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A NONLINEAR PARABOLIC-HYPERBOLIC SYSTEM FOR CONTACT INHIBITION AND A DEGENERATE PARABOLIC FISHER KPP EQUATION

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Abstract. We consider a mathematical model describing population dynamics of normal and abnormal cell densities with contact inhibition of cell growth from a theoretical point of view. In the first part of this paper, we discuss the global existence of a solution satisfying the *segregation property* in one space dimension for general initial data. Here, the term segregation property means that the different types of cells keep spatially segregated when the initial densities are segregated. The second part is devoted to singular limit problems for solutions of the PDE system and traveling wave solutions, respectively. Actually, the contact inhibition model considered in this paper possesses quite similar properties to those of the Fisher-KPP equation. In particular, the limit problems reveal a relation between the contact inhibition model and the Fisher-KPP equation.

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1. **Introduction.** In natural cell growth processes, contact inhibition of cells may occur when two different types of cells come into contact with each other (cf. [1]). A number of mathematical models (see for instance [13][19][20]) have been proposed for the theoretical understanding of the mechanism of contact inhibition. In [3], we have introduced a simple partial differential equation model, which describes contact inhibition between normal and abnormal cells with densities u and v. It includes the effect of pushing cells away from overcrowded regions so that each cell moves in the direction of lower cell density. In the case of one space dimension, the resulting model is given by

$$\begin{cases} u_t = (u(u+v)_x)_x + u(1-u-\alpha v) & \text{in } \mathbb{R}\times(0,\infty), \\ v_t = (v(u+v)_x)_x + \gamma v(1-\beta u - v/k) & \text{in } \mathbb{R}\times(0,\infty), \\ u(\cdot,0)=u_0 \text{ and } v(\cdot,0)=v_0 & \text{in } \mathbb{R}. \end{cases} \tag{1}$$

Here α, β, γ and k are positive constants, and the initial functions are bounded and nonnegative: $u_0, v_0 \in L^\infty(\mathbb{R})$ and $u_0, v_0 \geq 0$ in $\mathbb{R}$.

In mathematical ecology, the growth terms are regarded as of Lotka-Volterra competition type. For an introduction to the biological context we refer to [13] and the references therein.

In [3] we have shown that problem (1) has a solution (u,v) if the initial total density, $w_0 := u_0 + v_0$, is smooth and bounded away from zero (in [4] we have generalized this result to the case of higher spatial dimension). The solution satisfies the following *segregation property*:

$$u_0 v_0 = 0 \text{ a.e. in } \mathbb{R} \;\Rightarrow\; u(\cdot,t)v(\cdot,t) = 0 \text{ a.e. in } \mathbb{R} \text{ for all } t>0. \tag{2}$$

The segregation property is a consequence of the *parabolic-hyperbolic* character of problem (1), which becomes clear if we introduce $w := u+v$ as a new dependent variable:

$$\begin{cases} w_t = (ww_x)_x + f(w,v) + g(w,v) & \text{in } \mathbb{R}\times(0,\infty), \\ v_t = (vw_x)_x + g(w,v) & \text{in } \mathbb{R}\times(0,\infty), \\ w(\cdot,0)=w_0 \text{ and } v(\cdot,0)=v_0 & \text{a.e. in } \mathbb{R}, \end{cases}$$

where

$$f(w,v) = (w-v)(1-w+(1-\alpha)v), \quad g(w,v) = \gamma v(1-\beta w + (\beta - 1/k)v).$$

Since u and v are nonnegative, we remark that the following inequalities hold

$$(w-v)(1-\max\{1,\alpha\}w) \leq f(w,v) \leq (w-v)(1-\min\{1,\alpha\}w),$$
$$v(\gamma - \max\{\gamma\beta,\gamma/k\}w) \leq g(w,v) \leq v(\gamma - \min\{\gamma\beta,\gamma/k\}w).$$

Therefore, observe that

$$w(C^- - D^+ w) \leq (f+g)(w,v) \leq w(C^+ - D^- w), \tag{3}$$

where $C^+ = \max\{1,\gamma\}$, $C^- = \min\{1,\gamma\}$, $D^+ = \max\{1,\alpha,\gamma\beta,\gamma/k\}$ and $D^- = \min\{1,\alpha,\gamma\beta,\gamma/k\}$.

For several reasons the condition that w_0 is bounded away from zero is rather restrictive. First of all numerical tests are utterly important to understand the rich structure of the possible qualitative behavior of solutions, but often the simulations concern solutions with compactly supported initial data u_0 and v_0. (See Figure 1 and also [6] for some examples.) Our first main result is Theorem 2.2, which establishes the existence of solutions if merely $w_0 \geq 0$. Very recently, and independently similar results for the Neumann boundary value problem in 1D were obtained by

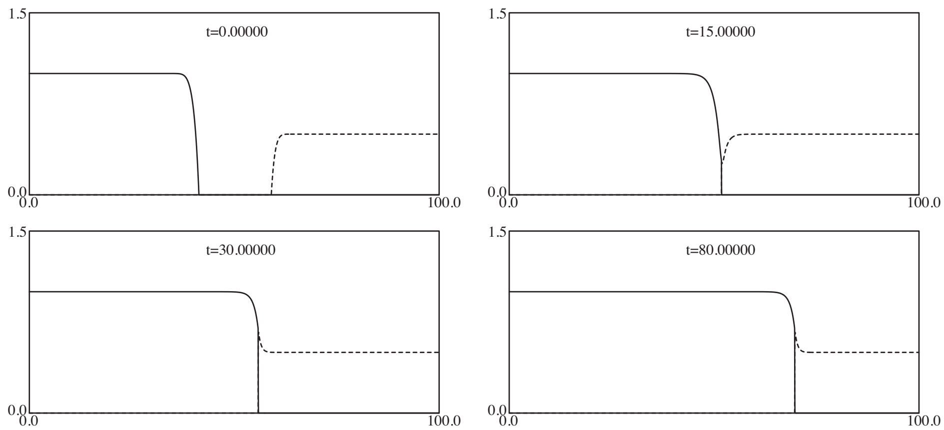

FIGURE 1. Snapshots of dynamics in (1) with compactly supported initial data. The parameter values are $\alpha = 4$, $\beta = 3$, $\gamma = 1$ and $k = 0.5$. The solid and dashed curves indicate u and v, respectively.

Carrillo et al([12]). The proofs in [12] are quite different from ours and are based on ideas from optimal transport theory.

The remaining results mainly concern the singular limit $k \to 0$. As $k \to 0$, the solution (u_k, v_k) of problem (1) converges to $(u, 0)$, where u is the unique weak solution of the *degenerate Fisher-KPP equation* (see Theorem 2.3):

$$\begin{cases} u_t = (uu_x)_x + u(1-u) & \text{in } \mathbb{R} \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}. \end{cases} \tag{4}$$

Probably the most interesting aspect of the limit $k \to 0$ concerns traveling wave solutions, i.e. solutions of the form $(U(x-ct), V(x-ct))$. Traveling wave solutions are natural candidates to describe the transient and/or large time behavior of classes of general solutions. In [6] we have shown that for all parameter values there exists a unique wave velocity for which there exists a unique (up to translation) segregated traveling wave solution, i.e. $U(z)V(z) = 0$ for all $z \in \mathbb{R}$. Observe that segregated traveling wave solutions do not depend on the specific values of α and β, the coefficients related to the interaction terms uv. For example, if $\alpha = 1$, $k = 2$ and $\beta = 1/k = 1/2$, the segregated traveling wave solutions seems to be a global attractor for a large class of solutions (see [6]).

Let (U_k^*, V_k^*, c_k^*) be the unique segregated traveling wave solution which satisfies

$$\begin{cases} (UU')' + cU' + U(1-U) = 0 \quad \text{and} \quad V = 0 & \text{if } z < 0, \\ (VV')' + cV' + \gamma V(1 - V/k) = 0 \quad \text{and} \quad U = 0 & \text{if } z > 0, \\ U(0^-) = V(0^+), \quad U'(0^-) = V'(0^+) = -c, \\ U(-\infty) = 1, \quad V(\infty) = k. \end{cases} \tag{5}$$

In Figure 2 we plotted the numerical values of the wave velocity c_k^* when the value of k is varied (it is easy to see that $c_k^* > 0 \;\; \Leftrightarrow 0 < k < 1$, see also [6]). In view of the equation for V_k^*, it is not surprising that $V_k^* \to 0$ as $k \to 0$. We shall show (see Theorem 2.4 for a more general statement) that (U_k^*, c_k^*) converges to

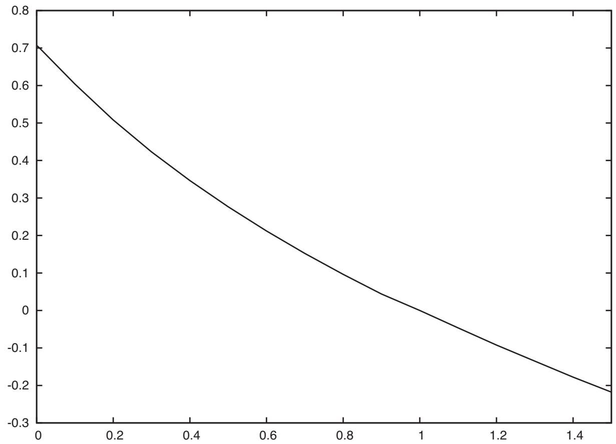

FIGURE 2. The relation between the parameter k and the wave velocity c_k^*, where the horizontal and vertical axes indicate k and c_k^*, respectively. The other parameter values are $\alpha = 4$, $\beta = 3$ and $\gamma = 1$.

$(U_0^*, c_0^*) := \left(\left(1 - e^{z/\sqrt{2}}\right)_+, 1/\sqrt{2}\right)$ as $k \to 0$, which solves

$$\begin{cases} (UU')' + cU' + U(1-U) = 0 \quad \text{in } \mathbb{R}, \\ U(-\infty) = 1, \quad U(\infty) = 0. \end{cases} \tag{6}$$

(See Figure 2: $c_0^* = 1/\sqrt{2} = 0.7071...$)

It is well-known ([9]) that c_0^* is a *minimal* wave velocity: the degenerate Fisher-KPP equation has a traveling wave solution if and only if $c \geq c_0^*$, and if $c > c_0^*$ the traveling wave solution is strictly positive for all z. A similar property as that of the degenerate Fisher-KPP equation holds for our system. If $\alpha = 1$ and $\beta = 1/k$, it turns out ([5]) that our system has a *non-segregated* or *overlapping* traveling wave solution for all wave velocities $c > c_k^* > 0$ (here we assume that $0 < k < 1$), i.e. a solution (U, V, c) of the problem

$$\begin{cases} (U(U+V)')' + cU' + U(1-U-V) = 0 \quad \text{in } \mathbb{R}, \\ (V(U+V)')' + cV' + \gamma V(1-(U+V)/k) = 0 \quad \text{in } \mathbb{R}, \\ U > 0, \quad V > 0 \quad \text{in } \mathbb{R}, \\ U(-\infty) = 1, \quad U(\infty) = 0, \quad V(-\infty) = 0, \quad V(\infty) = k. \end{cases} \tag{7}$$

Numerical evidence also suggests that c_k^* is a minimal wave velocity. Moreover, we numerically discussed in [5] that the exponential decay rate of an initial function u_0 is important for the large time behavior. Actually, a pair of initial functions (u_0, v_0) satisfying $(u, v) \to (1, 0)$ as $x \to -\infty$ and $(u, v) \to (0, k)$ with $u_0(x) = e^{-\xi x}$ as $x \to \infty$ evolves to the traveling wave solution with velocity

$$c(\xi) = \begin{cases} \frac{1-k}{\xi} & \text{if } \xi < \frac{1-k}{c_k^*}, \\ c_k^* & \text{if } \xi \geq \frac{1-k}{c_k^*}. \end{cases}$$

Roughly speaking, for an initial function u_0 with a large ξ, the solution in (1) converges to the segregated traveling wave solution, while for a moderate ξ, the solution converges to an overlapping one as time evolves. When we consider the evolution problem (1), the initial exponential decay rate of u_0 decides the large time behavior. So, at least if $\alpha = 1$ and $\beta = 1/k$, the analogy with the degenerate Fisher-KPP equation is striking. We shall make this more precise by showing that, if $\alpha = 1$ and $\beta = 1/k$, for all $c > c_0^*$ the corresponding overlapping traveling wave solutions converge to the one with velocity c of the degenerate Fisher-KPP equation as $k \to 0$ (see Theorem 2.4(ii)).

The paper concludes with two minor observations: we shall see that the case $k \to 0$ gives some insight in the case $k \to \infty$ (see Section 5), and we shall quantify the slope of the tangent line at $k = 1$, indicated in Figure 2 (see Theorem 2.4(iii)):

$$\frac{c_k^*}{k-1} \to -\frac{\sqrt{\gamma}}{1+\sqrt{\gamma}} \quad \text{as } k \to 1. \tag{8}$$

Finally we shall list some open problems for our system, mainly motivated by the analogy with the degenerate Fisher-KPP equation.

2. Main results.

We first define what we mean by a weak solution of problem (1):

Definition 2.1. A pair (u, v) is a solution of problem (1) if

(i) $u, v \in L^\infty(\mathbb{R} \times (0,\infty))$ and $u, v \geq 0$ a.e. in $\mathbb{R} \times (0,\infty)$;

(ii) $w := u + v \in L^2_{\text{loc}}([0,\infty); H^1_{\text{loc}}(\mathbb{R}))$;

(iii) for all test functions $\varphi \in C^1(\mathbb{R} \times [0,\infty))$ such that $\varphi = 0$ for large $|x|$ and t

$$\iint_{\mathbb{R}\times\mathbb{R}^+} (uw_x\varphi_x - u\varphi_t - u(1-u-\alpha v)\varphi) = \int_{\mathbb{R}} u_0\varphi(\cdot,0)$$

and

$$\iint_{\mathbb{R}\times\mathbb{R}^+} (vw_x\varphi_x - v\varphi_t - \gamma v(1-\beta u - v/k)\varphi) = \int_{\mathbb{R}} v_0\varphi(\cdot,0).$$

In this article, we will prove the following results. Problem (1) possesses a solution which satisfies the segregation property:

Theorem 2.2. *Let $\alpha, \beta, \gamma, k > 0$ and let $u_0, v_0 \in C(\mathbb{R})$ be such that $u_0, v_0 \geq 0$ in $\mathbb{R}$. Setting $w_0 := u_0 + v_0$, let either $w_0 \in L^1(\mathbb{R}^+)$ or*

$$\liminf_{x\to\infty} w_0(x) > 0 \text{ and } w_0 \in W^{1,\infty}(M,\infty) \text{ for some } M > 0, \tag{9}$$

and let either $w_0 \in L^1(\mathbb{R}^-)$ or

$$\liminf_{x\to-\infty} w_0(x) > 0 \text{ and } w_0 \in W^{1,\infty}(-\infty,-M) \text{ for some } M > 0. \tag{10}$$

Then problem (1) has a solution which satisfies property (2).

As $k \to 0$, the behavior of the solution is determined by the degenerate Fisher-KPP equation:

Theorem 2.3. *Let $u_0, v_0 \in C(\mathbb{R})$, and $u_0, v_0 \geq 0$ in $\mathbb{R}$. Let (u_k, v_k) be the corresponding solution defined by Theorem 2.2. Then*

$$u_k \to u, \ v_k \to 0 \quad \text{in } L^2_{\text{loc}}(\mathbb{R}\times[0,\infty)) \quad \text{as } k \to 0,$$

where u is the unique weak solution of problem (4).

Observe that the definition of solution of problem (4) is similar to Definition 2.1.

Also traveling wave solutions of our system converge to the corresponding ones of the degenerate Fisher-KPP equation:

Theorem 2.4. *Let $\gamma > 0$.*

(*i*) *Let* $(U_0^*, c_0^*) := \left(\left(1 - e^{z/\sqrt{2}}\right)_+, 1/\sqrt{2}\right)$ *be a traveling wave solution of the degenerate Fisher-KPP equation with minimal wave velocity* c_0^* *(see problem (6)). Let, for $k > 0$, (U_k^*, V_k^*, c_k^*) be the unique segregated traveling wave solution solving problem (5). Then*

$$U_k^* \to U_0^*,\ V_k^* \to 0 \text{ uniformly in } \mathbb{R},\ c_k^* \to c_0^* \quad \text{as } k \to 0.$$

(*ii*) *Let $\alpha = 1$, $\beta = 1/k$. Let $c > c_0^*$ and let $U_0^{(c)}$ be the unique traveling wave solution with wave velocity c solving problem (6) and the condition $U_0^{(c)}(0) = \frac{1}{2}$. Let $0 < k < 1$ be so small that $c > c_k^*$ and let $(U_k^{(c)}, V_k^{(c)})$ be an overlapping traveling wave solution of wave velocity c satisfying problem (7) and the conditions $U_k^{(c)}(0) = \frac{1}{2}$ and $U_k + V_k \leq 1$ in $\mathbb{R}$ (see [5]). Then*

$$U_k^{(c)} \to U_0^{(c)},\ V_k^{(c)} \to 0 \text{ uniformly in } \mathbb{R} \quad \text{as } k \to 0.$$

(*iii*) *c_k^* satisfies the convergence property (8).*

3. **The PDE system: Existence of solutions.** In this section, we prove Theorem 2.2. We consider the sequence of smooth problems

$$(P_n)\ \begin{cases} u_t = (u(u+v)_x)_x + u(1-u-\alpha v) & \text{in } (-n,n)\times(0,\infty) \\ v_t = (v(u+v)_x)_x + \gamma v(1-\beta u - v/k) & \text{in } (-n,n)\times(0,\infty) \\ (u+v)_x(\pm n, t) = 0 & \text{in } (0,\infty) \\ u(\cdot,0) = u_{0n},\ v(\cdot,0) = v_{0n} & \text{in } (-n,n), \end{cases}$$

where (u_{0n}, v_{0n}) is a sequence of smooth, nonnegative and uniformly bounded initial data, defined on $\mathbb{R}$, such that

$$u_{0n} \to u_0,\ v_{0n} \to v_0 \quad \text{uniformly in } \mathbb{R} \quad \text{as } n \to \infty,$$
$$1/n^2 < w_{0n} := u_{0n} + v_{0n} < B \quad \text{in } \mathbb{R},$$
$$\{w_{0n}\} \text{ are locally equicontinuous in } \mathbb{R}.$$

We will use the notations

$$\begin{aligned} a_n &:= \int_0^{-n} w_{0n} dx \to a := \int_0^{-\infty} w_0 dx \quad \text{as } n \to \infty, \\ b_n &:= \int_0^{n} w_{0n} dx \to b := \int_0^{\infty} w_0 dx \quad \text{as } n \to \infty. \end{aligned} \tag{11}$$

and remark that the constants a and b can be finite or infinite.

If $\liminf_{x\to\infty} w_0(x) > 0$ and $w_0 \in W^{1,\infty}(M,\infty)$, we assume that for some $C_1, C_2 > 0$ and $M_1 \geq M$

$$|(w_{0n})_x| \leq C_1, \quad w_{0n} > C_2 \quad \text{in } (M_1, n), \tag{12}$$

and make similar assumptions for $x \to -\infty$.

It follows as in [4] that Problem (P_n) possesses a smooth solution (u_n, v_n) such that $w_n := u_n + v_n$ satisfies

$$\min\left\{\frac{1}{n^2}, \frac{\min\{1,\gamma\}}{\max\{1,\alpha,\gamma\beta,\gamma/k\}}\right\} < w_n(x,t) < \max\left\{B, \frac{\max\{1,\gamma\}}{\min\{1,\alpha,\gamma\beta,\gamma/k\}}\right\}, \tag{13}$$

for all $(x,t) \in [-n,n]\times[0,\infty]$. Next we prove the following result.

Lemma 3.1. *For all $R, T > 0$, $0 < \mu < 1$ and $n > R + 1$*

$$\|w_n^{-\mu/2} w_{nx}\|_{L^2((-R,R))\times(0,T))} \le C_{R,T,\mu}, \tag{14}$$

where the constant $C_{R,T,\mu}$ does not depend on n.

Proof. We multiply the equation for w_n,

$$w_t = (ww_x)_x + u(1 - u - \alpha v) + \gamma v(1 - \beta u - v/k),$$

by $\zeta_R^2(x) w_n^{-\mu}(x,t)$ and integrate by parts over $(-n,n)\times(0,T)$, where $\zeta_R \in C^1(\mathbb{R})$ is a cut-off function satisfying:

$$0 \le \zeta_R \le 1 \text{ and } |\zeta_R'| \le 2 \text{ in } \mathbb{R}, \quad \zeta_R(x) = \begin{cases} 1 & \text{if } |x| < R \\ 0 & \text{if } |x| > R+1. \end{cases} \tag{15}$$

Hence

$$\frac{1}{1-\mu} \int_{-n}^{n} \zeta_R^2 w_n^{1-\mu} \Big|_{t=0}^{t=T} = \iint_{(-n,n)\times(0,T)} (\mu \zeta_R^2 w_n^{-\mu} w_{nx}^2 - 2\zeta_R w_n^{1-\mu} \zeta_R' w_{nx}) + \iint_{(-n,n)\times(0,T)} \zeta_R^2 w_n^{1-\mu} \left(\frac{u_n}{w_n}(1 - u_n - \alpha v_n) + \gamma \frac{v_n}{w_n}(1 - \beta u_n - v_n/k) \right),$$

and (14) follows from the uniform boundedness of u_n/w_n, v_n/w_n and w_n when we handle the term $\iint_{(-n,n)\times(0,T)} 2\zeta_R w_n^{1-\mu} \zeta_R' w_{nx}$ in a suitable way. Indeed, in order to obtain (14), we use the inequality

$$2\left| \iint_{(-n,n)\times(0,T)} \zeta_R w_n^{1-\mu} \zeta_R' w_{nx} \right| \le \frac{\mu}{2} \iint_{(-n,n)\times(0,T)} \zeta_R^2 w_n^{-\mu} w_{nx}^2 + \frac{2}{\mu} \iint_{(-n,n)\times(0,T)} w_n^{2-\mu} \zeta_R'^2,$$

so that there exists a positive constant $C(R, T, \mu)$ such that

$$\frac{\mu}{2} \iint_{(-n,n)\times(0,T)} \zeta_R^2 w_n^{-\mu} w_{nx}^2 \le C(R, T, \mu),$$

which in turn implies (14).

By [15], the functions w_n are locally equicontinuous in $\mathbb{R} \times [0, \infty)$. Hence there exist nonnegative functions $w \in C(\mathbb{R} \times [0,\infty)) \cap L^\infty(\mathbb{R} \times [0,\infty))$ and $v \in L^\infty(\mathbb{R} \times [0,\infty))$ and a subsequence (w_{n_j}, v_{n_j}) such that $w^{1-\mu/2} \in L^2_{\text{loc}}([0,\infty); H^1_{\text{loc}}(\mathbb{R}))$ and, as $j \to \infty$,

$$\begin{aligned} &w_{n_j} \to w \quad \text{in } C_{\text{loc}}(\mathbb{R} \times [0,\infty)) \\ &w_{n_j}^{1-\mu/2} \rightharpoonup w^{1-\mu/2} \quad \text{in } L^2_{\text{loc}}([0,\infty); H^1_{\text{loc}}(\mathbb{R})) \quad (\mu < 1) \\ &v_{n_j} \rightharpoonup v \quad \text{in } L^2_{\text{loc}}(\mathbb{R} \times [0,\infty)). \end{aligned}$$

Setting $u := w - v$, we shall show that (u, v) is a solution of Problem (1). Let

$$r_n := \frac{u_n}{w_n} \quad \text{in } (-n,n) \times [0,\infty).$$

Since r_n is bounded, it converges locally and weakly along a subsequence. To pass to the limit in the equations for u_n and v_n, we have to prove strong convergence, at least in the set where $w > 0$. Observe that r_n satisfies the equation

$$r_{nt} = w_{nx} r_{nx} + r_n(1 - r_n) G(r_n, w_n), \tag{16}$$

where

$$G(r,w) = (1 - rw - \alpha(1-r)w) - \gamma(1 - \beta rw - (1-r)w/k).$$

We introduce the characteristics associated to the velocity field $-w_{nx}$. Below we list some of their properties.

Lemma 3.2. *Let $X_n(y,0) \in [-n,n]$ be defined by*

$$y = \int_0^{X_n(y,0)} w_{0n}(s)ds \quad \text{for } a_n \le y \le b_n,$$

where $a_n = \int_0^{-n} w_{0n}(s)ds$ and $b_n = \int_0^n w_{0n}(s)ds$. For all $a_n \le y \le b_n$, there exists $X_n(y,\cdot) \in C^1([0,\infty))$ such that

$$X_{nt}(y,t) = -w_{nx}(X_n(y,t),t) \quad \text{for } t \ge 0,$$

and X_n has the following properties: for all $T > 0$ there exist positive constants c_T, C_T, a_T, A_T, B_T which do not depend on n, such that

(i) $0 < c_T \le w_n(X_n(y,t),t)X_{ny}(y,t) \le C_T$ in $(a_n,b_n)\times(0,T)$;
(ii) $X_{ny} \ge a_T > 0$ in $(a_n,b_n)\times(0,T)$;
(iii) X_n is uniformly bounded in $L^\infty((0,T);BV_{loc}(a_n,b_n)) \cap H^1((0,T);L^2_{loc}(a_n,b_n))$.

We will present the proof of Lemma 3.2 at the end of this section.

Next we use the notation

$$X_n(y,0) = X_{0n}(y), \quad y \in [a_n,b_n],$$

and remark that X_{0n} is the unique solution of the initial value problem

$$\begin{cases} X'_{0n}(y) = \dfrac{1}{w_{0n}(X_{0n}(y))} & y \in (0,b_n) \text{ and } y \in (a_n,0) \\ X_{0n}(0) = 0. \end{cases}$$

By Lemma 3.2(*iii*), Lemma A.1 in [16], and Aubin-Lions Lemma (cf. [10], Theorem II.5.16) there exist a subsequence $\{X_{n_j}\}$, which we denote again by $\{X_n\}$, and a function $X \in L^\infty_{\text{loc}}([0,\infty);BV_{\text{loc}}(a,b)) \cap H^1_{\text{loc}}([0,\infty);L^2_{\text{loc}}(a,b))$, such that, as $j \to \infty$,

$$X_n \to X \quad \text{in } C([0,T];L^2_{\text{loc}}(a,b)) \text{ and a.e. in } (a,b)\times(0,\infty). \tag{17}$$

Here we remark that a and b are either finite or infinite.

By Lemma 3.2(*ii*), for all $t \ge 0$, the inverse function $Y_n(\cdot,t)$ of $y \mapsto X_n(y,t)$ is well-defined for $|x| < n$ and $t \ge 0$, and it satisfies

$$Y_{nx}(x,t) = 1/X_{ny}(Y_n(x,t),t) \ \text{ and } \ Y_{nt} = Y_{nx}w_{nx}.$$

By Lemma 3.2, $0 \le Y_{nx} \le 1/a_T$ and $|Y_{nt}| \le {c_T}^{-1} w_n|w_{nx}|$. By (13) and (14), Y_{nt} is bounded in $L^2_{\text{loc}}(\mathbb{R}\times[0,\infty))$, uniformly in n. Next, we prove that Y_n is locally and uniformly Hölder continuous in $\mathbb{R}\times[0,\infty)$ and that

$$Y_n \to Y \quad \text{in } C_{\text{loc}}(\mathbb{R}\times[0,\infty)) \quad \text{as } n \to \infty,$$

where Y is the inverse function of $y \mapsto X(y,t)$ for each $t \ge 0$. We first remark that Y_n is uniformly Lipschitz continuous with respect to x, and that Y_{nt} is bounded in $L^2_{\text{loc}}(\mathbb{R}\times[0,\infty))$. We need to prove the uniform continuity with respect to t (at least locally).

Fix $\varepsilon > 0$, x_0 in a bounded interval $[-l,l]$ and $0 < t_2 < t_1 < T$. Let $\delta > 0$ and $\alpha > 0$ to be chosen below and let $|t_1 - t_2| < \delta$. Since $\int_{x_0-\delta^\alpha}^{x_0+\delta^\alpha}\int_0^T Y_{nt}^2 dxdt < C_1 = C_1(l,T)$, so there exists $\eta \in (x_0 - \delta^\alpha, x_0 + \delta^\alpha)$ such that $\int_0^T Y_{nt}^2(\eta,t)dt < C_1/(2\delta^\alpha)$. Since

$\{Y_n(\eta,\cdot)\}$ is bounded in $H^1(0,T)$ uniformly in n and since $H^1(0,T)\subset C^{1/2}([0,T])$, it follows that $Y_n(\eta,t)$ is Hölder continuous with respect to t, uniformly in n; more precisely, there holds $|Y_n(\eta,t_1)-Y_n(\eta,t_2)| < C_1|t_1-t_2|^{1/2}/\delta^{\alpha/2} = C_1\delta^{1/2-\alpha/2}$. Choosing $\alpha=1/3$, and x such that $|x-\eta|\le\delta^\alpha$, we deduce that

$$\begin{aligned}&|Y_n(x,t_1)-Y_n(x,t_2)|\\ \le&|Y_n(x,t_1)-Y_n(\eta,t_1)|+|Y_n(\eta,t_1)-Y_n(\eta,t_2)|+|Y_n(\eta,t_2)-Y_n(x,t_2)|\\ <&C\delta^\alpha+C_1\delta^{1/2-\alpha/2}+C\delta^\alpha\\ =&(2C+C_1)\delta^\alpha<\varepsilon\end{aligned}$$

if we choose δ sufficiently small. Therefore, since Y_n is locally uniformly Hölder continuous in $\mathbb{R}\times[0,\infty)$, we deduce that there exists a function $\overline{Y}$ and a subsequence of $\{Y_n\}$, which we denote again by $\{Y_n\}$ such that $Y_n\to\overline{Y}$ uniformly on compact sets of $\mathbb{R}\times[0,\infty)$ as $n\to\infty$. Next we show that $\overline{Y}(\cdot,t)$ is the inverse of $y\mapsto X(y,t)$ for all $t\ge0$. Indeed

$$|y-\overline{Y}(X(y,t),t)| = |Y_n(X_n(y,t),t)-\overline{Y}(X(y,t),t)|\le$$
$$|\overline{Y}(X_n(y,t),t)-\overline{Y}(X(y,t),t)|+|Y_n(X_n(y,t),t)-\overline{Y}(X_n(y,t),t)|,$$

which tends to zero as $n\to\infty$, so that $\overline{Y}=Y$.

Set $W_n(y,t):=w_n(X_n(y,t),t)$ and $W(y,t):=w(X(y,t),t)$. Since $w_n\to w$ in $C_{\text{loc}}(\mathbb{R}\times[0,\infty))$ and $X_n\to X$ in $C([0,T];L^2_{\text{loc}}(a,b))$ and a.e. in $(a,b)\times[0,\infty)$,

$$W_n\to W\quad\text{a.e. in } (a,b)\times[0,\infty) \text{ and in } L^1_{\text{loc}}((a,b)\times[0,\infty)).$$

Indeed, for each $t\ge0$,

$$|w_n(X_n(y,t),t)-w(X(y,t),t)|\le|w(X_n(y,t),t)-w(X(y,t),t)|+$$
$$|w_n(X_n(y,t),t)-w(X_n(y,t),t)|,$$

where the first term on the right-hand-side tends to zero as $n\to\infty$ by (17) and the second one converges to zero since w_n converges to w locally uniformly in $\mathbb{R}\times[0,\infty)$.

We may assume that w_0 is not identically equal to 0 (otherwise the solution is trivial: $(u,v)=(0,0)$). Consider, for $t\ge0$, the sets

$$\mathcal{P}_0^{(x)}(t):=\{x\in\mathbb{R}; w(x,t)=0\},\qquad \mathcal{P}_0^{(y)}(t):=\{Y(x,t); x\in\mathcal{P}_0^{(x)}(t)\}.$$

By (3) and standard theory on the porous media equation ([18]),

$$\mathcal{P}_0^{(x)}(t_1)\supseteq\mathcal{P}_0^{(x)}(t_2)\quad\text{ if } 0\le t_1\le t_2,$$

and, for all $R>0$,

$$\mathcal{P}_0^{(x)}(t)\cap[-R,R]=\emptyset\quad\text{if } t>T_R$$

for some $T_R\ge0$.

By construction, the set $\mathcal{P}_0^{(y)}(0)=\{y\in(a,b); w_0(X(y,0))=0\}$ is closed, and either finite or countable: for some $N\in\{1,2,\dots,\infty\}$

$$\mathcal{P}_0^{(y)}(0)=\{y_i; 1\le i<N\}.$$

We claim that in the (y,t) plane, the set where $W=0$ is the union of at most countable vertical segments: for all $y_i\in\mathcal{P}_0^{(y)}(0)$ there exists $\tau_i\ge0$ such that

$$\{(y,t)\in(a,b)\times[0,\infty); y\in\mathcal{P}_0^{(y)}(t), t\ge0\}=\{(y_i,t); 1\le i<N, 0\le t\le\tau_i\}. \tag{18}$$

This is an immediate consequence of the following result.

Lemma 3.3. *$Y(x,t)$ is constant ($= y_i$) in each connected component $\mathcal{P}_0^i$ of the set $\{(x,t) : w(x,t) = 0\}$.*

We postpone the proof to the end of this section.

By local regularity results (Ch. V Theorem 3.1 in [17]), $(w_{n_j})_x \to w_x \in C_{\text{loc}}(\mathcal{P}_+)$ as $j \to \infty$ in the set

$$\mathcal{P}_+ := \{(x,t) \in \mathbb{R} \times [0,\infty); w(x,t) > 0\},$$

and hence it follows from (18) that

$$(w_{n_j})_x(X_{n_j}(y,t),t) \to w_x(X(y,t),t) \quad \text{ a.e. in } (a,b) \times (0,T).$$

This implies that X represents a regular Lagrangian flow in the sense of [2] and [14]: the equation $X_t = -w_x(X,t)$ is satisfied in the sense of distributions and, for all $\psi \in L^\infty(\mathbb{R} \times [0,\infty))$ with bounded support,

$$\iint_{(a,b)\times(0,\infty)} \psi(X(y,t),t)dydt = \iint_{\mathbb{R}\times(0,\infty)} \psi(x,t)Y_x(x,t)dxdt.$$

We set

$$R_n(y,t) := r_n(X_n(y,t),t) \quad \text{for } a_n < y < b_n,\ t \geq 0.$$

Then, by (16),

$$\begin{cases} R_{nt} = R_n(1-R_n)G(R_n,W_n) & \text{in } (a_n,b_n) \times (0,\infty), \\ R_n(y,0) = R_{0n}(y) := u_{0n}(X_n(y,0))/w_{0n}(X_n(y,0)) & \text{in } (a_n,b_n). \end{cases}$$

Let $R(y,t)$ be defined by

$$\begin{cases} R_t = R(1-R)G(R,W) & \text{in } (a,b) \times (0,\infty) \\ R(y,0) = R_0(y) := u_0(X(y,0))/w_0(X(y,0)) & \text{in } (a,b). \end{cases} \tag{19}$$

Observe that

$$R_{0n} \to R_0 \quad \text{in } L^1_{\text{loc}}((a,b)) \text{ and a.e. in } (a,b) \quad \text{as } n \to \infty.$$

Indeed by construction, the convergence is locally uniform (and hence pointwise) in the set of the points $y \in (a,b)$ where $w_0(X(y,0)) > 0$, and $w_0(X(y,0)) > 0$ for a.e. $y \in (a,b)$. Arguing as in section 4.5 of [4], it follows that

$$R_{n_j} \to R \quad \text{in } L^1_{\text{loc}}((a,b) \times [0,\infty)) \quad \text{as } j \to \infty.$$

We set $r(x,t) := R(Y(x,t),t)$. Since X is a regular Lagrangian flow, it follows from Proposition 4.9 in [4], which extends Proposition 3.5 in [14], that r is a distributional solution of the transport equation

$$\begin{cases} (Y_x r)_t = (rY_x w_x)_x + Y_x r(1-r)G(r,w) & \text{in } \mathbb{R} \times (0,\infty), \\ [Y_x r](\cdot,0) = u_0 & \text{in } \mathbb{R}. \end{cases} \tag{20}$$

Observe that this is not surprising, since the product $Y_{nx} r_n$ satisfies

$$\begin{cases} (Y_{nx} r_n)_t = (r_n Y_{nx} w_{nx})_x + Y_{nx} r_n(1-r_n)G(r_n,w_n) & \text{in } (-n,n) \times (0,\infty), \\ [Y_{nx} r_n](\cdot,0) = r_{0n} w_{0n} = u_{0n} & \text{in } (-n,n). \end{cases}$$

Arguing as in section 4.6 of [4], we prove the strong convergence of r_{n_j} to r. First we show that for any test function $\varphi \in C^\infty(\mathbb{R} \times [0,\infty))$ with bounded support

$$\iint_{\mathbb{R}\times[0,\infty)} Y_{n_j x}(r_{n_j} - r)^2 \varphi \, dxdt \to 0 \quad \text{as } j \to \infty.$$

To this end, we prove that

$$r_{n_j} Y_{n_j x} \text{ converges weakly to } rY_x \text{ as } j \to \infty$$

and

$$Y_{n_j x} c_{n_j} \text{ converges weakly to } Y_x c \text{ as } j \to \infty,$$

where $c_n := r_n(1-r_n)G(r_n, w_n)$ and $c := r(1-r)G(r,w)$.

Let $\varphi(x,t)$ be a smooth test function with bounded support. Then, by the strong convergence of R_{n_j} and X_{n_j} as $j \to \infty$,

$$\iint_{(a,b)\times\mathbb{R}^+} R_{n_j}(y,t)\varphi(X_{n_j}(y,t),t)dydt \to \iint_{(a,b)\times\mathbb{R}^+} R(y,t)\varphi(X(y,t),t)dydt$$
$$= \iint_{\mathbb{R}\times\mathbb{R}^+} r(x,t)\varphi(x,t)Y_x(x,t)dxdt.$$

On the other hand, let ξ be the weak limit of $r_{n_j} Y_{n_j x}$ (up to subsequences). Then

$$\iint_{(a,b)\times\mathbb{R}^+} R_{n_j}(y,t)\varphi(X_{n_j}(y,t),t)dydt = \iint_{\mathbb{R}\times\mathbb{R}^+} r_{n_j}(x,t)\varphi(x,t)Y_{n_j x}(x,t)dxdt$$
$$\to \iint_{\mathbb{R}\times\mathbb{R}^+} \xi(x,t)\varphi(x,t)dxdt$$

and hence $\xi = rY_x$.

Next, let χ be the weak limit of $Y_{n_j x} c_{n_j}$. Taking the limit in

$$(Y_{n_j x} r_{n_j})_t = (r_{n_j} Y_{n_j x} w_{n_j x})_x + Y_{n_j x} c_{n_j} \text{ (in the sense of distributions)},$$

we find that

$$(Y_x r)_t = (rY_x w_x)_x + \chi \text{ (in the sense of distributions)}.$$

But we already know that

$$(Y_x r)_t = (rY_x w_x)_x + Y_x c \text{ (in the sense of distributions)},$$

so that $\chi = Y_x c$.

We repeat this procedure, replacing r_{n_j} by $r_{n_j}^2$. Since the strong convergence of R_{n_j} implies the strong convergence of $R_{n_j}^2$,

$$\iint_{(a,b)\times\mathbb{R}^+} R_{n_j}^2(y,t)\varphi(X_{n_j}(y,t),t)dydt \to \iint_{(a,b)\times\mathbb{R}^+} R^2(y,t)\varphi(X(y,t),t)dydt$$
$$= \iint_{\mathbb{R}\times\mathbb{R}^+} r^2(x,t)\varphi(x,t)Y_x(x,t)dxdt.$$

On the other hand, let $\tilde{\xi}$ be the weak limit of $r_{n_j}^2 Y_{n_j x}$. Then,

$$\iint_{(a,b)\times\mathbb{R}^+} R_{n_j}^2(y,t)\varphi(X_{n_j}(y,t),t)dydt = \iint_{\mathbb{R}\times\mathbb{R}^+} r_{n_j}^2(x,t)\varphi(x,t)Y_{n_j x}(x,t)dxdt$$
$$\to \iint_{\mathbb{R}\times\mathbb{R}^+} \tilde{\xi}(x,t)\varphi(x,t)dxdt.$$

Therefore, $\tilde{\xi} = r^2 Y_x$ and

$$r_{n_j}^2 Y_{n_j x} \text{ converges weakly to } r^2 Y_x \text{ as } j \to \infty.$$

Finally we consider $Y_{n_jx}(r_{n_j}-r)^2 = Y_{n_jx}r_{n_j}^2 + Y_{n_jx}r^2 - 2Y_{n_jx}r_{n_j}r$. We deduce from the weak convergences above that, for any test function $\varphi \in C^\infty(\mathbb{R}\times\mathbb{R}^+)$ with bounded support,

$$\iint_{\mathbb{R}\times\mathbb{R}^+} Y_{n_jx}(r_{n_j}-r)^2\varphi dxdt = \iint_{\mathbb{R}\times\mathbb{R}^+}(Y_{n_jx}r_{n_j}^2 + Y_{n_jx}r^2 - 2Y_{n_jx}r_{n_j}r)\varphi dxdt$$
$$\to \iint_{\mathbb{R}\times\mathbb{R}^+}(Y_x r^2 + Y_x r^2 - 2Y_x r^2)\varphi dxdt = 0$$

as $j\to\infty$.

We deduce from Lemma 3.2(i) that

$$\frac{w_n}{C_T} \le Y_{nx},$$

which, together with the inequaliy above and the locally uniform convergence of w_{n_j}, implies that

$$\iint_{\mathbb{R}\times\mathbb{R}^+} w(r_{n_j}-r)^2\varphi\ dxdt \to 0 \quad \text{as } j\to\infty,$$

and hence we have that $r_{n_j}\to r$ in $L^2_{\text{loc}}(\mathbb{R}\times[0,\infty))$ as $j\to\infty$. Combining this result and $w_{n_j}\to w$ in $C_{\text{loc}}(\mathbb{R}\times[0,\infty))$, we obtain that

$$u_{n_j}\to u,\ v_{n_j}\to v \quad \text{in } L^1_{\text{loc}}(\mathbb{R}\times[0,\infty)) \quad \text{as } j\to\infty.$$

This permits to pass to the limit in the equations for u_{n_j} and v_{n_j} and we have found a solution of Problem (1).

Arguing as in section 4.7 in [4], the segregation property (2) follows immediately from the equation for $R(y,t)$ (see (19)). That is, $R(1-R)$ satisfies

$$\begin{cases}(R(1-R))_t = R(1-R)(1-2R)G(R,W) & \text{in } (a,b)\times(0,\infty),\\ (R(1-R))(y,0) = 0 & \text{in } (a,b).\end{cases}$$

Since R and W are uniformly bounded, it follows from Gronwall's inequality that $R(y,t)(1-R(y,t)) = 0$ for all $t>0$ and a.e. in (a,b). Hence we have $u(x,t)v(x,t) = 0$ for all $t\ge 0$ and a.e. in $\mathbb{R}$.

To complete the proof of Theorem 2.2 we prove the two Lemmata in this section.

Proof of Lemma 3.2. Before proving the existence of $X_n(y,t)$, we prove the properties $(i)-(iii)$ as a priori estimates.

(i): The function $q_n(y,t) = w_n(X_n(y,t),t)X_{ny}(y,t)$ satisfies $q_n(y,0)=1$ and $q_{nt} = F_nq_n$, where F_n is the uniformly bounded function

$$F_n(y,t) = \left(\frac{u_n}{w_n}(1-u_n-\alpha v_n) + \frac{\gamma v_n}{w_n}(1-\beta u_n - v_n/k)\right)\Bigg|_{x=X_n(y,t)},$$

and the result follows from Gronwall's Lemma (see also (5) in [4]).

(ii) follows at once from (i) and the uniform boundedness of w_n.

(iii): First we consider the case that $w_0\in L^1(\mathbb{R})$. Then (see (14))

$$\iint_{(a_n,b_n)\times(0,T)} X_{nt}^2\zeta_R(X_n)dydt = \iint_{(-n,n)\times(0,T)}\frac{w_{nx}^2}{X_{ny}(Y_n(x,t),t)}\zeta_R dxdt$$
$$\le C\iint_{(-n,n)\times(0,T)} w_nw_{nx}^2\zeta_R dxdt \le C,$$

similarly,

$$\iint_{(a_n,b_n)\times(0,T)} X_n^2\zeta_R(X_n)dydt = \iint_{(-n,n)\times(0,T)} \frac{x^2}{X_{ny}(Y_n(x,t),t)}\zeta_R dxdt$$
$$\le C\iint_{(-n,n)\times(0,T)} w_n x^2\zeta_R dxdt \le C,$$

whence X_n is uniformly bounded in $H^1((0,T);L^2_{\text{loc}}(a_n,b_n))$ (observe that in this case $a_n \to a > -\infty$ and $b_n \to b < \infty$; see (11)). Hence, for all $\delta > 0$ and large enough n, there exist $y^-_{\delta,n} \in (a+\delta, a+2\delta)$ and $y^+_{\delta,n} \in (b-2\delta, b-\delta)$ such that

$$\int_0^T X_{nt}^2(y^\pm_{\delta,n},t)dt \le \frac{C}{\delta}. \tag{21}$$

This implies that X_n is uniformly bounded in $L^\infty((0,T);BV_{\text{loc}}(a,b))$: for all $0 < t \le T$ and $\delta > 0$

$$\int_{a+2\delta}^{b-2\delta} |X_{ny}(y,t)|dy = \int_{a+2\delta}^{b-2\delta} X_{ny}(y,t)dy = X_n(b-2\delta,t) - X_n(a+2\delta,t)$$
$$\le X_n(y^+_{\delta,n},t) - X_n(y^-_{\delta,n},t) \le C(\delta).$$

Next we consider the case that (9) and (10) are satisfied. If (12) is satisfied, it follows from the maximum principle and the gradient estimate ([17] Ch. V, Th. 3.1) that for all $T > 0$ there exist positive constants M_{1T}, C_{1T} and C_{2T} such that

$$|w_{nx}| < C_{1T}, \quad w_n > C_{2T} \quad \text{in } (M_{1T},n)\times(0,T). \tag{22}$$

By (22) and the similar property in $(-\infty, -M_{1T})$,

$$|X_{nt}(y,t)| \le C_{1T} \quad \text{if } |X_n(y,t)| > M_{1T},\ 0 \le t \le T. \tag{23}$$

Hence there exist y_T such that $X_n(-y_T,t) < -M_{1T}$ and $X_n(y_T,t) > M_{1T}$ for $0 \le t \le T$, and therefore

$$|X_n(y,t) - X_n(y,0)| < C \quad \text{for } y_T < |y| < n,\ 0 \le t \le T.$$

This implies the uniform boundedness of X_n in $L^\infty((0,T);BV_{\text{loc}}(\mathbb{R}))$. By (23) and part (i) of this lemma, uniform boundedness of $X_{nt}(y,t)$ in $L^2_{\text{loc}}(\mathbb{R}\times[0,T])$ is equivalent with uniform boundedness of $\sqrt{w_n}w_{nx}$ in $L^2_{\text{loc}}(\mathbb{R}\times[0,T])$, and hence, by (14), X_n is uniformly bounded in $H^1((0,T);L^2_{\text{loc}}(\mathbb{R}))$.

To complete the proof of (iii) we have to consider the cases that $w_0 \in L^1(\mathbb{R}^+)$ satisfies (10), and, respectively, that $w_0 \in L^1(\mathbb{R}^-)$ satisfies (9). It is enough to combine the ingredients of the proofs in the previous cases, and we leave the details to the interested reader.

Finally we observe that for all y the function $t \to X_n(y,t)$ is well-defined since, by part (iii), it is a priori bounded in $[0,T]$.

Proof of Lemma 3.3. By Lemma 3.2, $0 \le Y_x \le Cw$ and hence Y does not depend on x in $\mathcal{P}_0^i$. Let $t_i = \max\{t \ge 0; (x,t) \in \mathcal{P}_0^i\}$ and let $(x_i,t_i) \in \mathcal{P}_0^i$. Without loss of generality we may assume that $t_i > 0$ and $x_i = 0$.

Let $x_1 > 0$ and let $\mu_0 < \mu_1 < 1$. For all $\tau \in (0,t_i]$,

$$\left|\int_0^{x_1} Y(s,\tau)ds - \int_0^{x_1} Y(s,0)ds\right| \le \iint_{(0,x_1)\times(0,\tau)} |Y_t| \le C\iint_{(0,x_1)\times(0,\tau)} w|w_x|$$
$$\le C\left(\iint_{(0,x_1)\times(0,\tau)} w^{-\mu_0}w_x^2\right)^{1/2}\left(\iint_{(0,x_1)\times(0,\tau)} w^{2-\mu_0}\right)^{1/2}$$

Since $w^{1-\mu_0/2} \in L^2([0,T];H^1(-R,R)) \subseteq L^2([0,T];C^{1/2}(-R,R))$,

$$\iint_{(0,x_1)\times(0,\tau)} w^{2-\mu_0} = \iint_{(0,x_1)\times(0,\tau)} \frac{w^{2-\mu_0}(x,t)}{x} x \leq \int_0^{x_1} x \int_0^{\tau} \|w^{1-\mu_0/2}(t)\|^2_{C^{1/2}(-R,R))} \leq Cx_1^2.$$

On the other hand, since $\mu_0 < \mu_1 < 1$ and $\sup_{(0,x_1)\times(0,\tau)} w = o(1)$ as $x_1 \to 0$,

$$\iint_{(0,x_1)\times(0,\tau)} w^{-\mu_0} w_x^2 \leq o(1) \iint_{(0,x_1)\times(0,\tau)} w^{-\mu_1} w_x^2 = o(1) \quad \text{as } x_1 \to 0.$$

Hence

$$\frac{1}{x_1}\left| \int_0^{x_1} Y(s,\tau)ds - \int_0^{x_1} Y(s,0)ds \right| = o(1) \quad \text{as } x_1 \to 0,$$

and, since Y is continuous, $Y(0,\tau) = Y(0,0)$ for all $0 < \tau \leq t_1$.

4. **Singular limits: The degenerate KPP equation.** In this section we prove Theorems 2.3 and 2.4. First we consider the singular limit problem of the PDE-system (1).

Proof of Theorem 2.3. Let $T > 0$. In view of Lemma 3.1, it follows that

$$0 \leq u_k, v_k \leq w_k \leq C \quad \text{in } \mathbb{R}\times(0,T+1) \text{ for all } k,$$

where C is independent of k. Let $R > 0$ and let ζ_R be a cut-off function which satisfies (15).

Lemma 4.1. *Let $\mu > 0$. We have*

$$\iint_{\mathbb{R}\times(0,T+1)} \zeta_R^2 w_k^{\mu} w_{kx}^2 \leq C_{R,T,\mu} \tag{24}$$

and

$$\frac{\gamma}{k}\iint_{\mathbb{R}\times(0,T+1)} \zeta_R^2 v_k^2 \leq C_{R,T}. \tag{25}$$

Proof. Multiplying the equation for w_k, formally, by $\zeta_R^2 w_k^{\mu}$ with $\mu > 0$ and integrating by parts over $\mathbb{R}\times(0,T+1)$, we obtain that

$$\frac{1}{\mu+1}\int_{\mathbb{R}} \zeta_R^2 w_k^{\mu+1}\Big|_{t=0}^{t=T+1} = -2\iint_{\mathbb{R}\times(0,T+1)} \zeta_R\zeta_R' w_k^{\mu+1} w_{kx} - \mu \iint_{\mathbb{R}\times(0,T+1)} \zeta_R^2 w_k^{\mu} w_{kx}^2 + \iint_{\mathbb{R}\times(0,T+1)} \zeta_R^2 w_k^{\mu}\Big((1-u_k-\alpha v_k)u_k + \gamma(1-\beta u_k - \frac{v_k}{k})v_k\Big).$$

Here, we can estimate the first term of the right hand side by Young's inequality as follows:

$$\left|2\iint_{\mathbb{R}\times(0,T+1)} \zeta_R\zeta_R' w_k^{\mu+1} w_{kx}\right| \leq \frac{2}{\mu}\iint_{\mathbb{R}\times(0,T+1)} \zeta_R'^2 w_k^{2+\mu} + \frac{\mu}{2}\iint_{\mathbb{R}\times(0,T+1)} \zeta_R^2 w_k^{\mu} w_{kx}^2.$$

Therefore, thanks to this inequality and Lemma 3.1, we obtain

$$\frac{\mu}{2}\iint_{\mathbb{R}\times(0,T+1)} \zeta_R^2 w_k^{\mu} w_{kx}^2 + \frac{\gamma}{k}\iint_{\mathbb{R}\times(0,T+1)} \zeta_R^2 w_k^{\mu} v_k^2 \leq C_{R,T,\mu},$$

where the constant $C_{R,T,\mu}$ does not depend on k. This implies (24). Simply, multiplying the equation for w_k by ζ_R^2 and using the integration by parts over $\mathbb{R}\times(0,T+1)$, we obtain that

$$\int_{\mathbb{R}} \zeta_R^2 w_k \Big|_{t=0}^{t=T+1} = -2\iint_{\mathbb{R}\times(0,T+1)} \zeta_R\zeta_R' w_k w_{kx} + \iint_{\mathbb{R}\times(0,T+1)} \zeta_R^2 \left((1-u_k-\alpha v_k)u_k + \gamma(1-\beta u_k - \frac{v_k}{k})v_k\right).$$

The use of (24), Lemma 3.1 and the similar estimate

$$\left|2\iint_{\mathbb{R}\times(0,T+1)} \zeta_R\zeta_R' w_k w_{kx}\right| \le \frac{\mu}{2}\iint_{\mathbb{R}\times(0,T+1)} \zeta_R'^2 w_k^{2-\mu} + \frac{2}{\mu}\iint_{\mathbb{R}\times(0,T+1)} \zeta_R^2 w_k^{\mu} w_{kx}^2$$

give

$$\frac{\gamma}{k}\iint_{\mathbb{R}\times(0,T+1)} \zeta_R^2 v_k^2 \le C_{R,T}.$$

This leads (25). □

In order to obtain a strong convergence of $\{w_k\}$, we claim that

$$\{w_k^{3/2}\} \text{ is precompact in } L^2_{\text{loc}}(\mathbb{R}\times[0,\infty)). \tag{26}$$

In order to do that, we first prove the following Lemma:

Lemma 4.2. *Let $T,R>0$, $|\xi|\le 1$ and $0<\tau\le 1$, and set $h(s)=\frac{2}{3}s^{3/2}$. Then there exists a constant $C_{R,T}$ which does not depend on k such that*

$$\iint_{(-R,R)\times(0,T)} \left(h(w_k(x+\xi,t)) - h(w_k(x,t))\right)^2 dxdt \le C_{R,T}\xi^2 \tag{27}$$

and

$$\iint_{(-R,R)\times(0,T)} \left(h(w_k(x,t+\tau)) - h(w_k(x,t))\right)^2 dxdt \le C_{R,T}\tau. \tag{28}$$

Proof. The proof of (27) is immediate:

$$\begin{aligned}
&\iint_{(-R,R)\times(0,T)} \left(h(w_k(x+\xi,t)) - h(w_k(x,t))\right)^2 dxdt \\
=&\iint_{(-R,R)\times(0,T)} \left(\int_0^{\xi} (h'(w_k)w_{kx})(x+s,t)ds\right)^2 dxdt \\
=&\iint_{(-R,R)\times(0,T)} \left(\int_0^{1} (h'(w_k)w_{kx})(x+r\xi,t)\xi dr\right)^2 dxdt \\
\le&\xi^2 \iiint_{(0,1)\times(-R-1,R+1)\times(0,T)} w_k w_{kx}^2 dxdtdr \le C_{R,T}\xi^2.
\end{aligned}$$

Here we have used the estimate (24) with $\mu=1$.

Next we consider (28). Using (15), we have

$$\iint_{\mathbb{R}\times(0,T)} (h(w_k(x,t+\tau)) - h(w_k(x,t)))^2 \zeta_R(x)^2 dxdt$$
$$= \iint_{\mathbb{R}\times(0,T)} (h(w_k(x,t+\tau)) - h(w_k(x,t)))\zeta_R(x)^2 \int_0^\tau h_t(w_k(x,t+r))drdxdt$$
$$= \iint_{\mathbb{R}\times(0,T)} (h(w_k(x,t+\tau)) - h(w_k(x,t)))\zeta_R(x)^2 \int_0^\tau (w_k^{\frac{1}{2}}(x,t+r)(((w_k w_{kx})_x)(x,t+r)$$
$$+ f(w_k(x,t+r), v_k(x,t+r)) + g(w_k(x,t+r), v_k(x,t+r))))drdxdt$$

In the first term of the right hand side, we integrate by parts:

$$|\iiint_{(0,\tau)\times\mathbb{R}\times(0,T)} (h_x(w_k(x,t+\tau)) - h_x(w_k(x,t)))\zeta_R(x)^2(w_k^{\frac{3}{2}} w_{kx})(x,t+r)$$
$$+2(h(w_k(x,t+\tau)) - h(w_k(x,t)))\zeta_R(x)\zeta_R'(x)(w_k^{\frac{3}{2}} w_{kx})(x,t+r)$$
$$+\frac{1}{2}(h(w_k(x,t+r)) - h(w_k(x,t)))\zeta_R(x)^2(w_k^{\frac{1}{2}} w_{kx}^2)(x,t+r)drdxdt|$$
$$\leq C \iiint_{(0,\tau)\times\mathbb{R}\times(0,T+1)} (w_k w_{kx}^2)(x,t)\zeta_R^2(x)drdxdt$$
$$+ C \iiint_{(0,\tau)\times\mathbb{R}\times(0,T+1)} (\zeta_R'(x))^2 w_k^2(x,t)drdxdt$$
$$+ C \iiint_{(0,\tau)\times\mathbb{R}\times(0,T+1)} (w_k^{\frac{1}{2}} w_{kx}^2)(x,t)drdxdt \leq C_{R,T}\tau.$$

The second and third terms are estimated by using (25) as follows:

$$|\iiint_{(0,\tau)\times\mathbb{R}\times(0,T)} (h(w_k(x,t+\tau)) - h(w_k(x,t)))\zeta_R^2(x)w_k^{\frac{1}{2}}(x,t+r)$$
$$\times (f(w_k(x,t+r), v_k(x,t+r)) + g(w_k(x,t+r), v_k(x,t+r)))drdxdt|$$
$$\leq C_{R,T}\tau$$

which completes the proof of (28) and Lemma 4.2. □

Passage to the limit. It follows from (25) that

$$v_k \to 0 \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}\times[0,\infty))$$

as $k \to 0$. Moreover, we deduce from the boundedness of $\{w_k\}$ that there exists a function $w \in L^\infty(\mathbb{R}\times[0.\infty))$ and a subsequence $\{w_{k_j}\}$ such that

$$w_{k_j} \rightharpoonup w \text{ weakly in } L^2_{\text{loc}}(\mathbb{R}\times[0,\infty)) \tag{29}$$

as $k_j \to 0$. On the other hand, it follows from Lemma 4.2 and the Riesz-Fréchet-Kolmogoroff Theorem (Theorem IV.25 and Corollary IV.26 in [11]) that $\{w_k^{\frac{3}{2}}\}$ is relatively compact in $L^2_{\text{loc}}(\mathbb{R}\times[0,\infty))$. Therefore, there exists a subsequence of $\{w_{k_j}\}$ and a function $W \in L^2_{\text{loc}}([0,\infty); H^1_{\text{loc}}(\mathbb{R})) \cap L^\infty(\mathbb{R}\times[0,\infty))$ such that

$$w_{k_j}^{\frac{3}{2}} \to W \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}\times[0,\infty)) \text{ and a.e. in } \mathbb{R}\times[0,\infty)$$

as $k_j \to 0$. Hence since the function $h(s) = \frac{2}{3}s^{\frac{3}{2}}$ is continuous, we deduce that

$$w_{k_j} \to h^{-1}(\frac{2}{3}W) \text{ a.e. in } \mathbb{R}\times[0,\infty)$$

and it follows from (29) that $w = h^{-1}(\frac{2}{3}W)$ so that

$$w_{k_j} \to w \text{ a.e in } \mathbb{R} \times [0,\infty)$$

and

$$w_{k_j}^{\frac{3}{2}} \to w^{\frac{3}{2}} \text{ strongly in } L^2_{\rm loc}(\mathbb{R} \times [0,\infty)) \text{ and a.e. in } \mathbb{R} \times [0,\infty)$$
$$\text{and weakly in } L^2_{\rm loc}([0,\infty); H^1_{\rm loc}(\mathbb{R})).$$

Applying Lebesgue's dominated convergence theorem, we also deduce that

$$w_{k_j}^{\frac{1}{2}} \to w^{\frac{1}{2}} \text{ strongly in } L^2_{\rm loc}(\mathbb{R} \times [0,\infty)).$$

Taking the difference between the equations for w and v, we deduce that

$$(w_k - v_k)_t = ((w_k - v_k)w_{kx})_x + (w_k - v_k)(1 - w_k + (1-\alpha)v_k)$$

so that $w_k - v_k$ satisfies the weak form

$$\iint_{\mathbb{R}\times\mathbb{R}^+} ((w_k - v_k)w_{kx}\varphi_x - (w_k - v_k)\varphi_t - (w_k - v_k)(1 - w_k + (1-\alpha)v_k)\varphi) = \int_{\mathbb{R}} u_0\varphi(\cdot,0)$$

for all $\varphi \in C^1(\mathbb{R} \times [0,\infty))$ such that φ vanishes for large $|x|$ and t. We deduce from the convergence properties above and Lebesgue's dominated convergence theorem that

$$\iint_{\mathbb{R}\times\mathbb{R}^+} w_{k_j} w_{k_j x}\varphi_x = \iint_{\mathbb{R}\times\mathbb{R}^+} \frac{2}{3} w_{k_j}^{\frac{1}{2}} (w_{k_j}^{\frac{3}{2}})_x \varphi_x$$
$$\to \iint_{\mathbb{R}\times\mathbb{R}^+} \frac{2}{3} w^{\frac{1}{2}} (w^{\frac{3}{2}})_x \varphi_x = \iint_{\mathbb{R}\times\mathbb{R}^+} w w_x \varphi_x$$

as $k_j \to 0$. Moreover, there also holds that

$$\iint_{\mathbb{R}\times\mathbb{R}^+} (-(w_{k_j} - v_{k_j})\varphi_t - (w_{k_j} - v_{k_j})(1 - w_{k_j} + (1-\alpha)v_{k_j})\varphi)$$
$$\to \iint_{\mathbb{R}\times\mathbb{R}^+} (-w\varphi_t - w(1-w)\varphi)$$

as $k_j \to 0$.

Next, we consider the remaining term.

$$\left|\iint_{\mathbb{R}\times\mathbb{R}^+} v_{k_j} w_{k_j x}\varphi_x\right| = \left|\iint_{\{\mathbb{R}\times\mathbb{R}^+\}\cap\{(x,t);w_{k_j}\neq 0\}} v_{k_j}^{\frac{1}{2}} \frac{v_{k_j}^{\frac{1}{2}}}{w_{k_j}^{\frac{1}{2}}} w_{k_j}^{\frac{1}{2}} w_{k_j x}\varphi_x\right|$$
$$\le \iint_{\mathbb{R}\times\mathbb{R}^+} \left|v_{k_j}^{\frac{1}{2}} w_{k_j}^{\frac{1}{2}} w_{k_j x}\varphi_x\right|$$
$$\le \left(\iint_{\mathbb{R}\times\mathbb{R}^+} \left|v_{k_j}\varphi_x\right|\right)^{1/2} \left(\iint_{\mathbb{R}\times\mathbb{R}^+} \left|w_{k_j} w_{k_j x}^2 \varphi_x\right|\right)^{1/2}$$
$$\le C(\varphi)\left(\iint_{\mathbb{R}\times\mathbb{R}^+} \left|v_{k_j}\varphi_x\right|\right)^{1/2}$$

Since $L^2_{\rm loc}(\mathbb{R} \times [0,\infty)) \subset L^1_{\rm loc}(\mathbb{R} \times [0,\infty))$, it follows that

$$\|v_{k_j}\|_{L^1((-R,R)\times[0,T))} \le C\|v_{k_j}\|_{L^2((-R,R)\times[0,T))} \to 0$$

as $k_j \to 0$, so that

$$\iint_{\mathbb{R}\times\mathbb{R}^+} v_{k_j} w_{k_j x}\varphi_x \to 0$$

as $k_j \to 0$. Setting $u := w$, we deduce that u satisfies the weak form

$$\iint_{\mathbb{R}\times\mathbb{R}^+} (uu_x\varphi_x - u\varphi_t - u(1-u)\varphi) = \int_{\mathbb{R}} u_0\varphi(\cdot, 0)$$

for all $\varphi \in C^1(\mathbb{R}\times[0,\infty))$ such that φ vanishes for large $|x|$ and large t. Thus u coincide with the unique weak solution of the initial value problem

$$\begin{cases} u_t = (uu_x)_x + u(1-u) & \text{in } \mathbb{R}\times\mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } x \in \mathbb{R}, \end{cases}$$

and the whole sequence $\{u_k\}$ converges to u as $k \to 0$.

Next we consider the singular limit problem for traveling wave solutions. We begin with the case of segregated traveling wave solutions.

Proof of Theorem 2.4(*i*). Let $0 < k < 1$ and let (U_k^*, V_k^*, c_k^*) be the unique segregated traveling wave solution solving problem (5). Then $W_k^* := U_k^* + V_k^*$ satisfies

$$\begin{cases} (WW')' + c_k^* W' + Wh_k(W,z) = 0 & \text{in } \mathbb{R} \\ 0 < W \le 1 \text{ in } \mathbb{R}, \quad W(-\infty) = 1, \quad W(\infty) = k, \end{cases}$$

where

$$h_k(w,z) = \begin{cases} 1 - w & \text{if } z < 0 \\ \gamma(1 - w/k) & \text{if } z > 0. \end{cases}$$

We can prove from the phase plane analysis that

$$c_k^* \to c_0^* = 1/\sqrt{2} \quad \text{as } k \to 0. \tag{30}$$

The proof is given in the appendix.

Let $R > 0$ and let ζ_R be a cut-off function satisfying (15). Multiplying the equation for W_k^* by $\zeta_R^2 W_k^*$ and integrating by parts, we have

$$\begin{aligned} -2\int_{-\infty}^{\infty} (W_k^*)^2 (W_k^*)' \zeta_R \zeta_R' dz - \int_{-\infty}^{\infty} W_k^* (W_k^*)'^2 \zeta_R^2 dz \\ + \int_{-\infty}^{\infty} c_k^* W_k^* (W_k^*)' \zeta_R^2 dz + \int_{-\infty}^{\infty} (W_k^*)^2 h_k(W_k^*, z)\zeta_R^2 dz = 0 \end{aligned}$$

Using the estimates

$$\left| 2\int_{-\infty}^{\infty} (W_k^*)^2 (W_k^*)' \zeta_R \zeta_R' dz \right| \le 2\int_{-\infty}^{\infty} (W_k^*)^3 (\zeta_R')^2 dz + \frac{1}{2}\int_{-\infty}^{\infty} W_k^* (W_k^*)'^2 \zeta_R^2 dz \tag{31}$$

and

$$\left| \int_{-\infty}^{\infty} c_k^* W_k^* (W_k^*)' \zeta_R^2 dz \right| \le \frac{1}{4}\int_{-\infty}^{\infty} W_k^* (W_k^*)'^2 \zeta_R^2 dz + \int_{-\infty}^{\infty} (c_k^*)^2 W_k^* \zeta_R^2 dz, \tag{32}$$

we obtain that

$$\int_{-\infty}^{\infty} W_k^* (W_k^*)'^2 \zeta_R^2 dz + \frac{\gamma}{k}\int_0^{\infty} (W_k^*)^3 \zeta_R^2 dz \le C_R.$$

Here we have used that c_k^* and W_k^* are uniformly bounded (see also Appendix). Since $(W_k^*)^{3/2}$ is uniformly bounded in $H^1_{\mathrm{loc}}(\mathbb{R})$, there exist a function $U^{3/2} \in H^1_{\mathrm{loc}}(\mathbb{R})$ and a subsequence $\{k_j\}$ such that

$$(W_{k_j}^*)^{3/2} \to U^{3/2} \text{ weakly in } H^1_{\mathrm{loc}}(\mathbb{R}) \text{ and } W_{k_j}^* \to U \text{ in } C_{\mathrm{loc}}(\mathbb{R}) \text{ as } k_j \to 0.$$

In addition, since $W_k^* \to 0$ in $L^3_{\mathrm{loc}}([0,\infty))$,

$$U(z) = 0 \quad \text{if } z \geq 0.$$

Since W_k^* is decreasing in $\mathbb{R}$ (see [6]), U is also decreasing in $\mathbb{R}$.

Observe that, since $W_k^*((W_k^*)' + c_k^*)$ is decreasing in $(-\infty, 0]$ in the equation for W and since it vanishes at $z = 0$([6] and see also (35) in the appendix), $-c_k^* < (W_k^*)' < 0$ in $(-\infty, 0)$. Hence

$$(W_k^*)'' = -\frac{((W_k^*)' + c_k^*)(W_k^*)'}{W_k^*} + W_k^* - 1 \geq -1 \text{ in } (-\infty, 0).$$

Since $W_k^*(W_k^*)'' \geq -1$ and $(W_k^{*\prime})^2 \geq 0$ in $(-\infty, 0)$, we have $(W_k^{*2})'' \geq -2$ in $(-\infty, 0)$. On the other hand, since

$$W_k^*(W_k^*)'' = -(W_k^{*\prime})^2 - c_k^* W_k^{*\prime} - W_k^*(1 - W_k^*) \leq {c_k^*}^2 + 1,$$
$$(W_k^{*\prime})^2 \leq {c_k^*}^2,$$

in $(-\infty, 0)$, we obtain $(W_k^{*2})'' \leq 4{c_k^*}^2 + 2 < 4$ in $(-\infty, 0)$. Thus, there exists a function $U \in C^2((-\infty, 0))$ such that

$$ {W_{k_j}^*}^2 \to U^2 \text{ in } C^1_{\mathrm{loc}}((-\infty, 0))$$

and moreover,

$$W_{k_j}^* \to U \text{ in } C^1_{\mathrm{loc}}((-\infty, 0))$$

as $k_j \to 0$. It follows from the appendix and (30) that $(W_{k_j}^*)'(0^-) = -c_{k_j}^* \to -c_0^* = U'(0^-) < 0$ as $k_j \to 0$. Therefore $U > 0$ in a left-neighborhood of $z = 0$.

Finally, passing to the limit $j \to \infty$ in the equation for $W_{k_j}^*$, it follows that $U \in C(\mathbb{R})$ satisfies

$$\begin{cases} (UU')' + c_0^* U' + U(1 - U) = 0 \text{ and } 0 < U \leq 1 \text{ in } (-\infty, 0), \\ U = 0 \text{ in } [0, \infty). \end{cases}$$

Since U is decreasing, $U(-\infty) = 1$, and we have proved that $U = U_0^*$, the unique traveling wave solution with minimal wave velocity c_0^* and interface at $z = 0$.

Since the limit U is uniquely defined, it does not depend on the specific subsequence $\{k_j\}$.

Next we prove the convergence of overlapping traveling wave solutions.

Proof of Theorem 2.4(ii). Let $0 < k < 1$ be so small that $c > c_k^*$. We have shown in [5] that there exists an overlapping traveling wave solution, $(U_k^{(c)}, V_k^{(c)}, c)$, which satisfies

$$\begin{cases} (U(U+V)')' + cU' + U(1 - U - V) = 0 \quad \text{in } \mathbb{R} \\ (V(U+V)')' + cV' + \gamma V(1 - (U+V)/k) = 0 \quad \text{in } \mathbb{R} \\ U > 0, \quad V > 0, \quad U + V \leq 1 \quad \text{in } \mathbb{R} \\ U(-\infty) = 1, \quad U(\infty) = 0, \quad V(-\infty) = 0, \quad V(\infty) = k. \end{cases}$$

By translation invariance we assume that $U_k^{(c)}(0) = \frac{1}{2}$. Below we omit the superscript ${}^{(c)}$.

The function $W_k := U_k + V_k$ satisfies

$$\begin{cases} (WW')' + cW' + U(1 - W) + \gamma V(1 - W/k) = 0 \quad \text{in } \mathbb{R} \\ 0 < W \leq 1 \text{ in } \mathbb{R}, \quad W(-\infty) = 1, \quad W(\infty) = k. \end{cases}$$

Let $R > 0$. Multiplying the equation for W_k by $\zeta_R^2 W_k$, where ζ_R is a cut-off function satisfying (15), integrating by parts and using similar estimates to (31) and (32), we obtain that

$$\int_{-\infty}^{\infty} W_k (W_k')^2 \zeta_R^2 dz + \frac{\gamma}{k} \int_{-\infty}^{\infty} V_k W_k^2 \zeta_R^2 dz \le C_R.$$

Since $V_k \le W_k$ in $\mathbb{R}$, $V_k \to 0$ in $L^3_{\mathrm{loc}}(\mathbb{R})$. Since $W_k^{3/2}$ is uniformly bounded in $H^1_{\mathrm{loc}}(\mathbb{R})$, there exist a function $U^{3/2} \in H^1_{\mathrm{loc}}(\mathbb{R})$ and a subsequence $k_j \to 0$ such that

$$W_{k_j}^{3/2} \to U^{3/2} \text{ weakly in } H^1_{\mathrm{loc}}(\mathbb{R}) \quad \text{as } k_j \to 0$$

and hence

$$W_{k_j} \to U \text{ in } C_{\mathrm{loc}}(\mathbb{R}) \quad \text{as } k_j \to \infty.$$

Passing to the limit in the equation for U_k, we find that U is a solution of

$$(UU')' + cU' + U(1-U) = 0 \quad \text{in } \mathbb{R}$$

in the sense of distributions. Since $H^1_{\mathrm{loc}}(\mathbb{R}) \subset C_{\mathrm{loc}}(\mathbb{R})$, $U(0) = \frac{1}{2}$. To show that U is the unique traveling wave solution with wave velocity c solving problem (6) which satisfies $U(0) = \frac{1}{2}$, it is enough to prove that $U(-\infty) = 1$ and $U(\infty) = 0$. Since W_k is decreasing in $\mathbb{R}$ (see [5]), U is also decreasing in $\mathbb{R}$. So, $U(-\infty) = 1$ and $U(\infty) = 0$.

Since the limit U is uniquely defined, it does not depend on the specific subsequence $\{k_j\}$.

Finally we prove the behavior of the wave velocity of segregated traveling wave solutions as $k \to 0$.

Proof of Theorem 2.4(iii). We only consider the limit $k \to 1^+$. The proof for the limit $k \to 1^-$ is similar.

So let $k > 1$ and let (U_k^*, V_k^*, c_k^*) be the unique segregated traveling wave with its interface at $z = 0$. Let $w_k^* \in (1, k)$ be the value of $W_k^* = U_k^* + V_k^*$ at $z = 0$. Observe that $c_k^* < 0$ if $k > 1$. We set

$$p_k = \frac{1}{c_k^*}(W_k^*)_z$$

and

$$W_k^* = 1 + s(k-1) \quad \text{for } 0 < s < 1, \qquad w_k^* = 1 + s_k^*(k-1),$$

and consider p_k as a function of s (this is possible since $z \mapsto W_k^*(z)$ is strictly increasing if $k > 1$). Then

$$p_k(0^+) = p_k(1^-) = 0, \quad p_k(s_k^*) = -1, \quad -1 \le p_k(s) < 0 \text{ for } 0 < s < 1,$$

and

$$p_k' = -\frac{(k-1)(p_k+1)}{1+s(k-1)} + \begin{cases} \dfrac{(k-1)^2 s}{(c_k^*)^2 p_k} & \text{if } 0 < s < s_k^* \\ \dfrac{\gamma(k-1)^2(s-1)}{k(c_k^*)^2 p_k} & \text{if } s_k^* < s < 1. \end{cases}$$

Observe that

$$\frac{(k-1)p_k(s)(p_k(s)+1)}{1+s(k-1)} \to 0 \quad \text{as } k \to 1^+.$$

Hence, if $0 < s < s_k^*$,

$$\left(p_k^2\right)'(s) = 2\left(\frac{k-1}{c_k^*}\right)^2 s + o(1) \quad \text{as } k \to 1^+,$$

and, since $p_k(0) = 0$,

$$p_k^2(s) = \left(\frac{k-1}{c_k^*}\right)^2 s^2 + o(1) \quad \text{as } k \to 1^+.$$

Substitution of $s = s_k^*$ yields that

$$1 = \left(\frac{k-1}{c_k^*}\right)^2 (s_k^*)^2 + o(1) \quad \text{as } k \to 1^+,$$

whence, if $0 < s < s_k^*$,

$$\frac{c_k^*}{k-1} = -s_k^* + o(1), \quad p_k(s) = \frac{k-1}{c_k^*} s + o(1) \quad \text{as } k \to 1^+. \tag{33}$$

Similarly, if $s_k^* < s < 1$ we have that

$$p_k^2(s) = \frac{\gamma}{k}\left(\frac{k-1}{c_k^*}\right)^2 (1-s)^2 + o(1) = \gamma \left(\frac{k-1}{c_k^*}\right)^2 (1-s)^2 + o(1),$$

whence

$$1 = \gamma \left(\frac{k-1}{c_k^*}\right)^2 (1-s_k^*)^2 + o(1) \quad \text{as } k \to 1^+.$$

Therefore, as $k \to 1^+$

$$\frac{c_k^*}{k-1} = -\sqrt{\gamma}(1-s_k^*) + o(1), \quad p_k(s) = \frac{k-1}{c_k^*}\sqrt{\gamma}(1-s) + o(1). \tag{34}$$

By (33) and (34), $s_k^*(1+\sqrt{\gamma}) \to \sqrt{\gamma}$ as $k \to 1^+$:

$$s_k^* \to \frac{\sqrt{\gamma}}{1+\sqrt{\gamma}} \quad \text{and} \quad \frac{c_k^*}{k-1} \to -\frac{\sqrt{\gamma}}{1+\sqrt{\gamma}} \quad \text{as } k \to 1^+.$$

5. **Remarks and open problems.** In this paper, we have considered the existence of a solution with segregation property of Problem (1) for general initial data and singular limit problems to reveal a relation between Problem (1) and the degenerate Fisher-KPP equation. Though we considered the case $k \to 0$, this gives some insight in the case $k \to \infty$. Actually, the change of variables $\tilde{v} = u/k$, $\tilde{u} = v/k$, $\tilde{t} = \gamma t$, $\tilde{x} = \sqrt{\gamma/k}x$, $\tilde{\gamma} = 1/\gamma$, $\tilde{k} = 1/k$, $\tilde{\alpha} = \beta k$ and $\tilde{\beta} = \alpha k$ gives

$$\begin{cases} \tilde{u}_{\tilde{t}} = (\tilde{u}(\tilde{u}+\tilde{v})_{\tilde{x}})_{\tilde{x}} + \tilde{u}(1-\tilde{u}-\tilde{\alpha}\tilde{v}), \\ \tilde{v}_{\tilde{t}} = (\tilde{v}(\tilde{u}+\tilde{v})_{\tilde{x}})_{\tilde{x}} + \tilde{\gamma}\tilde{v}(1-\tilde{\beta}\tilde{u}-\tilde{v}/\tilde{k}), \end{cases}$$

which corresponds to the system (1).

In the first part, we proved the existence of a solution with segregation property for initial data merely $w_0 = u_0 + v_0 \geq 0$ in one space dimension. However, for arbitrary space dimensions, the problem is still open. In the second part, we treated two types of traveling wave solutions, say segregated and overlapping traveling waves. Recently, the existence of other traveling waves for large and small gamma, namely partially overlapping traveling wave solutions, and standing waves which possess extremely rapid decay tails was proven in [7][8]. From these results, Problem (1) possesses a surprising variety of mathematical structures depending on the parameter values even though we focus on traveling wave solutions. It appears that these reflect the parabolic-hyperbolic nature of Problem (1), which are completely different from those of a parabolic-parabolic system such as a reaction-diffusion system. As argued in this paper and in [5], in a special case, the structure of traveling wave solutions of Problem (1) is similar to that of the degenerate Fisher-KPP equation, noting that the first equation in Problem (1) coincides with a degenerate Fisher-KPP equation if we set $v = 0$, whereas setting $u = 0$ in the second equation of Problem (1) also yields a Fisher-KPP equation. We propose to study in future work the complete structure of the set of traveling wave solutions.

Appendix.

Appendix A. **Proof of** (30)**.** Let us consider

$$\begin{cases} (UU_z)_z + cU_z + (1-U)U = 0 \quad \text{in } \mathbb{R}_-, \\ U(z) > 0 \quad \text{in } \mathbb{R}_-, \\ U(-0) = h, \quad U_z(-0) = -c, \\ U(-\infty) = 1 \end{cases} \tag{35}$$

and

$$\begin{cases} \left(VV_z\right)_z + cV_z + \gamma\left(1 - \dfrac{V}{k}\right)V = 0 \quad \text{in } \mathbb{R}_+, \\ V(z) > 0 \quad \text{in } \mathbb{R}_+, \\ V(+0) = h, \quad V_z(+0) = -c, \\ V(+\infty) = k \end{cases} \tag{36}$$

with $0 < k < 1$ and $\gamma > 0$. Moreover, $h \geq 0$ is given a priori.

Proposition A.1 ([6])**.** *Assume that there exists a pair $(U(z), V(z), h, c)$ satisfying* (35) *and* (36) *for some $\gamma, k > 0$. Then, the following properties hold true*:

(i) *If $k = 1$, then $(U(z), V(z), h, c) = (1, 1, 1, 0)$.*
(ii) *If $k > 1$, then $h \in (1, k)$, $U_z > 0$, $V_z > 0$ and $c < 0$.*
(iii) *If $0 < k < 1$, then $h \in (k, 1)$, $U_z < 0$, $V_z < 0$ and $c > 0$.*

Theorem A.2. *Assume $0 < k < 1$. Then, for any $\gamma > 0$, there exists a unique (U, V, h, c) of* (35) *and* (36) *satisfying $0 < c < 1/\sqrt{2}$.*

Corollary A.3. *Assume Theorem A.2. For given $0 < k < 1$, denote c by c_k. Then,*

$$\lim_{k \to 0} c_k = c_0 = \frac{1}{\sqrt{2}}.$$

Remark A.4. The boundedness of the velocity c in Theorem A.2 and the convergence of c_k in Corollary A.3 are shown in the construction of (U, V, h, c).

A.1. **Proof of Theorem A.2.** We introduce

$$\begin{bmatrix} U_1(z) \\ U_2(z) \end{bmatrix} := \begin{bmatrix} U(z) \\ \dfrac{1}{c}U_z(z) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} := \begin{bmatrix} V(z) \\ \dfrac{1}{c}V_z(z) \end{bmatrix}.$$

Then, the following dynamical systems for (U_1, U_2) and (V_1, V_2) are derived from a usual calculation:

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}' = \begin{bmatrix} cU_2 \\ F(U_1, U_2, c) \end{bmatrix}, \quad \begin{bmatrix} U_1(-\infty) \\ U_2(-\infty) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{37}$$

and

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}' = \begin{bmatrix} cV_2 \\ G(V_1, V_2, c; k, \gamma) \end{bmatrix}, \quad \begin{bmatrix} V_1(+\infty) \\ V_2(+\infty) \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix}, \tag{38}$$

where

$$G(V_1, V_2, c; k, \gamma) := \frac{1}{ckV_1}\left[\gamma V_1(V_1 - k) - c^2 k V_2(1 + V_2)\right] \tag{39}$$

and $F(U_1, U_2, c) := G(U_1, U_2, c; 1, 1)$. If they satisfy the following matching condition

$$\begin{bmatrix} U_1(0) \\ U_2(0) \end{bmatrix} = \begin{bmatrix} V_1(0) \\ V_2(0) \end{bmatrix} = \begin{bmatrix} h \\ -1 \end{bmatrix},$$

then we obtain the desired solution (U, V, h, c) of (35) and (36).

Lemma A.5. *The following* (i) *and* (ii) *hold for* (37):

(i) *For any* $0 < c < 1/\sqrt{2}$*, there exists a unique solution* $(U_1(z; c), U_2(z; c))$ *of* (37) *and a continuous function*

$$h_U : (0, \sqrt{1/2}) \to (0, 1)\,; c \mapsto h_U(c)$$

such that $(U_1(0; c), U_2(0; c)) = (h_U(c), -1)$.

(ii) *For* $c \geq 1/\sqrt{2}$*, any local solution of* (37) *does not satisfy* $U_2(z; c) = -1$.

Remark A.6. The upper bound of the velocity of the segregated traveling wave solution coms from Lemma A.5.

Lemma A.7. *For any* $0 < c < 1/\sqrt{2}$ *the function* h_U *given in Lemma A.5 satisfies the following* (i) *and* (ii)*:*

(i) h_U *is monotone decreasing for any* $c > 0$,

(ii) $\lim_{c\to 0} h_U(c) = 1$ *and* $\lim_{c\to 1/\sqrt{2}} h_U(c) = 0$.

Lemma A.8. *Assume* $k > 0$ *and* $\gamma > 0$*. For any* $c > 0$*, there exists a unique solution* $(V_1(z; c), V_2(z; c))$ *of* (38) *and a continuous function*

$$h_V = h_V(\,\cdot\,; k, \gamma) : (0, +\infty) \to (k, +\infty); c \mapsto h_V(c)$$

such that $(V_1(0; c), V_2(0; c)) = (h_V(c), -1)$.

Lemma A.9. *For any* $k > 0$*,* $\gamma > 0$ *and* $c > 0$ *the function* h_V *given in Lemma A.8 satisfies the following* (i) *and* (ii):

(i) h_V *is monotone increasing for any* $c > 0$,

(ii) $\lim_{c\to 0} h_V(c) = k$ *and* $\lim_{c\to +\infty} h_V(c) = +\infty$.

Proofs of Lemmata A.5, A.7, A.8 and A.9 are essentially reduced into the study of the dynamcal system

$$\begin{bmatrix} \varphi \\ \psi \end{bmatrix}' = \begin{bmatrix} c\psi \\ G(\varphi, \psi, c; k, \gamma) \end{bmatrix} \tag{40}$$

For the sake of convenience, we display the phase portrait of (40) in Figure 3.

The proofs are the same as those in [6].

Lemma A.10. *Let* $h_V(c) = h_V(c; k, \gamma)$ *as in Lemma A.8. Then, it is satisfied that for any* $k > 0$ *and* $\gamma > 0$,

$$k < h_V(c) < k + c\sqrt{\frac{k}{\gamma}}.$$

Remark A.11. Lemma A.10 shows Corollary A.3.

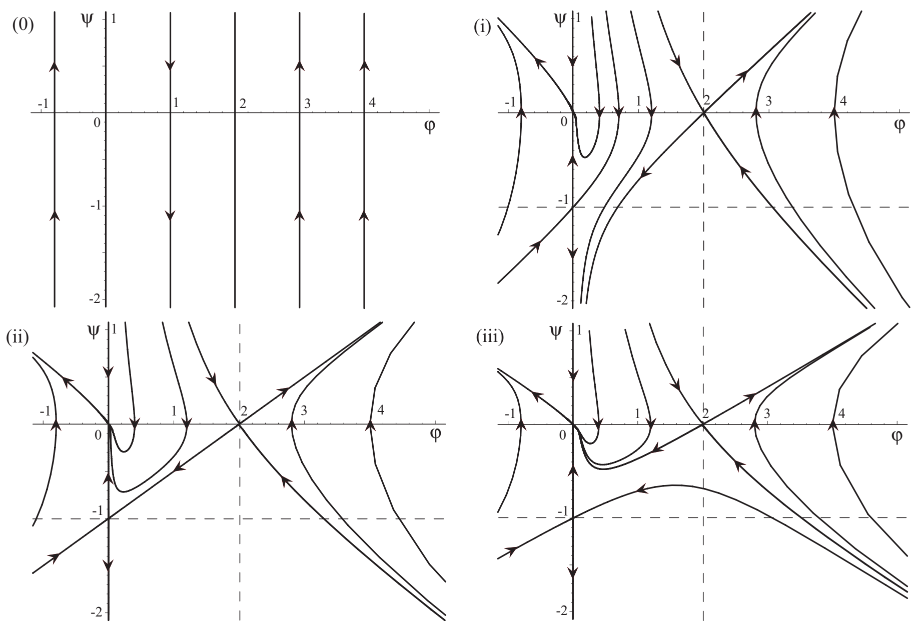

FIGURE 3. (φ, ψ)-phase planes for (40) with $k = 2$ and $\gamma = 1$: (0) $c = 0$, (i) $c = 0.8$, (ii) $c = 1$, (iii) $c = 1.2$.

Proof of Lemma A.10. Let (V_1, V_2) be the solution of (38) with $c > 0$. Then,

$$\begin{aligned} \frac{dV_2}{dV_1} &= \frac{G(V_1, V_2, c; k, \gamma)}{cV_2} \\ &= \frac{1}{c^2 k V_1 V_2}\left[\gamma V_1(V_1 - k) - c^2 k V_2(1 + V_2)\right] \\ &< \frac{\gamma(V_1 - k)}{c^2 k V_2}. \end{aligned} \tag{41}$$

By integrating the above inequality with respect to $V_1 \in (k, h_V(c))$ and combining it with $k < h_V(c)$, we are led to Lemma A.10. □

Proof of Theorem A.2. Assume $c \geq 1/\sqrt{2}$. Then, (ii) of Lemma A.5 implies that there is no solution (U, V, h, c) satisfying (35) and (36).

Assume $0 < c < 1/\sqrt{2}$. From Lemmata A.9 and A.7, we can apply the intermediate value theorem for continuous functions to show that the equation

$$h_U(c) - h_V(c; k, \gamma) = 0$$

has a unique solution $c^* = c^*(k, \gamma) \in (0, 1/\sqrt{2})$. Set $h(c^*) = h_U(c^*)$. Then,

$$(U(z), V(z), h, c) = (U_1(z; c^*), V_1(z; c^*), h(c^*), c^*)$$

is a desired unique solutions of (35) and (36). □

Proof of Corollary A.3. Fix $\gamma > 0$ arbitrarily. Denote $c^*(k, \gamma)$ by c_k^*. Since c_k^* is bounded above (by Lemma A.5), we can choose $\{k_j\}$ such that $k_j \to 0$ and $c_{k_j}^* \to \sigma \in [0, 1/\sqrt{2}]$ as $j \to \infty$, respectively.

Then,

$$h_U(c_{k_j}^*) - h_V(c_{k_j}^*; k_j, \gamma) = 0$$

and Lemma A.10 shows us that

$$k_j < h_V(c^*_{k_j}; k_j, \gamma) < k_j + c^*_{k_j}\sqrt{\frac{k_j}{\gamma}}.$$

Since h_U is continous with respect to c, we conclude by letting $j \to \infty$ that

$$h_U(\sigma) = 0.$$

It implies that $\sigma = 1/\sqrt{2}$.

Finally, since σ is independent of the choice of the subsequence $\{k_j\}$, we obtain a desired result. Thus it complete the proof. □

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