (hexagonal in this case). Namely,

if we denote $\mathcal{E}_{a,b} = $ then e.g. (2.5) and the l.h.s. of (2.7), (2.8) become

respectively, where in (2.5) we set $b = d$ and $t \times s = t \times 1_b$.

Remark 2.3.4. It is convenient to denote by $\mathcal{E}^{\mathrm{op}} : \times \Rightarrow \times^{\mathrm{op}}$ the **opposite braiding**, given by $\mathcal{E}^{\mathrm{op}}_{a,b} := \mathcal{E}^{-1}_{b,a}$. The terminology is motivated by the fact that we get $\mathcal{E}^{\mathrm{op}}_{b,a} \circ \mathcal{E}_{a,b} = 1_{a \times b}$ for all a, b, despite $\mathcal{E}_{b,a} \circ \mathcal{E}_{a,b}$ need not be $1_{a \times b}$.

In pictures, if we denote the opposite braiding by $\mathcal{E}^{\mathrm{op}}_{a,b} =:$

then the previous statement becomes intuitive

   and can happen that   .

The "square of braiding morphisms" $\mathcal{E}_{b,a} \circ \mathcal{E}_{a,b}$ is called **monodromy** of a, b in $\mathcal{C}$. Notice that it equals $1_{a \times b}$ if and only if $\mathcal{E}^{\mathrm{op}}_{b,a} = \mathcal{E}_{b,a}$, if and only if $\mathcal{E}^{\mathrm{op}}_{a,b} = \mathcal{E}_{a,b}$, if and only if the monodromy of b, a equals $1_{b \times a}$. In particular, the "vanishing" or "trivialization" of the monodromy morphisms is symmetric in a and b. This is *not* true in general for the braiding morphisms.

Notice that $\mathcal{E}$ is a braiding if and only if $\mathcal{E}^{\mathrm{op}}$ is a braiding and that they coincide if and only if the braided tensor category $\mathcal{C}$ is *symmetric*. It is an interesting exercise to show that (2.5) and (2.7) imply the usual relations of the braid group (Yang-Baxter equations, see [BK01, Ex. 1.2.6], namely the relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ represented via braiding isomorphisms as the easiest two possible arrows in $a \times b \times c \to c \times b \times a$).

The notions of *braided tensor functor* and *braided tensor natural transformation* are given by adding *constraints* involving the braiding, in the obvious way, and *not* by adding structure. We remark again that we want to keep the degree of freedom of having non-strict tensor functors even between strict tensor categories.

Definition 2.3.5. A **braided tensor functor** F, or better (F, μ, η), between two braided strict tensor categories $(\mathcal{C}, \times_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}, \mathcal{E}_{\mathcal{C}})$ and $(\mathcal{D}, \times_{\mathcal{D}}, \mathrm{id}_{\mathcal{D}}, \mathcal{E}_{\mathcal{D}})$ is

just a tensor functor $F : \mathcal{C} \to \mathcal{D}$ (in the sense of Definition 2.2.6) which respects the braiding.

Namely, the natural transformations involved make the following diagrams commute

$$
\begin{array}{ccc}
F(a) \times F(b) & \longrightarrow & F(a \times b) \\
\downarrow & & \downarrow \\
F(b) \times F(a) & \longrightarrow & F(b \times a)
\end{array}
$$

for every a, b in $\mathcal{C}$. Again we use improperly the symbols $\times$, id, $\mathcal{E}$ for both $\mathcal{C}$ and $\mathcal{D}$.

Notice that if F is *strict* then the previous conditions boil down to constraints on the action of the functor on braiding isomorphisms $F(\mathcal{E}_{a,b}) = \mathcal{E}_{F(a),F(b)}$ for every a, b in $\mathcal{C}$.

Definition 2.3.6. A **braided tensor natural transformation** τ between two braided tensor functors (F, μ_F, η_F) and (G, μ_G, η_G) running between two braided tensor categories $\mathcal{C}$ and $\mathcal{D}$, is just a tensor natural transformation $\tau : F \Rightarrow G$ with no additional constraint involving the braiding.

Similarly for **braided tensor natural isomorphism**, hence by definition two braided tensor functors F and G are braided tensor naturally isomorphic if and only if tensor naturally isomorphic, or better tensor 2-isomorphic.

Definition 2.3.7. Let $\mathcal{C}$ and $\mathcal{D}$ be two braided tensor categories. A braided tensor functor $F : \mathcal{C} \to \mathcal{D}$ is called an **equivalence of braided tensor categories** if there is another braided tensor functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \cong 1_{\mathcal{C}}$ and $F \circ G \cong 1_{\mathcal{D}}$ via (braided) tensor natural isomorphisms $\tau : G \circ F \Rightarrow 1_{\mathcal{C}}$ and $\beta : F \circ G \Rightarrow 1_{\mathcal{D}}$.

In this case the two categories are called **braided tensor equivalent**, and we write $\mathcal{C} \simeq \mathcal{D}$, or better $\mathcal{C} \simeq_{BTC} \mathcal{D}$.

Remark 2.3.8. Notice that $\mathcal{C} \simeq_{TC} \mathcal{D}$ does not imply $\mathcal{C} \simeq_{BTC} \mathcal{D}$. Indeed a tensor category can be equipped with mutually inequivalent braided structures, as show by the example of the *Ising tensor category* (arising e.g. as DHR representation category of the chiral Virasoro net with $c = 1/2$), which admits exactly four inequivalent braided structures, see [BKLR15, Ex. 3.1, 4.19].

On the other hand, similarly to the case of tensor equivalences, we have that a braided tensor functor is a braided tensor equivalence if and only if it is an equivalence of categories, see [EGNO15, Def. 8.1.7].

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2.4 Linearity, unitarity

In the previous sections we reviewed how tensor multiplications and braidings can be introduced in category theory. Besides these, the type of categories we deal with in this work has more structure, which is modeled on (and due to) operator algebras on complex Hilbert spaces. We will deal with a simple case of C^*-**categories**, see the seminal work of [DR89] for references, namely we restrict to those "finitely generated" or **rational**, i.e., those with finitely many isomorphism classes (in the sense of Definition 2.1.2) of irreducible objects (see Definition 2.4.3). They will arise from *completely rational* models in chiral CFT as the finitely many different *superselection sectors* of the theory of local observables.

Now, we have already a notion of multiplication on a category, given by composition of morphisms. In order to mimic an operator algebra (i.e. generalize it to an *operator algebroid*) we introduce two additional structures. One is *linearity*, namely morphisms can be added and multiplied by scalars from some field. The other is *unitarity*, namely we choose the field to be $\mathbb{C}$ and add an adjunction operation on morphisms. Notice that linear categories (with more structure) are of general interest, see e.g. [ENO05], [DMNO13] among many others, while the unitary case is usually motivated by mathematical physics. See the Appendix by M. Müger in [HM06].

Definition 2.4.1. A category $\mathcal{C}$ is called **linear** over $\mathbb{C}$, or $\mathbb{C}$-**linear category**, if its "Hom-spaces" (cf. Definition 2.1.2) have the following properties.

- $\mathrm{Hom}_{\mathcal{C}}(a, b)$ is a vector space over $\mathbb{C}$ for every pair $a, b \in \mathcal{C}^{(0)}$.

- The composition of arrows $\mathrm{Hom}_{\mathcal{C}}(a, b) \times \mathrm{Hom}_{\mathcal{C}}(b, c) \to \mathrm{Hom}_{\mathcal{C}}(a, c)$, $t, s \mapsto s \circ t$ is bilinear over $\mathbb{C}$ for every $a, b, c \in \mathcal{C}^{(0)}$.

Remark 2.4.2. Functors $F : \mathcal{C} \to \mathcal{D}$ between linear categories are always required to be compatible with the linear structure. Namely, for every $t, s : a \to b$ in $\mathcal{C}$ and $\lambda \in \mathbb{C}$ we require $F(\lambda t + s) = \lambda F(t) + F(s)$. In particular we consider only *bilinear tensor multiplications* $\times : (\mathcal{C}, \mathcal{C}) \to \mathcal{C}$.

In the fashion of Schur's Lemma in representation theory of groups or algebras, one defines

Definition 2.4.3. An object $a \in \mathcal{C}^{(0)}$ in a $\mathbb{C}$-linear category $\mathcal{C}$ is called **irreducible** if $\mathrm{Hom}_{\mathcal{C}}(a, a) = \mathbb{C}1_a$.

Similarly to the passage from $\mathbb{C}$-vector spaces to Hilbert spaces, we can pass from $\mathbb{C}$-linear categories to unitary ones.

Definition 2.4.4. A $\mathbb{C}$-linear category $\mathcal{C}$ is called **unitary**, or also ***-category**, if it is equipped with an *antilinear contravariant* functor $* : \mathcal{C} \to \mathcal{C}$, namely $(\lambda t + s)^* = \bar{\lambda} t + s$ and for every $t : a \to b$ then $t^* : b \to a$ [4], which is *trivial on objects*, i.e. $a^* = a$, *involutive*, i.e. squares to the identity, and *positive*, namely $t^* \circ t = 0$ implies $t = 0$ for every $t : a \to b$.

Example 2.4.5. As anticipated, the motivating example of unitary category is given by $Hilb$, the category of (not necessarily finite-dimensional) Hilbert spaces, where morphisms are given by bounded linear operators and the $*$ map is the usual adjunction of operators induced by the Hilbert sesquilinear product.

Remark 2.4.6. Functors $F : \mathcal{C} \to \mathcal{D}$ between unitary categories are always required to be compatible with the $*$-structure. Namely, for every $t : a \to b$ in $\mathcal{C}$ we require $F(t^*) = F(t)^*$. In particular we consider only $*$-*preserving tensor multiplications* in $\times : (\mathcal{C}, \mathcal{C}) \to \mathcal{C}$.
Moreover, isomorphisms $t : a \cong b$ in a unitary category and natural isomorphisms $\tau : F \Rightarrow G$ between $*$-functors are usually required to be unitary, in the sense that $t^{-1} = t^*$ and $\tau_a^{-1} = \tau_a^*$ for every a in $\mathcal{C}$. In particular, in the unitary case we consider only unitary equivalence classes of objects, unitary constraints on tensor structures and we admit only *unitary braidings*.
This last requirement is not very restrictive, indeed braidings are automatically unitary on a wide family of tensor categories, called *fusion categories* (see [ENO05] and later), thanks to [Gal14, Thm. 3.2].

Remark 2.4.7. The action of the $*$ map on morphisms of a unitary category $\mathcal{C}$ has an intuitive graphical expression, namely we can write $t : a \to b$, $t^* : b \to a$, $(s \circ t)^* = t^* \circ s^*$, with t, s composable morphisms, respectively as

$$
\begin{array}{c} a \\ | \\ \boxed{t} \\ | \\ b \end{array} , \qquad
\begin{array}{c} b \\ | \\ \boxed{t^*} \\ | \\ a \end{array} , \qquad
\left\{ \begin{array}{c} | \\ \boxed{t} \\ \boxed{s} \\ | \end{array} \right\}^* =
\begin{array}{c} | \\ \boxed{s^*} \\ \boxed{t^*} \\ | \end{array} .
$$

---

[4]hence is neither a functor (in the sense of Definition 2.1.4 which describes ordinary, or better, *covariant* functors) nor respects the previous remark.

Moreover, if we represent the $*$ map as a *mirror reflection* with respect to horizontal lines then the graphical notation can be used also in unitary braided tensor categories, because it matches with the definition of opposite braiding $\mathcal{E}_{a,b}^* = \mathcal{E}_{b,a}^{\mathrm{op}}$, i.e.

$$\left\{ \begin{array}{cc} a & b \\ & \\ & \\ b & a \end{array} \right\}^* = \begin{array}{cc} b & a \\ & \\ & \\ a & b \end{array} \quad .$$

On the other hand, to our knowledge there is no convincing graphical interpretation of addition of morphisms in the Hom-spaces (while multiplication by scalars $\lambda \in \mathbb{C}$ can be written as $t \mapsto \lambda 1_{\mathrm{id}} \times t$). So one is usually forced, in practice, to mix graphical and arabic notation, see e.g. [BKLR15] for a wide use of both. To our intuition, the graphical notation is a useful tool that simplifies in a coherent and essential way calculations and proofs. The reader is always invited to draw pictures.

Clearly, a unitary category with only one object is the same as a $*$-algebra with a positive involution. We conclude the section by making contact with the more interesting notion of C^*-category mentioned at the beginning. C^*-categories generalize C^*-algebras to the case with multiple objects.

Definition 2.4.8. A C^***-category** is a unitary category where every Hom-space is a Banach space and the collection of all norms is submultiplicative also between different objects, i.e., $\|s \circ t\| \leq \|s\| \|t\|$ whenever t and s are composable, and fulfills the C^*-identity, i.e., $\|t^* \circ t\| = \|t\|^2$.

Moreover, assuming to have finite-dimensional Hom-spaces, the notions of unitary category and of C^*-category are the same, see [Müg00, Prop. 2.1].

Proposition 2.4.9. [Müg00]. *Let $\mathcal{C}$ be a $\mathbb{C}$-linear category and assume that the Hom-spaces $\mathrm{Hom}_\mathcal{C}(a,b)$ are finite-dimensional vector spaces for every $a, b \in \mathcal{C}$. Then $\mathcal{C}$ is unitary if and only if it is a C^*-category.*

Remark 2.4.10. Notice that the family of C^*-categories described in the previous proposition is the multiple objects analogue of finite-dimensional C^*-algebras, i.e., of finite direct sums of complex matrix algebras (a very special class of C^*-algebras!).

In the context of C^*-categories with a *strict tensor structure*, see the previous sections, there is a stronger and more fundamental notion of "finite-dimensionality" for objects of $\mathcal{C}$, namely the *existence of conjugate objects in $\mathcal{C}$*, in the sense of [LR97]. Indeed, the latter implies finite-dimensionality of Hom-spaces as in Proposition 2.4.9, see the next section for a precise formulation.

Here we just mention that this "finite-dimensionality" condition is of fundamental importance in AQFT, namely it corresponds to the existence of *antiparticle states* in the sense of [DHR71], [DHR74]. The condition will be automatically satisfied in our work on *completely rational* chiral CFTs [KLM01, Cor. 39], once we restrict to *elementary particle states* (irreducible positive energy representations of the observable net) and finite direct sums of those. Moreover, the existence of conjugate objects can be generalized in several ways, see [Müg10b] for an overview and [ENO05] for the most famous class of categories with a possibly weaker notion of conjugation (fusion categories). Anyway, we shall deal with the unitary case only.

2.5 Subobjects, direct sums, conjugates

In the previous section we considered $*$-linear operations on the arrows of some category. Now we turn to objects (and arrows along with them) and require the existence of *subobjects*, *direct sums of objects*, *conjugate objects*, which generalize usual constructions available in categories of vector spaces with linear maps among them. In the following definitions, for which we refer to [DR89], [LR97], [BKLR15], let $\mathcal{C}$ be a ($\mathbb{C}$-linear) unitary category

Definition 2.5.1. An arrow $t : a \to b$ in $\mathcal{C}$ is called an **isometry** if $t^* \circ t = 1_a$, it is called a **unitary** if it is an isomorphism of the unitary category, i.e., $t^* \circ t = 1_a$ and $t \circ t^* = 1_a$. An arrow $p : a \to a$ in $\mathcal{C}$ is called a **projection** if $p^* \circ p = p$.

This generalizes the usual notions of orthogonal projections, isometries and unitaries between Hilbert spaces (i.e. in the category of), to the case where neither arrows need have kernel or range, nor objects need have vectors inside at all.

Definition 2.5.2. Consider two objects $a, b \in \mathcal{C}^{(0)}$. Then b is called a **subobject** of a if there is an isometry $v : b \to a$. We say that $\mathcal{C}$ **has**

subobjects if for every object a and every projection $p : a \to a$ there is an object b and an isometry $v : b \to a$, both in $\mathcal{C}$, such that $v \circ v^* = p$. Subobjects are denoted by $b \prec a$.

Definition 2.5.3. Consider two objects $a, b \in \mathcal{C}^{(0)}$. Then an object $c \in \mathcal{C}^{(0)}$ is called a **direct sum** of a and b if there are isometries $v : a \to c$, $w : b \to c$, both in $\mathcal{C}$, such that $v \circ v^* + w \circ w^* = 1_c$. We say that $\mathcal{C}$ **has (finite) direct sums** if for every pair of $a, b \in \mathcal{C}^{(0)}$ there is a direct sum $c \in \mathcal{C}^{(0)}$ of a and b. Direct sums are denoted by $c = a \oplus b$.

Remark 2.5.4. Notice that every object of $\mathcal{C}$ in the isomorphism class $[b]_\cong$ with $b \prec a$ is a subobject of a in $\mathcal{C}$. Similarly every object in $[a \oplus b]_\cong$ is a direct sum of a and b in $\mathcal{C}$.

In order to introduce the notion of *conjugate object* in the sense of [LR97], we need to assume, in addition, to have on $\mathcal{C}$ a strict *tensor multiplication* (as in Definition 2.2.1) and *C^*-structure* (as in Definition 2.4.8). Indeed conjugate objects are defined by specifying how the tensor unit (*vacuum sector*) should sit into the product of an object (some *superselection charge*) with its conjugate object (the *conjugate charge*). Conjugate objects have been first used in QFT to describe particle-antiparticle annihilation processes in a representation-theoretical way, see [DHR74, Sec. III].

Example 2.5.5. The motivating example to keep in mind in order to understand conjugates is given by the category $URep(G)$ of strongly continuous unitary representations of a compact group G on Hilbert spaces, together with its subcategory $URep_f(G)$ of finite-dimensional ones. The objects for both are G-modules and the arrows are bounded linear maps which intertwine the G-actions. Now, representation theory of compact groups is well known in mathematics. It is known that any (*finite-dimensional*) G-module is *completely reducible* into a (finite) direct sum of irreducible ones. This fact is essentially due to the compactness of G, more specifically it comes from the spectral decomposition of a self-adjoint operator which can be constructed thanks to the normalized Haar measure.

Forgetting now the group G and looking at the category $URep(G)$, which is its *dual*, the same conclusion can be drawn on completely different (and more general) grounds by replacing the finite-dimensionality assumption with the existence of conjugate objects, see [LR97]. The meaning of "finite-dimensionality" is then described by the *conjugate equations* in an *intrinsic*

way, i.e., one does not need linear bases inside objects anymore, neither their cardinality, nor vectors at all, cf. Remark 2.1.3.

Definition 2.5.6. Let $\mathcal{C}$ be a strict tensor C^*-category and let $a \in \mathcal{C}^{(0)}$. An object $b \in \mathcal{C}^{(0)}$ is called a **conjugate object** of a, or better a and b are called conjugate to each other in $\mathcal{C}$, if there exist morphisms $r : \mathrm{id} \to b \times a$, $s : \mathrm{id} \to a \times b$ in $\mathcal{C}$ which solve the **conjugate equations**, namely

$$(s^* \times 1_a) \circ (1_a \times r) = 1_a, \quad (r^* \times 1_b) \circ (1_b \times s) = 1_b \qquad (2.9)$$

or equivalently

$$(2.10)$$

the "zig-zag equations", if we let $r = \;$ $\; = \;$ $\;$ and $\; s = \;$ $.$

The category $\mathcal{C}$ is said to **have conjugate objects**, or equivalently called a **rigid category**, if every object in $\mathcal{C}$ has a conjugate (again in $\mathcal{C}$). Given an object a in $\mathcal{C}$, a conjugate object of a is usually denoted by $\bar{a}$ and a solution of (2.9) by $r : \mathrm{id} \to \bar{a} \times a$, $\bar{r} : \mathrm{id} \to a \times \bar{a}$. Notice that we can choose $\bar{\bar{a}} = a$ and the same morphisms provide a solution of (2.9).

Remark 2.5.7. Similarly to Remark 2.5.4, if a has a conjugate $\bar{a}$ in $\mathcal{C}$, then every object in the isomorphism class $[\bar{a}]_\cong$ serves as a conjugate for a. Moreover, given $[a]_\cong$ the conjugate isomorphism class $[\bar{a}]_\cong$ is uniquely determined.

For each pair of objects a, b in $\mathcal{C}$, both admitting conjugates in $\mathcal{C}$, there are linear isomorphisms between $\mathrm{Hom}_\mathcal{C}(a, b)$ and any of the following Hom-spaces $\mathrm{Hom}_\mathcal{C}(\mathrm{id}, b \times \bar{a})$, $\mathrm{Hom}_\mathcal{C}(\mathrm{id}, \bar{a} \times b)$, $\mathrm{Hom}_\mathcal{C}(\bar{b} \times a, \mathrm{id})$, $\mathrm{Hom}_\mathcal{C}(a \times \bar{b}, \mathrm{id})$. All these isomorphism can be easily represented via string diagrammatical calculus, but the coherence of the latter with the linear tensor structure, which guarantees the validity of the previous statements, is the content of the following general version of the *Frobenius Reciprocity Theorem*, see [LR97, Lem. 2.1].

Proposition 2.5.8. (Frobenius reciprocity). *Let $\mathcal{C}$ be a strict tensor C^*-category and let $a, \overline{a}$ be a pair of conjugate objects in $\mathcal{C}$ and $r, \overline{r}$ a solution of the conjugate equations (2.9). Then the map*

$$\mathrm{Hom}_{\mathcal{C}}(a \times b, c) \to \mathrm{Hom}_{\mathcal{C}}(b, \overline{a} \times c)$$

given by $t \mapsto (1_{\overline{a}} \times c) \circ (r \times 1_b)$ is a linear isomorphism and the inverse is $s \mapsto (\overline{r}^* \times 1_c) \circ (1_a \times s)$.

Similar isomorphisms hold for n-fold tensor products of objects $a_1 \times \ldots \times a_n$. Indeed if a and b admit conjugate objects in $\mathcal{C}$, so does $a \times b$ and a representative is given by $\overline{b} \times \overline{a}$. It is also worthwhile mentioning that

Proposition 2.5.9. [LR97]. *Let $\mathcal{C}$ be a strict tensor C^*-category and denote by $\mathcal{C}_f \subset \mathcal{C}$ the (full) subcategory of objects in $\mathcal{C}$ which admit a conjugate object in $\mathcal{C}$. Then $\mathcal{C}_f$ is closed under conjugates (trivial) and tensor products. If moreover $\mathcal{C}$ has finite direct sums and subobjects, then $\mathcal{C}_f$ is also closed under such operations.*

Conjugation of objects extends to arrows to an antilinear (covariant) functor. Namely for each a, b in $\mathcal{C}$ both admitting conjugates in $\mathcal{C}$ and for each $t : a \to b$, one can define an arrow $t^{\bullet} : \overline{a} \to \overline{b}$ by making covariant the *-operation : $t \mapsto t^*$ using the solution of the conjugate equations. There are two ways of doing so, as one can easily imagine from string diagrams, but they need not coincide in general, as pointed out by [BKLR15, Sec. 2.3], and the graphical calculus might lose coherence. For the purposes and needs of this work we can overcome the problem, relying on [BKLR15, Prop. 2.6], by assuming *irreducibility of the tensor unit*, i.e., $\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}, \mathrm{id}) = \mathbb{C}$, and choosing a special class of solutions of the conjugate equations (2.9), namely the *standard* ones, see Definition 2.5.11.

The irreducibility of the tensor unit, in the case of categories arising from chiral CFTs, corresponds to considering of endomorphisms of von Neumann factors, as we shall see later on in details, and is due to the uniqueness of the vacuum vector, see Proposition 1.1.8.
Going back to our motivating Example 2.5.5, we observe that the category $URep(G)$ does have a C^*-structure and a tensor multiplication which we can assume to be strict by Theorem 2.2.11, see also [DR89, Sec. 6]. Moreover, the role of the tensor unit is played by the trivial G-module, $\mathbb{C}$, which is obviously

irreducible. Hence, setting $\mathcal{C} = URep(G)$ (or better its strictification) we are in the case described by the following proposition [LR97, Lem. 3.2].

Proposition 2.5.10. [LR97]. *Let $\mathcal{C}$ be a strict tensor C^*-category with finite direct sums and subobjects. Assume that the tensor unit* id *is irreducible, i.e.,* $\text{Hom}_{\mathcal{C}}(\text{id}, \text{id}) = \mathbb{C}$, *and denote by $\mathcal{C}_f \subset \mathcal{C}$ the (full) subcategory of objects in $\mathcal{C}$ which admit a conjugate object in $\mathcal{C}$.*
Then for every pair of objects a, b in $\mathcal{C}_f$ the Hom-spaces $\text{Hom}_{\mathcal{C}}(a, b)$, $\text{Hom}_{\mathcal{C}}(a, a)$ are finite-dimensional vector spaces, in particular every object a in $\mathcal{C}_f$ is completely reducible into a (finite) direct sum $a = \bigoplus_i a_i$, where the a_i are irreducibles in $\mathcal{C}_f$. In other words, the subcategory $\mathcal{C}_f$ is automatically a **semisimple category**.

Moreover, on $\mathcal{C}_f$ there is an *intrinsic* notion on *dimension*, which is based on the conjugate equations and on the "correct" choice of solutions for them. Let $\mathcal{C}$ and $\mathcal{C}_f$ be as above and take an object a in $\mathcal{C}_f$, with conjugate object $\overline{a}$ together with a solution $r : \text{id} \to \overline{a} \times a, \overline{r} : \text{id} \to a \times \overline{a}$ of the conjugate equations (2.9). By definition, the morphisms $r^* \circ r$, $\overline{r}^* \circ \overline{r}$ are in $\text{Hom}_{\mathcal{C}}(\text{id}, \text{id})$ and by irreducibility of id they are both (real, positive) multiples of 1_{id}.
If in addition a is irreducible, then by Frobenius reciprocity $r, \overline{r}$ are both uniquely determined up to scalar multiples and we can normalize them such that $r^* \circ r = \overline{r}^* \circ \overline{r}$.

Definition 2.5.11. Let $\mathcal{C}$ and $\mathcal{C}_f$ as above and a an *irreducible* object in $\mathcal{C}_f$. A pair $r, \overline{r}$ as above is called a **normalized solution** of the conjugate equations (2.9), and the unique number $d_a \geq 0$ such that

$$r^* \circ r = \overline{r}^* \circ \overline{r} = d_a 1_{\text{id}} \tag{2.11}$$

or better

$$\overline{a} \bigcirc a \;\; = \;\; a \bigcirc \overline{a} \;\; = \;\; d_a 1_{\text{id}} \tag{2.12}$$

is called the (intrinsic) **dimension** of a in $\mathcal{C}$. By definition we have $d_a = d_{\overline{a}}$. Now, let a be a *reducible* object in $\mathcal{C}_f$, then the solution $r, \overline{r}$ is no longer unique up to scalars. So we consider direct sums of normalized solutions with respect to irreducible summands of a and call them **standard solutions** of the conjugate equations (2.9). The number $d_a \geq 0$ given by (2.11) does not depend on the choice of the standard solution, see [BKLR15, Prop. 2.4], and we call it again the dimension of a. If an object a lies in $\mathcal{C}$ and does not have a conjugate object we set $d_a = +\infty$.

Remark 2.5.12. Standard solutions have remarkable properties, namely they give rise to *traces* on the Hom-spaces, and they are "more left-right symmetric" than general solutions, i.e., they equip $\mathcal{C}$ with a *spherical structure*, namely

$$\tag{2.13}$$

in formulas

$$r^* \circ (1_{\overline{a}} \times t) \circ r = \overline{r}^* \circ (t \times 1_{\overline{a}}) \circ \overline{r} \tag{2.14}$$

for every $t : a \to a$ in $\mathcal{C}$. This property is also a characterization of standardness of the solution $r, \overline{r}$, see [LR97, Lem. 3.9]. Standard solutions can also be characterized as those solutions of (2.9) which are normalized and *minimize* (2.11), in the fashion of the index theory for subfactors. Indeed, with conjugate objects and a standard solution one can define the **standard left inverse** of a in $\mathcal{C}$ by setting for each $s : a \times b \to a \times c$

$$\varphi_a(s) := (d_a)^{-1} (r^* \times 1_c) \circ (1_{\overline{a}} \times s) \circ (r \times 1_b).$$

The standard left inverse is a linear map from $\mathrm{Hom}_{\mathcal{C}}(a \times b, a \times c)$ to $\mathrm{Hom}_{\mathcal{C}}(b, c)$ which is positive and faithful for $b = c$, and normalized, i.e., $\varphi_a(1_a) = 1_{\mathrm{id}}$ for $b = c = \mathrm{id}$. Furthermore, it does not depend neither on the choice of the standard solution, nor on the conjugate object $\overline{a}$, hence it is uniquely determined by $a \in \mathcal{C}_f$. The **trace property** for standard left inverses (up to dimensions) reads

$$d_a \, \varphi_a(s \circ t) = d_b \, \varphi_b(t \circ s) \tag{2.15}$$

for every a, b in $\mathcal{C}_f$ and $t : a \to b$, $s : b \to a$, see [BKLR15, Sec. 2.2] for proofs and clarifying pictures.

Remark 2.5.13. Standard left inverses have been discovered, prior to conjugate objects and conjugate equations, in [DHR71, Sec. III] in the context of charge conjugation in QFT. They are tightly connected with minimal faithful *conditional expectations* for inclusions of operator algebras $\rho(\mathcal{A}) \subset \mathcal{A}$ arising from DHR endomorphisms $\rho \in \mathrm{End}(\{\mathcal{A}\})$ of nets of local observables $\{\mathcal{A}(\mathcal{O})\}$, see [Lon89, Sec. 7] and Chapter 3 for definitions.

The terminology "dimension" in the previous definition is motivated by the following general fact.

Proposition 2.5.14. [LR97]. *Let $\mathcal{C}$ and $\mathcal{C}_f$ as above, then the dimension function $a \in \mathcal{C}_f^{(0)} \mapsto d_a \geq 0$ is additive (by definition) and tensor multiplicative, it is well defined on isomorphisms classes, moreover takes values $d_a \geq 1$ where $d_{\mathrm{id}} = 1$ and $d_a = 1$ if and only if a is invertible with respect to the tensor multiplication, namely $a \times \overline{a} \cong \mathrm{id} \cong \overline{a} \times a$*

Another motivation comes from our Example 2.5.5. In that case a G-module V has a conjugate object in $\mathcal{C} = URep(G)$, hence has (finite) intrinsic dimension in the sense of Definition 2.5.11, if and only if it is finite-dimensional as a vector space. In particular we have a consistent notation $\mathcal{C}_f = URep_f(G)$. Indeed a conjugate object of V is given by the complex conjugate vector space $\overline{V}$ with the conjugate transpose G-action and a solution of the equations (2.9) is provided by the linear maps

$$r : \lambda \in \mathbb{C} \mapsto \lambda \sum_{i=1,\dots,n} \overline{e}_i \otimes e_i \in \overline{V} \otimes V, \quad \overline{r} : \lambda \in \mathbb{C} \mapsto \lambda \sum_{i=1,\dots,n} e_i \otimes \overline{e}_i \in V \otimes \overline{V}$$

where $\{e_i\}$ is a basis of V and $\{\overline{e}_i\}$ is the dual basis of $\overline{V}$. Notice that we could have taken $Hilb$ as an example, which is recovered by considering $G = \{e\}$, but in that case the only irreducible object is $\mathbb{C}$ which has dimension 1.

Remark 2.5.15. In general, one does not expect to have $\mathcal{C} = \mathcal{C}_f$ because infinite-dimensional objects can be constructed, e.g., in $\mathcal{C} = Hilb$ or $\mathcal{C} = URep(G)$ by taking infinite sums of vector spaces.

Moreover, irreducible objects need not have finite dimension as in the case of $URep(G)$ with G compact. Indeed take the category $\mathcal{C} = URep(\widetilde{\mathcal{P}})$ of strongly continuous unitary representations of the Poincaré group (of its universal covering) and observe that it has many irreducible objects but none of them is finite-dimensional besides the trivial representation.

The same happens in the category $\mathcal{C} = \mathrm{DHR}\{\mathcal{A}_{c=1}\}$ of DHR superselection sectors (see Chapter 3) of the chiral Virasoro net with $c = 1$, were there are irreducible sectors which do not have a conjugate sector, see [Car03], [Reh94a].

The properties reviewed in this section guarantee that a strict tensor C^*-category with irreducible tensor unit, finite direct sums, subobjects and conjugate objects is automatically a *unitary fusion tensor category*, provided it is has finitely many isomorphisms classes of irreducible objects (**rationality**). The terminology *fusion category* is due to [ENO05], where the non-necessarily unitary case is taken into account. Notice that many authors, see e.g. [ENO05],

[DMNO13], work with skeletal categories (one object per isomorphism class). This makes clear that the size of this type of categories does not really matter in the analysis of their structure.

Definition 2.5.16. A **unitary fusion tensor category (UFTC)** is a unitary strict tensor category with finite direct sums, subobjects, conjugate objects, i.e., $\mathcal{C}_f = \mathcal{C}$, finitely many isomorphism classes of irreducible objects, such that $\text{Hom}_{\mathcal{C}}(\text{id}, \text{id}) = \mathbb{C}$ and $\text{Hom}_{\mathcal{C}}(a, b)$ is finite-dimensional for every a, b in $\mathcal{C}$.

Vice versa, a UFTC is automatically a rational C^*-category and notice that semisimplicity follows in either case.

Definition 2.5.17. Let $\mathcal{C}$ be a UFTC, the **spectrum** Δ of $\mathcal{C}$ is the set of its (finitely many) isomorphism classes of *irreducible* objects $\{[a_0], [a_1], \ldots, [a_n]\}$, where we assume $a_0 = \text{id}$, $[a_i] \neq [a_j]$ if $i \neq j$, and $n + 1$ is the **rank** of $\mathcal{C}$. The **fusion rules** of $\mathcal{C}$ are given by the non-negative integers defined by the fusion ring of the category, namely

$$[a_i] \times [a_j] = \bigoplus_k N_{i,j}^k [a_k] \tag{2.16}$$

where $i, j, k \in \{0, \ldots, n\}$ and $N_{i,j}^k$ is the *multiplicity* (number of representatives of the same class) of $[a_k]$ in $[a_i] \times [a_j]$.

It is clear from the definitions that

$$\sum_l N_{i,j}^l N_{l,k}^m = \sum_l N_{j,k}^l N_{i,l}^m, \quad N_{i,0}^j = \delta_{i,j} = N_{0,i}^j \tag{2.17}$$

respectively from associativity of $\times$ on objects [5] and unitality of id (also in the non-strict case, i.e., up to natural isomorphisms), where $i, j, k, m \in \{0, \ldots, n\}$ and $\delta_{i,j}$ is the Kronecker symbol. Conjugation of objects and Frobenius reciprocity, see Proposition 2.5.8, imply

$$N_{i,j}^k = N_{\bar{i},k}^j = N_{k,\bar{j}}^i = N_{\bar{j},\bar{i}}^{\bar{k}}, \quad N_{i,j}^0 = \delta_{i,\bar{j}} = N_{j,i}^0 \tag{2.18}$$

and the properties of the dimension function give in addition

$$d_i \, d_j = \sum_k N_{i,j}^k d_k \tag{2.19}$$

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[5]notice that the fusion rules $\{N_{i,j}^k\}$ say nothing about associativity of $\times$ on arrows, which is encoded in the *6-j symbols*, or *Wigner-Recah symbols*, of the UFTC, see [MS88], [Reh90a].

i.e., the dimension vector $(d_0, \ldots, d_n)^t$ is a simultaneous eigenvector for the *fusion matrices* $N_i := (N_{i,j}^k) \in M_{n+1 \times n+1}(\mathbb{Z}_{\geq 0})$ with respective eigenvector d_i, for every $i \in \{0, \ldots, n\}$.

We shall discuss again these numbers later, in the case of *braided* and *modular categories* where the tighter constraints on the tensor structure give rise to much more surprising relations among these (and other) numbers that can be extracted from the category.

2.6   Unitary modular tensor categories (UMTCs)

Modularity is one of the most exciting features of low-dimensional CFT. Its consequences have been first noticed in the physical literature (motivated by string theory in particular, see [Ver88], [MS88]) by direct inspection of models, and then put into a rigorous framework by [Reh90b], [FRS92]. It is also recognized as a fundamental feature in the description of certain two-dimensional statistical systems at critical temperature, quantum Hall effect, topological insulators and quantum computation.

Despite this variety of phenomena in which it shows up, not to mention pure mathematical examples, modularity can be clearly formulated in the abstract language of (unitary) braided tensor categories (UBTCs) as a minimality condition on the category with respect to its commutativity constraint, the braiding. A braided tensor category (in the sense employed in this chapter) is called *modular* (UMTC) if its *braided center* is trivial. In other words there are no (irreducible) objects having trivial monodromy with all the other objects besides those in the isomorphism class of the tensor unit. Now a rich source of examples of UBTCs is provided by algebraic QFT via the DHR construction (see Chapter 3). As we have already mentioned, the braiding arising as DHR braiding, i.e., the statistics operator which classifies states into Bose/Fermi particle states, does not show interesting modularity features in high dimensions. In that case it squarest to the identity, as the flip on vector spaces does, hence is maximally commutative or *symmetric*. Going down with the spacetime dimension from 3+1 to 1+1 or 1, the "causal topology" of points undergoes a transition, namely points causally disconnect spacetime in the sense that their space-like complements become topologically disconnected regions (left and right wedges or half-lines). This difference with the higher-dimensional case allows non-symmetric DHR braidings in QFT.

On the QFT side, many of the models described in the literature can be constructed by means of local algebras (see Chapter 1). Using this framework, once we restrict to *rational* CFTs, i.e. finitely many primary fields/positive energy non-vacuum representations/DHR representations, modularity comes as a *consequence* [KLM01]. The proof is general (model-independent) and relies only on *locality*, or more specifically on two strengthened versions of it: the first in the direction of inclusions of local algebras and their commutants (Haag duality, condition 1.5.2), the second in the direction of the vacuum sector representation (Split property, condition 1.6.9).

We shall review the result of [KLM01] in the next chapter, and build on top of it in Chapter 4. Now we define *abstract* UMTCs and later we discuss in some extent their "structure constants", the *modular data*, and the relation between the two.

Modular categories owe their name to the group $SL(2, \mathbb{Z})$, or better to its quotient $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\pm \mathbb{1}\}$, the *modular group*. This group (together with its representations) made its appearance in models of rational CFT in 1+1 dimensions, see [Ver88], as a group of transformations of the inverse temperature parameter acting covariantly on the Virasoro characters, see Remark 2.6.14 for explanations. The same relations (in a different context) have been found by Rehren in low-dimensional AQFT, see [Reh90b], independently of the specific model realization (and of conformal covariance!). There, it has been recognized that the group relations of $SL(2, \mathbb{Z})$ and the related formulas which are valid on physical grounds, can be understood in purely categorical language, independently also of QFT. Namely they characterize a special class of *braided* tensor categories, the modular ones (see Definitions 2.6.1, 2.6.4), in the sense made precise by Proposition 2.6.12. The generalization of this proposition to the non-unitary case, together with the terminology "modular" category, came afterwards in the work of [Tur92].

For more references see, e.g., the works of [Müg03], [Müg12], [DGNO10], [BNRW15] and books on the topic.

Definition 2.6.1. A unitary fusion tensor category (UFTC) $\mathcal{C}$ equipped with a braiding (as in Definition 2.3.1) is a **unitary modular tensor category (UMTC)** if the only isomorphism class of objects $[a]$ in $\mathcal{C}$ having trivial monodromy with every other class $[b]$ in $\mathcal{C}$, i.e., $\mathcal{E}_{b,a} \circ \mathcal{E}_{a,b} = 1_{a \times b}$ for every b in $\mathcal{C}$, is the isomorphism class of the tensor unit, i.e., $[a] = [\mathrm{id}]$, or a "multiple" of it, i.e., $[a] = N[\mathrm{id}]$.

Remark 2.6.2. Notice that the previous definition is well posed, indeed the triviality or "vanishing" of the monodromy is stable on isomorphism classes. Indeed let a, b in $\mathcal{C}$ having trivial monodromy, then $\tilde{a} \cong a$, $\tilde{b} \cong b$ have $\mathcal{E}_{\tilde{b},\tilde{a}} \circ \mathcal{E}_{\tilde{a},\tilde{b}} = (u \times v) \circ \mathcal{E}_{b,a} \circ \mathcal{E}_{a,b} \circ (u^* \times v^*)$ by naturality of the braiding, for suitable unitaries $u : a \to \tilde{a}$, $v : b \to \tilde{b}$ in $\mathcal{C}$.

We want to stress that the same is *not* true for the triviality of the braiding, indeed in the same notation as above one gets $\mathcal{E}_{\tilde{a},\tilde{b}} = (v \times u) \circ \mathcal{E}_{a,b} \circ (u^* \times v^*)$, but then $u \times v$ need not be equal to $v \times u$. In the case of the DHR category (see Chapter 3) the braiding is essentially given as *commutation relations* between suitable unitaries (charge transporters), see Definition 3.3.3. In the present case, the commutation relations are in the sense of the tensor multiplication $\times$ of arrows. In the case of the DHR braiding a simple observation (Remark 3.3.8) shows that they essentially boil down to commutation relations in the sense of the composition $\circ$ of arrows (pointwise multiplication in an algebra, in that case).

Remark 2.6.3. The previous definition depends only on the *spectrum* Δ of $\mathcal{C}$. So we could work with with skeletal UMTCs, similarly to what is usually done for UFTCs. Our impression is that one would anyway lose interesting pieces of information of the UMTCs forgetting its size and disregarding the arrows, e.g., when looking at *inclusions* of UMTCs. At least, this "size issues" are important in this work, see also the *braided bicommutant theorem* of Müger reported in Proposition 4.7.3.

Given the importance of modularity in this work, we define it twice, but before we introduce some notation. Let $\mathcal{C}$ be a UFTC with a braiding and denote by Vec $\subset \mathcal{C}$ the (full) subcategory of $\mathcal{C}$ whose objects are the tensor unit id and all the possible multiples of id in $\mathcal{C}$ [6] , with the inherited braiding. Then Vec is unitarily braided (non-strictly) tensor equivalent (non-isomorphic) to the category of all finite-dimensional complex vector spaces, with the flip as a braiding.

Let $\mathcal{D} \subset \mathcal{C}$ be any *subset* of $\mathcal{C}$, denote by

$$\mathcal{Z}_{\mathcal{C}}(\mathcal{D}) := \left\{ b \in \mathcal{C} : \mathcal{E}_{b,a} \circ \mathcal{E}_{a,b} = 1_{a \times b}, \, a \in \mathcal{D} \right\}$$

the **braided commutant of $\mathcal{D}$ in $\mathcal{C}$**, regarded as (full) subcategory of $\mathcal{C}$. It follows [Müg03, Lem. 2.8] that $\mathcal{Z}_{\mathcal{C}}(\mathcal{D}) \subset \mathcal{C}$ is a replete unital tensor

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[6]notice that this is more that asking mere closure under direct sums or multiples, in particular Vec contains the isomorphism classes (in $\mathcal{C}$) of each of its objects, i.e., is *replete* in $\mathcal{C}$.

subcategory of $\mathcal{C}$, closed under finite direct sums, subobjects and again semisimple, irrespectively of the properties of $\mathcal{D}$. If in addition $\mathcal{D}$ has conjugate objects (contains one representative conjugate object for each of its objects) then $\mathcal{Z}_\mathcal{C}(\mathcal{D})$ has conjugates by [Müg00, Eq. (2.17)].

Definition 2.6.4. Let $\mathcal{C}$ be a UFTC with a braiding, then $\mathcal{C}$ is a UMTC if and only if

$$\mathcal{Z}_\mathcal{C}(\mathcal{C}) \subset \text{Vec}$$

i.e., if and only if the **braided center of $\mathcal{C}$** is "trivial".

Remark 2.6.5. After [Reh90b], the objects of $\mathcal{C}$ which lie in $\mathcal{Z}_\mathcal{C}(\mathcal{C})$ are usually called *degenerate* objects. Also, $\mathcal{C}$ is a *symmetric* tensor category if and only if $\mathcal{Z}_\mathcal{C}(\mathcal{C}) = \mathcal{C}$. Moreover, Vec is both modular and symmetric, indeed plays the role of $\mathbb{C}$ in subfactor theory. Notice also that the inclusion in the previous definition is an equality due to the properties of the braided commutant.

In the presence of a braiding over a unitary fusion structure, i.e. for a UFTC $\mathcal{C}$ which is also a UBTC (but *not* necessarily UMTC), the numerical invariants defined in the previous section (fusion rules) become more symmetric. Namely for every $i, j, k \in \{0, \ldots, n\}$

$$N_{i,j}^k = N_{j,i}^k \tag{2.20}$$

and the fusion matrices become pairwise commuting $N_i N_k = N_k N_i$ for every $i, k \in \{0, \ldots, n\}$ by (2.17), conjugate symmetric, i.e., $(N_i)_j^k = (N_{\bar{i}})_k^j$ by (2.18). Notice that N_0 is the identity matrix. Vice versa, these symmetry properties for the fusion rules of a fusion category are necessary conditions for the existence of a braiding compatible with the tensor structure.

Example 2.6.6. The Ising UFTC mentioned in Remark 2.3.8, see [BKLR15, Ex. 3.1], has three inequivalent irreducible objects (rank 3), fusion rules given by $[\tau \times \tau] = [\text{id}]$, $[\tau \times \sigma] = [\sigma \times \tau] = [\sigma]$, $[\sigma \times \sigma] = [\text{id}] \oplus [\tau]$ and conjugation fixed by $[\bar{\sigma}] = [\sigma]$. Setting $\{[a_0 = \text{id}], [a_1 = \sigma], [a_2 = \tau]\}$ we have

$$N_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The dimensions (which need not be integers!) can be read out of the fusion rules, namely $d_\tau = 1$ and $d_\sigma = \sqrt{2}$.

There are only two unitary tensor structures which realize these Ising-like

fusion rules. One of them is the Ising category itself, the other is equivalent to the DHR category (see Chapter 3) of the $SU(2)$-currents at level 2. Each of the two has four possible inequivalent braided structures, see [BKLR15, Ex. 3.1, Ex. 4.19]. These facts can be shown by direct computation, realizing the fusion rules as tensor categories of endomorphisms of von Neumann factors, with Cuntz algebras of isometries specifying the tensor multiplication on arrows. Notice that this is no loss of generality by a result of [HY00], as we shall recall in Remark 4.2.5.

In a UFTC equipped with a braiding (*not* necessarily modular), along with the fusion rules one can consider other numerical invariants.

Definition 2.6.7. Let $\mathcal{C}$ be a UFTC with a braiding and a an object in $\mathcal{C}$. Consider the self-braiding unitary isomorphism $\mathcal{E}_{a,a}$ and choose a conjugate object $\overline{a}$ and a standard solution $r, \overline{r}$ of the conjugate equations (2.9). Then

$$\omega_a := (r^* \times 1_a) \circ (1_{\overline{a}} \times \mathcal{E}_{a,a}) \circ (r \times 1_a) = (1_a \times \overline{r}^*) \circ (\mathcal{E}_{a,a} \times 1_{\overline{a}}) \circ (1_a \times \overline{r}) \quad (2.21)$$

or better

$$(2.22)$$

is the **phase** of a in $\mathcal{C}$. If a is *irreducible* then ω_a is a unitary number which does not depend neither on the choice of $\overline{a}$ nor on $r, \overline{r}$ standard, and is well defined on isomorphism classes.

Unitarity can be visualized by writing, e.g. $\omega_a^* \circ \omega_a = 1_a$ as

while

need not be 1_a, unless e.g. $\mathcal{E}_{a,a} \circ \mathcal{E}_{a,a} = 1_{a \times a}$, in which case $\omega_a = \pm 1$ (which corresponds in physics to the Bose/Fermi alternative implemented in the category of DHR superselection sectors). These facts show that string diagrammatical calculus is not completely faithful with the calculus in a rigid UBTC. Indeed one-dimensional strings should be replaced by flat ribbons or strips, in order to give a meaning to the torsion or *twisting* operations above.

The twisting operation $a \mapsto \omega_a$ is not tensor, but only tensor up to monodromies in the following sense.

Proposition 2.6.8. [LR97], [Müg00]. *Let $\mathcal{C}$ as above and consider $a \mapsto \omega_a$ defined on arbitrary objects, the definition is well-posed and $\omega_a \in \mathrm{Hom}_{\mathcal{C}}(a, a)$ is a unitary isomorphism. Moreover, $\{\omega_a\}_{a \in \mathcal{C}^{(0)}}$ is a natural unitary isomorphism of the identity functor $1_{\mathcal{C}} \Rightarrow 1_{\mathcal{C}}$, namely*

$$\omega_b \circ t = t \circ \omega_a \tag{2.23}$$

for every $t : a \to b$ in $\mathcal{C}$, but is not tensor in general, indeed

$$\omega_{a \times b} = (\omega_a \times \omega_b) \circ \mathcal{E}_{b,a} \circ \mathcal{E}_{a,b} \tag{2.24}$$

for every a, b in $\mathcal{C}$. Moreover

$$(\omega_{\overline{a}} \times \mathrm{id}_a) \circ r = (\mathrm{id}_{\overline{a}} \times \omega_a) \circ r \tag{2.25}$$

for every a in $\mathcal{C}$ with conjugate object $\overline{a}$ and $r, \overline{r}$ a standard solution of the conjugate equations (2.9). The latter equation says that the twist is "tortile", cf. [Müg00, Def. 2.3]. In the irreducible case it says that $\omega_a = \omega_{\overline{a}}$ because we already have $d_a = d_{\overline{a}}$. [7]

Remark 2.6.9. In the context of braided tensor categories, both the dimensions and the phases of objects have been introduced by [DHR69b], [DHR71] in the

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[7]both these equalities are desirable in physics, when abstract conjugation, intrinsic dimensions and phases are used to describe particle-antiparticle symmetry.

analysis of superselection sectors coming from a global gauge symmetry. They first appeared as unitary and positive part of the numerical invariant which allows an (intrinsic) classification sectors, the *statistics parameter* (calculated as left inverse of the self-braidings). For this reason, in the physical literature, they are also referred to as *statistical dimension* and *statistical phase*. Later the dimension was recognized to be independent of the braiding (hence of the phase), as we presented it here, though tightly connected to it at least in unitary categories, see [LR97, Lem. 4.3]. Using this one can also show [HM06, Prop. A.45] that a *braided tensor functor* $F : \mathcal{C} \to \mathcal{D}$, between UBTCs with an underlying UFTC structure automatically preserves conjugate objects, standard solutions of the conjugate equations, dimensions and phases of objects.

The trace property of the standard left inverses mentioned in the previous section, together with the last statement of the proposition above, are the secret ingredients for the following results due to [Reh90b], [FRS92], to which modular categories owe their name. We recall first the definition of the *monodromy matrices* of [Reh90b, Sec. 5].

Definition 2.6.10. Let $\mathcal{C}$ be a UFTC equipped with a braiding, and denote its spectrum by $\Delta = \{[a_0], [a_1], \dots, [a_n]\}$. For every pair $i, j \in \{0, \dots, n\}$ choose $a_i \in [a_i]$, $a_j \in [a_j]$ and define numbers

$$Y_{i,j} := d_i d_j \varphi_i \varphi_j (\varepsilon_{i,j}^{\mathrm{op}} \circ \varepsilon_{j,i}^{\mathrm{op}}) \tag{2.26}$$

or better

$$Y_{i,j} \ := \ \bar{j} \bigcirc\!\!\!\bigcirc \bar{i} \quad \overset{j \ \ i}{} \tag{2.27}$$

where φ_i, φ_j are, respectively, the (normalized) standard left inverse of a_i, a_j. The number $Y_{i,j}$ does *not* depend on the choice of $a_i \in [a_i]$, $a_j \in [a_j]$ by naturality of the braiding and the trace property of left inverses.

The matrix $Y = (Y_{i,j})$ has the following properties.

Proposition 2.6.11. [Reh90b].

$$Y_{i,j} = \sum_k \frac{\omega_i \omega_j}{\omega_k} N_{i,j}^k d_k \tag{2.28}$$

$$Y_{i,j} = Y_{j,i} = \overline{Y_{i,\bar{j}}} = Y_{\bar{i},\bar{j}} \tag{2.29}$$

$$(d_j)^{-1} Y_{i,j} Y_{k,j} = \sum_m N_{i,k}^m Y_{m,j} \tag{2.30}$$

The first line (2.28) is essentially the definition of Y. It is derived from (2.23), (2.24) after choosing orthonormal bases of isometries $t_k^e : a_k \to a_j \times a_i$ in the Hom-spaces which realize the decomposition of $a_j \times a_i$ into subobjects a_k sitting in $[a_k]$. Indeed one gets in this way the *coefficients of the monodromy*, see [Reh90b, Eq. (2.39)], [FRS92, Lem. 3.3], namely

$$t_k^e = \quad\Longrightarrow\quad = \frac{\omega_i\,\omega_j}{\omega_k} \cdot \quad \cdot \delta_{e,e'}\delta_{k,k'} \tag{2.31}$$

The second line (2.29) can be visualized by giving an orientation to the circles (to distinguish 1_a and $1_{\bar{a}}$, respectively downwards and upwards directed, when labeling with a only). Then the second equality comes from rotating the j-ring around an horizontal axis by π (and taking the complex conjugate), the second by rotating the i-ring also, in the same manner.

The third line (2.30) means that the j-column vector of Y is a simultaneous eigenvector for each fusion matrix N_i for every i,j with eigenvalue $(d_j)^{-1}Y_{i,j}$. This is no surprise for $j = 0$ because $Y_{i,0} = Y_{0,i} = d_i$ and we already know by (2.19) that the dimension vector is a simultaneous eigenvector for each N_i. But for $j \neq 0$ we have possibly new eigenvectors and "quantum dimensions" defined by the monodromies (which need not be positive, nor real). In the case $\mathcal{E}_{i,j} = \mathcal{E}_{i,j}^{\mathrm{op}}$, i.e., if i and j have trivial monodromy, then the monodromy rings $Y_{i,j}$ split (vice versa also holds by [Müg03, Prop. 2.5]), namely

$$\Longrightarrow\quad Y_{i,j} = \bigcirc\bigcirc = d_i\, d_j\,,$$

and the eigenvalue of N_i given by (2.30) is again d_i. Notice that the first line (2.28) coincides with (2.19) in this case. If moreover j is *degenerate*, i.e.

$\mathcal{E}_{i,j} = \mathcal{E}_{i,j}^{\text{op}}$ for every i, then we get back also the usual dimension eigenvector for every N_i, equation (2.30) drops out j and boils down to (2.19) all the times, i.e., to the case $j = 0$.

The next proposition gives a characterization of *modularity* by means of the monodromy matrix Y and reveals a deep connection between braidings and fusion rings in abstract UMTCs. The motivation comes from a conjecture of Verlinde [Ver88] (in rational models of CFT) and is applicable in principle to general local nets of observables in low dimensions (without CFT).

Proposition 2.6.12. [Reh90b]. *Let $\mathcal{C}$ be a UFTC with a braiding, then the following are equivalent*

- *$\mathcal{C}$ is a UMTC (in the sense of Definition 2.6.4).*

- *The matrix Y is invertible.*

- *The complex number $\sigma := \sum_i d_i^2 \omega_i^{-1}$ fulfills $|\sigma|^2 \equiv (\sum_i d_i^2 \omega_i^{-1})(\sum_i d_i^2 \omega_i) = \sum_i d_i^2$ (the global dimension of $\mathcal{C}$) and the matrices*

$$S := |\sigma|^{-1} Y\,, \quad T := (\sigma/|\sigma|)^{1/3} \, diag(\omega_i)$$

satisfy the relations

$$SS^* = \mathbb{1} = TT^*$$

$$(ST)^3 = S^2 = C$$

$$TC = CT\,, \quad C = C^* = C^{-1}$$

where $C_{i,j} = \delta_{i,\bar{j}}$ is the conjugation matrix. In other words S, T form a unitary representations of $SL(2, \mathbb{Z})$, whose generators are $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$ on $\mathbb{C}^{n+1}$, where $n + 1$ is the rank of $\mathcal{C}$.
*In the case $C = \mathbb{1}$, i.e., if every object of $\mathcal{C}$ is self-conjugate, the representation factors through the **modular group** $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\pm \mathbb{1}\}$. Moreover, S completely determines the fusion rules and simultaneously diagonalizes the fusion matrices, namely*

$$N_{i,k}^m = \sum_j \frac{S_{i,j} S_{k,j} \overline{S_{m,j}}}{S_{0,j}}\,, \quad S^* N_i S = diag(S_{i,j}/S_{0,j}) \qquad (2.32)$$

where $S_{i,j}/S_{0,j} = (d_j)^{-1} Y_{i,j}$ are the quantum dimensions of i with respect to j.

Example 2.6.13. In the Example 2.6.6, the eight possible compatible UBTC structures with the Ising-like fusion rules have corresponding S,T matrices given by

$$S = (1/2) \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \,, \quad T = (1/\omega_\sigma)^{1/3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_\sigma & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

where $\omega_\sigma \in \{\exp(2\pi\imath\, h_\sigma)\,, \ h_\sigma = (2k+1)/16 \,, \ k = 0,\dots,7\}$. In this case, notice that S is always the same and invertible (hence these realizations are all modular) while T is different for each case (hence S,T distinguish the eight UMTCs up to braided tensor equivalence).

Remark 2.6.14. Relations analogous to (2.32) have been found by Verlinde [Ver88] in rational CFT, verified in some models and conjectured to hold true in general rational CFTs. The difference to our exposition is the origin of the S,T matrices themselves: for us they come from unitary braided tensor categories alone, in the other case they are defined by looking at how the conformal (Virasoro) characters of the CFT in non-vacuum sectors behave under modular transformations $\tau \mapsto \tau^{-1}$, $\tau \mapsto \tau+1$ of the inverse temperature parameter. A proof of the equality of these two constructions of the modular matrices is still missing. Anyway, the previous conjecture holds true in our setting by Proposition 2.6.12, and has been shown also in the rational CFT setting, with the respective S,T matrices, by [MS88]. Their proof relies on some polynomial equations among certain matrices which describe consistency conditions on the n-point functions (for chiral vertex operators) of the CFT. The datum of these matrices turns out to identify uniquely (up to braided tensor equivalence) a UMTC and, vice versa, they can be *non-canonically* extracted out of any such UMTC by choosing bases of isometries in the Hom-spaces $c \to a \times b$. The polynomial equations arise then by imposing the associativity and braiding constraints on objects *and* arrows of the category, see [FRS92, Sec. 3, App. A], [DHW13].

The two different pictures are both compatible with chiral "rational" conformal nets of local observables (in AQFT) by the following observation. On one hand, local algebras determine their positive energy representations via the DHR construction (see Chapter 3) and the representations form a category which is unitary, tensor and braided (by locality). Under natural "rationality" assumptions on the net, the DHR category is automatically modular [KLM01] (in the sense employed here). On the other hand, local algebras uniquely

determine also their vacuum sector and the dynamics up to unitary equivalence (Theorem 1.6.14), hence, *in principle*, the conformal Hamiltonian in non-vacuum representations and the Virasoro characters.

2.7 Modular data and "rigidity" of UMTCs

In the previous section we have seen how the presence of a *modular braiding* on $\mathcal{C}$ puts additional constraints on the fusion rules. These constraints in some sense "close the circle" in the process of starting from a spectrum Δ with $n+1$ elements and certain fusion rules N, adding a tensor structure, adding a braided structure.

Forgetting modularity for a moment, the following is know to hold for general fusion tensor categories over an algebraically closed field $\mathbb{K}$ of characteristic zero, hence a fortiori for UFTCs (in the sense of Definition 2.5.16). See [ENO05, Sec. 2.7], [BNRW15, Sec. 3.2] for a proof and the history on how ideas developed. The result is usually referred to as *Ocneanu rigidity*.

Proposition 2.7.1. (Ocneanu rigidity). *There are finitely many UFTCs (up to tensor equivalence, see Definition 2.2.9) compatible with a given set of fusion rules. There are finitely many UBTCs (up to braided tensor equivalence, see Definition 2.3.7) compatible with a given UFTC structure.*

In the modular case (again not necessarily unitary, but over $\mathbb{C}$) more is true thanks to [BNRW15, Thm. 3.1], namely a *rank finiteness theorem* holds.

Proposition 2.7.2. [BNRW15]. *For every fixed rank $n+1$ with $n \geq 0$, there are finitely many possible fusion rules which are compatible with some UMTC structure. In particular, there are finitely many UMTCs for fixed rank (up to braided tensor equivalence, see Definition 2.3.7).*

Remark 2.7.3. The proof of the previous result involves technologies from number theory and invokes Galois theory of field extensions, indeed there is a deep connection between fusion and modular categories over $\mathbb{C}$ and finite field extensions of $\mathbb{Q}$, see [DHW13, Sec. 4.2]. Here we just want to stress that modularity does enter the previous proof, and the connection between modularity and fusion rules is made possible by (2.32) in Proposition 2.6.12. See [BK01, Thm. 3.1.13] for a proof of (2.32) in general MTCs, not necessarily UMTCs, and for their definition. As far as the unitary case is concerned,

remember that every braiding on a UFTC is automatically unitary and there is a unique unitary twist structure [Gal14], namely the one canonically given by Proposition 2.6.8.

We also mention that modularity must play a role in the previous result, indeed for fixed n there are infinitely many possible fusion rules, see [Gan05, Sec. 2]. On the other hand, it is also known [BNRW15, Rmk. 3.25] that the number of UMTCs with fixed rank $n + 1$ is *not* polynomially bounded in n.

In virtue of the rank finiteness theorem, a *classification* of UMTCs (and MTCs as well) at fixed rank becomes a tractable problem. Now, the numbers we explicitly introduced in the previous sections (and the relations among them) depend only on the braided tensor equivalence class of the UMTC we start from. We summarize them in the following

Definition 2.7.4. Let $\mathcal{C}$ be a UMTC (in the sense of Definition 2.6.4), the collection of numerical invariants $\{n, \Delta, N, S, T\}$ extracted from $\mathcal{C}$, where $n + 1$ is the rank, Δ the spectrum, N the fusion matrices and S, T the unitary generators of the $SL(2, \mathbb{Z})$ representation, is called the **modular data of $\mathcal{C}$**.

Remark 2.7.5. Of course there are many redundancies in $\{n, \Delta, N, S, T\}$, for example n can be read in the dimension of the fusion matrices N_i, and the information about the conjugation of objects contained in Δ, is also encoded in $N_{i,j}^0$. Moreover, N can be derived from S by (2.32) and, vice versa, S can be computed using (2.28) from N and T. Indeed the dimensions of the objects are determined by the fusion matrices as their highest eigenvalues (by Perron-Frobenius theory of non-negative matrices), see [Gan05, Sec. 4], and the phases can be computed as $\omega_i = T_{i,i}/T_{0,0}$. A more economic set of invariants is given by $\{S, T\}$, and an even more economic one by $\{N, \omega\}$ (fusion rules and phases only!).

The following question is then natural, and widely shared among experts, see e.g. [FGV95a], [RSW09], [DHW13], [BNRW15].

Question 2.7.6. *Are the modular data $\{n, \Delta, N, S, T\}$ a complete invariant for UMTCs (resp. MTCs)? In other words, do they distinguish among braided tensor inequivalent (cf. Definition 2.3.7) UMTCs (resp. MTCs)?*

The answer is expected to be yes. Direct inspection of the low rank cases, see [FGV95b, Sec. 5], [RSW09], shows that the answer is indeed yes for MTCs up to rank 3 and for UMTCs up to rank 4, and there is work in progress

for rank 5. The answer is also yes for "pointed" MTCs (if all objects have dimension 1), see [FRS04], [EGNO15, Sec. 8.4], and for certain classes of MTCs with arbitrarily big rank, see [BGN$^+$16]. We shall not deal with the question of *realizability* of "abstract" modular data by means of MTCs and just refer to [BNRW15] for the interested reader.

We left out in the exposition some of the known relations among the entries of the modular data, notably *Vafa's theorem* which says that the T matrix has finite order, while $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, $m \in \mathbb{N}$.

In this work we want to say something about a weaker version of the previous question, which is still open to our knowledge, and mention a strategy to attack it. The starting point is a new proof of the formula for the *Frobenius-Schur indicator* in a UMTC $\mathcal{C}$. The indicator is a number attached to irreducible objects of $\mathcal{C}$ which takes the value 0 on non-self-conjugate ones, and respectively $+1$ or -1 on (self-conjugate) real or pseudo-real objects. We recall that for an irreducible *self-conjugate* object, any normalized solution $r, \overline{r}$ of the conjugate equations (2.9) fulfills either $r = \overline{r}$ or $r = -\overline{r}$. In the first case the object is called **real**, in the second **pseudo-real**, see [LR97, Sec. 5], using the same terminology as for group representations.

The formula for the Frobenius-Schur indicator appears in rational CFT in the work of [Ban97, Eq. (1)] without proof. The author shows only that the possible values are 0, ± 1 and that they fit with the conjugation structure. Afterwards, [NS07, Sec. 7] derived the same formula as a special case of more general "higher" indicators, in the context of MTCs. We give here a proof, in the case of UMTCs, which has the advantage of being simpler and closer to the lines of Bantay. In particular, we can show another formula, cf. [Ban97, Eq. (3)], which expresses the trace of the self-braidings $\varepsilon_{a,a}$ in an *arbitrary* "channel" $c \prec a \times a$ (not only $c = \text{id}$ as one needs for the determination of the Frobenius-Schur indicator). This more general trace formula for self-braidings is what we shall need later for our purposes. All these formulas have the remarkable property of depending *only* on modular data, hence they express intrinsic (invariant) quantities of the UMTC. The following proof has been found by K.-H. Rehren [Reh13], to whom we owe our gratitude.

Proposition 2.7.7. *Let $\mathcal{C}$ be a UMTC with modular data $\{n, \Delta, N, S, T\}$.*

Then for every $[a_i] \in \Delta$ and $[a_k] \in \Delta$ such that $m := N_{i,i}^k > 0$ we have [8]

$$\mathrm{Tr}_{a_k}(\mathcal{E}_{a_i,a_i}) := \sum_{e=1,\dots,m} \quad = \omega_i^{-1} \sum_{r,s} \overline{S_{r,k}} S_{s,0} N_{r,s}^i \frac{\omega_s^2}{\omega_r^2} \qquad (2.33)$$

where the sum is taken over an orthonormal basis of isometries $t_k^e : a_k \to a_i \times a_i$ for $e = 1, \dots, m$. Indeed, the number on the l.h.s does not depend on the choice of $a_i \in [a_i]$, $a_k \in [a_k]$ nor on that of the basis of isometries, by naturality of the braiding and by the trace property of left inverses. Hence it defines another invariant for the UMTC $\mathcal{C}$. In particular, for every $[a_i] \in \Delta$, the number

$$\nu_i := \sum_{r,s} S_{r,0} S_{s,0} N_{r,s}^i \frac{\omega_r^2}{\omega_s^2} \qquad (2.34)$$

takes the values 0, $+1$ or -1 respectively if $[a_i]$ is non-self-conjugate, real or pseudo-real in $\mathcal{C}$. The last line is Bantay's expression for the Frobenius-Schur indicator.

Proof. The following argument makes clear the advantages of using the string diagrammatical notation, indeed the proof written with usual sums, compositions and products of morphism would be (to us) almost unreadable. Now, the trace on the l.h.s. of equation (2.33) means

$$\mathrm{Tr}_{a_k}(\mathcal{E}_{a_i,a_i}) = \sum_{e=1,\dots,m} (t_k^e)^* \circ \mathcal{E}_{a_i,a_i} \circ t_k^e = \sum_{e=1,\dots,m}$$

where t_k^e, $e = 1, \dots, m$ is a linear basis of $\mathrm{Hom}_{\mathcal{C}}(a_k, a_i \times a_i)$ of orthonormal

---

[8]notice that equation (2.33) differs from [Ban97, Eq. (3)] by a conjugation, which does not affect the Frobenius-Schur indicator (2.34) because $k = 0$ is self-conjugate. If we denote $\tau_{k,i} := \mathrm{Tr}_{a_k}(\mathcal{E}_{a_i,a_i})$ then $\tau_{k,i} = \tau_{\overline{k},\overline{i}}$ by the trace property of left inverses and because the phases are a tortile unitary twist on $\mathcal{C}$.

isometries in the sense that $(t_k^e)^* \circ t_k^{e'} = \delta_{e,e'} 1_k$. We write

$$\sum_e \quad = \quad \sum_e \frac{1}{d_k} \quad = \quad \sum_e \frac{1}{d_k}$$
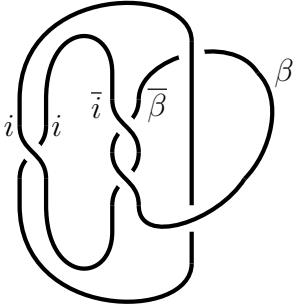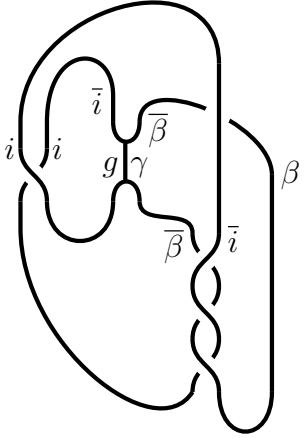
by the trace property (2.15) and considering the conjugation functor on arrows $t_k^e \mapsto (t_k^e)^\bullet$, see Proposition 2.5.8 and comments thereafter. Invertibility of the S matrix (equivalent to *modularity* by Proposition 2.6.12), or better $S^2 = C$, gives $\sum_{\beta=0,\ldots,n} S_{\alpha,\beta} S_{\beta,k} = \delta_{\alpha,\overline{k}}$ for every $\alpha = 0,\ldots,n$, hence the previous line can be rewritten as

$$= \sum_{f,\alpha,\beta} \frac{1}{d_{\overline{\alpha}}} S_{\alpha,\beta} S_{\beta,k} \quad = \quad \sum_{f,\alpha,\beta} \frac{1}{d_{\overline{\alpha}}} \frac{d_\alpha}{|\sigma|} S_{\beta,k}$$
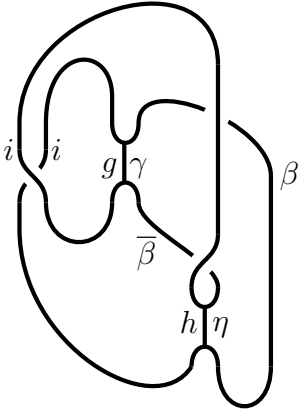
where t_α^f for $\alpha = 0,\ldots,n$ runs over orthonormal bases of isometries in $\mathrm{Hom}_{\mathcal{C}}(a_\alpha, a_{\overline{i}} \times a_{\overline{i}})$ which are mutually orthogonal and coincide with t_k^e for $\alpha = k$. The r.h.s. is obtained by definition of $S_{\alpha,\beta} = |\sigma|^{-1} Y_{\alpha,\beta}$, opening the α-ring up to multiplication with d_α. This previous insertion procedure is usually referred to as "killing-ring", after [BEK99]. Notice also that $d_\alpha = d_{\overline{\alpha}}$. By naturality and multiplicativity of the braiding we get

$$= \sum_{f,\alpha,\beta} \frac{1}{|\sigma|} S_{\beta,k} \quad = \quad \sum_{\beta} \frac{1}{|\sigma|} S_{\beta,k}$$

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where the equality comes from summing over $\sum_{f,\alpha} t_\alpha^f \circ (t_\alpha^f)^* = 1_{a_{\bar{i}} \times a_{\bar{i}}}$. Expanding the killing ring we obtain

$$= \sum_{\beta} \frac{1}{|\sigma|} S_{\beta,k} \qquad = \sum_{\beta,\gamma,g} \frac{1}{|\sigma|} S_{\beta,k} \frac{\omega_\gamma}{\omega_i \omega_\beta}$$

where the r.h.s. is given by the formula (2.31) for the coefficients of the monodromy, which are invariant and depend only on modular data (phases). Similarly we get

$$= \sum_{\beta,\gamma,g,\eta,h} \frac{1}{|\sigma|} S_{\beta,k} \frac{\omega_\gamma \omega_\eta}{\omega_i^2 \omega_\beta^2}$$

where β, γ, η run over the spectrum of $\mathcal{C}$ and g, h run over bases of isometries. The crucial step is to rewrite the previous (by naturality and multiplicativity

of the braiding) as

$$
= \sum_{\beta,\gamma,g,\eta,h} \frac{1}{|\sigma|} S_{\beta,k} \frac{\omega_\gamma \omega_\eta}{\omega_i^2 \omega_\beta^2} \quad \left[\text{diagram} \right] = \sum_{\beta,\gamma,g} \frac{1}{|\sigma|} S_{\beta,k} \frac{\omega_\gamma^2}{\omega_i \omega_\beta^2} \quad \left[\text{diagram} \right]
$$

where we have used $\omega_{\bar{i}} = \omega_i$ (the phases are a tortile unitary twist on $\mathcal{C}$ by Proposition 2.6.8) and $(t_\eta^h)^* \circ t_\gamma^g = \delta_{\gamma,\eta}\delta_{g,h}1_\gamma$. By the trace property of the left inverses we continue as

$$
= \sum_{\beta,\gamma,g} \frac{1}{|\sigma|} S_{\beta,k} \frac{\omega_\gamma^2}{\omega_i \omega_\beta^2} d_\gamma \quad \left[\text{diagram} \right] = \sum_{\beta,\gamma} \frac{1}{|\sigma|} S_{\beta,k} \frac{\omega_\gamma^2}{\omega_i \omega_\beta^2} d_\gamma N_{\beta,i}^{\bar{\gamma}}
$$

because $N_{\beta,i}^{\bar{\gamma}}$ is defined as the multiplicity of $[a_{\bar{\gamma}}]$ in $[a_\beta] \times [a_i]$. Moreover, $N_{\beta,i}^{\bar{\gamma}} = N_{\beta,\bar{\gamma}}^i$ by Frobenius reciprocity (Proposition 2.5.8), hence

$$
\text{Tr}_{a_k}(\mathcal{E}_{a_i,a_i}) = \sum_{\beta,\gamma} \frac{1}{|\sigma|} S_{\beta,k} \frac{\omega_\gamma^2}{\omega_i \omega_\beta^2} d_\gamma N_{\beta,\bar{\gamma}}^i = \omega_i^{-1} \sum_{\beta,\gamma} \overline{S_{\beta,k}} S_{\gamma,0} \frac{\omega_\gamma^2}{\omega_\beta^2} N_{\beta,\gamma}^i
$$

after changing the names of the summation indices, using $S_{\bar{\beta},k} = \overline{S_{\beta,k}}$ and the definition of $S_{\gamma,0}$. So we have shown (2.33). The proof of (2.34) now follows from [Ban97] by noticing that $S_{\beta,0}$ is real, being $k = 0$ self-conjugate, and we are done. □

Now, back to classification questions for UMTCs, we can ask the following weaker version of 2.7.6, namely

Question 2.7.8. *Do the modular data* $\{n, \Delta, N, S, T\}$ *distinguish among braided tensor inequivalent (cf. Definition 2.3.7) UMTCs once we fix the UFTC structure also?*

Assume we have a UMTC $\mathcal{C}$, the task is to "reconstruct" the braiding out of the underlying tensor (UFTC) structure and the modular data. So take two arbitrary irreducible objects a,b in $\mathcal{C}$, respectively in the isomorphism classes indexed by i,j in Δ, and consider the braiding unitary $\mathcal{E}_{a,b}$ of $\mathcal{C}$. Observe that it does *not* depend *only* on i,j in any sense, not even up to unitary conjugation (unless $i = j$). For every irreducible $[c]$ in $[a] \times [b]$ choose one $c \in [c]$ and an orthonormal basis of isometries $t_k^e : c \to a \times b$ where $e = 1, \dots, N_{i,j}^k$ if $[c]$ is indexed by k in Δ. We know by equation (2.31) the coefficients of the monodromy of i and j in the channel k, hence we can write it in diagonal form, i.e.

$$\mathcal{E}_{b,a} \circ \mathcal{E}_{a,b} = \frac{1}{\omega_i \omega_j} \sum_{k=0,\dots,n} \omega_k \sum_{e=1,\dots,N_{i,j}^k} t_k^e \circ (t_k^e)^*. \tag{2.35}$$

A first idea is to "take the square root" of the previous expression to obtain coefficients for the braiding, possibly canonical and invariant ones.

The monodromy is the difference between two braidings, namely $\mathcal{E}_{a,b}$ and the opposite $\mathcal{E}_{a,b}^{\mathrm{op}} = \mathcal{E}_{b,a}^*$. So if $a \neq b$ (as objects) we choose other orthonormal bases of isometries for every channel, i.e., $s_k^f : c \to b \times a$ where $f = 1, \dots, N_{j,i}^k (= N_{i,j}^k)$ and c is an irreducible which lies in the class indexed by k in Δ. The coefficients of the two braidings in each channel need not be diagonal, we denote them by

$$u_{e,f} := (s_k^f)^* \circ \mathcal{E}_{a,b} \circ t_k^e, \quad v_{e,f} := (s_k^f)^* \circ \mathcal{E}_{a,b}^{\mathrm{op}} \circ t_k^e$$

and write

$$\mathcal{E}_{a,b} = \sum_{k,e,f} u_{e,f} s_k^f \circ (t_k^e)^*, \quad \mathcal{E}_{a,b}^{\mathrm{op}} = \sum_{k,e,f} v_{e,f} s_k^f \circ (t_k^e)^*. \tag{2.36}$$

We trivially have $\sum_f u_{e,f} \overline{u_{e',f}} = \delta_{e,e'}$ and from equation (2.35) we get

$$\sum_f u_{e,f} \overline{v_{e',f}} = \frac{\omega_k}{\omega_i \omega_j} \delta_{e,e'}$$

i.e., $UU^* = \mathbb{1}$, $VV^* = \mathbb{1}$ and $U = \frac{\omega_k}{\omega_i \omega_j} V$ where U and V are respectively the matrices of the coefficients of $\mathcal{E}_{a,b}$ and of $\mathcal{E}_{a,b}^{\mathrm{op}}$ in the channel specified by the object c. Now, the monodromies $\mathcal{E}_{b,a} \circ \mathcal{E}_{a,b}$ and $\mathcal{E}_{a,b} \circ \mathcal{E}_{b,a}$ are in general different morphisms, but they have the same coefficients given by (2.31) in every channel, irrespectively of the choice of bases of isometries.

Hence, denoted by $U^{c,ab}$, $V^{c,ab}$ the previous matrices and by $U^{c,ba}$, $V^{c,ba}$ those obtained by exchanging the roles of a and b, we obtain as before $U^{c,ab}U^{c,ba} = \frac{\omega_k}{\omega_i\omega_j}\mathbb{1} = U^{c,ba}U^{c,ab}$ and $V^{c,ab}V^{c,ba} = \frac{\omega_i\omega_j}{\omega_k}\mathbb{1} = V^{c,ba}V^{c,ab}$. Notice that $\mathrm{Hom}_{\mathcal{C}}(c, a \times b)$ and $\mathrm{Hom}_{\mathcal{C}}(c, b \times a)$ can be identified thanks to the presence of a braiding symmetry. Commuting unitary matrices can be *simultaneously* diagonalized, i.e., we can assume $V^{c,ab} = \Lambda^{c,ab}$ and $V^{c,ba} = \Lambda^{c,ba}$ where $\Lambda^{c,ab}$, $\Lambda^{c,ba}$ are diagonal, up to a suitable choice of bases t_k^e and s_k^f in each channel. By the previous we have also $U^{c,ab} = \frac{\omega_k}{\omega_i\omega_j}\Lambda^{c,ab}$ and $U^{c,ba} = \frac{\omega_k}{\omega_i\omega_j}\Lambda^{c,ba}$ and changing one of the two bases, if needed, we can assume $\Lambda^{c,ab} = \Lambda^{c,ba} = diag(\mu_e)$ where $\mu_e := (\lambda_e^{c,ab})^{1/2}(\lambda_e^{c,ba})^{1/2}$, the λ's are the respective diagonal entries of the Λ matrices and $(\lambda)^{1/2}$ denotes the principal value of the square root. By the previous constraints on V we obtain $(\mu_e)^2 = \frac{\omega_i\omega_j}{\omega_k}$, hence $\mu_e = (\frac{\omega_i\omega_j}{\omega_k})^{1/2}$, absorbing the minus signs if possible (e.g., assuming $a \neq b$) by modifying one of the two bases of isometries in each channel. So, assuming $a \neq b$, we can write

$$U^{c,ab} = U^{c,ba} = (\frac{\omega_k}{\omega_i\omega_j})^{1/2}\mathbb{1}, \quad V^{c,ab} = V^{c,ba} = (\frac{\omega_i\omega_j}{\omega_k})^{1/2}\mathbb{1}$$

in every channel c, after suitable choices of bases of isometries for every triple a, b, c in $\mathcal{C}$ with $a \neq b$. Hence the braidings $\mathcal{E}_{a,b}$ and $\mathcal{E}_{a,b}^{\mathrm{op}}$ can be simultaneously put in a diagonal form with entries depending only on modular data.

The situation is more complicated when $a = b$ in $\mathcal{C}$ (not only $a \cong b$). In that case $V^{c,aa}$ can be diagonalized in every channel and its entries fulfill $(\lambda_e)^2 = \frac{\omega_i\omega_j}{\omega_k}$, however there is no consistent choice of the bases of isometries which fixes the ambiguity of the square root, in order to determine $U^{c,aa}$ (and $V^{c,aa}$) in every channel, hence the self-braiding $\mathcal{E}_{a,a}$.

But now, we know by equation (2.33) of Proposition 2.7.7 that the trace of $\mathcal{E}_{a,a}$ in every channel c is an invariant of the UMTC and, moreover, depends only on modular data. We have already seen in the previous discussion that $U^{c,aa}$ has only two possible eigenvalues, namely $\pm(\omega_k)^{1/2}\omega_i^{-1}$ where i and k denote respectively the indices of a and c in the spectrum Δ of $\mathcal{C}$. Moreover the dimension of the space of isometries is given by $N_{i,i}^k$, hence also the *multiplicities* of the eigenvalues are determined by modular data. This leads to the following observation

Proposition 2.7.9. *In a UMTC $\mathcal{C}$ the spectrum (including multiplicities) of the self-braiding morphisms $\mathcal{E}_{a,a}$, for every object a, is determined by the modular data $\{n, \Delta, N, S, T\}$ of $\mathcal{C}$.*
Equivalently, the braiding morphisms $\mathcal{E}_{a,b}$ (not only $a = b$) are determined by

modular data after a suitable choice of basis of isometries in every intertwiner space $\mathrm{Hom}_{\mathcal{C}}(c, a \times b)$ *and up to a suitable permutation of the indices in every channel* $c \prec a \times b$.

One way of giving a positive answer to our Question 2.7.8 is the following. Take two UMTCs $\mathcal{C}$ and $\mathcal{D}$ having equivalent UFTC structures and denote the respective braidings as $\mathcal{E}$ and $\tilde{\mathcal{E}}$. If the two categories have the same modular data, then there exist unitary morphisms $u_{a,b} : a \times b \to a \times b$ for every a, b in $\mathcal{C}$ (after identifying the two tensor structures) such that

$$\tilde{\mathcal{E}}_{b,a} \circ \tilde{\mathcal{E}}_{a,b} = u_{a,b} \circ \mathcal{E}_{b,a} \circ \mathcal{E}_{a,b} \circ u_{a,b}^*, \quad \tilde{\mathcal{E}}_{a,a} = u_{a,a} \circ \mathcal{E}_{a,a} \circ u_{a,a}^*. \tag{2.37}$$

We did not succeed in constructing a *braided* tensor equivalence between the two categories out of these relations yet, e.g., using these unitaries as multipliers of a *non-strict* tensor functor $\mathcal{C} \to \mathcal{D}$. We just mention that the obstructions encountered are, of course, of cohomological type, and that the problem of extending UFTC equivalence to UBTC equivalence is nontrivial, see e.g. [BNRW15, Sec. 5] for the classification of the rank 5 case. Moreover, looking at the *symmetric* fusion category case, it is known that there are non-isomorphic groups (the smallest are of order 64) whose representation categories are indeed equivalent *as tensor categories* but not as braided tensor categories (in this way they don't contradict Doplicher-Roberts's reconstruction theorem [DR89]). We thank R. Conti for pointing out this interesting fact.

Chapter 3

DHR representations of conformal nets

One of the main advantages of the algebraic description of QFT, and in principle of any other model-independent approach, is that a clear distinction can be made between QFTs (local observable fields) and their particle states. In AQFT we have (nets of) operator algebras on the observables side and (nets of) Hilbert space representations on the states side.

Mathematically speaking, the two sides are very close to each other, in the sense that every (abstract C^*-)operator algebra $\mathcal{A}$ can be realized on a Hilbert space, and vice versa every vector state has an Hilbert space independent counterpart given by the notion of "state" of the algebra, i.e., a positive linear normalized functional $\varphi : \mathcal{A} \to \mathbb{C}$. The bridge between the two is provided by the Gelfand-Naimark-Segal (GNS) construction [Ped89, E 4.3.16], [BR87, Thm. 2.3.16]: every "state" φ of $\mathcal{A}$ arises as a vector state $\varphi(\cdot) = (\Psi_\varphi | \pi_\varphi(\cdot) \Psi_\varphi)$ in some representation $(\pi_\varphi, \mathcal{H}_\varphi)$ of $\mathcal{A}$ from some unit vector $\Psi_\varphi \in \mathcal{H}_\varphi$.

A *representation* of a C^*-algebra $\mathcal{A}$ is given by assigning an Hilbert space on which $\mathcal{A}$ acts as bounded linear operators, in other words, it is a map $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$ which respects linear operations, algebra multiplication, adjunction and unit, i.e., a unital *-homomorphism. In the case of unital C^*-algebras such maps are automatically continuous in the norm topologies, see [Ped89, E 4.3.9].

In this chapter we review the famous DHR construction of AQFT, named after Doplicher, Haag and Roberts who wrote a series of papers [DHR69a], [DHR69b], [DHR71], [DHR74], 'in those five years when the Roman empire was built' (freely quoting Sergio Doplicher in [Dop14]).

Mathematically speaking this construction allows to associate to a local net (in our case a chiral conformal net, see Chapter 1) a unitary braided tensor category of representations of the net (fulfilling all the requirements of Chapter 2), and can be seen as a way of extracting from the local observables those representations which are compatible with locality in the strong sense explained below. Physically speaking, DHR representations (and DHR superselection sectors) are motivated by the need of finding criteria to select "physical" representations of the QFT, among the enormous variety of those provided by the theory of C^*-algebras. Namely, given a net of local observables $\{\mathcal{A}\}$, the DHR representations of $\{\mathcal{A}\}$ are those which describe particle excitations of the vacuum state (some background in our preparation procedures) which are *localizable in bounded regions of spacetime*, i.e., which cannot be detected by any measurement taking place sufficiently far-away in space. This rules out, e.g., states with nontrivial electric charge, by Gauss's law, but is of interest for charges which undergo confined interactions.

In the physical 3+1 dimensions, the DHR construction (or analysis) culminated in the classification of these representations into Bosonic or Fermionic ones, into a model-independent proof of the relation between spin and statistics, a well-defined scattering theory, an intrinsic determination of a global compact gauge group together with charged field operators which commute or anticommute at space-like distance and whose gauge fixed points is the given observable net. This construction (or analysis) relies only on Einstein's causality principle, and can be regarded for sure as one of the diamonds of AQFT. The study of DHR representations, their abstract categorical structure (but in the context of QFT) and the additional (more concrete) information given by their action on the observable net are the focal point of this thesis. In this chapter we mainly review DHR theory in the chiral (rational) CFT setting, where the representation-theoretical situation becomes easier and more complicated at the same time. Easier in the sense that irreducible representations have always finite dimension (in the sense of Definition 2.5.11), are automatically covariant (with respect to the Möbius group) and their conjugate representations can be characterized via reflections and the Tomita-Takesaki modular conjugations. More complicated in the sense that there is *no gauge group* of symmetries nor a complete set of field operators which generates DHR charges from the vacuum, essentially because the DHR category is *not symmetrically braided*, but modular.

In Chapter 4 we introduce a new invariant on top of the DHR category, which

is actually very natural, namely its *braided action* on some fixed local algebra. We shall see then a recipe to show the completeness of this new invariant on a suitable subclass of theories (which we call *prime*), i.e., to achieve a complete classification of them (up to isomorphisms of nets in the sense of Definition 1.6.1). This subclass of theories does not contain, e.g., those obtained as tensor products of other theories, and those known in CFT under the name of "holomorphic" (see Chapter 4). On the negative side, we do not know by now which are the models that fall in this class and which not, nor if there are models at all among the infinitely many provided by chiral CFT. On the positive side, this analysis led to some general structure results for rational CFTs, and the more specific ones which can be derived in the prime CFT case, rely essentially on the "rigidity" properties both of local algebras (injective type III_1 factors) and of representation categories (prime UMTCs). We conclude this introduction with two remarks. The first is to advocate our abstract recipe by mentioning that the most interesting obstructions we encountered appeared already in the physical literature a long time ago, see e.g. the section "reconstruction fantasies" in [MS88], where neither von Neumann factors nor modular categories were at the authors' disposal. Secondly, in the chiral CFT case the DHR selection criterion for "physical" representations is automatically fulfilled by those with positive (conformal) energy, and, remarkably, the superselection structures singled out by "localizability" properties exactly match those considered in the literature on RCFTs, which are handled by means of primary fields.

3.1 DHR states and representations

Superselection sectors are introduced in QFT to describe conservation laws of different types of "charges". States of a specific charge are described by vectors of a complex Hilbert space (to allow superposition) where the (observable) fields can be represented, i.e., act upon them as linear operators. Moreover elementary charges are described by *irreducible* representations of the observables, which are disjoint whenever the charges are different and decompose the full Hilbert space as direct summands. This allows to describe mathematically the fact that superposition of states of different charges is forbidden and that no transition between them can occur in a measurement process.

Adopting the dual picture, we have an observable net $\{\mathcal{A}\} = \{I \in \mathcal{I} \mapsto \mathcal{A}(I)\}$

and the states act upon it as linear functionals, i.e., give rise to Hilbert space representations via the GNS construction. Irreducibility of the GNS representation is then equivalent to *purity* of the state φ, see, e.g., [BR87, Thm. 2.3.19], namely to the fact that φ cannot be written as a nontrivial convex combination of two other states. In physical terms this says that φ is not a mixture state which comes from our ignorance in the preparation procedure of the ensemble of particles.

But now the question is, representations (i.e., states) of what? In order to describe all the local observables at once one can consider the following "global" objects

Definition 3.1.1. Let $\{\mathcal{A}\}$ be a local net on the line as in Definition 1.1.1, consider the unital *-algebra $\mathcal{A}_{\mathrm{loc}}$ generated by all local algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$ (i.e., the set theoretic union because $\mathcal{I}$ is directed and all local algebras have the same unit). The unital C^*-algebra $\mathcal{A}$ generated by $\mathcal{A}_{\mathrm{loc}}$ (i.e., its norm closure in $\mathcal{B}(\mathcal{H})$ in this case) is called the **quasilocal algebra** of the net $\{\mathcal{A}\}$.

Remark 3.1.2. Notice that $\{\mathcal{A}\}$ is not assumed to be Möbius covariant, indeed DHR representations can be defined kinematically and the spacetime dependence enters only through *locality* of the net, at a first stage. Similarly in 3+1 dimensions with or without Poincaré covariance. Notice also that $\mathcal{A}$ can be defined independently of the representation on $\mathcal{H}$ as the C^*-inductive limit of the (directed) net $\{\mathcal{A}\}$.

Without energy-momentum spectrum conditions, the defining *vacuum representation* $\mathcal{H}$ of the net, hence of the quasilocal algebra by continuity, can be viewed as some fixed representation $\pi_0 : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, $a \mapsto \pi_0(a) := a$.

In typical cases, cf. Proposition 1.1.8, $\pi_0(\mathcal{A}) = \mathcal{A}$ is not $\mathcal{B}(\mathcal{H})$, but it sits irreducibly into it, namely $\mathcal{A}' = \mathbb{C}\mathbb{1}$, i.e., $\mathcal{A}'' = \mathcal{B}(\mathcal{H})$. Under further physically motivated conditions on the net $\{\mathcal{A}\}$, the algebra $\mathcal{A}$ is known to be simple and of a universal type, as an abstract C^*-algebra, after [HKK70], [Tak70].

The task is to recognize among the uncountably many representations of $\mathcal{A}$ (which do not even form a set, but a proper class, see e.g. [Dyb12, Sec. 1.1.5]) those with a physical meaning, e.g., which describe some particle excitations. The representations considered in the following, after [DHR71], aim to describe a particular type of excitations of the vacuum π_0 (some reference background for our instruments) which can be "localized" in *bounded* regions of spacetime

(intervals $I \in \mathcal{I}$ in the chiral case, double cones in higher dimensions), i.e., which cannot be detected (distinguished from the vacuum background) by performing measurements in the (space-like) complement $I' = \mathbb{R} \smallsetminus \bar{I}$, with respect to the observables of $\{\mathcal{A}\}$. Moreover, "localizability" should be possible for *every* interval, if there is no minimal localization region, or at least for a family of intervals stable under arbitrary spacetime translations.

Definition 3.1.3. Given a local net $\{\mathcal{A}\}$ as in Definition 3.1.1, a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$ is **DHR localizable** if for *every* interval $I \in \mathcal{I}$ it fulfills

$$\pi_{\restriction \mathcal{A}(I')} \cong \pi_{0 \restriction \mathcal{A}(I')} \tag{3.1}$$

where $\mathcal{A}(I')$ denotes the C^*-subalgebra of $\mathcal{A}$ generated by all local algebras $\mathcal{A}(J)$, $J \subset I'$ (i.e., $I \cap J = \emptyset$ in the chiral case) and $\cong$ denotes unitary equivalence of the restricted representations. A representation π fulfilling the selection criterion (3.1) is also called a **DHR representation** of the net $\{\mathcal{A}\}$.

Remark 3.1.4. Unitary equivalence means that equality holds in (3.1) up to a unitary between the two Hilbert spaces $v_{\pi, I} : \mathcal{H} \to \mathcal{H}_\pi$, for *each* interval $I \in \mathcal{I}$, which *intertwines* the two representations on the subalgebra $\mathcal{A}(I')$, i.e.

$$v_{\pi, I} \pi_0(a) = \pi(a) v_{\pi, I} \tag{3.2}$$

for every $a \in \mathcal{A}(I')$. Notice also that if (3.1) holds for $I \in \mathcal{I}$ by means of a unitary $v_{\pi, I}$, then it holds for every $J \in \mathcal{I}$, $I \subset J$ by means of the same unitary.

Taking the C^*-closure for $\mathcal{A}(I')$ is a matter of convenience, see Remark 1.5.3. Indeed C^*-representations are *-homomorphisms by definition and automatically norm continuous. On the other hand, the distinction between the norm topology and the weak operator topology is fundamental. The DHR representations are locally normal, i.e., normal on local algebras $\mathcal{A}(I)$, as easily seen from (3.1), but neither globally normal nor extendible to $\mathcal{B}(\mathcal{H})$ in general. Moreover, under additional assumptions they are known to be equivalent to the vacuum on relative commutants $\mathcal{A}(I)^c = \mathcal{A}(I)' \cap \mathcal{A}$ only if they are globally equivalent to the vacuum or to multiples of it, see Proposition 4.3.5, and the C^*-inclusion $\mathcal{A}(I') \subset \mathcal{A}(I)^c$, see Corollary 4.3.9, is known to measure the number of nontrivial DHR representations of the given net.

The DHR selection criterion rules out theories with long-range forces, but is generally fulfilled in the case of observable nets arising as fixed points of a field net under a *global* compact gauge group of automorphisms, see [DHR69a, Thm. 6.1].

As observed by [BMT88, Sec. 1A], see also [Reh15, Sec. 4.2], for *conformally covariant* theories (in any spacetime dimension) the DHR representations cover the spectrum of physically interesting representations.

Proposition 3.1.5. [BMT88]. *Let $\{\mathcal{A}\}$ be a local conformal net on the line (Definition 1.1.3) fulfilling in addition Haag duality on $\mathbb{R}$ (Definition 1.5.2). Then every representation π which is Möbius covariant with positive (conformal) energy is automatically* DHR *localizable in the sense of Eq. (3.1).*

A similar statement holds [GL96, Sec. 2.1], [Müg10a, Lem. 2.11] replacing positivity of the energy with separability of $\mathcal{H}_\pi$. Indeed local algebras are type *III* factors by Proposition 1.1.8 and the complement of an interval $I \in \mathcal{I}$, once imbedded in $\mathbb{S}^1$, is again a bounded interval of $\mathbb{R}$ up to a conformal transformation. Similarly to Lemma 1.2.1, DHR representations of $\{\mathcal{A}\}$ can be extended from $\mathbb{R}$ to $\mathbb{S}^1$ and vice versa restricted from $\mathbb{S}^1$ to $\mathbb{R}$, see [KLM01, Prop. 50]. Notice that we always assume Haag duality on $\mathbb{R}$, or equivalently strong additivity by Lemma 1.5.4, in order to handle DHR theory.

Definition 3.1.6. In the previous notation, a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$ of $\mathcal{A}$ on $\mathcal{H}_\pi$ is called **irreducible** if the commutant of $\pi(\mathcal{A})$ in $\mathcal{B}(\mathcal{H}_\pi)$ is trivial, i.e., $\pi(\mathcal{A})' = \mathbb{C}1_\pi$, where $1_\pi = \mathbb{1} \in \mathcal{B}(\mathcal{H}_\pi)$.

Let π_1, π_2 be two representations of $\mathcal{A}$ respectively on $\mathcal{H}_1$ and $\mathcal{H}_2$, denote by $\mathrm{Hom}_{\mathrm{Rep}(\mathcal{A})}(\pi_1, \pi_2) := \{t \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) : t\pi_1(a) = \pi_2(a)t, a \in \mathcal{A}\}$ the complex vector space of **intertwining operators**, or simply **intertwiners**. Here $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Remark 3.1.7. Notice that the DHR representations of a local net $\{\mathcal{A}\}$ form a $\mathbb{C}$-linear category which is also unitary, as in Definition 2.4.4, once the arrows are equipped with the usual adjunction of bounded linear operators between Hilbert spaces. The composition of arrows is also given by the usual composition product of operators.

Definition 3.1.8. Let $\{\mathcal{A}\}$ be a local net as in Definition 3.1.1. Given an *irreducible* DHR representation π, the unitary equivalence class of π, denoted by $[\pi]_{\cong}$, is called a **DHR superselection sector** of the net $\{\mathcal{A}\}$. The

DHR superselection **charges** of the theory are then the labels for different isomorphisms classes of DHR representations.

In the next section, all these purely representation-theoretical notions are merged again into *algebra*. This observation, due to [DHR71], allows to establish a more intimate connection with the underlying *local net* $\{\mathcal{A}\}$, rather than with the quasilocal algebra $\mathcal{A}$ only, and to define on the DHR representations a tensor multiplication together with a canonical braiding (up to a left/right choice) in the sense of Chapter 2.

3.2   DHR endomorphisms

The crucial assumption in DHR theory is *Haag duality (on $\mathbb{R}$)*, as in Definition 1.5.2. Haag duality is an algebraic strengthening of locality, moreover every essentially dual net, e.g. a chiral CFT as in Proposition 1.4.1, can be turned into a Haag dual net, cf. Remark 1.5.6.

Let $\{\mathcal{A}\}$ be a local net realized on $\mathcal{H}$, as in Definition 1.1.1, not necessarily conformal but fulfilling Haag duality on $\mathbb{R}$ and irreducible in the sense that $\mathcal{A}' = \mathbb{C}\mathbb{1}$. Take a DHR representation of a $\{\mathcal{A}\}$, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$, and for each $I \in \mathcal{I}$ denote by $v_{\pi,I} : \mathcal{H} \to \mathcal{H}_\pi$ the unitary intertwining π_0 with π on $\mathcal{A}(I')$. We can promote the intertwining relation (3.2) to the whole quasilocal algebra $\mathcal{A}$, namely define

$$\rho_I(a) := v_{\pi,I}^* \pi(a) v_{\pi,I} \tag{3.3}$$

for every $a \in \mathcal{A}$. Then $\rho_I : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation of $\mathcal{A}$ unitarily equivalent to π, in the sense of Definition 3.1.6, but again on the vacuum space $\mathcal{H}$. In particular it is DHR and irreducible if and only if π is irreducible. Moreover $\rho_I(a) = a$ if $a \in \mathcal{A}(I')$, i.e., ρ_I is *localizable in I*.

Notice that the vacuum representation $\pi = \pi_0$ is also a DHR representation of $\{\mathcal{A}\}$ and in this case the unitaries $v_{\pi_0,I} \in \mathcal{A}(I')' = \mathcal{A}(I)$ by Haag duality on $\mathbb{R}$, i.e., they are observables which sit in $\mathcal{A}_{\mathrm{loc}}$. On the other hand, for every $I \in \mathcal{I}$ the unitary $v_{\pi_0,I} = \mathbb{1}$ also fulfills (3.2) for $\pi = \pi_0$, and we get $\rho_I = \mathrm{id}$. In this case every representation ρ_I constructed as in (3.3) for each I is an endomorphisms of $\mathcal{A}$, more specifically an *inner automorphism* $\rho_I = \mathrm{Ad}_u$ for some unitary $u \in \mathcal{A}_{\mathrm{loc}}$. Moreover the intertwining operators between $\rho_I = \mathrm{Ad}_{u_{\pi_0,I}}$ and $\rho_J = \mathrm{Ad}_{u_{\pi_0,J}}$, in the sense of Definition 3.1.6, also belong to $\mathcal{A}_{\mathrm{loc}}$. Namely $u_{\pi_0,J} u_{\pi_0,I}^* \in \mathcal{A}(I) \vee \mathcal{A}(J) \subset \mathcal{A}(K) \subset \mathcal{A}_{\mathrm{loc}}$ whenever $I, J \subset K$

for big enough $K \in \mathcal{I}$.

More generally consider ρ_I as in (3.3) for a non-vacuum DHR representation. Then for every $J \in \mathcal{I}$ and $a \in \mathcal{A}(J)$, consider an arbitrary element $b \in \mathcal{A}(K')$ such that $I, J \subset K$ and calculate $\rho_I(a)b = \rho_I(ab) = \rho_I(ba) = b\rho_I(a)$, hence $\rho_I(a) \in \mathcal{A}(K')' = \mathcal{A}(K)$. We deduce that ρ_I maps $\mathcal{A}_{\text{loc}}$ into itself and by norm continuity extends to an *endomorphism* of $\mathcal{A}$. By a similar argument we get that every intertwining operator $t \in \mathcal{B}(\mathcal{H}, \mathcal{H}) = \mathcal{B}(\mathcal{H})$ between different ρ_I and ρ_J necessarily belongs to some $\mathcal{A}(K)$ for big enough $K \in \mathcal{I}$. More specifically it belongs to $(\mathcal{A}(I') \cap \mathcal{A}(J'))'$ but in general *not* to $\mathcal{A}(I) \vee \mathcal{A}(J)$ if $I \cap J = \emptyset$.

In other words the representations ρ_I arising from (3.3) varying the localization interval $I \in \mathcal{I}$ and the DHR representation π of $\{\mathcal{A}\}$, with the associated intertwining operators as arrows, form a category of representations of $\mathcal{A}$ which embeds faithfully and fully into the endomorphism category $\text{End}(\mathcal{A})$. Equations (3.3) and (3.1) guarantee that *unitary* intertwiners exist for every pair of intervals $I, J \in \mathcal{I}$, hence we have copies of the representation π localizable in every interval of the line and they are all pairwise unitarily equivalent, i.e., they are *transportable* to every other interval. Summarizing

Definition 3.2.1. Let $\{\mathcal{A}\}$ be a local net realized on $\mathcal{H}$, as in Definition 1.1.1, assume irreducibly of the defining vacuum representation $\pi_0 = \text{id}$ and Haag duality on $\mathbb{R}$. An endomorphism $\rho \in \text{End}(\mathcal{A})$, i.e. a *-homomorphisms from $\mathcal{A}$ into itself, is called a **DHR endomorphism** of $\{\mathcal{A}\}$ if it is **localizable** in some $I \in \mathcal{I}$, i.e., $\rho_{\restriction \mathcal{A}(I')} = \text{id}_{\restriction \mathcal{A}(I')}$, and **transportable** to every other $J \in \mathcal{I}$, i.e., there exists a unitary u such that $\text{Ad}_u \, \rho = u\rho(\cdot)u^*$ is localizable in J.

Remark 3.2.2. Notice that DHR endomorphisms are automatically unital and injective, i.e., *-isomorphisms onto their image, as one can easily check using transportability (or simplicity of $\mathcal{A}$).

Definition 3.2.3. Let $\{\mathcal{A}\}$ as above. The **DHR category** of $\{\mathcal{A}\}$, denoted by $\text{DHR}\{\mathcal{A}\}$, is the collection of the DHR endomorphism of $\{\mathcal{A}\}$ as objects and intertwining operators in $\mathcal{A}$ as arrows. Two DHR endomorphisms ρ, σ are unitarily equivalent as representations of $\mathcal{A}$ if and only if $\rho = \text{Ad}_u \, \sigma$ for some unitary $u \in \mathcal{U}(\mathcal{A})$, in this case we write $\rho \cong \sigma$. The DHR superselection sectors of $\{\mathcal{A}\}$ (Definition 3.1.8) are in one-to-one correspondence with unitary equivalence classes of DHR endomorphisms through the formula $\pi \cong \pi_0 \circ \rho$.

Remark 3.2.4. Notice that the arrows of $\text{DHR}\{\mathcal{A}\}$ belong necessarily to $\mathcal{A}_{\text{loc}}$,

due to the sharp localization of the DHR endomorphisms. Moreover $\mathrm{Ad}_u\,\rho$ is not necessarily DHR if $u \in \mathcal{U}(\mathcal{A})$ but $u \in \mathcal{A} \smallsetminus \mathcal{A}_{\mathrm{loc}}$.

3.3   Statistics of particles and DHR braiding

One of the guiding principles of the DHR analysis is the following. The physical content of the theory is determined by measurable quantities, the net of local observables in our setting. In particular the net itself (in the vacuum sector) should determine its own states of physical interest (particle excitations), their superselection quantum numbers (charges), their fusion and conjugation structure, their statistics (e.g. Bose/Fermi permutation symmetry or more general "exotic" ones). The charge structure is determined without *charged fields*, which can then be introduced for convenience at a later stage, at least in 3+1 dimensions [DR90], as a (non-local) extension of the observable net. They generate the DHR ρ-charges from the vacuum sector $\pi_0 = \mathrm{id}$ through the formula $\psi_\rho a = \rho(a)\psi_\rho$, $a \in \mathcal{A}$, i.e., they are charged (non-observable) intertwiners among the representations π_0 and $\pi = \pi_0 \circ \rho$. The structure of the DHR charges is naturally formulated (and analysed) in the language of unitary braided tensor categories (UBTCs), as introduced in Chapter 2 from an abstract point of view. Anyway in the DHR setting we have essentially two features which have no abstract categorical counterpart (to our knowledge), namely commutation relations of tensor products between objects $\rho\,\sigma = \sigma\rho$ and trivialization of braiding operators $\mathcal{E}_{\rho,\sigma} = \mathbb{1}$, on which the analysis of Chapter 4 is based.

In the following we give an account of the DHR superselection structure structure as it emerges from chiral CFTs. Notice however that this case differs from the high-dimensional one because the charged fields are in general not enough to implement DHR endomorphism, see [Reh94b, Sec. 2], and there is no intrinsic gauge group explaining the charge structure as in 3+1 dimensions. Indeed the duality result [DR89] does not apply whenever the DHR category has more complicated statistics than Bose or Fermi ones.

Endomorphisms of an algebra, unlike representations, can be composed. The composition $\rho \times \sigma := \rho \circ \sigma$ of DHR endomorphisms is again localizable, by taking a sufficiently big interval on which both ρ, σ are localizable. Moreover it is again transportable because endomorphisms act on the arrows and

$u \times v := u\rho(v)$ defines a unitary charge transporter from $\rho \times \sigma$ to $\tilde{\rho} \times \tilde{\sigma}$, where $\tilde{\rho} = \mathrm{Ad}_u\, \rho$, $\tilde{\sigma} = \mathrm{Ad}_v\, \sigma$ and $u : \rho \to \tilde{\rho}$, $v : \sigma \to \tilde{\sigma}$ are intertwining unitaries. Notice that the composition product and the product of operators in $\mathcal{A}$ are both strictly associative. Moreover the equality $u\rho(v) = \tilde{\rho}(v)u$ holds, cf. (2.1), and allows to construct a strict, *-preserving and bilinear tensor multiplication functor, namely

Proposition 3.3.1. [DHR69b], [DHR71]. *Let $\{\mathcal{A}\}$ be a local net realized on $\mathcal{H}$, as in Definition 1.1.1, assume irreducibly of the defining vacuum representation $\pi_0 = \mathrm{id}$ and Haag duality on $\mathbb{R}$. The category $\mathrm{DHR}\{\mathcal{A}\}$ is a strict tensor C^*-category with irreducible unit id, in the sense of Definition 2.2.1 and 2.4.8, which embeds faithfully and fully in the endomorphism category of the quasilocal algebra $\mathrm{End}(\mathcal{A})$.*

Unlike charged fields, DHR endomorphisms necessarily commute (as endomorphisms!) whenever they can be localized in disjoint intervals of $\mathbb{R}$. This fact [DHR69b, Lem. 2.2] reflects locality of the net at the level of DHR superselection sectors.

Lemma 3.3.2. [DHR69b]. *Let $\{\mathcal{A}\}$ be as above. If ρ, σ are DHR endomorphisms of $\{\mathcal{A}\}$, localizable respectively in $I, J \in \mathcal{I}$ and $I \cap J = \emptyset$, then $\rho \times \sigma = \sigma \times \rho$.*

The previous lemma, together with transportability, allows to exchange the order of tensor products of endomorphisms up to unitary equivalence. The unitary operator which implements the *exchange symmetry* encodes intrinsic information about the DHR sectors, in particular it contains their statistics.

Definition 3.3.3. Let $\{\mathcal{A}\}$ be as above and let ρ, σ be DHR endomorphisms of $\{\mathcal{A}\}$. Choose DHR endomorphisms $\tilde{\rho}, \tilde{\sigma}$ respectively equivalent to ρ, σ and localizable in two intervals $I_1, I_2 \in \mathcal{I}$ such that $I_1 \cap I_2 = \emptyset$ and I_1 is to the left of I_2 [1]. Choose unitary charge transporters $u : \rho \to \tilde{\rho}$, $v : \sigma \to \tilde{\sigma}$ and consider the unitary intertwiner $\mathcal{E}_{\rho,\sigma} : \rho \times \sigma \to \tilde{\rho} \times \tilde{\sigma} = \tilde{\sigma} \times \tilde{\rho} \to \sigma \times \rho$ defined by

$$\mathcal{E}_{\rho,\sigma} := v^* \times u^* \circ u \times v = \sigma(u^*)v^*u\rho(v).$$

Call it the **left braiding** of the DHR endomorphisms ρ, σ. If ρ is already localizable to the left of σ, one may choose $u = v = \mathbb{1}$, hence

$$\mathcal{E}_{\rho,\sigma} = \mathbb{1}.$$

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[1]In [FRS92] the opposite right/left convention is adopted for the DHR braiding; this is related to a different convention for the Cayley map given in Remark 1.1.4.

Remark 3.3.4. Notice that $\mathcal{E}_{\rho,\sigma}$ belongs to the algebra of observables $\mathcal{A}_{\text{loc}}$ of the given net $\{\mathcal{A}\}$ (as a unitary operation that we can perform on the system, taking our localizable particle-like excitations in left/right far apart regions of the line).

The notation $\mathcal{E}_{\rho,\sigma}$ is consistent because the intertwiner does not depend on any of the choices made before, but on ρ, σ and on the left/right order of the target intervals only. This is the content of the following *key* lemma [DHR71, Lem. 2.6], see also [DHR69b, Lem. 2.3], [Bor65, Thm. V-5] and [FRS89, Sec. 2] for the low-dimensional generalization.

Lemma 3.3.5. [DHR71]. *The unitary intertwiner $\mathcal{E}_{\rho,\sigma}$ considered in Definition 3.3.3 is invariant under small perturbations of the localization intervals of $\tilde{\rho}, \tilde{\sigma}$ which respect the above constraints and does not depend on the specific choice of unitary charge transporters u, v.*

Using the degrees of freedom exploited in the previous lemma, the localization properties of DHR endomorphisms and their intertwiners, hence locality, one can check *naturality* of the family $\{\mathcal{E}_{\rho,\sigma}\}_{\rho,\sigma}$, in the sense of Definition 2.1.7. This is the crucial step in showing that the DHR braiding is indeed a braiding on DHR endomorphisms, in the abstract sense of Definition 2.3.1

Proposition 3.3.6. [DHR71], [FRS89]. *Let $\{\mathcal{A}\}$ be a local net realized on $\mathcal{H}$, as in Definition 1.1.1, and assume Haag duality on $\mathbb{R}$. The category $\text{DHR}\{\mathcal{A}\}$ is a UBTC, in the sense of Definition 2.4.4 and 2.3.1.*

Proof. The statement is contained in [DHR71, Thm. 4.3], [FRS89, Eq. (4.26)] and is formulated in [FRS92, Sec. 2.2] in the low-dimensional case. Here we give a simple proof of naturality, due to its importance in our work and because we could not find it explicitly in the literature, see comments after [FRS89, Eq. (4.26)]. Let ρ, σ be DHR endomorphisms of $\{\mathcal{A}\}$ and let $\mathcal{E}_{\rho,\sigma} = \sigma(u^*)v^*u\rho(v)$ as in Definition 3.3.3. Naturality is easy on unitary intertwiners, namely let r, s be unitaries in $\mathcal{A}_{\text{loc}}$ and $\rho_1 := \text{Ad}_r\,\rho$, $\sigma_1 := \text{Ad}_s\,\sigma$. Then $ur^* : \rho_1 \to \tilde{\rho}$ and $vs^* : \sigma_1 \to \tilde{\sigma}$ hence, by Lemma 3.3.5, $\mathcal{E}_{\rho_1,\sigma_1} = s\sigma(ru^*)s^*sv^*ur^*r\rho(vs^*)r^* = s\sigma(r)\sigma(u^*)v^*u\rho(v)\rho(s^*)r^* = s \times r \circ \mathcal{E}_{\rho,\sigma} \circ r^* \times s^*$ which reads as equation (2.5). More generally let $t : \tau \to \rho$ be an intertwiner (not necessarily unitary) between DHR endomorphisms. Then $\mathcal{E}_{\rho,\sigma} \circ t \times 1_\sigma = \sigma(u^*)v^*u\rho(v)t = \sigma(u^*)v^*ut\tau(v) = \sigma(u^*)v^*utw^*w\tau(v)$ where $w : \tau \to \tilde{\tau}$ is a unitary such that $\tilde{\tau}$ is localizable left to $\tilde{\sigma}$. Now $utw^* : \tilde{\tau} \to \tilde{\rho}$, hence belongs to a local algebra arbitrarily far away to the left, up to changing the localization regions of $\tilde{\rho}, \tilde{\tau}$

thanks to Lemma 3.3.5, and commutes with v^*. Then $\sigma(u^*)v^*utw^*w\tau(v) = \sigma(u^*)utw^*v^*w\tau(v) = \sigma(u^*)utw^*\sigma(w)\sigma(w^*)v^*w\tau(v)$. As before we can assume that utw^* belongs to a local algebra on which σ acts trivially, hence we get $\sigma(u^*)\sigma(utw^*)\sigma(w)\sigma(w^*)v^*w\tau(v) = \sigma(t)\sigma(w^*)v^*w\tau(v) = 1_\sigma \times t \circ \mathcal{E}_{\tau,\sigma}$. With this one can show (2.5) in full generality for the DHR category of $\{\mathcal{A}\}$. Lemma 3.3.5 guarantees also multiplicativity of the DHR braiding (2.7) on both sides, and the proof is complete. □

In view of the invariance properties of the left braiding explained by Lemma 3.3.5, one can only interchange the *left/right* position of the localization intervals of the target endomorphisms to obtain something different. In this way one defines the *right braiding* $\widetilde{\mathcal{E}}_{\rho,\sigma}$, for which the analogous of Lemma 3.3.5 holds and one can show as in Proposition 3.3.6 that DHR $\mathcal{A}$ is braided also with respect to $\{\widetilde{\mathcal{E}}_{\rho,\sigma}\}_{\rho,\sigma}$. However, it is easily seen that the right braiding coincides with the *opposite* of the left braiding as in Remark 2.3.4, namely $\widetilde{\mathcal{E}}_{\rho,\sigma} = \mathcal{E}^{\mathrm{op}}_{\rho,\sigma} = \mathcal{E}^*_{\sigma,\rho}$.

Remark 3.3.7. Notice that the distinction between left braiding and (opposite) right braiding comes from a choice in the orientation (positive or negative) of the real line $\mathbb{R}$. Given a local net $\{\mathcal{A}\}$ as in Proposition 3.3.6 with left braiding $\mathcal{E}$, reversing the orientation of $\mathbb{R}$ leads to another net with left braiding $\mathcal{E}^{\mathrm{op}}$, cf. [KLM01]. However the realization of opposite braided tensor categories in CFT is an interesting nontrivial task, as pointed out to us by R. Longo, because reversing the orientation of $\mathbb{R}$ leads to a generator of the translations with negative spectrum, hence to a negative conformal Hamiltonian due to Proposition 1.1.5. See [Bis16, Prop. 4.3, 4.4] for a general strategy in this direction and for its application to the case of the $SU(2)$-currents at any level. See also [Bis15, Prop. 4.7] for the connection of this problem to the realization issue of subfactors in chiral CFT.

Another interesting interpretation of the braiding operators is the following

Remark 3.3.8. Let $\{\mathcal{A}\}$ be as in Proposition 3.3.6 and take two DHR endomorphisms $\rho, \sigma \in \mathrm{DHR}\{\mathcal{A}\}$. We may assume that ρ, σ are both localizable in some interval $I \in \mathcal{I}$ and let the transported charges $\tilde{\rho}, \tilde{\sigma}$ as in Definition 3.3.3 be localizable in two intervals respectively to the left and to the right of I. In this scenario, the braiding $\mathcal{E}_{\rho,\sigma}$ has the significance of *commutation relations* between each *pair* of left/right *unitary charge transporters* $u \in \mathrm{Hom}_{\mathrm{DHR}\{\mathcal{A}\}}(\rho, \tilde{\rho}), v \in \mathrm{Hom}_{\mathrm{DHR}\{\mathcal{A}\}}(\sigma, \tilde{\sigma})$ chosen as in Definition 3.3.3.

Namely

$$vu = uv\mathcal{E}_{\rho,\sigma}.$$

Indeed by the very definition $\mathcal{E}_{\rho,\sigma} = \sigma(u^*)v^*u\rho(v) = v^*\tilde{\sigma}(u)^*\tilde{\rho}(v)u = v^*u^*vu$ because $\tilde{\rho}$ and $\tilde{\sigma}$ act trivially on v and u respectively. This observation is one of the bases of our intuition on the results of Chapter 4, cf. [Wie95] for similar ideas.

In the next section we shall review known consequences of conformal covariance on the structure of the DHR category. Notably the so called "property B" after Borchers, see [DHR71], is a consequence of type III factoriality of the net, hence DHR endomorphisms admit subobjects and direct sums [DHR71, Lem. 2.5] in the sense of Definition 2.5.2 and 2.5.3, realized by Cuntz algebras of isometries living inside $\mathcal{A}_{\mathrm{loc}}$.

3.4 DHR categories of completely rational conformal nets

For **completely rational** models of chiral CFT (see Definition 3.4.1 below), the respective categories DHR$\{\mathcal{A}\}$ have *finitely many* inequivalent irreducible endomorphisms and they are all *finite-dimensional* [KLM01, Cor. 10, 39] in the sense of the intrinsic dimension (Definition 2.5.11) defined by the conjugate equations. Moreover every DHR endomorphism is *reducible* as a (possibly infinite) direct sum of irreducible ones [KLM01, Cor. 39]. These three instances motivate the terminology "rationality" in this context.
In particular, the subcategories of finite-dimensional objects DHR$\{\mathcal{A}\}_f \subset$ DHR$\{\mathcal{A}\}$ coincide with those generated by finite direct sums of irreducibles, and we have examples of UFTCs (see Definition 2.5.16) realized by DHR endomorphisms. Moreover the DHR braiding turns out to be non-degenerate in a natural way [KLM01, Cor. 37] hence DHR$\{\mathcal{A}\}_f$ has the abstract structure of a UMTC (see Definition 2.6.4). We first recall the definition of complete rationality (in the conformal case) and then state the *modularity* result of the respective DHR categories. See [KLM01], [Müg10a].

Definition 3.4.1. A local conformal net on the line $\{\mathcal{A}\}$ as in Definition 1.1.3 is called **completely rational** if the following conditions are satisfied.

(a) *Haag duality on $\mathbb{R}$* (cf. Definition 1.5.2): $\mathcal{A}(I')' = \mathcal{A}(I)$ for all $I \in \mathcal{I}$.

(b) *Split property* (cf. Definition 1.6.9): for every $I, J \in \mathcal{I}$, $I \Subset J$ there exists a type I factor $\mathcal{F}$ such that $\mathcal{A}(I) \subset \mathcal{F} \subset \mathcal{A}(J)$.

(c) *Finite index two-interval subfactor*: $\mathcal{A}(I_1 \cup I_2) \subset \mathcal{A}(I)' \cap \mathcal{A}(J)$ has finite Jones index, where $I, J \in \mathcal{I}$, $I \Subset J$ and $I' \cap J = I_1 \cup I_2$ for $I_1, I_2 \in \mathcal{I}$.

With conformal covariance, see [GLW98], condition (a) is equivalent to

(a)' *Strong additivity*: $\mathcal{A}(I_1 \cup I_2) = \mathcal{A}(I)$ where $I \in \mathcal{I}$, $p \in I$ and $\{p\}' \cap I = I \smallsetminus \{p\} = I_1 \cup I_2$ for $I_1, I_2 \in \mathcal{I}$.

Remark 3.4.2. Conditions (a) and (b) strengthen the locality assumption on the net, they are natural and fulfilled in many models, as we have seen in Chapter 1. Condition (c) is the characteristic feature of "rational" theories, i.e., those with finitely many superselection sectors, namely it controls the size of the DHR category, i.e., the number of inequivalent DHR superselection sectors. It is the only one not motivated by general physical intuition, see [Reh15]. For example, the $U(1)$-currents [BMT88] have uncountably many inequivalent sectors. At the same time, there is a dichotomy between theories with finitely many or uncountably many inequivalent irreducible sectors, as found by [LX04] in the diffeomorphism covariant case.
Notice that complete rationality, in the conformal setting, is a local condition, i.e., can be checked inside one arbitrarily fixed local algebra.

In the case of completely rational chiral CFTs, Kawahigashi, Longo and Müger have shown that

Proposition 3.4.3. [KLM01]. *Let $\{\mathcal{A}\}$ be a completely rational conformal net on the line as in Definition 3.4.1. Then every irreducible DHR endomorphism ρ has a conjugate DHR endomorphism $\bar{\rho}$, there are finitely many different DHR sectors $[\mathrm{id}], [\rho_1], \ldots, [\rho_n]$ and $\mathrm{DHR}\{\mathcal{A}\}_f$ is a UFTC which is also a UMTC once equipped with the DHR braiding introduced in Definition 3.3.3.*
In particular, every $\rho \in \mathrm{DHR}\{\mathcal{A}\}_f$ has a well defined dimension $d_\rho \geq 1$ and phase $\omega_\rho \in \mathbb{C}$, $|\omega_\rho| = 1$ as introduced in the abstract setting of Sec. 2.5, 2.6.

Remark 3.4.4. Statements analogous to Proposition 3.4.3 hold for the **local DHR subcategories** $\mathrm{DHR}^I\{\mathcal{A}\}_f$ of finite-dimensional endomorphisms localizable in some fixed interval $I \in \mathcal{I}$, indeed $\mathrm{DHR}^I\{\mathcal{A}\}_f \simeq \mathrm{DHR}\{\mathcal{A}\}_f$ as abstract UBTCs. Both these facts follow easily from transportability of DHR endomorphism in any arbitrarily small interval. In the following we shall

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consider only the subcategories of finite-dimensional objects (because they contain already all the DHR (irreducible) sectors) and denote them again by DHR$\{\mathcal{A}\}$ and DHR$^I\{\mathcal{A}\}$ with abuse of notation.

Remark 3.4.5. Notice that beyond completely rational models, take e.g. the Virasoro net with central charge $c = 1$, the case of irreducible DHR endomorphisms with infinite intrinsic dimension is typical, see [Car03], [Reh94a].

Despite the conjugation of DHR charges has been first understood globally [DHR71] via a shifting the localization regions of endomorphisms to space-like infinity, conjugate endomorphisms hence the notion of dimension, cf. [LR97], are local notions, i.e., can be read inside any local algebra, see [Lon89], [Lon90]. We recall the result in the (completely rational) chiral CFT setting [GL96, Cor. 3.7]

Proposition 3.4.6. [Lon89], [GL96] (**Index-dimension theorem**). *Let* $\{\mathcal{A}\}$ *be as above and* ρ *a* DHR *endomorphism localizable in an interval* $I \in \mathcal{I}$, *then*

$$d_\rho{}^2 = \mathrm{Ind}(\rho(\mathcal{A}(I)), \mathcal{A}(I))$$

where the r.h.s denotes the minimal index of the subfactor $\rho(\mathcal{A}(I)) \subset \mathcal{A}(I)$. *The equality holds in the infinite index case also, by setting* $d_\rho := \infty$ *when a conjugate in the sense of Definition 2.5.6 does not exist.*

Conformal (Möbius) covariance of the net in the sense of Definition 1.1.3 means conformal covariance of the vacuum sector $\pi_0 = \mathrm{id}$, by definition. The latter carries over to finite-dimensional DHR representations by [GL92, Thm. 6.4], using [DHR71, Cor. 6.2], and to arbitrary DHR representations using [KLM01, Cor. 40]. The key ingredient is the Bisognano-Wichmann property (Theorem 1.3.2) of the net in the vacuum sector, as exploited by Guido and Longo in [GL92, Thm. 7.1].

Proposition 3.4.7. [GL92]. *Let* $\{\mathcal{A}\}$ *be as above and* ρ *a* DHR *endomorphism, then* ρ *is Möbius covariant with positive (conformal) energy. Namely there is a strongly continuous unitary representation* U_ρ *of* $\widetilde{\mathsf{M\ddot{o}b}}$ *on* $\mathcal{H}$, *with positive generator* H_ρ *of the rotations subgroup, such that* $\rho \circ \alpha_{\dot{g}} = \mathrm{Ad}_{U_\rho(g)} \circ \rho$, $g \in \widetilde{\mathsf{M\ddot{o}b}}$, [2]. *Here* $\alpha_g = \mathrm{Ad}_{U(g)}$ *refers to the representation of* $\widetilde{\mathsf{M\ddot{o}b}}$ *which implements covariance in the vacuum sector and, for every* $g \in \widetilde{\mathsf{M\ddot{o}b}}$, *equality is meant on the local algebras* $\mathcal{A}(I)$, $I \in \mathcal{I}$ *such that* $gI \in \mathcal{I}$, *as in (1.1).*

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[2]$\widetilde{\mathsf{M\ddot{o}b}}$ denotes the universal (simply connected) covering of $\mathsf{M\ddot{o}b}$ and $g \mapsto \dot{g}$ the covering map, see [GL92, Sec. 8].

Remarkably, the geometric action of the modular conjugation associated to half-lines and the vacuum vector, e.g., to the positive half-line as in Theorem 1.3.2, gives an explicit formula for the conjugate DHR endomorphisms, see [GL92, Thm. 4.1] and [GL96, Thm. 2.11] for the formulation in chiral CFT. This deep fact unveils the connection between DHR charge conjugation and inversion of spacetime coordinates (interpreting the real line as a light-ray in $\mathbb{R}^{1+1}$ spacetime) on purely algebraic grounds.

Proposition 3.4.8. [GL92]. *Let $\{\mathcal{A}\}$ be as above and ρ a DHR endomorphism with finite dimension, then $\overline{\rho} = j \circ \rho \circ j$ is a conjugate DHR endomorphism of ρ. Here $j = \mathrm{Ad}_{J_{\mathbb{R}_+,\Omega}}$ denotes the adjoint action of the modular conjugation $J_{\mathbb{R}_+,\Omega}$ associated to the positive half-line and the vacuum vector.*

Remark 3.4.9. Notice that $j = \mathrm{Ad}_{J_{\mathbb{R}_+,\Omega}}$ gives rise to an anti-automorphism $j : \mathcal{A} \to \mathcal{A}$ of the quasilocal algebras which preserves the local substructure, and the same is true for every other half-line of $\mathbb{R}$.

In the same spirit of Proposition 3.4.6, which connects the (more algebraic) notion of dimension of a DHR endomorphism with the (more analytical) one of minimal index of subfactor, there is a connection between *spin* and *statistics* of DHR sectors, namely

Proposition 3.4.10. [FRS92], [GL96] (**Spin-statistics theorem**). *Let $\{\mathcal{A}\}$ be as above and ρ a DHR endomorphism with finite dimension, then $\omega_\rho = e^{i2\pi h_\rho}$ holds, where ω_ρ is the phase of $[\rho]$ defined by means of the self-braiding operator $\mathcal{E}_{\rho,\rho}$ in the DHR category (see Definition 2.6.7) and $h_\rho \pmod{\mathbb{Z}}$ is the scaling dimension, or conformal spin, of $[\rho]$ (i.e., the lowest eigenvalue of the conformal Hamiltonian H_ρ in the sector determined by ρ).*

Chapter 4

Abstract points in CFT

A point of an interval $I_0 \subset \mathbb{R}$ is uniquely determined by the two connected components I_1 and I_2 in which it splits I_0, which are again intervals. Now assume to have a quantum field theory over I_0, or, very roughly speaking, a collection of "operators which depend on points of I_0". Then we can lift the two intervals I_1 and I_2 to the two respective families of field operators, or better to the local algebras $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ when the QFT is formulated as an AQFT. Our purpose is to invert the machine and, fixed an AQFT, "recover points" out of field operators, despite there need not be point-like localized fields (neither bounded or unbounded) as this clashes, e.g., with covariance [HM06, Ch. 6], nor fields at all inside the local algebras. The additional information is contained in the DHR representation category of the field theory (as considered in Chapter 3), which plays the role of a dual object with respect to local algebras, it has an action on them and a way of recovering "local information" through this action via the braiding operators (which are defined by space-like separation of charges, i.e., by choosing points of I_0, cf. Definition 3.3.3). In Section 4.2 we shall make these ideas precise in the notion of *braided action* of the DHR category on model independent algebras, and formulate it as an invariant for local nets of observables in the sense of Definition 4.2.4.

As reviewed in Chapter 1, the vacuum vector Ω of a chiral CFT (which is a dynamical object and very difficult to construct in non-free theories) contains enough information to completely reconstruct the net $\{\mathcal{A}\}$ out of two or three suitably chosen local algebras (see respectively Proposition 1.5.5 and 1.5.1). These deep facts are based on modular theory (Section 1.3) and exploit the enormous amount of information contained in the vacuum state. In partic-

ular these data uniquely determine the chiral CFT $\{\mathcal{A}\}$ up to isomorphism (Definition 1.6.1) as explained in Section 1.6. On the other hand the vacuum vector (and the vacuum sector) are unique (under natural assumptions) and uniquely determined by the local algebras thanks to Theorem 1.6.14 of [Wei11].

Now, *without* assuming the knowledge of the vacuum state, in this chapter we want to explore to which extent the DHR category allows to reconstruct the underlying QFT. The answer cannot be unique because two QFTs may easily share the same DHR category up to equivalence. E.g., by tensoring a QFT with another one which has no nontrivial sectors ("holomorphic CFT", in the context of chiral conformal QFT) does not change its DHR category. By invoking its braided action, however, the distinction is revealed, see Section 4.7, and we offer a sufficient criterion to exclude the presence of holomorphic factors. This criterion seems to be the right one to grasp the information about localization (left/right separation) of charges, hence *dually* of observables, out of the DHR braiding, in the sense of Proposition 4.8.5. It is also a good candidate to be a necessary condition, in view of Proposition 4.7.10.
We shall restrict ourselves to *chiral conformal* QFTs, because in this case *complete rationality* (Definition 3.4.1) implies *non-degeneracy* of the DHR braiding, i.e., the DHR category has the abstract structure of a *unitary modular tensor category* (Definition 2.6.4) in view of Proposition 3.4.3. This means that the braiding of DHR endomorphisms, beyond being a braiding on the collection of objects in an abstract tensor category, encodes a sharp distinction between left and right. Our basic idea is to start with either the *global* C^*-algebra $\mathcal{A}$ of quasilocal observables, or a single *local* von Neumann algebra $\mathcal{A}(I_0)$ where I_0 is an arbitrarily fixed bounded interval of the line $\mathbb{R}$ (or equivalently of the circle $\mathbb{S}^1$). The local picture is technically advantageous, but not essential, see Sections 4.3 and 4.4. Indeed neither $\mathcal{A}$, nor $\mathcal{A}(I_0)$, carry any specific information about the models, by well known results of [Haa87], [Tak70], and thus serve as a universal environment ("blanc canvas") to let the DHR category act on.
Either locally or globally, relative commutants have a geometric interpretation both on half-intervals (strong additivity) or half-lines (relative essential duality), see Proposition 4.1.4. Also the structure of the two-interval subfactor can be extended verbatim to a unital C^*-inclusion of algebras in the real line picture, see Corollary 4.3.9. Moreover the *action* of the DHR category on the observables behaves similarly locally or globally: compare modularity

with Proposition 4.3.5, and the duality relations between observables and endomorphisms localizable in half-lines (Proposition 4.3.3) or intervals (Proposition 4.3.7), either on $\mathbb{R}$ or confined in some fixed interval I_0. The latter proposition gives also an affirmative answer (in the chiral conformal setting) to a conjecture of S. Doplicher [Dop82] (in (3+1)-dimensional theories), see Remark 4.3.8.

Our main tool to reconstruct the *local substructure* of the net are *abstract points* of the braided action of the DHR category, see Section 4.5. The crucial observation is that the DHR category possesses, by its very definition based on the underlying local structure, a characteristic property: its braiding trivializes $\mathcal{E}_{\rho,\sigma} = \mathbb{1}$ whenever ρ, σ are localizable in mutually left/right separated regions of the real line. Since points are responsible for left/right splittings of the line, this motivates our definition of abstract points as suitable pairs of subalgebras that trivialize the braiding.

Using algebraic deformation techniques, abstract points can be carried wildly far-away from the naive geometric picture of two half-interval algebras, see Section 4.6. We therefore need to understand what is required to identify abstract point as geometric points, up to unitary equivalence. In Section 4.9 we show a way of deriving the *completeness* of the braided action as an invariant for local nets, but on a subclass of completely rational conformal nets which we call *prime conformal nets*, see Definition 4.7.5. Primality of a conformal net rules out holomorphic and tensor products cases, and relies on the notion of *prime UMTC* due to [Müg03]. In order to state the classification result we actually need two further assumptions, see Section 4.9, hence the content of Proposition 4.9.1 is still an abstract recipe, as we do not know which examples fit into the classification. Yet the recipe is quite surprising and natural, in the sense that it is essentially based on two facts about completely rational nets: the structure of the two-interval subfactor ([KLM01, Thm. 33]) and of the fixed points of the local DHR subcategories (Proposition 4.3.7).

In principle our techniques apply to general rational BTCs, in particular to UMTCs, thanks to realization results of [HY00] by means of endomorphisms. Hence solving the previous trivialization constraints $\mathcal{E}_{\rho,\sigma} = \mathbb{1}$ and then applying our machinery, can be viewed as a possible way to *realize* abstract UMTCs by means of suitable, e.g., prime (see Definition 4.7.5), conformal nets via the DHR construction. We do not discuss this "exoticity" problem for abstract

UMTCs in this work, and we refer to [Kaw15] for more explanations, and to [Bis16] for a systematic positive answer on the realization of Drinfeld doubles of subfactors with index less than 4.

This chapter is joint work with K.-H. Rehren and is submitted online (almost identical, but in the next introductory section) as [GR15].

4.1 Conformal nets and points on the line

The purpose of this section is to collect structure properties of QFT models that shall be used for the reconstruction of local algebras from an action of the DHR category in later sections. Although these results are well known (except Proposition 4.1.4), it is worthwhile to exhibit them in due context. We also exploit here some subtleties about the correspondence between the real line picture and the circe picture for local conformal nets (see Section 1.2), due to their conceptual importance in the present work, namely Lemma 4.1.2 and 4.1.3.

Let $\{\mathcal{A}\}$ be a local conformal net on the line (given on bounded intervals of $\mathbb{R}$ as in Definition 1.1.3) we can consider algebras for *arbitrary* regions $S \subset \mathbb{R}$ as follows. Define $\mathcal{A}(S)$ to be the von Neumann algebra, respectively C^*-algebra, generated by all local algebras $\mathcal{A}(I)$ such that $I \subset S$, depending on whether S is a *bounded*, respectively *unbounded*, region of $\mathbb{R}$. In the first case notice that $\mathcal{A}(S) \subset \mathcal{A}(J)$ for a sufficiently big $J \in \mathcal{I}$, in the second case let $\mathcal{R}(S) := \mathcal{A}(S)''$.

In this way we get the *quasilocal C^*-algebra* $\mathcal{A} := \mathcal{A}(\mathbb{R})$, the algebras of "space-like" complements of intervals $\mathcal{A}(I')$ where $I' := \mathbb{R} \smallsetminus \bar{I}$ and $I \in \mathcal{I}$, the half-line ("wedge") algebras $\mathcal{A}(W)$ where $W \subset \mathbb{R}$ is a non-empty open half-line, left or right oriented.

The two different notations are motivated by the fact that $\mathcal{A}(S) \subset \mathcal{A}$ in both cases by definition, while, if S is an unbounded region, $\mathcal{R}(S)$ is in general not contained in $\mathcal{A}$, not even in the conformal case, see [BGL93, Sec. 1].

Remark 4.1.1. The distinction between norm and weak closure is not just technical, it is essential to understand the structure of local nets and their DHR representation theory. Assume Haag duality on $\mathbb{R}$ and consider for instance $I \Subset J$, i.e., $\bar{I} \subset J$ where $I, J \in \mathcal{I}$. Then $I' \cap J = I_1 \cup I_2$ and

$\mathcal{A}(I_1 \cup I_2) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2) \subset \mathcal{A}(I)' \cap \mathcal{A}(J)$ is the two-interval subfactor considered by [KLM01], and $\vee$ is a short-hand notation for the von Neumann algebra generated. The previous inclusion is proper in many examples, in particular DHR charge transporters from I_1 to I_2 do not belong to $\mathcal{A}(I_1 \cup I_2)$. On the other hand, take $I' = W_1 \cup W_2$, $I \in \mathcal{I}$ and observe that

$$\mathcal{A}(W_1 \cup W_2) = C^*\{\mathcal{A}(W_1) \cup \mathcal{A}(W_2)\} \subset \mathcal{R}(W_1 \cup W_2) = \mathcal{A}(W_1) \vee \mathcal{A}(W_2)$$

is by Haag duality on $\mathbb{R}$ the inclusion $\mathcal{A}(I') \subset \mathcal{A}(I)'$, again proper in general. In this case DHR charge transporters from W_1 to W_2 are again not in $\mathcal{A}(W_1 \cup W_2)$ but they belong to the weak closure $\mathcal{R}(W_1 \cup W_2)$. Geometrically speaking, half-lines W_1 and W_2 "weakly touch at infinity" and allow charge transportation.

Chiral Rational CFTs (*chiral RCFTs*) correspond, in the algebraic setting, to a class of local conformal nets singled out by the conditions (a), (b) and (c) of Definition 3.4.1 imposed on the local algebras. Throughout this chapter we will restrict to the completely rational case whenever representation theoretical issues are concerned.

As we have seen in Lemma 1.2.1, local conformal nets on the line $\{\mathcal{A}\}$ can be uniquely extended to local conformal nets on the circle. Denote by $\{\tilde{\mathcal{A}}\}$ the extension and recall the additivity property of the net seen in Proposition 1.4.3, namely $\mathcal{A}(I) = \bigvee_{J \Subset I} \mathcal{A}(J)$ where $I \in \mathcal{I}$ and all $J \in \mathcal{I}$, after [FJ96, Eq. (24)]. Then the two definitions one might give of weakly closed half-line algebras (either on the line or on the circle) are the same, namely

Lemma 4.1.2. *In the assumptions above, it holds*

$$\tilde{\mathcal{A}}(W) = \mathcal{R}(W)$$

where $W \subset \mathbb{R}$ is a half-line.

Proof. Take $g \in \mathsf{M\ddot{o}b}$ such that $gI = W$ where $I \in \mathcal{I}$ is some fixed bounded interval. Then $\tilde{\mathcal{A}}(W) = U(g)\mathcal{A}(I)U(g)^* = U(g)(\vee_{J \Subset I}\mathcal{A}(J))U(g)^* = \vee_{K \Subset W, K \in \mathcal{I}} \mathcal{A}(K) = \mathcal{A}(W)'' = \mathcal{R}(W)$. Observe that $J \Subset I$, i.e., $\overline{J} \subset I$ in [FJ96, Eq. (24)] says that boundary points don't matter at the level of local algebras. $\qquad\square$

Haag duality on $\mathbb{R}$ (assumption (a)) allows to give a purely algebraic interpretation of the extension/restriction procedure of conformal nets from $\mathbb{R}$ to $\mathbb{S}^1$, cf. [KLM01, App. B], namely

Lemma 4.1.3. *Let $\{\mathcal{A}\}$ be a local conformal net on the line which fulfills assumption (a) in addition. Then the extended local conformal net on the circle $\{\tilde{\mathcal{A}}\}$ is strongly additive and is given by extending Haag duality to $\mathbb{S}^1$, i.e.*

$$\tilde{\mathcal{A}}(I) = \mathcal{A}(I')'$$

for all $I \in \tilde{\mathcal{I}}$, where $\tilde{\mathcal{I}}$ denotes the family of open non-empty non-dense intervals $I \subset \mathbb{S}^1$ as in Lemma 1.2.1.

Proof. Observe first that if $I \in \tilde{\mathcal{I}}$ also $I' \in \tilde{\mathcal{I}}$ where $I' = \mathbb{S}^1 \smallsetminus \overline{I}$, so $\tilde{\mathcal{A}}(I')$ is weakly closed by assumption in the circle picture and we have a priori more local algebras. Haag duality on $\mathbb{R}$ says this is not the case, or better that they are determined by local algebras on the line. Take $I \in \tilde{\mathcal{I}}$ then by definition $\tilde{\mathcal{A}}(I) = \mathcal{A}(I) = \mathcal{A}(I')'$ if $I \subset \mathbb{R}$ and $I \in \mathcal{I}$. If $I \subset \mathbb{R}$ is a open half-line $I = W$ we have already seen that $\tilde{\mathcal{A}}(W) = \mathcal{A}(W)'' = \mathcal{A}(W')'$, the last equality being essential duality. It remains the case where $I \not\subset \mathbb{R}$, i.e., I contains the infinity point $\infty \in \mathbb{S}^1$. Then $\tilde{\mathcal{A}}(I) = \tilde{\mathcal{A}}(I_1 \cup I_2) \equiv \tilde{\mathcal{A}}(I_1) \vee \tilde{\mathcal{A}}(I_2)$ by conformal covariance and strong additivity, where $\{\infty\}' \cap I = I_1 \cup I_2$ and $I_1, I_2 \subset \mathbb{R}$ are open half-lines. From the previous step $\tilde{\mathcal{A}}(I) = \mathcal{A}(I_1)'' \vee \mathcal{A}(I_2)'' = \mathcal{A}(I_1 \cup I_2)'' = \mathcal{A}(J')''$ where $J = I' \in \mathcal{I}$ and $J' = \mathbb{R} \smallsetminus \overline{J}$, then by Haag duality on $\mathbb{R}$ we conclude $\tilde{\mathcal{A}}(I) = \mathcal{A}(J)' = \mathcal{A}(I')'$.

By conformal covariance, see Lemma 1.5.4, Haag duality on $\mathbb{R}$ is equivalent to strong additivity, in particular it is a local condition and trivially extends to the circle. $\qquad\square$

With the split property (assumption (b)) both the local algebras $\mathcal{A}(I)$ for all $I \in \mathcal{I}$ and the quasilocal algebra $\mathcal{A}$ are *canonical* objects, in the sense that they are universal (independent of the specific model) up to spatial isomorphism. The first as the unique injective ("hyperfinite") type III_1 factor by [Haa87], the second by a general result of [Tak70]. In particular, they contain no specific information about the models. Moreover *locality* of the net is not needed neither in [Tak70] nor to apply the result of [Haa87]. In the first only isotony enters, for the second we know that Bisognano-Wichmann's modular covariance holds regardless of locality [DLR01].

The entire information about the chiral CFT is then encoded in the inclusions and relative commutation relations among different local algebras, i.e., in the *local algebraic structure* of the net. This statement is made precise by the algebraic version of Haag's theorem due to M. Weiner [Wei11] and reviewed

in Theorem 1.6.14.

Moreover inclusions of local algebras $\mathcal{A}(I) \subset \mathcal{A}(J)$ for $I, J \in \mathcal{I}$, $I \subset J$ are known to be *normal* and *conormal*, i.e., respectively

$$\mathcal{A}(I)^{cc} = \mathcal{A}(I), \quad \mathcal{A}(I) \vee \mathcal{A}(I)^c = \mathcal{A}(J) \tag{4.1}$$

where $\mathcal{N}^c := \mathcal{N}' \cap \mathcal{M}$ denotes the *relative commutant* of the inclusion $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras. Notice that the normality and conormality relations above do not depend on the specific geometric position of I inside J, nor on Haag duality (assumption (a)).

With Haag duality on $\mathbb{R}$ (assumption (a)), there is a geometric interpretation of the relative commutant and of the normality and conormality relations (4.1) for inclusions of local algebras which arise for the choice of *points*. Namely let $I \in \mathcal{I}$, take $p \in I$ and let $\{p\}' \cap I = I \smallsetminus \{p\} = I_1 \cup I_2$, $I_1, I_2 \in \mathcal{I}$. The relative commutant of $\mathcal{A}(I_1) \subset \mathcal{A}(I)$ is then given by

$$\mathcal{A}(I_1)^c := \mathcal{A}(I_1)' \cap \mathcal{A}(I) = \mathcal{A}(I_2). \tag{4.2}$$

It follows from conformal covariance, cf. [GLW98], that the relations (4.2) are actually equivalent to assumption (a).

Now a **point of an interval**, $p \in I$, is uniquely determined by two intervals $I_1, I_2 \in \mathcal{I}$ as above, the *relative complements* of p in I. Algebraically, $p \in I$ splits $\mathcal{A}(I)$ into a pair of commuting subalgebras $\mathcal{A}(I_1), \mathcal{A}(I_2) \subset \mathcal{A}(I)$ which in the Haag dual case are each other's relative commutants.
Similarly a **point of the line**, $p \in \mathbb{R}$, is uniquely determined by two half-lines $W_1, W_2 \subset \mathbb{R}$, the relative complements of p in $\mathbb{R}$, and determines two "global" unital C^*-inclusions $\mathcal{A}(W_1), \mathcal{A}(W_2) \subset \mathcal{A} := \mathcal{A}(\mathbb{R})$. Our first main structure result, see Proposition 4.1.4, shows that the same geometric interpretation of relative commutants holds in the global case. The proof is independent of assumption (a), but as a technical tool we need to assume (b). Merging the standard terminology of "relative commutant" and "essential duality" for local algebras we can call this property *relative essential duality*.

Proposition 4.1.4. *Let $\{\mathcal{A}\}$ be a local conformal net on the line as in Definition 1.1.3, which fulfills the split property (assumption (b)). Consider*

the inclusion of unital C^*-algebras $\mathcal{A}(W) \subset \mathcal{A}$, where $W \subset \mathbb{R}$ is a half-line, left or right oriented, then

$$\mathcal{A}(W)^c := \mathcal{A}(W)' \cap \mathcal{A} = \mathcal{A}(W')$$

where $W' = \mathbb{R} \setminus \overline{W}$ is the opposite half-line.

Proof. Observe first that $\mathcal{A}(W)' = \mathcal{R}(W')$, hence the statement is equivalent to $\mathcal{A}(W) = \mathcal{R}(W) \cap \mathcal{A}$. This does not boil down to essential duality $\mathcal{R}(W)' = \mathcal{R}(W')$, because typically $\mathcal{A}(W) \subset \mathcal{R}(W)$ is proper and $\mathcal{R}(W) \not\subset \mathcal{A}$, see [BGL93, Sec. 1].

By the split property we have that $\mathcal{R}(W)$ is the injective factor of type III_1 and the same holds for its commutant. Consider then a norm continuous conditional expectation

$$E : \mathcal{B}(\mathcal{H}) \to \mathcal{R}(W)'$$

given by averaging over the adjoint action of the unitary group $G := \mathcal{U}(\mathcal{R}(W))$ of $\mathcal{R}(W)$, equipped with the ultraweak topology or equivalently with any of the other weak operator topologies.

Now, injectivity is equivalent to amenability of the unitary group, i.e., to the existence of a left invariant state ("mean") on the unital C^*-subalgebra $\mathcal{C}_{ru}(G)$ of right uniformly continuous functions in $L^\infty(G)$, see [dlH79], [Pat92]. Similar to [Arv74] one can define an integral $E(b) := \int_G \mathrm{Ad}_u(b)\, \mathrm{d}u$ with respect to such a mean m, for every $b \in \mathcal{B}(\mathcal{H})$, as the unique element in $\mathcal{B}(\mathcal{H})$ such that

$$\langle \varphi, \int_G \mathrm{Ad}_u(b)\, \mathrm{d}u \rangle = \int_G \langle \varphi, \mathrm{Ad}_u(b) \rangle\, \mathrm{d}u \quad \forall \varphi \in \mathcal{B}(\mathcal{H})_*$$

where $\mathcal{B}(\mathcal{H})_*$ is the predual, and the r.h.s. is defined by the mean on functions

$$\int_G \langle \varphi, \mathrm{Ad}_u(b) \rangle\, \mathrm{d}u = m(f_{\varphi,b}), \quad f_{\varphi,b}(u) := \langle \varphi, \mathrm{Ad}_u(b) \rangle.$$

One can easily see by formal computations that $E(b)u = uE(b)$ for all $u \in G$ hence $E(b) \in \mathcal{R}(W)'$, see also [dlH79, Lem. 1, 2]. Moreover, E is a norm one projection *onto* $\mathcal{R}(W)'$, i.e., $\|E(b)\| \leq \|b\|$ and $E(b) = b$ if $b \in \mathcal{R}(W)'$, hence a conditional expectation by [Tom57]. Observe that E cannot be normal because $\mathcal{R}(W)$ is type III, see [Tak03, Ex. IX.4].

The next step is to show that E preserves the local structure of the net, i.e., maps local algebras into local algebras and $\mathcal{A}$ into itself. So take a

bounded interval I containing the origin of W, we want to show that

$$E : \mathcal{A}(I) \to \mathcal{A}(I) \cap \mathcal{R}(W)'.$$

First, assume in addition that Haag duality on $\mathbb{R}$ holds. Take $a \in \mathcal{A}(I)$ and $\mathcal{A}(I) = \mathcal{A}(I')' = (\mathcal{R}(W_1) \vee \mathcal{R}(W_2))'$ where $I' = W_1 \cup W_2$ and W_1, W_2 are half-lines. If for instance $W_2 \subset W$, then every $x \in \mathcal{R}(W_2)$ commutes with $E(a) \in \mathcal{R}(W)'$. Take now any $y \in \mathcal{R}(W_1) \subset \mathcal{R}(W')$, then

$$E(a)y = \int_G \mathrm{Ad}_u(a)y \, \mathrm{d}u = \int_G y \, \mathrm{Ad}_u(a) \, \mathrm{d}u = yE(a)$$

because $uy = yu$, $u \in \mathcal{R}(W)$ and $ay = ya$, $a \in \mathcal{A}(I)$ by locality. Hence $E(a)$ commutes with $\mathcal{R}(W_2)$ and with $\mathcal{R}(W_1)$, and we can conclude that $E(a) \in \mathcal{A}(I)$.

In general, a more refined and purely algebraic argument [dlH79, Lem. 2 (iii)] shows directly that $E(a) \in \mathcal{A}(I) \vee \mathcal{R}(W)$ which coincides with $\mathcal{R}(W_1')$ by additivity, hence $E(a) \in \mathcal{R}(W_1' \cap W')$ where $W_1' \cap W' = I \cap W' \in \mathcal{I}$ and

$$E : \mathcal{A}(I) \to \mathcal{A}(I \cap W') = \mathcal{A}(I) \cap \mathcal{R}(W)'.$$

Exhausting $\mathbb{R}$ with a sequence of intervals I_n containing the origin of W, by norm continuity of E we get $E : \mathcal{A} \to \mathcal{A}$ and

$$C^* \{\bigcup_n \mathcal{A}(I_n \cap W')\} = E(\mathcal{A}) = \mathcal{A}(W)^c.$$

But also $C^* \{\bigcup_n \mathcal{A}(I_n \cap W')\} = \mathcal{A}(W')$, hence $\mathcal{A}(W)^c = \mathcal{A}(W')$ follows. $\qquad \square$

Remark 4.1.5. The techniques employed here are similar to those used in [Dop82, Sec. 5]. There, however, local algebras $\mathcal{A}(I)$ are considered instead of half-line algebras and one does not need additivity nor essential duality to show that conditional expectations on $\mathcal{A}(I)'$ preserve the local substructure of $\mathcal{A}$.

As a consequence of Proposition 4.1.4, assuming the split property we can take the relative commutant of the inclusion $\mathcal{A}(W') \subset \mathcal{A}(W)^c \subset \mathcal{A}(W)'$ and obtain

$$\mathcal{A}(W) = \mathcal{A}(W)^{cc} = \mathcal{R}(W) \cap \mathcal{A} \tag{4.3}$$

where the relative commutants refer to the inclusions $\mathcal{A}(W) \subset \mathcal{A}$.

This is similar to the case of local algebras $\mathcal{A}(I) \subset \mathcal{A}$, $I \in \mathcal{I}$ if we assume Haag duality on $\mathbb{R}$, indeed

$$\mathcal{A}(I) = \mathcal{A}(I)^{cc} \tag{4.4}$$

follows by taking relative commutants of the inclusion $\mathcal{A}(I') \subset \mathcal{A}(I)^c \subset \mathcal{A}(I)'$, cf. [DHR69a, Sec. V]. The relations (4.3) and (4.4) are a global version of the normality relations (4.1) encountered before.

Heuristically speaking, we regard *normality* as an algebraic fingerprint of *connectedness* in the following sense. Algebras associated to intervals $\mathcal{A}(I)$ or half-lines $\mathcal{A}(W)$ are "connected", relative commutants $\mathcal{A}(I)^c$ are also "connected" in a broader sense, e.g., on the circle, because $\mathcal{A}(I)^c = \mathcal{A}(I)^{ccc}$ always holds. On the other hand, algebras $\mathcal{A}(S) \subset \mathcal{A}$ associated to disconnected regions, e.g., $S = I'$, $I \in \mathcal{I}$, need not be normal. Indeed, assuming (a), the inclusion

$$\mathcal{A}(I') \subset \mathcal{A}(I')^{cc} = \mathcal{A}(I)^c \tag{4.5}$$

is proper in many examples, see Corollary 4.3.9. In the case of *holomorphic* nets there is no algebraic distinction (in the sense of normality relations) between "connected" and "disconnected" regions at the level of nets, cf. [RT13] for an explicit isomorphism between interval and two-interval algebras in the case of graded-local Fermi nets. Notice that the unital C^*-inclusion (4.5) is a "global" version of the *two-interval subfactor* $\mathcal{A}(I_1 \cup I_2) \subset \mathcal{A}(I_1 \cup I_2)^{cc} = \mathcal{A}(I)^c$ considered by [KLM01], where relative commutants are taken in $\mathcal{A}(J)$ for $I \Subset J$, $I' \cap J = I_1 \cup I_2$. Indeed $((\mathcal{A}(I_1) \vee \mathcal{A}(I_2))' \cap \mathcal{A}(J))' \cap \mathcal{A}(J) = (\mathcal{A}(I_1)' \cap \mathcal{A}(I_1 \cup I))' \cap \mathcal{A}(J) = \mathcal{A}(I)' \cap \mathcal{A}(J)$.

In the following we shall concentrate on local conformal nets on the line $\{\mathcal{A}\}$, see Definition 1.1.3, which are in addition *completely rational*, as in Definition 3.4.1. In this case we recalled in Proposition 3.4.3 that the category of finitely reducible (finite-dimensional) DHR *representations* of the net, denoted again by DHR$\{\mathcal{A}\}$, has the abstract structure of a *unitary modular tensor category* (UMTC, see Definition 2.6.4).

Modularity is the essentially new feature of DHR categories arising in low-dimensional models. Moreover, the key ingredient in the proof of modularity is the discovery of a deep connection between the algebraic structure of the net and the structure of its representation category. More precisely, the two-interval subfactor [KLM01, Thm. 33] is a Longo-Rehren subfactor [LR95,

Prop. 4.10] and is uniquely determined up to isomorphism by the tensor structure of the category (forgetting the braiding), see [KLM01, Cor. 35]. Hence the DHR *braiding* can be seen as an additional ingredient whose definition requires, in the low-dimensional case, the choice of a point (irrespectively of its position) in order to separate the localization of DHR endomorphisms.

We close the section by mentioning that complete rationality is realized by several models: Wess-Zumino-Witten $SU(N)$-currents [Was98], Virasoro nets with central charge $c < 1$ [Car04], [KL04], lattice models [DX06], [Bis12b], the Moonshine vertex operator algebra [KL06]. Further candidates come from more general loop groups [GF93] and vertex operator algebras [CKLW15]. Moreover, complete rationality passes to tensor products [KLM01], group-fixed points [Xu00a], finite index extensions and finite index subnets [Lon03].

4.2   Braided actions of DHR categories

The motivation of our work is the following: in the variety of completely rational models, one can easily find non-isomorphic ones, see Definition 1.6.1, having equivalent DHR categories in the sense of abstract UBTCs, see [EGNO15, Def. 8.1.7, Rmk. 9.4.7]. Examples of this can be constructed by looking at completely rational **holomorphic nets**, i.e., nets with only one irreducible DHR sector: the vacuum. In this case the DHR category coincides with Vec, the category of finite-dimensional complex vector spaces, up to unitary braided tensor equivalence. Take now a completely rational conformal net $\{\mathcal{A}\}$ and tensor it with a nontrivial holomorphic net $\{\mathcal{A}_{\mathrm{holo}}\}$, then [1]

$$\mathrm{DHR}\{\mathcal{A} \otimes \mathcal{A}_{\mathrm{holo}}\} \simeq \mathrm{DHR}\{\mathcal{A}\} \boxtimes \mathrm{DHR}\{\mathcal{A}_{\mathrm{holo}}\} \simeq \mathrm{DHR}\{\mathcal{A}\}$$

but $\{\mathcal{A}\} \not\cong \{\mathcal{A} \otimes \mathcal{A}_{\mathrm{holo}}\}$, because tensoring with nontrivial holomorphic nets increases the central charge by a multiple of 8. Hence the UBTC equivalence class of the DHR category is *not* a complete invariant for nets, i.e., the correspondence between completely rational conformal nets (up to isomorphism) and their DHR categories (up to UBTC equivalence)

$$\{\mathcal{A}\} \mapsto \mathrm{DHR}\{\mathcal{A}\} \tag{4.6}$$

---

[1]Here $\simeq$ denotes UBTC equivalence and $\boxtimes$ is the Deligne product (the "tensor product" in the category of semisimple linear categories).

is not one-to-one. We might replace equivalence of categories with the much stronger notion of isomorphism of categories, see [ML98], but this is not what we want to do. Instead we consider the **action** of the DHR category on the net as additional structure, i.e., consider its realization as a *braided tensor category of endomorphisms of the net*. For technical reasons, we look at the action on a local algebra rather than the "global" defining action $\mathrm{DHR}\{\mathcal{A}\} \subset \mathrm{End}(\mathcal{A})$ on the quasilocal algebra. Namely, fix an arbitrary interval $I_0 \in \mathcal{I}$ and consider the "local" full subcategory $\mathrm{DHR}^{I_0}\{\mathcal{A}\} \subset \mathrm{DHR}\{\mathcal{A}\}$ whose objects are the DHR endomorphisms ρ localizable in I_0, i.e., $\rho_{\restriction \mathcal{A}(I_0')} = \mathrm{id}_{\restriction \mathcal{A}(I_0')}$.

Notice that the inclusion functor in this case is also an equivalence, i.e., essentially surjective in addition

$$\mathrm{DHR}^{I_0}\{\mathcal{A}\} \simeq \mathrm{DHR}\{\mathcal{A}\} \qquad (4.7)$$

because I_0 is open and there is by definition (and by Möbius covariance) no minimal localization length. Considering the action on local algebras means considering the *restriction functor* $\rho \mapsto \rho_{\restriction \mathcal{A}(I_0)}$

$$\mathrm{DHR}^{I_0}\{\mathcal{A}\} \hookrightarrow \mathrm{End}(\mathcal{A}(I_0)) \qquad (4.8)$$

which is well-defined, *strict tensor* and *faithful* by Haag duality on $\mathbb{R}$. Recall that the arrows of the endomorphism category on the right hand side are defined as

$$\mathrm{Hom}_{\mathrm{End}(\mathcal{A}(I_0))}(\hat{\rho}, \hat{\sigma}) := \big\{ t \in \mathcal{A}(I_0) : t\hat{\rho}(a) = \hat{\sigma}(a)t \,, a \in \mathcal{A}(I_0) \big\}$$

where $\hat{\rho}, \hat{\sigma} \in \mathrm{End}(\mathcal{A}(I_0))$. With conformal symmetry [GL96] have shown that the restriction functor is also *full* (i.e., local intertwiners are global, see also [Rob74] for a proof in higher spacetime dimension), hence an *embedding* of categories. The restriction functor is by no means essentially surjective, i.e., not every (finite index) endomorphism of the injective type III_1 factor $\mathcal{A}(I_0)$ is realized by DHR endomorphisms of $\{\mathcal{A}\}$. But it has *replete image*, i.e., it is closed under unitary isomorphism classes in $\mathrm{End}(\mathcal{A}(I_0))$.

The first interesting point concerning the embedding (4.8) is the following

Remark 4.2.1. Forgetting the braiding, the remaining *abstract* structure of $\mathrm{DHR}^{I_0}\{\mathcal{A}\}$ is the one of a *unitary fusion tensor category* (UFTC). Functors between unitary categories (or *-categories) will always be assumed to preserve

the *-structure. A result of Popa [Pop95] states that an embedding $\mathcal{C} \hookrightarrow \mathrm{End}(\mathcal{M})$ as above, where $\mathcal{C}$ is a UFTC and $\mathcal{M}$ is the unique injective type III_1 factor, is canonical in the following sense. Take two equivalent UFTCs realized as endomorphisms of injective type III_1 factors $\mathcal{C} \subset \mathrm{End}(\mathcal{M})$ and $\mathcal{D} \subset \mathrm{End}(\mathcal{N})$ where we can assume $\mathcal{M}, \mathcal{N} \subset \mathcal{B}(\mathcal{H})$. By [Pop95, Cor. 6.11], see also [KLM01, Cor. 35], there exists a spatial isomorphism $\mathrm{Ad}_U : \mathcal{M} \to \mathcal{N}$ where U is unitary in $\mathcal{B}(\mathcal{H})$ which implements an equivalence $\mathcal{C} \simeq \mathcal{D}$ as follows

$$\hat{\rho}_i \mapsto \mathrm{Ad}_U \circ \hat{\rho}_i \circ \mathrm{Ad}_{U^*} \simeq \hat{\sigma}_i \tag{4.9}$$

for all $i = 0, \ldots, n$ where $\{\hat{\rho}_0, \ldots, \hat{\rho}_n\}$ and $\{\hat{\sigma}_0, \ldots, \hat{\sigma}_n\}$ are generating sets for $\mathcal{C}$ and $\mathcal{D}$ respectively and $\simeq$ stands for unitary isomorphism in $\mathrm{End}(\mathcal{N})$. If both embeddings are *replete* as in (4.8), we can extend the equivalence (4.9) to an isomorphism of categories $\mathcal{C} \cong \mathcal{D}$ and every $\hat{\sigma} \in \mathcal{D}$ can be written as

$$\hat{\sigma} = \mathrm{Ad}_U \circ \hat{\rho} \circ \mathrm{Ad}_{U^*} =: {}^U\hat{\rho}$$

for a unique $\hat{\rho} \in \mathcal{C}$, moreover $t \mapsto \mathrm{Ad}_U(t) =: {}^U t$ gives a *-linear bijection of the Hom-spaces $\mathrm{Ad}_U : \mathrm{Hom}(\hat{\rho}_i, \hat{\rho}_j) \to \mathrm{Hom}({}^U\hat{\rho}_i, {}^U\hat{\rho}_j)$. This isomorphism is manifestly strict tensor.

Take two nets $\{\mathcal{A}\}$, $\{\mathcal{B}\}$ and consider as in (4.8) the replete embeddings of the respective DHR categories

$$\mathrm{DHR}^{I_0}\{\mathcal{A}\} \hookrightarrow \mathrm{End}(\mathcal{A}(I_0)), \quad \mathrm{DHR}^{I_0}\{\mathcal{B}\} \hookrightarrow \mathrm{End}(\mathcal{B}(I_0))$$

for some fixed interval $I_0 \in \mathcal{I}$. As we said, it may happen that $\mathrm{DHR}\{\mathcal{A}\} \simeq \mathrm{DHR}\{\mathcal{B}\}$ as UBTCs, hence as UFTCs forgetting the braiding. By Remark 4.2.1, there is a spatial isomorphism $\mathrm{Ad}_U : \mathcal{A}(I_0) \to \mathcal{B}(I_0)$ which implements a strict tensor isomorphism between the images of the two restrictions, hence between the respective local DHR subcategories.

However, the latter isomorphism $F_U : \mathrm{DHR}^{I_0}\{\mathcal{A}\} \to \mathrm{DHR}^{I_0}\{\mathcal{B}\}$ need *not* preserve the braidings

$$\varepsilon^{\mathcal{A}}_{\rho_1,\rho_2} = v_2^* \times u_1^* \cdot u_1 \times v_2 = \rho_2(u_1^*)v_2^* u_1 \rho_1(v_2) \in \mathrm{Hom}_{\mathrm{DHR}\{\mathcal{A}\}}(\rho_1\rho_2, \rho_2\rho_1)$$

where $\rho_1, \rho_2 \in \mathrm{DHR}^{I_0}\{\mathcal{A}\}$ and u_1, v_2 are unitaries in $\mathcal{A}(I_0)$ such that $\mathrm{Ad}_{u_1} \rho_1$ is localizable left to $\mathrm{Ad}_{v_2} \rho_2$ inside I_0. Indeed

$$F_U(\varepsilon^{\mathcal{A}}_{\rho_1,\rho_2}) = \mathrm{Ad}_U(\rho_2(u_1^*)v_2^* u_1 \rho_1(v_2)) = F_U(v_2^*) \times F_U(u_1^*) \cdot F_U(u_1) \times F_U(v_2)$$

is in the correct intertwiner space

$$F_U(\mathcal{E}^{\mathcal{A}}_{\rho_1,\rho_2}) \in \mathrm{Hom}_{\mathrm{DHR}\{\mathcal{B}\}}(F_U(\rho_1)F_U(\rho_2), F_U(\rho_2)F_U(\rho_1))$$

but can be $F_U(\mathcal{E}^{\mathcal{A}}_{\rho_1,\rho_2}) \neq \mathcal{E}^{\mathcal{B}}_{F_U(\rho_1),F_U(\rho_2)}$ because, for instance, $F_U(u_1), F_U(v_2)$ need not be charge transporters which take the respective endomorphisms one left to the other inside I_0.

Take now two *isomorphic* nets $\{\mathcal{A}\}$, $\{\mathcal{B}\}$ (see Definition 1.6.1). Then there is a unitary W which implements spatial isomorphisms $\mathrm{Ad}_W : \mathcal{A}(I) \to \mathcal{B}(I)$ for *every* $I \in \mathcal{I}$, hence for I_0 and all of its subintervals. The resulting strict tensor isomorphism $F_W : \mathrm{DHR}^{I_0}\{\mathcal{A}\} \to \mathrm{DHR}^{I_0}\{\mathcal{B}\}$ defined on objects as $\rho \mapsto \mathrm{Ad}_W \circ \rho \circ \mathrm{Ad}_{W^*}$ is *braided* in addition. Indeed F_W respects the localization regions of the DHR endomorphisms, by definition, hence $F_W(\mathcal{E}^{\mathcal{A}}_{\rho_1,\rho_2}) = \mathcal{E}^{\mathcal{B}}_{F_W(\rho_1),F_W(\rho_2)}$. More generally

Definition 4.2.2. Let $\mathcal{C}$ be an abstract strict UMTC and $\mathcal{M}$ a von Neumann factor. A strict tensor replete embedding

$$G : \mathcal{C} \hookrightarrow \mathrm{End}(\mathcal{M})$$

will be called a **braided action** of $\mathcal{C}$ on $\mathcal{M}$.

Remark 4.2.3. The previous notion is purely *tensor* categorical, indeed the category $\mathrm{End}(\mathcal{M})$ is an enormous object which does not have a "global" braiding. However any braided action can be promoted to an actual *braided* functor by endowing the (replete tensor) image $G(\mathcal{C}) \subset \mathrm{End}(\mathcal{M})$ with the braiding $\hat{\mathcal{E}}_{G(\rho),G(\sigma)} := G(\mathcal{E}_{\rho,\sigma})$. Our terminology is motivated by the importance of the realization of $\mathcal{C}$ as a *braided* tensor category of endomorphism of $\mathcal{M}$, see Definition 4.2.4 below for the precise formulation of this statement. The endomorphisms in the range of the embedding have automatically finite index. Moreover if $\mathcal{M}$ is type III, they are automatically normal and injective (unital).

In our case at hand, $\mathcal{C} := \mathrm{DHR}^{I_0}\{\mathcal{A}\}$ for some fixed $I_0 \in \mathcal{I}$ and the **braided action of the DHR category**, remember the equivalence (4.7), on $\mathcal{M}_0 := \mathcal{A}(I_0)$ is given by the *restriction functor* (4.8).

Definition 4.2.4. Let $\mathcal{C}$, $\mathcal{D}$ be two abstract strict UMTCs and $\mathcal{M}$, $\mathcal{N}$ two von Neumann factors. Two braided actions $G_1 : \mathcal{C} \hookrightarrow \mathrm{End}(\mathcal{M})$ and

$G_2 : \mathcal{D} \hookrightarrow \mathrm{End}(\mathcal{N})$ will be called **isomorphic** if there is a spatial isomorphism $\mathrm{Ad}_U : \mathcal{M} \to \mathcal{N}$ implementing a strict tensor isomorphism between the respective images which is also braided. Equivalently, the unique strict tensor isomorphism $F_U : \mathcal{C} \to \mathcal{D}$ which makes the following diagram commute

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\;G_1\;} & \mathrm{End}(\mathcal{M}) \\
{\scriptstyle F_U}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Ad}_U} \\
\mathcal{D} & \xrightarrow{\;G_2\;} & \mathrm{End}(\mathcal{N})
\end{array}
$$

is in addition a UBTC isomorphism.

Take two nets $\{\mathcal{A}\}$, $\{\mathcal{B}\}$, their respective DHR categories together with their braided actions respectively on $\mathcal{A}(I_0)$, $\mathcal{B}(I_0)$ for some fixed I_0. Clearly from the previous discussion, if $\{\mathcal{A}\}$ and $\{\mathcal{B}\}$ are isomorphic nets (see Definition 1.6.1) then $\mathrm{DHR}^{I_0}\{\mathcal{A}\}$ and $\mathrm{DHR}^{I_0}\{\mathcal{B}\}$ have isomorphic braided actions (see Definition 4.2.4) hence we have an *invariant*.

Remarkably, the situation described in Definition 4.2.2 is general for UMTCs, in the sense that every abstract UMTC $\mathcal{C}$ admits a braided action on the injective type III_1 factor $\mathcal{M}$.

Remark 4.2.5. As in Remark 4.2.1, we drop the braiding on $\mathcal{C}$ and consider its UFTC structure first. Without loss of generality, i.e., up to a (non-strict) tensor equivalence [ML98, Thm. 1, §XI.3], we can assume that $\mathcal{C}$ is strict. Relying on a deep result of [HY00], we know that the presence of conjugates (rigidity) and the C^*-structure guarantee the existence of a (non-strict) tensor embedding $G : \mathcal{C} \hookrightarrow \mathrm{End}(\mathcal{M})$, where $\mathcal{M}$ is the unique injective type III_1 factor. Now the image of $\mathcal{C}$ in $\mathrm{End}(\mathcal{M})$ can be endowed with the braiding which promotes G to a braided embedding, taking care of the nontrivial multiplicativity constraints of the functors, and can be completed to a UMTC $\hat{\mathcal{C}}$ realized and replete in $\mathrm{End}(\mathcal{M})$, which is equivalent to $\mathcal{C}$ as an abstract UMTC. The inclusion functor gives then a braided action of $\hat{\mathcal{C}}$ on $\mathcal{M}$ in the strong sense employed in Definition 4.2.2. We thank R. Longo for pointing out the existence of such results, and for motivating discussions.
Moreover, similarly to Remark 4.2.1 but in this more general context, the (non-strict) tensor embedding $G : \mathcal{C} \hookrightarrow \mathrm{End}(\mathcal{M})$ of a UFTC $\mathcal{C}$ is also expected to be unique (in a suitable sense, cf. [HP15, Conj. 3.6]).

4.3 Duality relations

Motivated by [Dop82] we consider the duality pairing

$$\mathcal{A} \xleftrightarrow{\perp} \mathrm{DHR}\{\mathcal{A}\} \tag{4.10}$$

between the DHR category and the algebra $\mathcal{A}$ of quasilocal observables of a given (Haag dual) local conformal net $\{\mathcal{A}\}$, defined by the action $(a, \rho) \mapsto \rho(a)$.

Definition 4.3.1. Given a unital C^*-subalgebra $\mathcal{N} \subset \mathcal{A}$ we define its **dual** as

$$\mathcal{N}^\perp := \big\{\rho \in \mathrm{DHR}\{\mathcal{A}\} : \rho(n) = n,\, n \in \mathcal{N}\big\}$$

and $\mathrm{Hom}_{\mathcal{N}^\perp}(\rho, \sigma) := \mathrm{Hom}_{\mathrm{DHR}\{\mathcal{A}\}}(\rho, \sigma)$ for every $\rho, \sigma \in \mathcal{N}^\perp$. In other words, $\mathcal{N}^\perp \subset \mathrm{DHR}\{\mathcal{A}\}$ is a full subcategory, i.e., specified by its objects only.

$\mathcal{N}^\perp$ is automatically a unital tensor category of endomorphisms of $\mathcal{A}$. Conversely

Definition 4.3.2. Given a unital tensor full subcategory $\mathcal{C} \subset \mathrm{DHR}\{\mathcal{A}\}$ we define its **dual** as

$$\mathcal{C}^\perp := \big\{a \in \mathcal{A} : \sigma(a) = a,\, \sigma \in \mathcal{C}\big\}.$$

$\mathcal{C}^\perp$ is automatically a unital C^*-subalgebra of $\mathcal{A}$. We have the following

Proposition 4.3.3. *Let $\{\mathcal{A}\}$ be a local conformal net on the line fulfilling in addition Haag duality on $\mathbb{R}$ (assumption (a)). Take $\mathcal{A}(W) \subset \mathcal{A}$ where $W \subset \mathbb{R}$ is a half-line, left or right oriented, then*

$$\mathcal{A}(W)^\perp = \mathrm{DHR}^{W'}\{\mathcal{A}\}$$

where $\mathrm{DHR}^{W'}\{\mathcal{A}\}$ is the full subcategory of $\mathrm{DHR}\{\mathcal{A}\}$ whose objects are the endomorphisms localizable in the half-line W', opposite to W.

Proof. $(\supset)$: assume ρ localizable in $I_\rho \subset W'$, $I_\rho \in \mathcal{I}$, then by definition ρ acts trivially on $\mathcal{A}(I'_\rho) \supset \mathcal{A}(W)$.

$(\subset)$: take $\rho \in \mathrm{DHR}\{\mathcal{A}\}$ such that $\rho_{\restriction \mathcal{A}(W)} = \mathrm{id}$, we may assume ρ localizable in $I_\rho \in \mathcal{I}$ sufficiently big, such that it contains the origin of W. Let now $I'_\rho = W_1 \cup W_2$, where W_1, W_2 are half-lines and $W_1 \subset W$. Then ρ acts trivially on $\mathcal{A}(W)$ and on $\mathcal{A}(W_2)$ hence on the C^*-algebra they generate, being a (norm continuous) *-homomorphism, which equals $\mathcal{A}(J')$ for $J \in \mathcal{I}$ such that $J' = W \cup W_2$. Hence $J \subset W'$ and $\rho \in \mathrm{DHR}^{W'}\{\mathcal{A}\}$. $\qquad\square$

Combining Proposition 4.1.4 and 4.3.3 we obtain

Corollary 4.3.4. *Let $\{\mathcal{A}\}$ be a local conformal net on the line fulfilling Haag duality on $\mathbb{R}$ (assumption (a)) and the split property (assumption (b)). Then $\mathcal{A}(W)^{c\perp} = \mathrm{DHR}^W\{\mathcal{A}\}$ for every half-line $W \subset \mathbb{R}$, left or right oriented. In particular*

$$\mathcal{A}(W)^{\perp} \simeq \mathrm{DHR}\{\mathcal{A}\} \simeq \mathcal{A}(W)^{c\perp}$$

as UBTCs.

Also, by definition, we have trivial braiding operators

$$\varepsilon_{\rho\sigma} = \mathbb{1} \tag{4.11}$$

whenever $\rho \in \mathrm{DHR}^W\{\mathcal{A}\}$, $\sigma \in \mathrm{DHR}^{W'}\{\mathcal{A}\}$ and W is a *left* half-line, hence W' a *right* half-line. Equation (4.11) is one of the characteristic features of the DHR braiding coming from spacetime localization of charges in QFT. An abstract UBTCs need *not* have this kind of trivialization property for braiding operators at all.

The situation is different for local algebras $\mathcal{A}(I) \subset \mathcal{A}$, $I \in \mathcal{I}$, as shown by Doplicher in [Dop82, Prop. 2.3] with the split property (assumption (b)):

Proposition 4.3.5. [Dop82]. *Let $\{\mathcal{A}\}$ be a local conformal net on the line fulfilling in addition assumptions (a) and (b), then*

$$\mathcal{A}(I)^{c\perp} = \langle \mathrm{Inn}^I\{\mathcal{A}\}\rangle_{\oplus}$$

for every $I \in \mathcal{I}$, where $\mathrm{Inn}^I\{\mathcal{A}\}$ is the full subcategory of $\mathrm{DHR}\{\mathcal{A}\}$ whose objects are the inner automorphisms localizable in I and $\langle - \rangle_{\oplus}$ denotes the completion under (finite) direct sums in $\mathcal{A}(I)$, i.e., the inner endomorphisms localizable in I.

In particular,

$$\mathcal{A}(I)^{\perp} \simeq \mathrm{DHR}\{\mathcal{A}\}, \quad \mathcal{A}(I)^{c\perp} \simeq \mathrm{Vec}. \tag{4.12}$$

Remark 4.3.6. The previous proposition has a deep insight in the theory of DHR superselection sectors in any spacetime dimension, see also [Bor65, Lem. III-1 (erratum)], [DHR69a, Sec. V], [Rob11, Sec. 1.9] and discussions therein. Notice also that the proof in [Dop82] is formulated in 3+1 dimensions and holds in the case of Abelian gauge symmetry, i.e., DHR automorphisms only. See [Müg99, Prop. 4.2] for the adaptation to the general case, and [Dri79] for related arguments. Notice also that by definition $\mathrm{DHR}^I\{\mathcal{A}\} = \mathcal{A}(I')^{\perp}$.

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Furthermore, using now all the assumptions of complete rationality (a), (b), (c), we can prove our second main structure result

Proposition 4.3.7. *Let $\{\mathcal{A}\}$ be a completely rational conformal net on the line, then*

$$\mathrm{DHR}^I\{\mathcal{A}\}^\perp = \mathcal{A}(I')$$

for every $I \in \mathcal{I}$.

Proof. ($\supset$): trivial by definition of DHR localization.

($\subset$): take $a \in \mathcal{A}$ such that $\rho(a) = a$ for all $\rho \in \mathrm{DHR}^I\{\mathcal{A}\}$. It follows easily that $a \in \mathcal{A}(I)^c = \mathcal{A}(I)' \cap \mathcal{A}$ by using inner automorphisms localizable in I, the task is to show that $a \in \mathcal{A}(I')$. We divide the proof into three steps.

We first assume that (i) $a \in \mathcal{A}_{\mathrm{loc}}$, i.e., $a \in \mathcal{A}(K)$ for some sufficiently big interval $I \Subset K$ and that (ii) all DHR endomorphisms have dimension $d_\rho = 1$ (pointed category case).

Then the inclusion $\mathcal{A}(I') \subset \mathcal{A}(I)^c$ is locally the two-interval subfactor $\mathcal{A}(I_1 \cup I_2) \subset \mathcal{A}(I)' \cap \mathcal{A}(K) = \mathcal{A}(I)^c$ where $I' \cap K = I_1 \cup I_2$ and $I_1, I_2 \in \mathcal{I}$. Hence $a \in \mathcal{A}(I)^c$ has a unique "harmonic" expansion [LR95, Eq. (4.10)]

$$a = \sum_{i=0,\dots,n} a_i \overline{R}_i \tag{4.13}$$

where $a_i \in \mathcal{A}(I_1 \cup I_2)$ are uniquely determined coefficients and $\overline{R}_i \in \mathcal{A}(I)^c$ are (fixed) generators of the extension. The computation of this extension is the core of [KLM01]. The extension has finite index by assumption (c) and the generators are uniquely determined, up to multiplication with elements of $\mathcal{A}(I_1 \cup I_2)$, by the DHR category of $\{\mathcal{A}\}$. Indeed

$$\overline{R}_i \in \mathrm{Hom}_{\mathrm{DHR}\{A(I)\}}(\mathrm{id}, \rho_i^1 \overline{\rho}_i^2)$$

are solutions of the conjugate equations [LR97, Sec. 2] for the i-th sector $[\rho_i]$ where ρ_i^1 is localizable in I_1 and $\overline{\rho}_i^2$ is localizable in I_2, and n is the number of DHR sectors of the theory different from the vacuum $[\rho_0] = [\mathrm{id}]$. By Frobenius reciprocity [LR97, Lem. 2.1] and up to multiplication with elements of $\mathcal{A}(I_1 \cup I_2)$, the generators $\overline{R}_i$ can be thought as unitary $[\rho_i]$-charge transporters from I_2 to I_1, equivalently as unitary $[\overline{\rho}_i]$-charge transporters from I_1 to I_2. By assumption, for all $\rho \in \mathrm{DHR}^I\{\mathcal{A}\}$ we have

$$a = \sum_i a_i \overline{R}_i = \rho(a) = \sum_i a_i \rho(\overline{R}_i)$$

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To fix ideas, from now on we assume I_1 left to I and I_2 right to I. By naturality and tensoriality of the braiding, see [DHR71, Lem. 2.6], [FRS92, Sec. 2.2], we have

$$\mathcal{E}_{\rho_i^1,\rho}\,\rho_i^1(\mathcal{E}_{\overline{\rho}_i^2,\rho})\,\overline{R}_i = \rho(\overline{R}_i)$$

which reduces to

$$\rho(\overline{R}_i) = \mathcal{E}_{\overline{\rho}_i^2,\rho}\,\overline{R}_i$$

because of the respective localization properties of the endomorphisms. In this special case we have $\mathcal{E}_{\overline{\rho}_i^2,\rho} = \lambda_{\overline{\rho}_i,\rho}\mathbb{1}$ where $\lambda_{\overline{\rho}_i,\rho} \in \mathbb{T}$ is a complex phase, hence $a_i\,\mathcal{E}_{\overline{\rho}_i^2,\rho} \in \mathcal{A}(I_1 \cup I_2)$ and by uniqueness of the previous expansion, if $a_i \neq 0$ we must have $\mathcal{E}_{\overline{\rho}_i^2,\rho} = \mathbb{1}$ for all $\rho \in \mathrm{DHR}^I\{\mathcal{A}\}$. But also $\mathcal{E}_{\rho,\overline{\rho}_i^2} = \mathbb{1}$ for all $\rho \in \mathrm{DHR}^I\{\mathcal{A}\}$, hence $[\overline{\rho}_i]$ is degenerate. By modularity of the category all coefficients $a_i = 0$ for $i = 1, \ldots, n$ and we are left with $a = a_0$ because $\overline{R}_0 = \mathbb{1}$ can be chosen without loss of generality. In particular, $a \in \mathcal{A}(I_1 \cup I_2)$.

We now relax the assumption (ii) about the category and allow DHR endomorphisms of dimension $d_\rho > 1$. As above we have

$$a = \rho(a) = \sum_i a_i\,\mathcal{E}_{\overline{\rho}_i^2,\rho}\,\overline{R}_i$$

for all $\rho \in \mathrm{DHR}^I\{\mathcal{A}\}$ but now the coefficients have different localization properties and we need a more refined argument. Then rewrite

$$a = \sum_i a_i\,\rho_i^1(\mathcal{E}_{\rho,\overline{\rho}_i^2}\,\mathcal{E}_{\overline{\rho}_i^2,\rho})\overline{R}_i$$

and consider for all $\rho \in \mathrm{DHR}^I\{\mathcal{A}\}$ a conjugate endomorphism $\overline{\rho}$ again localizable in I and operators $\overline{R}_\rho \in \mathrm{Hom}_{\mathrm{DHR}\{A(I)\}}(\mathrm{id}, \rho\overline{\rho})$ as before. The latter are $\overline{R}_\rho \in \mathcal{A}(I)$ and can be normalized such that $\overline{R}_\rho^*\overline{R}_\rho = d_\rho\mathbb{1}$. Then we can write

$$a = d_\rho^{-1}\overline{R}_\rho^*\overline{R}_\rho a = d_\rho^{-1}\overline{R}_\rho^* a\overline{R}_\rho = d_\rho^{-1}\sum_i a_i\,\overline{R}_\rho^*\,\rho_i^1(\mathcal{E}_{\rho,\overline{\rho}_i^2}\,\mathcal{E}_{\overline{\rho}_i^2,\rho})\overline{R}_i\overline{R}_\rho$$

by locality, and using $\rho_i^1\overline{\rho}_i^2(\overline{R}_\rho^*) = \overline{R}_\rho^*$ we have also

$$a = \rho(a) = d_\rho^{-1}\sum_i a_i\,\rho_i^1\overline{\rho}_i^2(\overline{R}_\rho^*)\,\rho_i^1(\mathcal{E}_{\rho,\overline{\rho}_i^2}\,\mathcal{E}_{\overline{\rho}_i^2,\rho})\overline{R}_i\overline{R}_\rho$$

where on the right hand side we have formed a "killing-ring", after [BEK99, Sec. 3], in order to exploit *modularity*. Then choose one representative for each sector $\rho_j \in \mathrm{DHR}^I\{\mathcal{A}\}$ where $j = 0, \ldots, n$ and consider

$$\left(\sum_j d_{\rho_j}^2\right) a = \sum_j d_{\rho_j}^2\,\rho_j(a) = \sum_{i,j} a_i\,d_{\rho_j}\rho_i^1\overline{\rho}_i^2(\overline{R}_{\rho_j}^*)\,\rho_i^1(\mathcal{E}_{\rho_j,\overline{\rho}_i^2}\,\mathcal{E}_{\overline{\rho}_i^2,\rho_j})\overline{R}_i\overline{R}_{\rho_j}$$

$$= \sum_i a_i \left(\sum_k d_{\rho_k}^2 \right) \delta_{[\overline{\rho}_i],[\mathrm{id}]} \overline{R}_i = \left(\sum_k d_{\rho_k}^2 \right) a_0 \overline{R}_0$$

by unitarity of the S-matrix, as shown by [Reh90b] in the case of UMTCs. As before we conclude $a = a_0 \in \mathcal{A}(I_1 \cup I_2)$.

It remains the case when $a \in \mathcal{A} \smallsetminus \mathcal{A}_{\mathrm{loc}}$ relaxing assumption (i). By the split property (assumption (b)) we have that $\mathcal{A}(I)$ is injective hence generated by an amenable group of unitaries. Averaging over its adjoint action (cf. proof of Proposition 4.1.4) we get a conditional expectation $E : \mathcal{B}(\mathcal{H}) = \mathcal{A}(I) \vee \mathcal{A}(I)' \to \mathcal{A}(I)'$ mapping for all $I \Subset K$, $K \in \mathcal{I}$

$$E(\mathcal{A}(K)) = \mathcal{A}(K) \cap \mathcal{A}(I)', \quad E(\mathcal{A}) = \mathcal{A}(I)^c.$$

Since E is norm continuous we have

$$\mathcal{A}(I)^c = C^*(\cup_{n \in \mathbb{N}} \mathcal{A}(K_n) \cap \mathcal{A}(I)'), \quad I \Subset \mathcal{K}_n \nearrow \mathbb{R}, \ K_n \in \mathcal{I}$$

hence we can write $a = \lim_n a_n$ where $a_n \in \mathcal{A}(K_n) \cap \mathcal{A}(I)'$. As in the previous steps we get

$$a_n = \sum_i a_{n,i} \overline{R}_i$$

where we can choose $\overline{R}_i$ independently of n (at least for big n). From the assumptions and norm continuity of $\rho \in \mathrm{DHR}^I\{\mathcal{A}\}$ we have

$$a = \rho(a) = \lim_n \rho(a_n) = \lim_n \sum_i a_{n,i} \, \mathcal{E}_{\overline{\rho}_i^2, \rho} \, \overline{R}_i.$$

Now we show that for all i the sequences $(a_{n,i})_n$ converge to some $b_i \in \mathcal{A}(I')$. Indeed the coefficients are explicitly given [LR95, Eq. (4.10)] as

$$a_{n,i} = \lambda \, E_n(a_n \overline{R}_i^*)$$

where λ is the μ_2-index of the two-interval subfactor and we denoted by $E_n : \mathcal{A}(K_n) \cap \mathcal{A}(I)' \to \mathcal{A}(K_n \cap I')$ the minimal conditional expectations, see [KLM01, Prop. 5]. Compute

$$\|a_{n,i} - a_{m,i}\| = \lambda \, \|E_n(a_n \overline{R}_i^*) - E_m(a_m \overline{R}_i^*)\|$$

but now it holds [KLM01, Lem. 11] that $E_{m \restriction \mathcal{A}(K_n) \cap \mathcal{A}(I)'} = E_n$ if $m > n$, thus

$$\lambda \, \|E_m((a_n - a_m)\overline{R}_i^*)\| \leq \lambda \, (d_{\rho_i})^{1/2} \|a_n - a_m\| \longrightarrow 0$$

for $n, m \to \infty$. Then $(a_{n,i})_n$ are Cauchy sequences. Since $\mathcal{A}(I')$ is by definition norm closed, the limit points $b_i \in \mathcal{A}(I')$ exist. Hence we have shown that the (local) unique expansion formula (4.13) makes sense also in the quasilocal limit for the inclusion $\mathcal{A}(I') \subset \mathcal{A}(I)^c$

$$ a = \sum_i b_i \overline{R}_i. \tag{4.14} $$

With the same argument as in the (local) two-interval case we can show that $\rho(a) = a$ for all $\rho \in \mathrm{DHR}^I\{\mathcal{A}\}$ implies $b_i = 0$ whenever $i \neq 0$, hence $a = b_0 \in \mathcal{A}(I')$ and the proof is complete. $\qquad\square$

Remark 4.3.8. A statement similar to the previous proposition appears in [Dop82] as a "natural conjecture" which explains the shape of the inclusion $\mathcal{A}(\mathcal{O}') \subset \mathcal{A}(\mathcal{O})^c$ where $\mathcal{O}$ is any open double cone region in Minkowski spacetime $\mathbb{R}^{3+1}$. The generators of the extension can be interpreted in that case as local measurements of (global Abelian) superselection charges, see also [DL83b]. The situation here is much different: DHR superselection charges in low dimensions have non-degenerately braided statistics (opposite to permutation group), the category is modular instead of symmetric, there is no global gauge symmetry and the generators of the extension $\mathcal{A}(I') \subset \mathcal{A}(I)^c$, where $I \in \mathcal{I}$, seem to have a purely topological nature. Surprisingly (in the light of the previous facts) the proof of the statement relies essentially on modularity. To our knowledge, by now there is no other proof of the statement in different contexts.

From the previous proof, we also get the following

Corollary 4.3.9. *With the assumptions of Proposition 4.3.7, every element* $a \in \mathcal{A}(I)^c = \mathcal{A}(I)' \cap \mathcal{A}$ *admits a unique "harmonic" expansion, cf. [LR95, Eq. (4.10)]*

$$ a = \sum_{i=0,\ldots,n} b_i \overline{R}_i $$

where $b_i \in \mathcal{A}(I')$ *are uniquely determined coefficients and* $\overline{R}_i \in \mathrm{Hom}(\mathrm{id}, \rho_i^1 \overline{\rho}_i^2) \subset \mathcal{A}(I)^c$ *are (fixed) generators of the extension of unital C^*-algebras*

$$ \mathcal{A}(I') \subset \mathcal{A}(I)^c. $$

In particular, for holomorphic conformal nets it holds (cf. Proposition 4.1.4)

$$ \mathcal{A}_{\mathrm{holo}}(I') = \mathcal{A}_{\mathrm{holo}}(I)^c. $$

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Remark 4.3.10. Relations analogous to Proposition 4.3.7 hold for half-lines $W \subset \mathbb{R}$, namely $\mathrm{DHR}^W\{\mathcal{A}\}^\perp = \mathcal{A}(W')$ as one can easily show using Proposition 4.1.4. We shall see later a more general argument, see Proposition 4.5.5.

4.4 Local duality relations

We turn now to the local picture, i.e., consider as environment some local algebra $\mathcal{A}(I_0)$ for arbitrarily fixed $I_0 \in \mathcal{I}$ instead of the quasilocal algebra $\mathcal{A}$. Similarly to (4.10) we consider the *local* duality pairing

$$\mathcal{A}(I_0) \xleftrightarrow{\;\perp\;} \mathrm{DHR}^{I_0}\{\mathcal{A}\}. \tag{4.15}$$

The local version of all the statements we made in Section 4.3 follows analogously, thanks to strong additivity, by considering *local* interval algebras $\mathcal{A}(I) \subset \mathcal{A}(I_0)$ if $I \Subset I_0$, $I \in \mathcal{I}$, and *local* half-line algebras $\mathcal{A}(I_1) \subset \mathcal{A}(I_0)$ if $I_1 = W \cap I_0$, $W \subset \mathbb{R}$ is any half-line with origin $p \in I_0$.

In the following the symbol $\perp$ will refer to (4.15). Similarly to the notion of relative commutant for unital inclusions of algebras, i.e., $\mathcal{N}^c = \mathcal{N}' \cap \mathcal{A}(I_0)$ if $\mathcal{N} \subset \mathcal{A}(I_0)$, we introduce relative commutants of subcategories

Definition 4.4.1. Let $\mathcal{C} \subset \mathrm{DHR}^{I_0}\{\mathcal{A}\}$ be a unital full inclusion of tensor categories, we define the **relative commutant** as

$$\mathcal{C}^c := \big\{\rho \in \mathrm{DHR}^{I_0}\{\mathcal{A}\} : \rho\,\sigma = \sigma\rho,\ \sigma \in \mathcal{C}\big\}$$

where the equality sign means pointwise equality as endomorphisms of $\mathcal{A}(I_0)$, or equivalently of $\mathcal{A}$. We define $\mathcal{C}^c \subset \mathrm{DHR}^{I_0}\{\mathcal{A}\}$ as a full subcategory, i.e., $\mathrm{Hom}_{\mathcal{C}^c}(\rho, \sigma) := \mathrm{Hom}_{\mathrm{DHR}\{\mathcal{A}\}}(\rho, \sigma)$ for every $\rho, \sigma \in \mathcal{C}^c$.

$\mathcal{C}^c$ is automatically a unital tensor category of endomorphisms of $\mathcal{A}(I_0)$. Now combining relative commutants and duals, given a subalgebra $\mathcal{N} \subset \mathcal{A}(I_0)$ we define a unital tensor full subcategory $\mathcal{C}_\mathcal{N} \subset \mathrm{DHR}^{I_0}\{\mathcal{A}\}$ as

$$\mathcal{C}_\mathcal{N} := \mathcal{N}^{c\perp}$$

where by definition $\mathrm{Hom}_{\mathcal{C}_\mathcal{N}}(\rho, \sigma) = \mathrm{Hom}_{\mathrm{DHR}\{\mathcal{A}\}}(\rho, \sigma)$ for every $\rho, \sigma \in \mathcal{C}_\mathcal{N}$.

Remark 4.4.2. Despite we use the term "local" for the duality pairing (4.15) and for the respective subcategories of $\mathrm{DHR}^{I_0}\{\mathcal{A}\}$ defined as above, it should

be kept in mind that both $\mathcal{C}_{\mathcal{N}}$ and $\mathrm{DHR}^{I_0}\{\mathcal{A}\}$ are categories of globally defined endomorphisms of the quasilocal algebras $\mathcal{A}$, which then are "localizable" in smaller regions, e.g., I_0, i.e., act trivially on every local algebra $\mathcal{A}(J)$, $J \subset I_0'$ and on $\mathcal{N}^c$.

Summarizing the previous results, we have

Corollary 4.4.3. *Let $p \in I_0$ and $I_0 \smallsetminus \{p\} = I_1 \cup I_2$. Let $\mathcal{N} := \mathcal{A}(I_1)$, then $\mathcal{N}^c = \mathcal{A}(I_2)$, $\mathcal{C}_{\mathcal{N}} = \mathrm{DHR}^{I_1}\{\mathcal{A}\}$, $\mathcal{C}_{\mathcal{N}^c} = \mathrm{DHR}^{I_2}\{\mathcal{A}\}$. Moreover, if I_1 is to the left of I_2, then $\varepsilon_{\rho,\sigma} = \mathbb{1}$ whenever $\rho \in \mathcal{C}_{\mathcal{N}}$, $\sigma \in \mathcal{C}_{\mathcal{N}^c}$.*

Remark 4.4.4. It is well known that a point as the localization of an observable is an over-idealization, forcing fields to be distributions, and making the intersections of local algebras corresponding to regions intersecting at a point trivial. In contrast, the proper way of "lifting" points to quantum field theory rather seems to be their role as separators between local algebras, trivializing the braiding as in Corollary 4.4.3.

4.5 Abstract points

Let $\{\mathcal{A}\}$ be a completely rational conformal net on the line (Definition 3.4.1). In the previous two sections we essentially used the action of the DHR category, and its abstract structure of UMTC. Now we employ the DHR braiding as well, see equation (4.11) and comments thereafter, hence the braided action (Definition 4.2.2) given by the *restriction functor*

$$\mathcal{C} := \mathrm{DHR}^{I_0}\{\mathcal{A}\} \hookrightarrow \mathrm{End}(\mathcal{M}_0)$$

where $\mathcal{M}_0 := \mathcal{A}(I_0)$ and $I_0 \in \mathcal{I}$ is an arbitrarily fixed interval.

Definition 4.5.1. We call **abstract point of $\mathcal{M}_0$** an ordered pair of algebras $(\mathcal{N}, \mathcal{N}^c)$ where $\mathcal{N} \subset \mathcal{M}_0$ such that

(i) $\mathcal{N}$ and $\mathcal{N}^c$ are injective type III_1 factors.

(ii) $\mathcal{N} = \mathcal{N}^{cc}$ and $\mathcal{N} \vee \mathcal{N}^c = \mathcal{M}_0$.

(iii) $\mathcal{C}_{\mathcal{N}} \simeq \mathcal{C}$ and $\mathcal{C}_{\mathcal{N}^c} \simeq \mathcal{C}$ as UBTCs.

(iv) $\varepsilon_{\rho,\sigma} = \mathbb{1}$ whenever $\rho \in \mathcal{C}_{\mathcal{N}}$, $\sigma \in \mathcal{C}_{\mathcal{N}^c}$.

With abuse of notation we denote abstract points by $p := (\mathcal{N}, \mathcal{N}^c)$, and call $\mathcal{N}$, $\mathcal{N}^c$ respectively the **left, right relative complement of p in $\mathcal{M}_0$**.

More generally, given an "abstract" UMTC $\mathcal{C}$ together with a braided action on the injective type III_1 factor $\mathcal{M}$, see Definition 4.2.2 and Remark 4.2.5, we can analogously define abstract points of $\mathcal{M}$ (with respect to the braided action $\mathcal{C} \hookrightarrow \mathrm{End}(\mathcal{M})$). In the case of a UMTC coming from a completely rational conformal net, $\mathcal{C} = \mathrm{DHR}^{I_0}\{\mathcal{A}\}$ together with its *canonical* braided action on $\mathcal{M}_0$, the existence of those is the content of the previous sections.

Remark 4.5.2. Condition (iii) is indeed equivalent to essential surjectivity of the inclusion functors $\mathcal{C}_\mathcal{N} \subset \mathcal{C}$ and $\mathcal{C}_{\mathcal{N}^c} \subset \mathcal{C}$. In fact $\mathcal{C}_\mathcal{N} \subset \mathcal{C} \subset \mathrm{DHR}\{\mathcal{A}\}$ are full inclusions by definition, the latter also essentially surjective, and the inclusion functor is trivially unitary strict tensor and braided.

Remark 4.5.3. Condition (iv) consists a priori of uncountably many constraints on braiding operators. We shall see in Proposition 4.5.11 that it is indeed equivalent to a finite system of equations. This makes (iv) a more tractable ("rational") condition.

Remark 4.5.4. From Corollary 4.4.3 we know that ordered pairs of local algebras $(\mathcal{A}(I_1), \mathcal{A}(I_2))$, associated respectively to the left and right relative complements I_1, I_2 of some $p \in I_0$, are also abstract points of $\mathcal{M}_0 = \mathcal{A}(I_0)$. We shall refer to them as *honest* points of $\mathcal{M}_0$ (with respect to the net $\{\mathcal{A}\}$). The converse is not true in general, see in Sections 4.6 and 4.7.

At the level of generality of Definition 4.5.1 we can show the following

Proposition 4.5.5. *Let $p = (\mathcal{N}, \mathcal{N}^c)$ be an abstract point of $\mathcal{M}_0$, then the quadruple $(\mathcal{N}, \mathcal{N}^c, \mathcal{C}_\mathcal{N}, \mathcal{C}_{\mathcal{N}^c})$ is uniquely determined by any one of its elements.*

Proof. It is sufficient to show that $\mathcal{C}_{\mathcal{N}^c}$ determines $\mathcal{N}$. By definition $\mathcal{C}_{\mathcal{N}^c}{}^\perp = \mathcal{N}^{cc\perp\perp} = \mathcal{N}^{\perp\perp}$ holds and the inclusion $\mathcal{N} \subset \mathcal{N}^{\perp\perp}$ is trivial. The opposite inclusion also holds for algebras of the form $\mathcal{N} = \mathcal{P}^c$, where $\mathcal{P} \subset \mathcal{M}_0$ is any unital C^*-subalgebra of $\mathcal{M}_0$, cf. [Dop82, Sec. 5], in our case $\mathcal{P} = \mathcal{N}^c$. Let $a \in \mathcal{N}^{\perp\perp}$ and consider the unitary group $\mathcal{U}(\mathcal{P})$, then $\mathrm{Ad}_u \in \mathcal{N}^\perp$ for all $u \in \mathcal{U}(\mathcal{P})$ hence $\mathrm{Ad}_u(a) = a$ and we conclude $a \in \mathcal{U}(\mathcal{P})'$. Now $\mathcal{U}(\mathcal{P})$ linearly spans $\mathcal{P}$, hence $a \in \mathcal{M}_0 \cap \mathcal{P}' = \mathcal{P}^c = \mathcal{N}$. $\qquad\square$

The gain in considering together pairs of subfactors or pairs of subcategories is that we can use the braiding operators between endomorphisms as a remnant of their localization properties (left/right separation) hence, dually, of the net. The first interesting consequence of Definition 4.5.1 is however the following

Proposition 4.5.6. *Let $(\mathcal{N}, \mathcal{N}^c)$ be a pair of subfactors of $\mathcal{M}_0$ fulfilling conditions (i) and (ii) in the Definition 4.5.1 of abstract points.*

If we consider for instance $\mathcal{N} \subset \mathcal{M}_0$ and the associated $\mathcal{C}_\mathcal{N} \subset \mathcal{C}$, we have

- *if $\rho \in \mathcal{C}_\mathcal{N}$ then $\rho \in \mathrm{End}(\mathcal{N})$.*

- *if $t \in \mathrm{Hom}_{\mathcal{C}_\mathcal{N}}(\rho, \sigma)$ where $\rho, \sigma \in \mathcal{C}_\mathcal{N}$, then $t \in \mathcal{N}$.*

- *if $t \in \mathcal{N}$ and $t\rho(n) = \sigma(n)t$ for all $n \in \mathcal{N}$ where $\rho, \sigma \in \mathcal{C}_\mathcal{N}$, then $t \in \mathrm{Hom}_{\mathcal{C}_\mathcal{N}}(\rho, \sigma)$.*

In other words, we have a well-defined, faithful and full restriction functor $\rho \mapsto \rho_{|\mathcal{N}}$

$$\mathcal{C}_\mathcal{N} \hookrightarrow \mathrm{End}(\mathcal{N}).$$

- *if $\rho \in \mathcal{C}_\mathcal{N}$ and $u \in \mathcal{U}(\mathcal{N})$ then $\mathrm{Ad}_u\, \rho \in \mathcal{C}_\mathcal{N}$.*

Hence the restriction functor has replete image, in particular it is specified by its sectors (unitary isomorphism classes of objects) only.

Proof. First, take $\rho \in \mathcal{C}_\mathcal{N} = \mathcal{N}^{c\perp}$ and $n \in \mathcal{N}$, then $\rho(n)m = \rho(nm) = m\rho(n)$ for all $m \in \mathcal{N}^c$ and we get $\rho(n) \in \mathcal{M}_0 \cap \mathcal{N}^{c\prime} = \mathcal{N}^{cc} = \mathcal{N}$.

Second, take $t \in \mathcal{M}_0$ such that $t\rho(a) = \sigma(a)t$ for all $a \in \mathcal{M}_0$, where $\rho, \sigma \in \mathcal{C}_\mathcal{N}$. Now, letting $a \in \mathcal{N}^c$ we have $ta = at$ hence $t = \mathcal{N}^{cc} = \mathcal{N}$.

Third, we have $t \in \mathcal{N}$ and $t\rho(n) = \sigma(n)t$ if $n \in \mathcal{N}$ by definition and $t\rho(m) = \sigma(m)t$ if $m \in \mathcal{N}^c$ because $tm = mt$. Now, every $a \in \mathcal{M}_0 = \mathcal{N} \vee \mathcal{N}^c$ can be written as an ultra-weak limit of finite sums $a = uw\text{-}\lim \sum_i n_i m_i$ where $n_i \in \mathcal{N}$ and $m_i \in \mathcal{N}^c$. Also, ρ, σ are automatically normal on $\mathcal{M}_0$, see [Tak02, p. 352], being $\mathcal{M}_0$ non-type I and $\mathcal{H}$ separable. Normality on $\mathcal{M}_0 = \mathcal{A}(I_0)$ can also be derived by DHR transportability of the endomorphisms, but we prefer the previous argument which is intrinsic and local. From these two facts we conclude that $t\rho(a) = \sigma(a)t$ for all $a \in \mathcal{M}_0$, hence as DHR endomorphisms because local intertwiners are global, i.e., $\mathcal{C} \hookrightarrow \mathrm{End}(\mathcal{M}_0)$ is full.

The last point is trivial to show, but has interesting consequences (see Proposition 4.5.7). $\qquad\square$

The conditions stated in Definition 4.5.1 contain many redundancies. Out of the operator algebraic assumptions (i) and (ii) on $\mathcal{N}$ and $\mathcal{N}^c$, one can derive properties of their dual categories $\mathcal{C}_\mathcal{N}$ and $\mathcal{C}_{\mathcal{N}^c}$ which are custom assumptions in C^* tensor category theory, see, e.g., [LR97]. Nevertheless, assumptions (iii) and (iv) cannot be derived from the previous, see Proposition 4.3.3 and 4.3.5, unless the net $\{\mathcal{A}\}$ is holomorphic.

Proposition 4.5.7. *Let $(\mathcal{N}, \mathcal{N}^c)$ be a pair of subfactors of $\mathcal{M}_0$ fulfilling conditions (i) and (ii) in the Definition 4.5.1 of abstract points. Then the subcategories $\mathcal{C}_\mathcal{N}$ and $\mathcal{C}_{\mathcal{N}^c}$ automatically have irreducible tensor unit, subobjects, finite direct sums and conjugate objects.*

In other words, they are C^* tensor categories which are also fusion and rigid.

Proof. The restriction functor $\mathcal{C}_\mathcal{N} \hookrightarrow \mathrm{End}(\mathcal{N})$ is full and faithful by Proposition 4.5.6, hence irreducibility of the tensor unit of $\mathcal{C}_\mathcal{N}$ is equivalent to factoriality of $\mathcal{N}$.

In general the existence of subobjects in $\mathrm{DHR}\{\mathcal{A}\}$ follows because we have a *net* of type *III* factors, i.e., $\mathcal{A}(I_0)$ alone being type *III* is not sufficient to construct DHR subendomorphisms. In our case we need again Proposition 4.5.6 together with $\mathcal{N}$ being type *III*. Let $\rho \in \mathcal{C}_\mathcal{N}$ and $e \in \mathrm{Hom}_{\mathcal{C}_\mathcal{N}}(\rho, \rho) \subset \mathcal{N}$ a non-zero orthogonal projection. Choose $v \in \mathcal{N}$ such that $v^*v = \mathbb{1}$, $vv^* = e$ and let $\sigma(n) := v^*\rho(n)v$, $n \in \mathcal{N}$, then $\sigma \in \mathrm{End}(\mathcal{N})$ by definition. In order to show $\sigma \prec \rho$ in $\mathcal{C}_\mathcal{N}$ we need to extend σ to $\mathcal{M}_0$ and then to the quasilocal algebra $\mathcal{A}$, in such a way that the intertwining relation $v \in \mathrm{Hom}_{\mathcal{C}_\mathcal{N}}(\sigma, \rho)$ holds, cf. Remark 4.4.2. Now $\sigma(m) := v^*\rho(m)v = m$, $m \in \mathcal{N}^c$, and ρ is normal on $\mathcal{M}_0$ hence σ extends to $\mathrm{End}(\mathcal{M}_0)$ with $\sigma_{\restriction \mathcal{N}^c} = \mathrm{id}$ and $v \in \mathrm{Hom}_{\mathrm{End}(\mathcal{M}_0)}(\sigma, \rho)$. On the other hand $\rho \in \mathcal{C}$ and $\mathcal{C}$ has subobjects, hence let $w \in \mathcal{M}_0$ and $\tau \in \mathcal{C}$ such that $w^*w = \mathbb{1}$, $ww^* = e$ and $w \in \mathrm{Hom}_\mathcal{C}(\tau, \rho) = \mathrm{Hom}_{\mathrm{End}(\mathcal{M}_0)}(\tau, \rho)$. Now w^*v is unitary in $\mathrm{Hom}_{\mathrm{End}(\mathcal{M}_0)}(\sigma, \tau)$ hence we can extend $\sigma \in \mathcal{C}$ because $\mathcal{C} \hookrightarrow \mathrm{End}(\mathcal{M}_0)$ is replete. Thus $\sigma \in \mathcal{C}_\mathcal{N}$ and $v \in \mathrm{Hom}_{\mathcal{C}_\mathcal{N}}(\sigma, \rho)$ because $\mathcal{C}_\mathcal{N} \hookrightarrow \mathrm{End}(\mathcal{N})$ is full.

Along similar lines one can show the existence of direct sums in $\mathcal{C}_\mathcal{N}$.

To show existence of conjugates in $\mathcal{C}_\mathcal{N}$ we need, in addition, results from the theory of infinite subfactors with finite index. Let $\rho \in \mathcal{C}_\mathcal{N}$ be an irreducible DHR endomorphism, hence with finite (minimal) index $\mathrm{Ind}(\rho(\mathcal{M}_0), \mathcal{M}_0) < \infty$ [KLM01, Cor. 39], i.e., finite statistical dimension $d_\rho < \infty$ [GL96, Cor. 3.7]. Let Φ be the unique left inverse of ρ, see [GL96, Cor. 2.12], which is normal on $\mathcal{M}_0$ and localizable in I_0, hence in particular $\Phi(\mathcal{M}_0) \subset \mathcal{M}_0$. For every $n \in \mathcal{N}$, $m \in \mathcal{N}^c$ we have $\Phi(m) = \Phi(\rho(m)) = m$ and $\Phi(n)m = \Phi(n\rho(m)) = \Phi(nm) = m\Phi(n)$ hence $\Phi_{\restriction \mathcal{N}^c} = \mathrm{id}$ and $\Phi(\mathcal{N}) \subset \mathcal{N}^{cc} = \mathcal{N}$.

Again by Proposition 4.5.6, irreducibility of ρ is equivalent to irreducibility of the subfactor $\rho(\mathcal{N}) \subset \mathcal{N}$, then $E_{\restriction \mathcal{N}} := \rho \circ \Phi_{\restriction \mathcal{N}}$ coincides with the unique normal faithful (minimal) conditional expectation given by [Lon89, Thm. 5.5]. After setting $\lambda := \mathrm{Ind}(\rho(\mathcal{M}_0), \mathcal{M}_0)^{-1}$, we have the Pimnser-Popa bound

[Lon89, Thm. 4.1]

$$E(a^*a) \geq \lambda a^*a, \quad a \in \mathcal{M}_0 \tag{4.16}$$

where λ is the best constant fulfilling equation (4.16). In particular, it holds for all $a \in \mathcal{N} \subset \mathcal{M}_0$ and if we let $\mu := \mathrm{Ind}(\rho(\mathcal{N}), \mathcal{N})^{-1}$ by the same argument on $\rho(\mathcal{N}) \subset \mathcal{N}$ and by uniqueness of $E_{|\mathcal{N}}$ we get $\mu \geq \lambda$, hence $\mathrm{Ind}(\rho(\mathcal{N}), \mathcal{N}) < \infty$. Now we turn to the construction of the conjugate endomorphism of ρ in $\mathcal{C}_{\mathcal{N}}$. As before we begin "locally", i.e., by construction of the restriction of the conjugate as an object of $\mathrm{End}(\mathcal{N})$, and then extend. Let $\rho_{\mathcal{N}} := \rho_{|\mathcal{N}} \in \mathrm{End}(\mathcal{N})$ and $\overline{\rho} := (\rho_{\mathcal{N}})^{-1} \circ \gamma \in \mathrm{End}(\mathcal{N})$ where γ is a canonical endomorphisms of $\mathcal{N}$ into $\rho(\mathcal{N})$ [Lon90, Thm. 3.1]. By finiteness of the index of $\rho(\mathcal{N}) \subset \mathcal{N}$ [Lon90, Thm. 4.1 and 5.2] we have a solution $R \in \mathrm{Hom}_{\mathrm{End}(\mathcal{N})}(\mathrm{id}, \overline{\rho}\rho_{\mathcal{N}})$, $\overline{R} \in \mathrm{Hom}_{\mathrm{End}(\mathcal{N})}(\mathrm{id}, \rho_{\mathcal{N}}\overline{\rho})$ of the conjugate equations [LR97, Sec. 2] in $\mathrm{End}(\mathcal{N})$. First, we extend $\overline{\rho}$ to $\mathcal{M}_0$ by making use of another formula for the canonical endomorphism [LR95, Eq. (2.19)]

$$\gamma(n) = \lambda d_\rho^{-1} E(\overline{R}n\overline{R}^*), \quad n \in \mathcal{N}. \tag{4.17}$$

By (4.17) γ extends normally to $\mathcal{M}_0$ and to the quasilocal algebra $\mathcal{A}$. Also, for $m \in \mathcal{N}^c$ we get $\gamma(m) = \lambda d_\rho^{-1} E(\overline{R}m\overline{R}^*) = \lambda d_\rho^{-1} E(\overline{R}\overline{R}^*)m = m$ by [LR95, Eq. (4.1)], hence $\gamma_{|\mathcal{N}^c} = \mathrm{id}$ and $\gamma(\mathcal{M}_0) \subset \rho(\mathcal{M}_0)$. It follows that we can extend normally $\overline{\rho} := \rho^{-1} \circ \gamma \in \mathrm{End}(\mathcal{M}_0)$ because ρ is injective hence bicontinuous onto its image in the ultraweak topology [Ped79, p. 59]. Moreover we have $\overline{\rho}_{|\mathcal{N}^c} = \mathrm{id}$ and $R \in \mathrm{Hom}_{\mathrm{End}(\mathcal{M}_0)}(\mathrm{id}, \overline{\rho}\rho)$, $\overline{R} \in \mathrm{Hom}_{\mathrm{End}(\mathcal{M}_0)}(\mathrm{id}, \rho\overline{\rho})$.

On the other hand $\rho \in \mathcal{C}$ and let $\tilde{\rho} \in \mathcal{C}$ be a DHR conjugate of ρ, hence by irreducibility and [Lon90, Thm. 3.1] we have a unitary $u \in \mathrm{Hom}_{\mathrm{End}(\mathcal{M}_0)}(\overline{\rho}, \tilde{\rho})$. As above we extend $\overline{\rho} \in \mathcal{C}$ by repleteness of $\mathcal{C} \hookrightarrow \mathrm{End}(\mathcal{M}_0)$, hence $\overline{\rho} \in \mathcal{C}_{\mathcal{N}}$ together with $R \in \mathrm{Hom}_{\mathcal{C}_{\mathcal{N}}}(\mathrm{id}, \overline{\rho}\rho)$, $\overline{R} \in \mathrm{Hom}_{\mathcal{C}_{\mathcal{N}}}(\mathrm{id}, \rho\overline{\rho})$, and we have the statement in the irreducible case.

Now $R, \overline{R}$ can be normalized in such a way $R^*R = \overline{R}^*\overline{R}$ gives the (intrinsic) dimension of ρ in $\mathcal{C}_{\mathcal{N}}$. The latter does not depend on the choice of normalized solutions in $\mathcal{C}$, and equals the statistical dimension d_ρ on one side and $\mathrm{Ind}(\rho(\mathcal{N}), \mathcal{N})^{1/2}$ on the other by [LR97, p. 121]. In particular, it holds $\lambda = \mu$ and $d_\rho{}^2 = \mathrm{Ind}(\rho(\mathcal{N}), \mathcal{N})$.

The construction of conjugates extends to finite direct sums, concluding the proof of the proposition for $\mathcal{C}_{\mathcal{N}}$. Similarly for $\mathcal{C}_{\mathcal{N}^c}$ interchanging the roles of $\mathcal{N}$ and $\mathcal{N}^c$. □

Remark 4.5.8. See [GL92, Thm. 2.2, Cor. 2.4] for a similar discussion on the conjugation of endomorphisms of *sub*factors.

Going back to the duality between subalgebras and subcategories, under assumption (iii) we can lift the normality relations contained in (ii) from $\mathcal{N}, \mathcal{N}^c$ to $\mathcal{C}_\mathcal{N}, \mathcal{C}_{\mathcal{N}^c}$, in the sense of Definition 4.4.1.

Proposition 4.5.9. *Let $(\mathcal{N}, \mathcal{N}^c)$ be a pair of subfactors of $\mathcal{M}_0$ fulfilling conditions (i), (ii) and (iii) in the Definition 4.5.1 of abstract points. Then*

$$(\mathcal{C}_\mathcal{N})^c = \mathcal{C}_{\mathcal{N}^c}, \quad (\mathcal{C}_{\mathcal{N}^c})^c = \mathcal{C}_\mathcal{N}$$

and the operations in the diagram

$$
\begin{array}{ccc}
\mathcal{N} & \stackrel{\perp}{\longmapsto} & \mathcal{C}_{\mathcal{N}^c} \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\
\mathcal{N}^c & \stackrel{\perp}{\longmapsto} & \mathcal{C}_\mathcal{N}
\end{array}
$$

are commutative and invertible.

Proof. Take $\rho \in \mathcal{C}_{\mathcal{N}^c}$ and first assume (iv) in addition, then $\mathcal{E}_{\sigma,\rho} = \mathbb{1}$ for all $\sigma \in \mathcal{C}_\mathcal{N}$ gives in particular $\rho\sigma = \sigma\rho$ and we can conclude $\rho \in (\mathcal{C}_\mathcal{N})^c$. But we want the statement independent of braiding operators, hence we use Proposition 4.5.6 to draw the same conclusion. Indeed $\rho(\sigma(m)) = \rho(m) = \sigma(\rho(m))$ for all $\sigma \in \mathcal{C}_\mathcal{N}$ and $m \in \mathcal{N}^c$, and the same holds for $n \in \mathcal{N}$. As before, by assumption (i) and (ii) we have $\mathcal{M}_0 = \mathcal{N} \vee \mathcal{N}^c$ and ρ, σ are normal on $\mathcal{M}_0$. Hence $\rho\sigma = \sigma\rho$ for all $\sigma \in \mathcal{C}_\mathcal{N}$ and again $\rho \in (\mathcal{C}_\mathcal{N})^c$.

Vice versa, if $\rho \in (\mathcal{C}_\mathcal{N})^c$ then in particular $\rho \operatorname{Ad}_u = \operatorname{Ad}_u \rho$ for all $u \in \mathcal{U}(\mathcal{N})$, explicitly $\rho(uau^*) = u\rho(a)u^*$ for all $a \in \mathcal{M}_0$. Then we have $u^*\rho(u) \in \operatorname{Hom}_{\operatorname{End}(\mathcal{M}_0)}(\rho, \rho) = \operatorname{Hom}_\mathcal{C}(\rho, \rho)$. If ρ is irreducible, then $u^*\rho(u) = \lambda_u$ where $\lambda_u \in \mathbb{T}$ is a complex phase. The map $u \in \mathcal{U}(\mathcal{N}) \mapsto \lambda_u \in \mathbb{T}$ is a norm continuous unitary character, hence trivial by [Kad52, Thm. 1] because $\mathcal{N}$ is a non-type I factor by assumption (i), and we have $\rho(u) = u$ for all $u \in \mathcal{U}(\mathcal{N})$. In this case, we conclude $\rho \in \mathcal{N}^\perp = \mathcal{C}_{\mathcal{N}^c}$.

In general, if $\rho \in (\mathcal{C}_\mathcal{N})^c$ is (finitely) reducible, we can write ρ as a finite direct sum of irreducibles $\rho = \oplus_{i=1,\dots,n}\rho_i$ with $\rho_i \in \mathcal{C}_{\mathcal{N}^c}$ by assumption (iii). Notice that we already have the inclusion $\mathcal{C}_{\mathcal{N}^c} \subset (\mathcal{C}_\mathcal{N})^c$. Let $\rho, \sigma \in (\mathcal{C}_\mathcal{N})^c$ and $t \in \operatorname{Hom}_{(\mathcal{C}_\mathcal{N})^c}(\rho, \sigma)$, then one has

$$\operatorname{Ad}_u(t)\rho(\operatorname{Ad}_u(a)) = \sigma(\operatorname{Ad}_u(a))\operatorname{Ad}_u(t)$$

for every $u \in \mathcal{U}(\mathcal{N})$, because $\operatorname{Ad}_u \in \mathcal{C}_\mathcal{N}$. But every Ad_u is an automorphisms of $\mathcal{M}_0$ hence we get $\operatorname{Ad}_u(t) \in \operatorname{Hom}_{(\mathcal{C}_\mathcal{N})^c}(\rho, \sigma)$ and $u \in \mathcal{U}(\mathcal{N}) \mapsto \operatorname{Ad}_u$

is a group representation of $\mathcal{U}(\mathcal{N})$ on the finite-dimensional vector space $V := \mathrm{Hom}_{(\mathcal{C}_\mathcal{N})^c}(\rho, \sigma)$, see [LR97, Lem. 3.2]. Now, $V^*V = \mathrm{Hom}_{(\mathcal{C}_\mathcal{N})^c}(\rho, \rho)$ is isomorphic to a finite-dimensional block-diagonal matrix algebra, e.g., if $n = 2$ then $\mathrm{Hom}_{(\mathcal{C}_\mathcal{N})^c}(\rho_1 \oplus \rho_2, \rho_1 \oplus \rho_2)$ is either the full matrix algebra $M_2(\mathbb{C}) \cong \mathbb{C}^4$ if $\rho_1 \cong \rho_2$ or diagonal matrices $\Lambda_2(\mathbb{C}) \cong \mathbb{C}^2$ if $\rho_1 \ncong \rho_2$. Hence we can consider the Hilbert inner product on V given by the (non-normalized) trace of V^*V, i.e.

$$(t|s) := \mathrm{Tr}(t^*s) = \sum_{i=1,\ldots,n} t_i^*(t^*s)t_i$$

where $t, s \in V$ and $\{t_1, \ldots, t_n\} \subset \mathcal{M}_0$ is a Cuntz algebra of isometries defining $\rho = \oplus_i \rho_i$, namely $t_i^* t_j = \delta_{i,j}$ and $\sum_i t_i t_i^* = \mathbb{1}$ and $t_i \in \mathrm{Hom}_{(\mathcal{C}_\mathcal{N})^c}(\rho_i, \rho)$. The definition of trace does not depend on the choice of $\{t_1, \ldots, t_n\}$ and that matrix units of V^*V form an orthonormal basis of V^*V with respect to the previous inner product. Now, given $t, s \in V$ and $u \in \mathcal{U}(\mathcal{N})$ compute

$$(\mathrm{Ad}_u(t)|Ad_u(s)) = \mathrm{Tr}(ut^*su^*) = \mathrm{Tr}(\rho(u)\rho(u^*)ut^*su^*\rho(u)\rho(u^*))$$

$$= \sum_{i=1,\ldots,n} t_i^*(\rho(u)\rho(u^*)ut^*su^*\rho(u)\rho(u^*))t_i = u\,\mathrm{Tr}(\rho(u^*)ut^*su^*\rho(u))u^*$$

$$= \mathrm{Tr}(t^*s) = (t|s)$$

because $\rho_i(u) = u$, being $\rho_i \in \mathcal{C}_{\mathcal{N}^c}$, and $u^*\rho(u) \in V^*V$ so we can use the trace property. Hence the representation of $\mathcal{U}(\mathcal{N})$ on V is unitary with respect to the previous inner product, and norm continuous, as one can easily check with respect to the induced C^*-norm of $V \subset \mathcal{M}_0$ and then using the equivalence of norms for finite-dimensional vector spaces. Again by [Kad52] and assumption (i) the representation must be trivial, i.e., $\mathrm{Ad}_u(t) = t$ for all $u \in \mathcal{U}(\mathcal{N})$, hence $t \in \mathcal{N}' \cap \mathcal{M}_0 = \mathcal{N}^c$ and we have shown $\mathrm{Hom}_{(\mathcal{C}_\mathcal{N})^c}(\rho, \sigma) \subset \mathcal{N}^c$.

In conclusion, we get that every Cuntz algebra of isometries defining the direct sum $\rho = \oplus_i \rho_i$ lies in $\mathcal{N}^c$, hence we conclude $\rho \in \mathcal{C}_{\mathcal{N}^c}$. Both subcategories $\mathcal{C}_{\mathcal{N}^c}$ and $(\mathcal{C}_\mathcal{N})^c$ are full by definition, hence they have the same Hom-spaces, and the proof is complete. $\qquad\square$

We are indebted with Y. Tanimoto for pointing out a mistake in a preliminary proof of Proposition 4.5.9, for carefully proof-reading a previous version of this manuscript and in general for sharing his deep and clear insights on the topic.

Concerning condition (iv) in Definition 4.5.1, the following shows that the braiding contains *all* the information about the subcategories $\mathcal{C}_\mathcal{N}$, $\mathcal{C}_{\mathcal{N}^c}$ and charge transportation among them.

Lemma 4.5.10. *Let $p = (\mathcal{N}, \mathcal{N}^c)$ be an abstract point of $\mathcal{M}_0$. Let $\rho \in \mathcal{C}$, then*

- *$\rho \in \mathcal{C}_\mathcal{N}$ if and only if $\mathcal{E}_{\rho, \mathrm{Ad}_u} = \mathbb{1}$ for all $u \in \mathcal{U}(\mathcal{N}^c)$.*

*Let $\rho \in \mathcal{C}$, $v \in \mathcal{U}(\mathcal{M}_0)$ and set $\tilde{\rho} := \mathrm{Ad}_v \rho$. We call v an **abstract ρ-charge transporter to $\mathcal{C}_{\mathcal{N}^c}$** if it holds $\sigma(v) = v\mathcal{E}_{\sigma, \rho}$ for all $\sigma \in \mathcal{C}_\mathcal{N}$. The terminology is motivated by the following equivalence*

- *$\tilde{\rho} \in \mathcal{C}_{\mathcal{N}^c}$ if and only if v is an abstract ρ-charge transporter to $\mathcal{C}_{\mathcal{N}^c}$.*

Analogous statements hold interchanging $\mathcal{N}$ with $\mathcal{N}^c$ and $\mathcal{E}$ with $\mathcal{E}^{\mathrm{op}}$. [2]

Proof. By naturality of the braiding and using the convention $\mathcal{E}_{\rho, \mathrm{id}} = \mathbb{1}$ we see that triviality of braiding operators with inner automorphisms Ad_u is triviality of the action of the endomorphism on u. Hence the first statement follows.

For the second, take $\rho \in \mathcal{C}$ and $v \in \mathcal{U}(\mathcal{M}_0)$ an abstract ρ-charge transporter to $\mathcal{C}_{\mathcal{N}^c}$. For every $\sigma \in \mathcal{C}_\mathcal{N}$, $a \in \mathcal{M}_0$ compute $\sigma\tilde{\rho}(a) = \sigma(v)\sigma\rho(a)\sigma(v^*) = v\mathcal{E}_{\sigma,\rho}\sigma\rho(a)\mathcal{E}_{\sigma,\rho}^*v^* = \tilde{\rho}\,\sigma(a)$ hence $\tilde{\rho} \in (\mathcal{C}_\mathcal{N})^c = \mathcal{C}_{\mathcal{N}^c}$ by Proposition 4.5.9. Vice versa, if $\tilde{\rho} = \mathrm{Ad}_v \rho \in \mathcal{C}_{\mathcal{N}^c}$ for some $v \in \mathcal{U}(\mathcal{M}_0)$ then $\mathcal{E}_{\sigma, \tilde{\rho}} = \mathbb{1}$ for every $\sigma \in \mathcal{C}_\mathcal{N}$ by (iv). Hence $v\mathcal{E}_{\sigma, \rho}\sigma(v^*) = \mathbb{1}$ and we obtain the second statement. $\square$

On the other hand, after defining $\mathcal{C}_\mathcal{N}$, $\mathcal{C}_{\mathcal{N}^c}$ by duality from $\mathcal{N}$, $\mathcal{N}^c$, condition (iv) turns out to be equivalent to a finite system of equations.

Proposition 4.5.11. *Let $(\mathcal{N}, \mathcal{N}^c)$ be a pair of subfactors of $\mathcal{M}_0$ fulfilling conditions (i), (ii) and (iii) in the Definition 4.5.1 of abstract points. For each sector labelled by $i = 0, \ldots, n$ choose (assumption (iii)) irreducible representatives $\rho_i \in \mathcal{C}_\mathcal{N}$ and $\sigma_i \in \mathcal{C}_{\mathcal{N}^c}$ respectively in $\mathcal{C}_\mathcal{N}$ and $\mathcal{C}_{\mathcal{N}^c}$, such that $[\rho_i] = [\sigma_i]$. Then*

$$\mathcal{E}_{\rho_i, \sigma_j} = \mathbb{1}, \quad i, j = 0, \ldots, n$$

is equivalent to condition (iv).

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[2]The *opposite* braiding of $\mathcal{C}$ is defined as $\mathcal{E}_{\rho,\sigma}^{\mathrm{op}} := \mathcal{E}_{\sigma,\rho}^*$, or equivalently by interchanging left and right localization in the DHR setting.

Proof. In order to show the nontrivial implication, we first take $\rho \in \mathcal{C}_\mathcal{N}$ and $\sigma \in \mathcal{C}_{\mathcal{N}^c}$ irreducible. By Proposition 4.5.6 we have $\mathrm{Ad}_{u_i}\, \rho = \rho_i$ and $\mathrm{Ad}_{v_j}\, \sigma = \sigma_j$ for some $i, j \in \{0, \ldots, n\}$ and $u_i \in \mathcal{U}(\mathcal{N})$, $v_j \in \mathcal{U}(\mathcal{N}^c)$. Naturality of the braiding gives

$$\mathcal{E}_{\rho,\sigma} = \sigma(u_i^*) v_j^* \mathcal{E}_{\rho_i,\sigma_j} u_i \rho(v_j)$$

hence $\mathcal{E}_{\rho,\sigma} = \sigma(u_i^*) v_j^* u_i \rho(v_j) = \mathbb{1}$ because, e.g., $u_i \rho(v_j) = u_i v_j = v_j u_i$. Hence we have shown (iv) in the irreducible case.

In the reducible case, we can write direct sums $\rho = \sum_a s_a \rho_a s_a^*$ and $\sigma = \sum_b t_b \sigma_b t_b^*$ where $a, b \in \{0, \ldots, n\}$ and $\rho_a \in \mathcal{C}_\mathcal{N}$, $\sigma_b \in \mathcal{C}_{\mathcal{N}^c}$ run in our choice of representatives and $\{s_a\}_a$, $\{t_b\}_b$ are Cuntz algebras of isometries respectively in $\mathcal{N}$, $\mathcal{N}^c$, again by Proposition 4.5.6. As before

$$\mathcal{E}_{\rho,\sigma} = \sum_{a,b} \sigma(s_a) t_b \mathcal{E}_{\rho_a,\sigma_b} s_a^* \rho(t_b^*) = \sum_{a,b} s_a s_a^* t_b t_b^* = \mathbb{1}$$

so we conclude (iv) for all $\rho \in \mathcal{C}_\mathcal{N}$, $\sigma \in \mathcal{C}_{\mathcal{N}^c}$. $\qquad\square$

Remark 4.5.12. Thinking in terms of DHR localization properties of the endomorphisms, if we have $\rho \in \mathcal{C}_\mathcal{N}$, $[\rho] \neq [\mathrm{id}]$, the previous statement says that it cannot be localizable in some interval I_ρ which is to the right of some localization intervals I_j of $\sigma_j \in \mathcal{C}_{\mathcal{N}^c}$ as above, for all $j = 0, \ldots, n$, for every choice of such $\sigma_j \in \mathcal{C}_{\mathcal{N}^c}$. This would imply degeneracy of $[\rho]$, hence contradict modularity of $\mathrm{DHR}\{\mathcal{A}\}$. Despite this naive left/right separation picture, and the results of the last section, we shall see next how abstract points can become wildly non-geometric or "fuzzy". This is a typical situation in QFT where points of spacetime are replaced by (field) operators.

4.6   Fuzzy abstract points

Let $\{\mathcal{A}\}$ be a completely rational conformal net on the line, let $I_0 \in \mathcal{I}$, $\mathcal{M}_0 = \mathcal{A}(I_0)$ and $\mathcal{C} = \mathrm{DHR}^{I_0}\{\mathcal{A}\}$. Inside $\mathcal{M}_0$ we can find honest points (those associated to geometric points $p \in I_0$, see Remark 4.5.4), but also uncountably many families of abstract points which are *fuzzy*, in the sense that they are not honest anymore (with respect to $\{\mathcal{A}\}$) and do not resemble any kind of geometric interpretation. The following examples give algebraic deformations of abstract points into abstract points, and of honest points into possibly fuzzy ones.

Example 4.6.1. Let $p = (\mathcal{A}(I_1), \mathcal{A}(I_2))$ be an honest point of $\mathcal{M}_0$ and consider localizable unitaries $u \in \mathcal{U}(\mathcal{M}_0)$. Then $upu^* := (\mathrm{Ad}_u(\mathcal{A}(I_1)), \mathrm{Ad}_u(\mathcal{A}(I_2)))$ is an abstract point of $\mathcal{M}_0$, see Definition 4.5.1. Indeed conditions (i) and (ii) follow because $\mathrm{Ad}_u : \mathcal{M}_0 \to \mathcal{M}_0$ is a normal automorphism, in particular $\mathrm{Ad}_u(\mathcal{A}(I_1)^c) = \mathrm{Ad}_u(\mathcal{A}(I_1))^c$. Now if $\rho \in \mathcal{C}_{\mathcal{A}(I_1)}$ then ${}^u\rho := \mathrm{Ad}_u \circ \rho \circ \mathrm{Ad}_{u^*}$ is again in $\mathcal{C}$ because $\mathrm{Ad}_u \circ \rho \circ \mathrm{Ad}_{u^*} = u\rho(u^*)\rho(\cdot)\rho(u)u^*$ and $u\rho(u^*) \in \mathcal{U}(\mathcal{M}_0)$. Moreover it acts trivially on $\mathrm{Ad}_u(\mathcal{A}(I_1))^c$ hence $\rho \mapsto {}^u\rho$ defines a bijection between the objects of $\mathcal{C}_{\mathcal{A}(I_1)}$ and $\mathcal{C}_{\mathrm{Ad}_u(\mathcal{A}(I_1))}$, and (iii) follows. One easily checks that $\rho \mapsto {}^u\rho$ respects the tensor structure of $\mathcal{C}$, where the action on arrows $s \in \mathrm{Hom}_{\mathcal{C}}(\rho, \sigma)$, $\rho, \sigma \in \mathcal{C}$ is given by ${}^us := \mathrm{Ad}_u(s)$. Condition (iv) is also fulfilled because $\rho \mapsto {}^u\rho$ respects the braiding of $\mathcal{C}$, namely

$$\mathcal{E}_{{}^u\rho, {}^u\sigma} = u\sigma(u^*)\sigma(u\rho(u^*))\mathcal{E}_{\rho,\sigma}\rho(\sigma(u)u^*)\rho(u)u^* = {}^u\mathcal{E}_{\rho,\sigma}$$

by naturality, hence $\mathcal{E}_{\rho,\sigma} = \mathbb{1}$ if and only if $\mathcal{E}_{{}^u\rho, {}^u\sigma} = \mathbb{1}$. In other words $u \in \mathcal{U}(\mathcal{M}_0)$, $\rho \mapsto {}^u\rho$ gives rise to a group of UBTC autoequivalences of $\mathcal{C}$ which are also strict tensor and automorphic.

It can happen that $upu^* = p$, e.g., if u is localizable away from the cut geometric point $p \in I_0$. Otherwise u and p need not "commute" and upu^* can be viewed as a "fat" point of $\mathcal{M}_0$.

Example 4.6.2. Let $p = (\mathcal{A}(I_1), \mathcal{A}(I_2))$ as in the previous example and consider the modular group of $\mathcal{M}_0$ with respect to any faithful normal state φ, e.g., the vacuum state $\omega(\cdot) = (\Omega| \cdot \Omega)$ of $\{\mathcal{A}\}$. Denote by Δ_φ and $\sigma_t^\varphi = \mathrm{Ad}_{\Delta_\varphi^{it}}$, $t \in \mathbb{R}$ respectively the modular operator and the modular group of $(\mathcal{M}_0, \varphi)$. Then $\Delta_\varphi^{it} p \Delta_\varphi^{-it}$ is an abstract point of $\mathcal{M}_0$, for every $t \in \mathbb{R}$. Indeed (i) and (ii) follow as before, while (iii) is guaranteed by the existence of localizable Connes cocycles $u_{\rho,t} \in \mathcal{U}(\mathcal{M}_0)$, as shown by [Lon97, Prop. 1.1], which fulfill the intertwining relation ${}^t\rho = \mathrm{Ad}_{u_{\rho,t}} \rho$ on $\mathcal{M}_0$ for ${}^t\rho := \sigma_t^\varphi \circ \rho \circ \sigma_{-t}^\varphi$. Hence ${}^t\rho$ is again DHR and $t \mapsto {}^t\rho$ gives a tensor autoequivalence of $\mathcal{C}$, defined on arrows as ${}^ts := \sigma_t^\varphi(s)$. Using more advanced technology we can show that $t \mapsto {}^t\rho$ respects the braiding of $\mathcal{C}$. Namely

$$\mathcal{E}_{{}^t\rho, {}^t\sigma} = u_{\sigma,t}\sigma(u_{\rho,t})\mathcal{E}_{\rho,\sigma}\rho(u_{\sigma,t}^*)u_{\rho,t}^* = u_{\sigma\rho,t}\mathcal{E}_{\rho,\sigma}u_{\rho\sigma,t}^* = \sigma_t^\varphi(\mathcal{E}_{\rho,\sigma}) = {}^t\mathcal{E}_{\rho,\sigma}$$

where the first equality follows by naturality of the braiding, the second and third by tensoriality and naturality of the Connes cocycles associated to the modular action of $\mathbb{R}$, see respectively [Lon97, Prop. 1.4, 1.3]. In particular, $\mathcal{E}_{\rho,\sigma} = \mathbb{1}$ if and only if $\mathcal{E}_{{}^t\rho, {}^t\sigma} = \mathbb{1}$, hence condition (iv) is satisfied. As before

$t \in \mathbb{R}$, $\rho \mapsto {}^t\rho$ gives rise to a group of UBTC autoequivalences of $\mathcal{C}$ which are again strict tensor and automorphic. The point $\Delta_\varphi^{it} p \Delta_\varphi^{-it}$ is not honest in general, but highly fuzzy.

In the special case of the vacuum state $\varphi = \omega$, the modular action is geometric and coincides with the dilations subgroup $t \mapsto \Lambda_{I_0}^t$ of Möb which preserve I_0 (Bisognano-Wichmann property [GL96, Prop. 1.1]), hence $\Delta_\omega^{it} p \Delta_\omega^{-it} = \Lambda_{I_0}^{-2\pi t}(p)$ is just a Möbius transformed honest point (with respect to $\{\mathcal{A}\}$).

In the terminology of [Tur10, App. 5] due to M. Müger, see also [Lon97, App. A], we have found that $\mathcal{U}(\mathcal{M}_0)$ (and all of its subgroups) and $\mathbb{R}$ (for every choice of faithful normal state on $\mathcal{M}_0$) *act* on $\mathcal{C}$ (as UBTC strict automorphisms), and the actions are strict. One can then define the category of "G-fixed points", $\mathcal{C}^G$, where G denotes one of these groups with the associated action. In our case $\mathcal{C}^G = \mathcal{C}$ because all the objects ρ of $\mathcal{C}$ are "G-equivariant", i.e., admit a *cocycle* for the G-action, i.e., unitary isomorphisms $v_{\rho,g} : \rho \to {}^g\rho$, $g \in G$, such that $v_{\rho,gh} = {}^g(v_{\rho,h}) \circ v_{\rho,g}$. In Example 4.6.1 the cocycle identity follows because ρ are *-homomorphisms, in Example 4.6.2 it coincides with the characterization of the Connes cocycles.

In our case these actions are also implemented by unitaries $U_g \in \mathcal{U}(\mathcal{H})$, hence we have examples of (groups of) automorphisms of the braided action $\mathcal{C} \hookrightarrow \mathrm{End}(\mathcal{M}_0)$ in the sense of Definition 4.2.4.

4.7 Prime UMTCs and prime conformal nets

There are other types of abstract points, living inside completely rational nets that *factorize* as tensor products, which are abstract but neither honest nor fuzzy, in the sense that they are almost geometric, or better, geometric in 1+1 dimensions. Ruling out these cases will lead us to the notion of *prime conformal nets*.

Example 4.7.1. Consider a completely rational conformal net on the line of the form $\{I \in \mathcal{I} \mapsto \mathcal{A}(I) = \mathcal{A}_1(I) \otimes \mathcal{A}_2(I)\} = \{\mathcal{A}_1 \otimes \mathcal{A}_2\}$, where $\{\mathcal{A}_1\}$, $\{\mathcal{A}_2\}$ are two nontrivial nets, then $\mathrm{DHR}\{\mathcal{A}\} \simeq \mathrm{DHR}\{\mathcal{A}_1\} \boxtimes \mathrm{DHR}\{\mathcal{A}_2\}$ as UBTCs. An equivalence is given by $\rho \boxtimes \sigma \mapsto \rho \otimes \sigma$, $T \boxtimes S \mapsto T \otimes S$ where essential surjectivity follows from [KLM01, Lem. 27] and the braiding on the l.h.s. is defined as $\varepsilon_{\rho \boxtimes \sigma, \tau \boxtimes \eta} = \varepsilon_{\rho,\tau}^{\mathcal{A}_1} \boxtimes \varepsilon_{\sigma,\eta}^{\mathcal{A}_2}$. We can consider as before a local algebra $\mathcal{M}_0 := \mathcal{A}_1(I_0) \otimes \mathcal{A}_2(I_0)$ for some interval $I_0 \in \mathcal{I}$, and take two honest

points $p_1 = (\mathcal{A}_1(I_1), \mathcal{A}_1(I_2))$ in $\mathcal{A}_1(I_0)$ and $p_2 = (\mathcal{A}_2(J_1), \mathcal{A}_2(J_2))$ in $\mathcal{A}_2(I_0)$ we respectively in the two components. Now setting $\mathcal{N} := \mathcal{A}_1(I_1) \otimes \mathcal{A}_2(J_1)$ we have that irreducibles in $\mathcal{C}_\mathcal{N}$ are given by $\mathrm{Ad}_u \, \rho \otimes \sigma$ for some $\rho \in \mathrm{DHR}^{I_1}\{\mathcal{A}_1\}$, $\sigma \in \mathrm{DHR}^{J_1}\{\mathcal{A}_2\}$ and $u \in \mathcal{U}(\mathcal{N})$. Moreover, the pair of algebras $q = (\mathcal{N}, \mathcal{N}^c)$ is an abstract point of $\mathcal{M}_0$, but *not* honest unless $I_1 = J_1$. In other words, $q = p_1 \otimes p_2$ is an honest point of $\mathcal{M}_0$ if and only if $p_1 = p_2$ as geometric points of I_0.

We recall the following definition due to [Müg03], see also [DMNO13].

Definition 4.7.2. A UMTC $\mathcal{C}$ is called a **prime UMTC** if $\mathcal{C} \not\simeq \mathrm{Vec}$ and every full unitary fusion subcategory $\mathcal{D} \subset \mathcal{C}$ which is again a UMTC is either $\mathcal{D} \simeq \mathcal{C}$ or $\mathcal{D} \simeq \mathrm{Vec}$ as UBTCs.

The terminology is motivated by the following proposition, which is among the deepest results on the structure of UMTCs. It establishes prime UMTCs as building blocks in the classification program of UMTCs, see [RSW09].

Proposition 4.7.3. [Müg03], [DGNO10]. *Let $\mathcal{C}$ be a UMTC, let $\mathcal{D} \subset \mathcal{C}$ be a unitary full fusion subcategory and consider the centralizer of $\mathcal{D}$ in $\mathcal{C}$* [3] *defined as the full subcategory of $\mathcal{C}$ with objects*

$$\mathcal{Z}_\mathcal{C}(\mathcal{D}) := \big\{ x \in \mathcal{C} : \mathcal{E}_{x,y} = \mathcal{E}_{x,y}^{\mathrm{op}}, \, y \in \mathcal{D} \big\}.$$

It holds

- $\mathcal{Z}_\mathcal{C}(\mathcal{D})$ *is a unitary (full) fusion subcategory of $\mathcal{C}$, which is also replete, and $\mathcal{Z}_\mathcal{C}(\mathcal{Z}_\mathcal{C}(\mathcal{D})) = \overline{\mathcal{D}}$ where $\overline{\mathcal{D}}$ denotes the repletion of $\mathcal{D}$ in $\mathcal{C}$.*

If $\mathcal{D}$ is in addition a UMTC, i.e., $\mathcal{Z}_\mathcal{D}(\mathcal{D}) \simeq \mathrm{Vec}$, then

- $\mathcal{Z}_\mathcal{C}(\mathcal{D})$ *is also a UMTC and $\mathcal{C} \simeq \mathcal{D} \boxtimes \mathcal{Z}_\mathcal{C}(\mathcal{D})$ as UBTCs.*

In particular, every UMTC admits a finite prime factorization, i.e.

$$\mathcal{C} \simeq \mathcal{D}_1 \boxtimes \ldots \boxtimes \mathcal{D}_n$$

as UBTCs, where $\mathcal{D}_i$, $i = 1, \ldots, n$ are prime UMTCs, fully realized in $\mathcal{C}$.

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[3] or braided relative commutant of $\mathcal{D} \subset \mathcal{C}$. Cf. the definition of relative commutant $\mathcal{D}^c$ we introduced in Section 4.3 for full inclusions of tensor categories. Cf. also the definition [HP15, Def. 2.9] of relative commutant in the sense of Drinfeld.

Remark 4.7.4. Observe that assuming DHR$\{\mathcal{A}\}$ to be prime as an abstract UMTC rules out holomorphic nets. Moreover the examples seen in 4.7.1 cannot arise, unless one of the two tensor factors is holomorphic, i.e., $\{\mathcal{A}\} = \{\mathcal{A}_1 \otimes \mathcal{A}_{\text{holo}}\}$. The following definition is aimed to rule out also this case.

Definition 4.7.5. Let $\{\mathcal{A}\}$ be a completely rational conformal net on the line. Fix arbitrarily $I_0 \in \mathcal{I}$ and let $\mathcal{M}_0 = \mathcal{A}(I_0)$, $\mathcal{C} = \text{DHR}^{I_0}\{\mathcal{A}\}$. We call $\{\mathcal{A}\}$ a **prime conformal net** if the following conditions are satisfied.

- $\mathcal{C} \simeq \text{DHR}\{\mathcal{A}\}$ is a prime UMTC.

- For every ordered pair $p = (\mathcal{N}, \mathcal{N}^c)$, $q = (\mathcal{M}, \mathcal{M}^c)$ of abstract points of $\mathcal{M}_0$, if $\mathcal{N} \vee \mathcal{M}^c$ is normal in $\mathcal{M}_0$ then $\mathcal{M} \subset \mathcal{N}$, in particular $\mathcal{N} \vee \mathcal{M}^c = \mathcal{M}_0$.

Remark 4.7.6. Notice that the primality assumption on $\mathcal{C} \simeq \text{DHR}\{\mathcal{A}\}$ is purely categorical, i.e., invariant under equivalence of UBTCs, hence contains no information about the actual size of the category. By definition of prime UMTCs, holomorphic nets are *not* prime conformal nets.

Remark 4.7.7. If p, q mutually fulfill, e.g., $\mathcal{R} = (\mathcal{R} \cap \mathcal{S}) \vee (\mathcal{R} \cap \mathcal{S}^c)$ for $\mathcal{R}, \mathcal{S} \in \{\mathcal{N}, \mathcal{N}^c, \mathcal{M}, \mathcal{M}^c\}$ (resembling strong additivity), then the statements $\mathcal{M} \subset \mathcal{N}$ and $\mathcal{N} \vee \mathcal{M}^c = \mathcal{M}_0$ are actually equivalent.

It is easy to see that *prime* conformal nets cannot factor through nontrivial holomorphic subnets.

Example 4.7.8. Let $\{\mathcal{A}\}$ be a prime conformal net on the line, hence not holomorphic, but factoring through a holomorphic subnet, $\{\mathcal{A}\} = \{\mathcal{A}_1 \otimes \mathcal{A}_{\text{holo}}\}$. Considering points $p_1 \otimes p_2$ of $\mathcal{M}_0$ like in Example 4.7.1, it is easy to construct $\mathcal{N} \vee \mathcal{M}^c$ which are normal in $\mathcal{M}_0$ but neither exhaust $\mathcal{M}_0$ nor have $\mathcal{M} \subset \mathcal{N}$, e.g., enlarging $\mathcal{M}$ in the holomorphic component. Then $\{\mathcal{A}\}$ cannot be prime unless $\{\mathcal{A}_{\text{holo}}\} = \{\mathbb{C}\}$.

Remark 4.7.9. Both the notion of primality for completely rational conformal nets and the property of not factorizing through holomorphic subnets are *invariant* under isomorphism of nets.

Concerning the converse of the implication seen in Example 4.7.8, let $\{\mathcal{A}\}$ be a completely rational net, not necessarily prime, take p, q as in Definition 4.7.5. The idea is that $(\mathcal{N} \vee \mathcal{M}^c)^c = \mathcal{N}^c \cap \mathcal{M}$ are abstract "interval algebras" which lie in the "holomorphic part" of the net whenever $\mathcal{N} \vee \mathcal{M}^c$ is normal in $\mathcal{M}_0$. More precisely, we can show that they necessarily factor out in a

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tensor product subalgebra of $\mathcal{M}_0$, and that the local subcategories associated to them à la DHR are trivial, namely $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}} \subset \mathrm{Vec}$. [4]

Proposition 4.7.10. *Let $\{\mathcal{A}\}$ be a completely rational conformal net on the line, fix $I_0 \in \mathcal{I}$ and let $\mathcal{M}_0 = \mathcal{A}(I_0)$, $\mathcal{C} = \mathrm{DHR}^{I_0}\{\mathcal{A}\}$. Consider the family $\mathcal{F}$ of ordered pairs of abstract points $p = (\mathcal{N}, \mathcal{N}^c)$, $q = (\mathcal{M}, \mathcal{M}^c)$ such that $\mathcal{N} \vee \mathcal{M}^c$ is normal in $\mathcal{M}_0$, then the following holds.*

- *For every $(p, q) \in \mathcal{F}$ we have $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}} \subset \mathrm{Vec}$.*
- *Consider the subalgebra of $\mathcal{M}_0$ defined as*

$$\mathcal{M}_0^{\mathrm{holo}} := \bigvee_{(p,q)\in\mathcal{F}} \mathcal{N}^c \cap \mathcal{M}$$

then $\mathcal{M}_0^{\mathrm{holo}}$ is either $\mathbb{C}$ or a type III_1 subfactor of $\mathcal{M}_0$, and the same holds for the relative commutant

$$(\mathcal{M}_0^{\mathrm{holo}})^c = \bigcap_{(p,q)\in\mathcal{F}} \mathcal{N} \vee \mathcal{M}^c.$$

Moreover we have a splitting

$$\mathcal{M}_0^{\mathrm{holo}} \vee (\mathcal{M}_0^{\mathrm{holo}})^c \cong \mathcal{M}_0^{\mathrm{holo}} \otimes (\mathcal{M}_0^{\mathrm{holo}})^c$$

as von Neumann algebras.

Proof. Normality of $\mathcal{N} \vee \mathcal{M}^c$ in $\mathcal{M}_0$ means $\mathcal{N} \vee \mathcal{M}^c = (\mathcal{N} \vee \mathcal{M}^c)^{cc}$, equivalently $(\mathcal{N}^c \cap \mathcal{M})^c = \mathcal{N} \vee \mathcal{M}^c$, but there is a more useful characterization. Without assuming normality, let $\rho \in \mathcal{C}_{\mathcal{N}}$, $\tilde{\rho} \in \mathcal{C}_{\mathcal{M}^c}$ and u a unitary charge transporter from ρ to $\tilde{\rho}$. For every $a \in \mathcal{N}^c \cap \mathcal{M}$ we have $ua = u\rho(a) = \tilde{\rho}(a)u = au$ hence $u \in (\mathcal{N}^c \cap \mathcal{M})^c = (\mathcal{N} \vee \mathcal{M}^c)^{cc}$. Denoting by

$$\mathcal{U}_{\mathcal{C}}(\mathcal{N}, \mathcal{M}^c) := \mathrm{vN}\{u \in \mathrm{Hom}_{\mathcal{C}}(\rho, \tilde{\rho}) \cap \mathcal{U}(\mathcal{M}_0), \ \rho \in \mathcal{C}_{\mathcal{N}}, \ \tilde{\rho} \in \mathcal{C}_{\mathcal{M}^c}\}$$

the von Neumann algebra generated by the charge transporters, we have

$$\mathcal{N} \vee \mathcal{M}^c \subset \mathcal{U}_{\mathcal{C}}(\mathcal{N}, \mathcal{M}^c) \subset (\mathcal{N} \vee \mathcal{M}^c)^{cc} \tag{4.18}$$

where the first inclusion holds because the unitaries in $\mathcal{U}(\mathcal{N})$ and $\mathcal{U}(\mathcal{M}^c)$ generate inner automorphisms from the vacuum. Normality of $\mathcal{N} \vee \mathcal{M}^c$ in

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[4]We identify Vec with the full subcategory of $\mathcal{C}$ whose objects are the *inner* endomorphisms, cf. Proposition 4.3.5.

$\mathcal{M}_0$ turns out to be *equivalent* to $\mathcal{U}_\mathcal{C}(\mathcal{N}, \mathcal{M}^c) = \mathcal{U}_\mathcal{C}(\mathcal{N}, \mathcal{M}^c)^{cc} = \mathcal{N} \vee \mathcal{M}^c$. Using this we can show that $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M} \subset \text{Vec}$. Let $\rho \in \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M}$ and observe that $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M} = \mathcal{N}^\perp \cap \mathcal{M}^{c\perp} = (\mathcal{N} \vee \mathcal{M}^c)^\perp$ because endomorphisms in $\mathcal{C}$ are normal. Now by normality of $\mathcal{N} \vee \mathcal{M}^c$ in $\mathcal{M}_0$ we have that $\rho \in \mathcal{U}_\mathcal{C}(\mathcal{N}, \mathcal{M}^c)^\perp$, i.e., $\rho(u) = u$ for every unitary generator $u \in \mathcal{U}_\mathcal{C}(\mathcal{N}, \mathcal{M}^c)$. On the other hand for every $\sigma \in \mathcal{C}_\mathcal{N}$ and $\tilde{\sigma} := \text{Ad}_u \sigma \in \mathcal{C}_{\mathcal{M}^c}$ we have $\mathcal{E}_{\rho,\tilde{\sigma}} = \mathbb{1}$ by assumption (iv), i.e., $\rho(u) = u\mathcal{E}_{\rho,\sigma}$ by naturality of the braiding, hence $\mathcal{E}_{\rho,\sigma} = \mathbb{1}$. Again by (iv) we have $\mathcal{E}_{\sigma,\rho} = \mathbb{1}$ and by (iii) $\mathcal{C}_\mathcal{N} \simeq \mathcal{C}$ from which we can conclude that ρ has vanishing monodromy with every sector, hence $\rho \in \text{Vec}$ by modularity of $\mathcal{C}$, showing the first statement.

The second statement follows using modular theory on abstract points of $\mathcal{M}_0$, see Example 4.6.2, [Reh00, Prop. 2.8]. Let $\sigma_t^\omega := \text{Ad}_{\Delta_\omega^{it}}$, $t \in \mathbb{R}$ be the modular group of $\mathcal{M}_0$ associated to the vacuum state ω of the net, we know that if p is an abstract point of $\mathcal{M}_0$ then $\sigma_t^\omega(p)$, $t \in \mathbb{R}$ are also abstract points. Furthermore $t \mapsto \sigma_t^\omega$ respects $\mathcal{M}_0$ and the normality property for subalgebras of $\mathcal{M}_0$, hence maps $\mathcal{F}$ onto $\mathcal{F}$ because $(\sigma_t^\omega)^{-1} = \sigma_{-t}^\omega$ and we conclude $\sigma_t^\omega(\mathcal{M}_0^{\text{holo}}) = \mathcal{M}_0^{\text{holo}}$, $t \in \mathbb{R}$. By Takesaki's theorem [Tak72] we have a faithful normal conditional expectation $E : \mathcal{M}_0 \to \mathcal{M}_0^{\text{holo}}$ intertwining $E \circ \sigma_t^\omega = \sigma_t^\varphi \circ E$, $t \in \mathbb{R}$, where φ is the faithful normal state obtained by restricting ω to $\mathcal{M}_0^{\text{holo}}$ and σ_t^φ is the associated modular group, see [Str81, Sec. 10]. Now the vacuum state ω is given by the unique vector invariant under the group of I_0-preserving dilations by [GL96, Cor. B.2]. This, together with the Bisognano-Wichmann property [GL96, Prop. 1.1], imply that $t \mapsto \sigma_t^\omega$ is ergodic on $\mathcal{M}_0$, hence $t \mapsto \sigma_t^\varphi$ is ergodic on $\mathcal{M}_0^{\text{holo}}$. In other words, φ has trivial centralizer, then by [Lon08b, Prop. 6.6.5] $\mathcal{M}_0^{\text{holo}}$ is a *factor* of type III_1 or trivial $\mathcal{M}_0^{\text{holo}} = \mathbb{C}$. The same holds for $(\mathcal{M}_0^{\text{holo}})^c$. In particular, $\mathcal{M}_0^{\text{holo}}$ being a subfactor of $\mathcal{M}_0$, we can apply [Tak72, Cor. 1] to get the splitting of $\mathcal{M}_0^{\text{holo}} \vee (\mathcal{M}_0^{\text{holo}})^c$ as von Neumann tensor product, completing the proof of the second statement. $\square$

4.8   Comparability of abstract points

In the previous sections we analysed the braiding condition (iv) in Definition 4.5.1: $\mathcal{E}_{\rho,\sigma} = \mathbb{1}$ on honest and abstract points of a net $\{\mathcal{A}\}$, see Eq. (4.11), Lemma 4.5.10, Proposition 4.5.11, and showed how it can be led far away from geometry in Section 4.6.

In this section we draw some of its consequences, as in the proof Proposition

4.7.10, and to do so we introduce *comparability* $p \sim q$ of abstract points, along with an order relation $p < q$ compatible with the geometric ordering of honest points. The terminology is motivated by the fact that two abstract points $p \sim q$ in a *prime* conformal net are necessarily $p < q$ or $q < p$ or $p = q$, see Proposition 4.8.5. The order symbols should be intended as inclusions of relative complement algebras of p, q in $\mathcal{M}_0$.

Let $p = (\mathcal{N}, \mathcal{N}^c)$, $q = (\mathcal{M}, \mathcal{M}^c)$ be two abstract points of $\mathcal{M}_0$ as in Definition 4.5.1 and $(\mathcal{R}, \mathcal{S})$ be any pair of elements from $\{\mathcal{N}, \mathcal{N}^c, \mathcal{M}, \mathcal{M}^c\}$. Similarly to Eq. (4.18) we have that the von Neumann algebras of unitary charge transporters

$$\mathcal{U}_\mathcal{C}(\mathcal{R}, \mathcal{S}) := \mathrm{vN}\{u \in \mathrm{Hom}_\mathcal{C}(\rho, \tilde{\rho}) \cap \mathcal{U}(\mathcal{M}_0), \ \rho \in \mathcal{C}_\mathcal{R}, \ \tilde{\rho} \in \mathcal{C}_\mathcal{S}\} \qquad (4.19)$$

always sit in between

$$\mathcal{R} \vee \mathcal{S} \subset \mathcal{U}_\mathcal{C}(\mathcal{R}, \mathcal{S}) \subset (\mathcal{R} \vee \mathcal{S})^{cc},$$

in particular $\mathcal{U}_\mathcal{C}(\mathcal{R}, \mathcal{S})^{cc} = (\mathcal{R} \vee \mathcal{S})^{cc}$. Hence asking *normality* of (4.19) in $\mathcal{M}_0$ is equivalent to asking that charge transporters *generate* as von Neumann algebras the relative commutants, cf. [Müg99, Cor. 4.3], [KLM01, Thm. 33], i.e., $\mathcal{U}_\mathcal{C}(\mathcal{R}, \mathcal{S}) = (\mathcal{R} \vee \mathcal{S})^{cc} = (\mathcal{R}^c \cap \mathcal{S}^c)^c$.

Notice that, e.g., $\mathcal{U}_\mathcal{C}(\mathcal{N}, \mathcal{N})$ and $\mathcal{U}_\mathcal{C}(\mathcal{N}, \mathcal{N}^c)$ are always normal in $\mathcal{M}_0$ by (ii) and that $\mathcal{U}_\mathcal{C}(\mathcal{R}, \mathcal{S}) = \mathcal{U}_\mathcal{C}(\mathcal{S}, \mathcal{R})$ by definition.

Lemma 4.8.1. *In the above notation, assume that $\mathcal{U}_\mathcal{C}(\mathcal{R}, \mathcal{S})$ is normal in $\mathcal{M}_0$ for every pair $(\mathcal{R}, \mathcal{S})$ of elements in $\{\mathcal{N}, \mathcal{N}^c, \mathcal{M}, \mathcal{M}^c\}$, then*

- *$\mathcal{C}_{\mathcal{N} \cap \mathcal{M}} = \mathcal{C}_\mathcal{N} \cap \mathcal{C}_\mathcal{M}$ and $\mathcal{C}_{\mathcal{N}^c \cap \mathcal{M}^c} = \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}^c}$.*

- *$\mathcal{C}_{\mathcal{N} \cap \mathcal{M}^c} \subset \mathcal{C}_\mathcal{N} \cap \mathcal{C}_{\mathcal{M}^c}$ and $\rho \in \mathcal{C}_{\mathcal{N} \cap \mathcal{M}^c}$ if and only if ρ is an inner endomorphism of $\mathcal{C}$; in symbols: $\mathcal{C}_{\mathcal{N} \cap \mathcal{M}^c} = (\mathcal{C}_\mathcal{N} \cap \mathcal{C}_{\mathcal{M}^c}) \cap \mathrm{Vec}$. Similarly $\mathcal{C}_{\mathcal{M} \cap \mathcal{N}^c} = (\mathcal{C}_\mathcal{M} \cap \mathcal{C}_{\mathcal{N}^c}) \cap \mathrm{Vec}$.*

Proof. Consider the intersection of left-left relative complements $\mathcal{C}_\mathcal{N} \cap \mathcal{C}_\mathcal{M}$. The inclusion $\mathcal{C}_{\mathcal{N} \cap \mathcal{M}} \subset \mathcal{C}_\mathcal{N} \cap \mathcal{C}_\mathcal{M}$ reads $(\mathcal{N} \cap \mathcal{M})^{c\perp} \subset \mathcal{N}^{c\perp} \cap \mathcal{M}^{c\perp} = (\mathcal{N}^c \vee \mathcal{M}^c)^\perp$ hence follows easily by taking duals of $\mathcal{N}^c \vee \mathcal{M}^c \subset (\mathcal{N}^c \vee \mathcal{M}^c)^{cc} = (\mathcal{N} \cap \mathcal{M})^c$. The opposite inclusion follows from the braiding condition and normality assumption on charge transporters. Take $\rho \in \mathcal{C}_\mathcal{N} \cap \mathcal{C}_\mathcal{M}$ then by (iv) we have $\varepsilon_{\rho, \tilde{\sigma}} = \mathbb{1}$ for every $\tilde{\sigma} := \mathrm{Ad}_u \sigma \in \mathcal{C}_{\mathcal{M}^c}$ where $\sigma \in \mathcal{C}_{\mathcal{N}^c}$ and u is a unitary generator of $\mathcal{U}_\mathcal{C}(\mathcal{N}^c, \mathcal{M}^c)$. Hence $\rho(u) = u \varepsilon_{\rho, \sigma}$ by naturality of the braiding.

But also $\mathcal{E}_{\rho,\sigma} = \mathbb{1}$ by assumption (iv) and $\rho \in \mathcal{U}_\mathcal{C}(\mathcal{N}^c, \mathcal{M}^c)^\perp = (\mathcal{N} \cap \mathcal{M})^{c\perp}$ follows, hence we have the first statement. The right-right case follows similarly.

In the left-right case the inclusion $\mathcal{C}_{\mathcal{N} \cap \mathcal{M}^c} \subset \mathcal{C}_\mathcal{N} \cap \mathcal{C}_{\mathcal{M}^c}$ can be proper, as shown by Proposition 4.3.5 in the honest case. Take $\rho \in \mathcal{C}_\mathcal{N} \cap \mathcal{C}_{\mathcal{M}^c}$, by normality $\rho \in \mathcal{C}_{\mathcal{N} \cap \mathcal{M}^c}$ if and only if $\rho(u) = u$ for every unitary generator $u \in \mathcal{U}_\mathcal{C}(\mathcal{N}^c, \mathcal{M})$. But now by (iv) we have $\mathcal{E}_{\tilde{\sigma},\rho} = \mathbb{1}$ for every $\tilde{\sigma} := \mathrm{Ad}_u\, \sigma \in \mathcal{C}_\mathcal{M}$ where $\sigma \in \mathcal{C}_{\mathcal{N}^c}$, $u \in \mathcal{U}_\mathcal{C}(\mathcal{N}^c, \mathcal{M})$, hence $\rho(u) = u\mathcal{E}_{\sigma,\rho}^*$ together with $\mathcal{E}_{\rho,\sigma} = \mathbb{1}$. By assumption (iii) $\mathcal{C}_{\mathcal{N}^c} \simeq \mathcal{C}$ and modularity of $\mathcal{C}$, we can conclude that $\rho \in \mathcal{C}_{\mathcal{N} \cap \mathcal{M}^c}$ if and only if $\rho \in \mathrm{Vec}$, and the proof is complete. $\qquad\square$

As already remarked, given a pair of abstract points $p = (\mathcal{N}, \mathcal{N}^c)$, $q = (\mathcal{M}, \mathcal{M}^c)$ of $\mathcal{M}_0$, the algebras $\mathcal{N} \cap \mathcal{M}^c$ can be viewed as abstract "interval algebras" of $\mathcal{M}_0$ with associated "local" DHR subcategories $\mathcal{C}_\mathcal{N} \cap \mathcal{C}_{\mathcal{M}^c}$.

Denote by $\Delta(\mathcal{C})$ the *spectrum* of $\mathcal{C}$ and let $\mathcal{U}_{\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M}}(\mathcal{N}, \mathcal{M}^c) \subset \mathcal{U}_\mathcal{C}(\mathcal{N}, \mathcal{M}^c)$ be the subalgebra generated by ρ-charge transporters associated to sectors $[\rho] \in \Delta(\mathcal{C}_\mathcal{N} \cap \mathcal{C}_{\mathcal{M}^c})$. The vacuum [id] is always in the spectrum, hence $\mathcal{U}_{\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M}}(\mathcal{N}, \mathcal{M}^c)$ is also intermediate in $\mathcal{N} \vee \mathcal{M}^c \subset (\mathcal{N} \vee \mathcal{M}^c)^{cc}$.

Lemma 4.8.2. *In the above notation, assume that $\mathcal{U}_{\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M}}(\mathcal{N}, \mathcal{M}^c)$ and $\mathcal{U}_{\mathcal{C}_{\mathcal{M}^c} \cap \mathcal{C}_\mathcal{N}}(\mathcal{M}, \mathcal{N}^c)$ are normal in $\mathcal{M}_0$, then $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M}$ and $\mathcal{C}_{\mathcal{M}^c} \cap \mathcal{C}_\mathcal{N}$ have "modular spectrum", i.e.*

$$\mathcal{Z}_{\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M}}(\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M}) \subset \mathrm{Vec}, \quad \mathcal{Z}_{\mathcal{C}_{\mathcal{M}^c} \cap \mathcal{C}_\mathcal{N}}(\mathcal{C}_{\mathcal{M}^c} \cap \mathcal{C}_\mathcal{N}) \subset \mathrm{Vec}\,.$$

Proof. Let $\rho \in \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M}$ such that $\mathcal{E}_{\rho,\sigma} = \mathcal{E}_{\rho,\sigma}^{\mathrm{op}}$ for all $\sigma \in \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M}$. Inspired by [Müg99, Lem. 3.2] we can write $\mathcal{E}_{\rho,\sigma} = u^*\rho(u)$ and $\mathcal{E}_{\rho,\sigma}^{\mathrm{op}} = x^*\rho(x)$ where u and x are unitaries transporting σ respectively to $\mathcal{C}_{\mathcal{M}^c}$ and $\mathcal{C}_\mathcal{N}$, see Lemma 4.5.10. Hence triviality of the monodromy $\mathcal{E}_{\rho,\sigma} = \mathcal{E}_{\rho,\sigma}^{\mathrm{op}}$ is triviality of the action $\rho(ux^*) = ux^*$. Moreover every generator w of $\mathcal{U}_{\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M}}(\mathcal{N}, \mathcal{M}^c)$ can be written as $w = ux^*$ with u and x as above. By normality $\mathcal{U}_{\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M}}(\mathcal{N}, \mathcal{M}^c) = (\mathcal{N} \vee \mathcal{M}^c)^{cc}$ hence, reversing the argument, one can drop the restriction $\sigma \in \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M}$ and get $\mathcal{E}_{\rho,\sigma} = \mathcal{E}_{\rho,\sigma}^{\mathrm{op}}$ for all $\sigma \in \mathcal{C}$. By modularity of $\mathcal{C}$ we get $\rho \in \mathrm{Vec}$. Analogously interchanging $\mathcal{N}$ and $\mathcal{M}$. $\qquad\square$

Normality of $\mathcal{U}_{\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M}}(\mathcal{N}, \mathcal{M}^c)$ obviously implies normality of $\mathcal{U}_\mathcal{C}(\mathcal{N}, \mathcal{M}^c)$. We are now ready to introduce the notion of comparability of two abstract points p, q mentioned in the beginning of this section.

Definition 4.8.3. Let $\{\mathcal{A}\}$ be a completely rational conformal net on the line. In the notation of Definition 4.5.1, two abstract points $p = (\mathcal{N}, \mathcal{N}^c)$, $q = (\mathcal{M}, \mathcal{M}^c)$ of $\mathcal{M}_0$ are called **comparable** if they fulfill the following

- $\mathcal{U}_{\mathcal{C}_{\mathcal{R}^c} \cap \mathcal{C}_{\mathcal{S}^c}}(\mathcal{R}, \mathcal{S}) = \mathcal{U}_{\mathcal{C}_{\mathcal{R}^c} \cap \mathcal{C}_{\mathcal{S}^c}}(\mathcal{R}, \mathcal{S})^{cc}$.
- $\mathcal{R} \vee \mathcal{S} = (\mathcal{R} \vee \mathcal{S})^{\perp\perp}$.

for every pair $(\mathcal{R}, \mathcal{S})$ in $\{\mathcal{N}, \mathcal{N}^c, \mathcal{M}, \mathcal{M}^c\}$. In this case, we write $p \sim q$.

Observe that $\mathcal{U}_{\mathcal{C}_{\mathcal{R}^c} \cap \mathcal{C}_{\mathcal{S}^c}}(\mathcal{R}, \mathcal{S})$ and $(\mathcal{C}_{\mathcal{R}^c} \cap \mathcal{C}_{\mathcal{S}^c})^\perp = (\mathcal{R} \vee \mathcal{S})^{\perp\perp}$ are both intermediate algebras in the inclusions $\mathcal{R} \vee \mathcal{S} \subset (\mathcal{R} \vee \mathcal{S})^{cc}$. Hence comparability means that these bounds are maximally, respectively minimally, saturated.

Remark 4.8.4. We have already motivated the normality condition on charge transporters. Concerning biduality, it easily holds for left or right local half-line algebras, see Proposition 4.3.3, Remark 4.3.10, and for two-interval algebras, as we have shown in Proposition 4.3.7. Notice also that comparability is manifestly reflexive, symmetric and invariant under isomorphism of nets (but not manifestly transitive).

Proposition 4.8.5. *Let $\{\mathcal{A}\}$ be a prime conformal net on the line (Definition 4.7.5) and take two abstract points $p = (\mathcal{N}, \mathcal{N}^c)$, $q = (\mathcal{M}, \mathcal{M}^c)$ of $\mathcal{M}_0$. If $p \sim q$ then either $p < q$ or $q < p$ or $p = q$, i.e., respectively $\mathcal{N} \subset \mathcal{M}$ or $\mathcal{M} \subset \mathcal{N}$ or $\mathcal{N} = \mathcal{M}$.*

In particular, in the case of a prime conformal net, comparability of p and q can be checked on the two pairs $(\mathcal{N}, \mathcal{M}^c)$, $(\mathcal{M}, \mathcal{N}^c)$.

Proof. The idea of the proof is that $\mathcal{N}^c \cap \mathcal{M}$ and $\mathcal{M}^c \cap \mathcal{N}$ are, a priori, abstract interval algebras of two *different* tensor factors of the net. Call for short $\mathcal{C}_1 := \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}}$ and $\mathcal{C}_2 := \mathcal{C}_{\mathcal{M}^c} \cap \mathcal{C}_{\mathcal{N}}$ and observe that

$$\mathcal{C}_1 \subset \mathcal{Z}_{\mathcal{C}}(\mathcal{C}_2), \quad \mathcal{C}_2 \subset \mathcal{Z}_{\mathcal{C}}(\mathcal{C}_1) \tag{4.20}$$

because for every $\rho \in \mathcal{C}_1$, $\sigma \in \mathcal{C}_2$ we have $\mathcal{E}_{\rho, \sigma} = \mathbb{1} = $ and $\mathcal{E}_{\sigma, \rho} = \mathbb{1}$ by condition (iv), in particular $\mathcal{E}_{\sigma, \rho} \mathcal{E}_{\rho, \sigma} = \mathbb{1}$. We also have

$$\mathcal{Z}_{\mathcal{C}_1}(\mathcal{C}_1) \subset \mathrm{Vec}, \quad \mathcal{Z}_{\mathcal{C}_2}(\mathcal{C}_2) \subset \mathrm{Vec} \tag{4.21}$$

by Lemma 4.8.2. Notice that it can be $\mathcal{C}_1 = \mathcal{C}_2 = \{\mathrm{id}\}$, e.g., if $\mathcal{N} = \mathcal{M}$. In order to invoke primality of the DHR category $\mathcal{C}$ as a UMTC, we take the closures of $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{C}$ under conjugates, subobjects, finite direct sums,

tensor products and unitary isomorphism classes. Denote them respectively by $\tilde{\mathcal{C}}_1$, $\tilde{\mathcal{C}}_2$. In other words, they are the smallest replete fusion subcategories of $\mathcal{C}$ containing $\mathcal{C}_1$, $\mathcal{C}_2$ respectively. Thanks to [Müg03, Thm. 3.2], see also [DGNO10, Thm. 3.10], they are characterized as double braided relative commutant subcategories of $\mathcal{C}$, i.e.

$$\tilde{\mathcal{C}}_1 = \mathcal{Z}_\mathcal{C}(\mathcal{Z}_\mathcal{C}(\tilde{\mathcal{C}}_1)), \quad \tilde{\mathcal{C}}_2 = \mathcal{Z}_\mathcal{C}(\mathcal{Z}_\mathcal{C}(\tilde{\mathcal{C}}_2)).$$

Now inclusions (4.20) and (4.21) clearly extend to subobjects, direct sums, tensor products and unitary isomorphism classes, because the vanishing of the monodromy is a condition stable under such operations, see [Müg00, Sec. 2.2], and Vec is a replete fusion subcategory of $\mathcal{C}$. We need to check that (4.20) and (4.21) extend to conjugates because neither of the two sides of (4.20) nor the l.h.s. of (4.21) are a priori rigid. Let $\rho \in \mathcal{C}_1$, $\sigma \in \mathcal{C}_2$ and choose a conjugate $\overline{\rho} \in \mathcal{C}$ of ρ, we want to show that $\mathcal{E}_{\sigma,\overline{\rho}}\mathcal{E}_{\overline{\rho},\sigma} = \mathbb{1}$. By condition (iii) we can assume $\overline{\rho} \in \mathcal{C}_{\mathcal{N}^c}$ up to unitary isomorphism, equivalently we could have assumed $\overline{\rho} \in \mathcal{C}_\mathcal{M}$. By Proposition 4.5.6 we have that every solution of the conjugate equations $R \in \mathrm{Hom}_\mathcal{C}(\mathrm{id}, \overline{\rho}\rho)$, $\overline{R} \in \mathrm{Hom}_\mathcal{C}(\mathrm{id}, \rho\overline{\rho})$ for ρ, $\overline{\rho}$, see [LR97, Sec. 2], lies in $\mathcal{N}^c$, in particular $\sigma(R) = R$, $\sigma(\overline{R}) = \overline{R}$. Hence we get $\mathcal{E}_{\overline{\rho},\sigma} = R^* \overline{\rho}(\mathcal{E}^*_{\rho,\sigma})\overline{\rho}\sigma(\overline{R}) = R^*\overline{\rho}(\overline{R}) = \mathbb{1}$ and similarly $\mathcal{E}_{\sigma,\overline{\rho}} = \overline{\rho}\sigma(\overline{R}^*)\overline{\rho}(\mathcal{E}^*_{\sigma,\rho})R = \overline{\rho}(\overline{R}^*)R = \mathbb{1}$. In particular, $\overline{\rho}$ and σ have vanishing monodromy.

Summing up we have $\tilde{\mathcal{C}}_1 \subset \mathcal{Z}_\mathcal{C}(\mathcal{C}_2)$ and similarly $\tilde{\mathcal{C}}_2 \subset \mathcal{Z}_\mathcal{C}(\mathcal{C}_1)$. Moreover, given $\sigma \in \mathcal{C}_2$ choose a conjugate $\overline{\sigma} \in \mathcal{C}$ and observe that the vanishing of the monodromy of $\overline{\sigma}$ and every ρ in $\tilde{\mathcal{C}}_1$ is equivalent to the vanishing of the monodromy of σ and every ρ, by rigidity of $\tilde{\mathcal{C}}_1$, see [Müg00, Eq. (2.17)]. Hence we have

$$\tilde{\mathcal{C}}_1 \subset \mathcal{Z}_\mathcal{C}(\tilde{\mathcal{C}}_2), \quad \tilde{\mathcal{C}}_2 \subset \mathcal{Z}_\mathcal{C}(\tilde{\mathcal{C}}_1) \tag{4.22}$$

and the two inclusions are equivalent by the double braided relative commutant theorem. We can extend also inclusions (4.21) by observing that $\mathcal{Z}_{\mathcal{C}_1}(\tilde{\mathcal{C}}_1) \subset \mathcal{Z}_{\mathcal{C}_1}(\mathcal{C}_1) \subset \mathrm{Vec}$ and that, given $\rho \in \mathcal{C}_1$ and a conjugate $\overline{\rho} \in \mathcal{C}$, the vanishing of the monodromy of $\overline{\rho}$ and every σ in $\tilde{\mathcal{C}}_1$ is equivalent, as above, to the vanishing of the monodromy of ρ and every σ. Thus we have $\rho \in \mathrm{Vec}$, hence $\overline{\rho} \in \mathrm{Vec}$, and we conclude

$$\mathcal{Z}_{\tilde{\mathcal{C}}_1}(\tilde{\mathcal{C}}_1) = \mathrm{Vec}, \quad \mathcal{Z}_{\tilde{\mathcal{C}}_2}(\tilde{\mathcal{C}}_2) = \mathrm{Vec} \tag{4.23}$$

which means modularity for the replete fusion subcategories $\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2 \subset \mathcal{C}$. By primality of $\mathcal{C}$ as a UMTC, see Definition 4.7.2, the two subcategories are either $\mathcal{C}$ or Vec and by the inclusions (4.22) we can assume $\tilde{\mathcal{C}}_1 = \mathrm{Vec}$, up to exchanging the roles of $\mathcal{N}$ and $\mathcal{M}$.

In particular, we obtain $\mathcal{C}_1 = \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}} \subset \mathrm{Vec}$, hence

$$\mathcal{C}_{\mathcal{N}^c \cap \mathcal{M}} = \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}}$$

by Lemma 4.8.1, i.e., $(\mathcal{N}^c \cap \mathcal{M})^{c\perp} = (\mathcal{N} \vee \mathcal{M}^c)^\perp$. Now by comparability we have a biduality relation $(\mathcal{N} \vee \mathcal{M}^c)^{\perp\perp} = \mathcal{N} \vee \mathcal{M}^c$, while $(\mathcal{N}^c \cap \mathcal{M})^{c\perp\perp} = (\mathcal{N}^c \cap \mathcal{M})^c$ follows by the same argument as in Proposition 4.5.5. By taking duals we have that $\mathcal{N} \vee \mathcal{M}^c$ is normal in $\mathcal{M}_0$, hence $\mathcal{M} \subset \mathcal{N}$ by the primality assumption on the net. In particular, $\mathcal{C}_1 = \{\mathrm{id}\}$, and the proof is complete. $\square$

As said before, normality of $\mathcal{U}_{\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}}}(\mathcal{N}, \mathcal{M}^c)$ is equivalent to saying that the inclusion $\mathcal{N} \vee \mathcal{M}^c \subset (\mathcal{N} \vee \mathcal{M}^c)^{cc}$ is generated by charge transporters associated to sectors $[\rho] \in \Delta(\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}})$. We could strengthen this assumption by asking that the inclusion has the structure of a Longo-Rehren inclusion associated with $\{[\rho] \in \Delta(\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}})\}$. This amounts to specifying not only the generators of the extension, but also the algebraic relations among them [KLM01, Eq. (15), Prop. 45].

We show next that the latter can be derived, in our language of abstract points, from the *fusion* structure of the intersection categories. Notice however that we don't require, a priori, $\mathcal{N} \vee \mathcal{M}^c$ to split as a von Neumann tensor product, nor $\mathcal{N}$ and $\mathcal{M}^c$ to be commuting algebras.

Proposition 4.8.6. *Let $\{\mathcal{A}\}$ be a completely rational conformal net on the line and take two abstract points $p = (\mathcal{N}, \mathcal{N}^c)$, $q = (\mathcal{M}, \mathcal{M}^c)$, in the notation of Definition 4.5.1. If we assume that*

- *$\mathcal{U}_{\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}}}(\mathcal{N}, \mathcal{M}^c)$ and $\mathcal{U}_{\mathcal{C}_{\mathcal{M}^c} \cap \mathcal{C}_{\mathcal{N}}}(\mathcal{M}, \mathcal{N}^c)$ are normal in $\mathcal{M}_0$,*
- *$\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}}$ and $\mathcal{C}_{\mathcal{M}^c} \cap \mathcal{C}_{\mathcal{N}}$ are UFTCs in $\mathcal{C}$,*
- *$\mathcal{C}_{\mathcal{N}} \cap \mathcal{C}_{\mathcal{M}} \simeq \mathcal{C}$ and $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}^c} \simeq \mathcal{C}$*

then $\mathcal{N} \vee \mathcal{M}^c \subset (\mathcal{N} \vee \mathcal{M}^c)^{cc}$ and $\mathcal{M} \vee \mathcal{N}^c \subset (\mathcal{M} \vee \mathcal{N}^c)^{cc}$ have the structure of Longo-Rehren inclusions, in the sense that the generators of the extensions fulfill the relations [KLM01, Eq. (15)].

Proof. Consider the inclusion $\mathcal{N} \vee \mathcal{M}^c \subset (\mathcal{N} \vee \mathcal{M}^c)^{cc}$. Being $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}}$ a UFTC we can arrange its irreducible sectors $\{[\rho] \in \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}}\}$ in a *rational system* $\{[\rho_i]\}_i$, in the terminology of [KLM01], see also [Reh90b], [BEK99]. By assumption, for each $[\rho_i]$ we can choose $\overline{\rho}_i \in \mathcal{C}_{\mathcal{N}} \cap \mathcal{C}_{\mathcal{M}}$, $\rho_i \in \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}^c}$ and $R_i \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{id}, \overline{\rho}_i \rho_i)$ such that $R_i^* R_i = d_{\rho_i} \mathbb{1}$ and $R_0 = \mathbb{1}$. In particular, $R_i a = \overline{\rho}_i \rho_i(a) R_i$ for all $a \in \mathcal{N} \vee \mathcal{M}^c$ and $R_i \in (\mathcal{N}^c \cap \mathcal{M})^c = (\mathcal{N} \vee \mathcal{M}^c)^{cc}$.

Now, $R_i R_j \in \mathrm{Hom}_\mathcal{C}(\mathrm{id}, \overline{\rho}_i \rho_i \overline{\rho}_j \rho_j) = \mathrm{Hom}_\mathcal{C}(\mathrm{id}, \overline{\rho}_i \overline{\rho}_j \rho_i \rho_j)$ because, e.g., $\mathcal{C}_\mathcal{N}$ and $\mathcal{C}_{\mathcal{N}^c}$ commute in the sense of Proposition 4.5.9, and

$$R_i R_j = \sum_{k,\alpha,\beta} (w_\alpha w_\alpha^* \times v_\beta v_\beta^*) \cdot (R_i \times R_j)$$

where k runs over irreducible components $[\rho_k] \prec [\rho_i][\rho_j]$ and α, β over orthonormal bases of isometries $w_\alpha \in \mathrm{Hom}_{\mathcal{C}_\mathcal{N}}(\overline{\rho}_k, \overline{\rho}_i \overline{\rho}_j)$, $v_\beta \in \mathrm{Hom}_{\mathcal{C}_{\mathcal{M}^c}}(\rho_k, \rho_i \rho_j)$. Then $\sum_{k,\alpha,\beta} w_\alpha w_\alpha^* \times v_\beta v_\beta^* \cdot R_i \times R_j = \sum_{k,\alpha,\beta} w_\alpha v_\beta \, \lambda_{\alpha,\beta}^k R_k$ where $\lambda_{\alpha,\beta}^k \in \mathbb{C}$ because $[\rho_k]$ is irreducible, hence $[\mathrm{id}] \prec [\overline{\rho}_k][\rho_k]$ with multiplicity one, and $\overline{\rho}_k(v_\beta) = v_\beta$. Setting $C_{ij}^k := \sum_{\alpha,\beta} w_\alpha v_\beta \, \lambda_{\alpha,\beta}^k$ we have (non-canonical) intertwiners in $\mathrm{Hom}_\mathcal{C}(\overline{\rho}_k \rho_k, \overline{\rho}_i \overline{\rho}_j \rho_i \rho_j) = \mathrm{Hom}_\mathcal{C}(\overline{\rho}_k \rho_k, \overline{\rho}_i \rho_i \overline{\rho}_j \rho_j)$ which lie in $\mathcal{N} \vee \mathcal{M}^c$ and fulfill

$$R_i R_j = \sum_k C_{ij}^k R_k.$$

In particular, we have $C_{\overline{i}i}^0 \in \mathrm{Hom}_\mathcal{C}(\mathrm{id}, \overline{\rho}_{\overline{i}} \rho_{\overline{i}} \overline{\rho}_i \rho_i)$ again in $\mathcal{N} \vee \mathcal{M}^c$, hence $R_{\overline{i}}^* C_{\overline{i}i}^0$ is a multiple of R_i, i.e., we get

$$R_i^* = \lambda C_{\overline{i}i}^{0*} R_{\overline{i}}$$

for some $\lambda \in \mathbb{C}$, and we have shown up to normalization constants the algebraic relations of [KLM01, Eq. (15)].

On the other hand, by Frobenius reciprocity [LR97, Lem. 2.1] the R_i generate the extension $\mathcal{N} \vee \mathcal{M}^c \subset (\mathcal{N} \vee \mathcal{M}^c)^{cc}$ because every unitary charge transporter $u \in \mathrm{Hom}_\mathcal{C}(\rho, \tilde{\rho})$, $\rho \in \mathcal{C}_\mathcal{N}$, $\tilde{\rho} \in \mathcal{C}_{\mathcal{M}^c}$ such that $[\rho] = [\rho_i]$ for some i, can be written as $u = \lambda v \rho_i(r^*) R_{\overline{i}} = \lambda v r^* R_{\overline{i}}$ for suitable $\lambda \in \mathbb{C}$, $v \in \mathcal{M}^c$ unitary and $r \in \mathcal{N}$ isometric. In particular, every $b \in (\mathcal{N} \vee \mathcal{M}^c)^{cc}$ admits a (not necessarily unique) "harmonic" expansion

$$b = \sum_i b_i R_i \tag{4.24}$$

where $b_i \in \mathcal{N} \vee \mathcal{M}^c$, cf. [LR95, Eq. (4.10)], [KLM01, Prop. 45], and we are done. $\qquad\square$

Corollary 4.8.7. *With the assumptions of the previous proposition, $\mathcal{N} \vee \mathcal{M}^c$ is bidual in $\mathcal{M}_0$, i.e., $(\mathcal{N} \vee \mathcal{M}^c)^{\perp\perp} = \mathcal{N} \vee \mathcal{M}^c$. Moreover $\mathcal{N} \vee \mathcal{M}^c$ is normal in $\mathcal{M}_0$ if and only if $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M} \subset \mathrm{Vec}$, and $\mathcal{N} \vee \mathcal{M}^c = \mathcal{M}_0$ if and only if $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_\mathcal{M} = \{\mathrm{id}\}$. Analogous statements hold interchanging $\mathcal{N}$ and $\mathcal{M}$, hence in particular $p \sim q$.*

Proof. The category $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}}$ is automatically modular with the braiding inherited from $\mathcal{C}$, thanks to Lemma 4.8.2. The first statement follows by the same argument leading to Proposition 4.3.7 which relies on the (not necessarily unique) harmonic expansion (4.24), on rigidity of $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}}$ and on unitarity of its modular S-matrix.

Normality of $\mathcal{N} \vee \mathcal{M}^c$ implies $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}} \subset \mathrm{Vec}$ as we have seen in Proposition 4.7.10, the converse follows from the normality assumption on charge transporters.

The nontrivial implication in the last statement follows from biduality. $\qquad\square$

4.9 Abstract points and (Dedekind's) completeness

In the following we show a way of deriving completeness of the invariant introduced in Section 4.2, Eq. (4.8), on the class of *prime* conformal nets. This section is rather speculative, in the sense that it relies on two assumptions on the "good behaviour" of abstract point (in the prime CFT case). The first is horizontal and concerns *transitivity* of the comparability relation $p \sim q$, the second is vertical and asks *totality* of the unitary equivalence $p = UqU^*$ encountered in Section 4.6. Here we do not discuss about the issue of deriving them, nor strengthening Definition 4.5.1 or 4.8.3 in order to do so, nor deciding how do they constrain models. We just show how the structure of the real line (Dedekind's completeness axiom) and of a conformal net can cooperate in the reconstruction of the latter up to isomorphism from its abstract points, thanks to Proposition 4.8.5.

Proposition 4.9.1. *Let $\{\mathcal{A}\}$ be a prime conformal net on the line (Definition 4.7.5), fix arbitrarily $I_0 \in \mathcal{I}$ and assume in addition that comparability $p \sim q$ is transitive, and unitary equivalence $p = UqU^*$ is total on the abstract points of $\mathcal{M}_0 = \mathcal{A}(I_0)$. Then $\{\mathcal{A}\}$ is uniquely determined up to isomorphism by its abstract points inside $\mathcal{M}_0$.*

Proof. Take first an honest abstract point $p = (\mathcal{A}(I_1), \mathcal{A}(I_2))$ of $\mathcal{M}_0$ with respect to $\{\mathcal{A}\}$, as in Remark 4.5.4. By Remark 4.8.4 all the other honest points are equivalent to p. We want to show that they exhaust the comparability equivalence class. Let $q = (\mathcal{N}, \mathcal{N}^c)$ be an abstract point of $\mathcal{M}_0$ such that $q \sim p$, hence by transitivity $q \sim r$ for every honest point $r = (\mathcal{A}(J_1), \mathcal{A}(J_2))$,

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and by Proposition 4.8.5 either $r \leq q$ or $q < r$. Consider the maximum over the first family, i.e., the von Neumann algebra generated by the left relative complements, and the minimum over the second, i.e., the intersection of the left relative complements. The resulting algebras are again honest points because the net is additive and they coincide because the real line is Dedekind complete, thus q is also honest with respect to $\{\mathcal{A}\}$.

Now take an arbitrary abstract point $s = (\mathcal{M}, \mathcal{M}^c)$ of $\mathcal{M}_0$. By the totality assumption there is a unitary $U \in \mathcal{U}(\mathcal{H})$ such that $s = UpU^*$ where $p = (\mathcal{A}(I_1), \mathcal{A}(I_2))$ as above. Now every unitary is eligible as an isomorphism of local conformal nets, because positivity of the energy is preserved by unitary conjugation, hence call $\{\tilde{\mathcal{A}}\}$ the net defined on algebras by $\tilde{\mathcal{A}}(I) := U\mathcal{A}(I)U^*$, $I \in \mathcal{I}$, and observe that $s = (\tilde{\mathcal{A}}(I_1), \tilde{\mathcal{A}}(I_2))$ is an honest point of $\tilde{\mathcal{A}}(I_0) = \mathcal{A}(I_0)$ with respect to the new net. As before, r determines all the other honest points (because the comparability relation and its transitivity property are invariant under isomorphisms of nets), hence all the local interval algebras $\tilde{\mathcal{A}}(I) \subset \tilde{\mathcal{A}}(I_0)$, $I \subset I_0$ by taking intersections. By Proposition 1.6.14 the latter determine $\{\tilde{\mathcal{A}}\}$ up to isomorphism, hence $\{\mathcal{A}\}$ as well, and the proof is complete. $\square$

4.10 Conclusions

In chiral conformal QFT, the DHR category $\mathcal{C} = \mathrm{DHR}\{\mathcal{A}\}$ is a unitary braided tensor category corresponding to the positive-energy representations of the model. In completely rational models, the braiding is non-degenerate, hence it is a modular tensor category (UMTC). While abstract UMTCs are rigid structures and cannot distinguish the underlying CFT model uniquely, we have studied the question to which extent the *braided action* of this category on a single (local or global) algebra $\mathcal{A}$ is a complete invariant of the model. The strategy is to exploit the trivialization of the braiding, which is a characteristic feature of the DHR braiding, in certain geometric constellations to identify pairs of subalgebras (called "abstract points"). They are candidates for subalgebras of local observables associated to regions (half-intervals or half-lines) separated by a geometric point. Modularity is needed to distinguish the left from the right complement, and enters in our analysis through the stronger categorical notion of primality for UMTCs. As the main tool in this direction, we established powerful duality relations between subalgebras of $\mathcal{A}$ and subcategories of $\mathcal{C}$, and a characterization of "prime" CFT models

that do not factor through nontrivial subnet, either holomorphic or not. We formulate a unitary equivalence relation and a comparability relation between abstract points. Assuming that the former is total and the latter is transitive, we showed that the action of the DHR category is a complete invariant for prime CFT models, i.e., it allows (in principle) to reconstruct the local QFT up to unitary equivalence.

We assumed throughout that the action does come from a CFT, so that we only have to decide whether two inequivalent CFTs can give rise to the same action. We did not address the more ambitious question of how to characterize those actions which possibly come from a CFT, thus leaving the realization problem of braided actions of abstract UMTCs by DHR categories of some local net for future research.

Acknowledgements

It is a pleasure to express deep gratitude to my supervisor, Karl-Henning Rehren. Beyond being a great example and a singularity in several respects, he always listens to other people's opinions, with patience, and never dispenses with sharing his striking intuitions, personal interests and questions about physics. I thank him for giving me the chance of working together during this experience in Göttingen.

I am greatly indebted to Marcel Bischoff for uncountably many explanations, for his wisdom, constant care and availability, for his knowledge of models, for making possible a series of seminars during his stay in Göttingen, for his motivating interest in my work and for sharing his own activity.

I also thank Yoh Tanimoto for answering many doubts with clear insights, for sharing a deep understanding of mathematical physics and some of his questions and interests, and for teaching us the importance of locality.

I thank Roberto Longo for asking questions and giving important suggestions for our work, for many illuminating conversations during his visits to Göttingen and for pointing out interesting open problems. I also thank him for accepting to be a member of my thesis committee.

My gratitude goes also to Federico Dradi for becoming a good friend, for giving support and for his catching overall curiosity.

I thank Gennaro Tedesco for being a rough and unique officemate, for teaching me some realism, and some physics.

I am indebted to Chiara Entradi for helping with time concerns during the preparation of talks, for her friendship, for raising non-conventional interesting questions on my issues and for helping with exercise classes on electro-magnetism.

I also thank Simone Dresti for being with me always pragmatic and honest, as much as a physicist who reads W. Rudin can be.

I thank Dorothea Bahns for motivating me to give talks in our research training

school and for immediately accepting to be my second thesis supervisor.

I thank the other members of the committee as well, Laura Covi, Stefan Kehrein, Ralf Meyer, for their availability and for asking questions in some talks given during my PhD studies.

I am grateful to Pieter Naaijkens for an invitation to Hannover, and for an enjoyable day spent together with other colleagues. I am also indebted to Daniela Cadamuro, Detlev Buchholz, Jens Mund, Wei Yuan, Giuseppe Ruzzi, Fabio Ciolli, Roberto Conti, Adriana Schiani (the latter during some stays in Frascati and Rome) and Pierre Martinetti, Vincenzo Morinelli, Francesco Fidaleo, Luca Tomassini (when they came to Germany for a while).

I want to thank Rohit Holkar for pleasant conversations about maths and life, Olivier Gabriel, Lara Wimmer, Riccardo Catena, Tobias Binder, Marcel Langenberg, Ulrich Welling, and Michael Gustafsson for his experienced and kind advices, Maritza Ullrich-Totz and Karin Pulmer for teaching me German (and beyond), Saskia Borchers and Ina Röbbelen for being reliable friends and Roberta Aralla for giving support and sharing scientific interests.

I thank Martina for her being with me, though very often far away in space, for her patience and listening, for supporting in every occasion, for making the effort of taking part to my local life, and for (indeed useful) encouragements like 'se non ti piace quello che stai facendo, trovati un altro lavoro'. All the people who are missing here, friends (Giacomo in particular, the only good guy who came to see Göttingen in the end), family and my parents Nilva and Pietro (who supported me in the best way one could ever wish), will be acknowledged elsewhere.

I also want to thank the GRK1493 (research training school of Göttingen) and the ESI (Erwin Schrödinger Institute of Wien) for providing stimulating environments for joint research and science communication.

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