Extended scaling relations for planar lattice models

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Abstract

It is widely believed that the critical properties of several planar lattice systems, like the Eight Vertex or the Ashkin-Teller models, are well described by an effective continuum fermionic theory obtained as a formal scaling limit. On the basis of this assumption several extended scaling relations among their indices were conjectured. We prove the validity of some of them, among which the ones predicted by Kadanoff [16] and by Luther and Peschel [20].

1 Introduction and main results

Integrable models in statistical mechanics, like the Ising or the Eight vertex (8V) models in two dimensions, provide conceptual laboratories for the understanding of phase transitions. Integrability is however a rather delicate property requiring very special features, and it is usually lost in more realistic models.

The principle of *universality*, phenomenologically quite well verified, says that the singularities for second order phase transitions should be insensitive to the specific details of the model. From the theoretical side, a mathematical justification of universality in planar lattice models is rather difficult to provide. Only very recently Pinson and Spencer established, see [32, 29], a form of universality for the 2D Ising model; they added to the Ising Hamiltonian a perturbation breaking the integrability and showed that the indices they can compute are *exactly the same* as the Ising model ones.

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While the critical indices of the Ising model are expressed by *pure numbers*, there are other lattice models in which some of the critical exponents vary continuously with the parameters appearing in the Hamiltonian. A celebrated example is provided by the Eight vertex model, solved by Baxter in [2]; even though it can be mapped in two Ising models coupled by a quartic interaction, its critical indices are different from the Ising ones.

Several authors, starting from Kadanoff and collaborators [16, 17, 18] and Luther and Peschel [20], have argued that many models, like the *Askhin-Teller* (AT) model and several others, belongs to the class of universality of the 8V model. The notion of universality in this case is much more subtle; it does not mean that the indices are the same for all the models in the same class (on the contrary, the indices depend on all the details of the Hamiltonian), rather it means that there are *scaling relations* between them, such that all the indices of a single model can be expressed in terms of one of the indices of the same model.

The notion of universality for models with continuously varying indices has been deeply investigated over the years, see for instance [18, 27, 28, 33]; it has been pointed out that such models are well described in the scaling limit by an effective continuum fermionic theory, and on the basis of this assumption several extended scaling relations between their indices were derived. While the assumption of a continuum scaling limit description of planar lattice models is very powerful, it is well known that a mathematical justification of it is very difficult, see *e.g.* [31].

The aim of this paper is to provide a mathematical proof of some of the exact scaling relations derived in the literature for planar lattice models. We will focus mainly on the 8V and AT models, but, as we shall explain after the main theorem below, our result can be extended to several other models.

We start from the well known (see [3]) Ising formulation of the 8V and the AT models. Let Λ be a square subset of $\mathbb{Z}^2$ of side L; if $\mathbf{x} = (x_0, x) \in \Lambda$ and $\mathbf{e}_0 = (1, 0)$, $\mathbf{e}_1 = (0, 1)$, we consider two independent configurations of spins, $\{\sigma_\mathbf{x} = \pm 1\}_{\mathbf{x}\in\Lambda}$ and $\{\sigma'_\mathbf{x} = \pm 1\}_{\mathbf{x}\in\Lambda}$ and the Hamiltonian

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$$H(\sigma, \sigma') = H_J(\sigma) + H_{J'}(\sigma') - J_4 V(\sigma, \sigma') , \tag{1}$$

where $J > 0$ and $J' > 0$ are two parameters, H_J is the (ferromagnetic) Ising Hamiltonian in the lattice Λ,

$$H_J(\sigma) = -J \sum_{j=0,1} \sum_{\mathbf{x}\in\Lambda} \sigma_\mathbf{x} \sigma_{\mathbf{x}+\mathbf{e}_j} , \tag{2}$$

V is the quartic interaction and $-J_4$ is the coupling. In the AT model, J and J' can be different (in which case the model is called *anisotropic*) and $V = V_{AT}$, with (see Fig. 1.1)

$$V_{AT}(\sigma, \sigma') = \sum_{j=0,1} \sum_{\mathbf{x}\in\Lambda} \sigma_\mathbf{x} \sigma_{\mathbf{x}+\mathbf{e}_j} \sigma'_\mathbf{x} \sigma'_{\mathbf{x}+\mathbf{e}_j} . \tag{3}$$

In the 8V model $J = J'$ and $V = V_{8V}$, with (see Fig. 1.1)

$$V_{8V}(\sigma, \sigma') = \sum_{j=0,1} \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}+j(\mathbf{e}_0+\mathbf{e}_1)}\sigma_{\mathbf{x}+\mathbf{e}_0}\sigma'_{\mathbf{x}+j(\mathbf{e}_0+\mathbf{e}_1)}\sigma'_{\mathbf{x}+\mathbf{e}_1} \; . \qquad (4)$$

In this paper we will focus our attention on two observables,

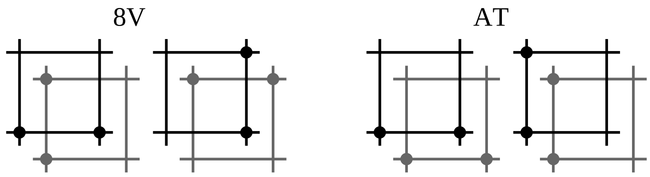

Figure 1: : The quartic interaction in the 8V and in the AT case. The gray and the black square are the same square of the lattice.

$$O^\varepsilon_{\mathbf{x}} = \sum_{j=0,1} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j} + \varepsilon \sum_{j=0,1} \sigma'_{\mathbf{x}}\sigma'_{\mathbf{x}+\mathbf{e}_j} \quad , \quad \varepsilon = \pm \; , \qquad (5)$$

and their truncated correlations in the *thermodynamic limit*

corr

$$G^\varepsilon(\mathbf{x}-\mathbf{y}) = \lim_{\Lambda\to\infty} \langle O^\varepsilon_{\mathbf{x}} O^\varepsilon_{\mathbf{y}}\rangle_\Lambda - \langle O^\varepsilon_{\mathbf{x}}\rangle_\Lambda \langle O^\varepsilon_{\mathbf{y}}\rangle_\Lambda \quad , \quad \varepsilon = \pm \; , \qquad (6)$$

where $\langle \cdot \rangle_\Lambda$ is the average over all configurations of the spins with statistical weight $e^{-\beta H(\sigma,\sigma')}$. In the AT model, $\langle O^+_{\mathbf{x}}\rangle$ is called the *energy*, while $\langle O^-_{\mathbf{x}}\rangle$ is called the *crossover*; and viceversa in the 8V model, see *e.g.* [27].

Despite their similarity, an exact solution exists for the 8V model but *not* for the AT model. In recent times the methods of fermionic Renormalization Group (RG) (introduced in [7]; see *e.g.* [25] for an updated introduction) has been applied to such models (for J_4 small), using the well known representation of such correlations in terms of *Grassmann integrals*, see *e.g.* [30]. It was proved in [21, 22] that both the 8V and the isotropic ($J = J'$) AT systems have a nonzero *critical temperature*, T_c, such that, if $T \neq T_c$, $G^\varepsilon(\mathbf{x}-\mathbf{y})$ decays faster than any power of $\xi^{-1}|\mathbf{x}-\mathbf{y}|$, with

$$\xi^{-1} \sim C\,|T - T_c|^\nu \text{ , as } T \to T_c \; . \qquad (7)$$

Moreover, for $T \to T_c$, there are two constants C_ε, $\varepsilon = \pm$, such that

xpm

$$G^\varepsilon(\mathbf{x}-\mathbf{y}) \sim \frac{C_\varepsilon}{|\mathbf{x}-\mathbf{y}|^{2x_\varepsilon}} \text{ , as } |\mathbf{x}-\mathbf{y}| \to \infty \; , \qquad (8)$$

where $x_\pm$ are critical indices expressed by *convergent series* in J_4. The analysis in [22] allows to compute the indices $\nu, x_\pm$ with arbitrary precision (by an explicit computation of the lowest orders and a rigorous bound on the remainder); however, the complexity of such expansions makes essentially impossible to see directly from them the scaling relations.

In the case of the anisotropic AT model, it was proven in [13, 14] that there are two critical temperatures, $T_{1,c}$ and $T_{2,c}$, and the corresponding critical indices are the same as those of the Ising model. However as $J - J' \to 0$:

$$|T_{1,c} - T_{2,c}| \sim |J - J'|^{x_T} , \tag{9}$$

with a *transition index*, x_T, different from 1 if $J_4 \neq 0$.

In this paper we will prove the following Theorem.

Theorem 1.1 *If the coupling is small enough, the critical indices of the 8V or AT models verify*

$$x_- = \frac{1}{x_+} , \tag{10}$$

$$\nu = \frac{1}{2 - x_+} ; \tag{11}$$

and, in the case of the anisotropic AT model,

$$x_T = \frac{2 - x_+}{2 - x_-} . \tag{12}$$

Remarks

1. Equation (10) is the *extended scaling law* first conjectured by Kadanoff for the AT and 8V models (see eq.(13b) and (15b) of [16]). Eq.(11) was conjectured by Luther and Peschel in [20] (see eq.(16) and table I of that paper).

2. The scaling relation (12) was never conjectured before.

3. All the critical indices we consider can be expressed as simple functions of one of them, in agreement with the general belief.

4. The exact value of the index ν for the Eight Vertex model has been obtained by Baxter, see (10.12.23b) of [3]; hence the values of $x_\pm$, which cannot be computed from the exact solution, can be obtained from (10) and (11).

5. We have considered just the 8V or the AT models for definiteness, but our theorem is valid for any Hamiltonian of the form (1), if the quartic interaction is small enough and verifies some symmetry conditions,

listed in App. O of [22]; in particular, our result is valid for a generic interaction of the form

$$V(\sigma,\sigma') = \sum_{j=0,1}\sum_{\mathbf{x},\mathbf{y}\in\Lambda} v(\mathbf{x}-\mathbf{y})\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j}\sigma'_{\mathbf{y}}\sigma'_{\mathbf{y}+\mathbf{e}_j} \ , \tag{13}$$

with $v(\mathbf{x}-\mathbf{y})$ a rotational invariant short range potential ($|v(\mathbf{x}-\mathbf{y})| \leq Ce^{-\kappa|\mathbf{x}-\mathbf{y}|}$, C,κ positive constants).

6. Our analysis can be extended to a large class of (integrable or non integrable) quantum spin chains or fermion models known as *Luttinger liquids* [15]. For instance in the case of the XYZ spin chain with a magnetic field, analyzed in [9], calling x_- the index appearing in the oscillating part of the spin-spin correlation along the z direction (see (1.20) of [9]), and α the index appearing in the decay rate (see (1.19) of [9]), it is a simple corollary of the proof of Theorem 1.1 that $\nu = (2 - x_-^{-1})^{-1}$.

7. Several other relations are conjectured in the literature, concerning critical indices, which are much more difficult to study with our methods, like the indices of the polarization correlations. New ideas seems to be required to treat such cases.

The main steps of the proof are the following.
(i) The correlations of the spin models are written in terms of *interacting* one dimensional fermion models, whose correlations can be computed in terms of Grassmann integrals;
(ii) the critical indices are expressed in terms of convergent expansions by a Renormalization Group analysis of the functional integrals;
(iii) the original theory is shown to be equivalent (in the sense that their critical indices coincide) to an effective continuum fermionic model, defined as the formal scaling limit of the original one, provided that the coupling λ_∞ of the equivalent model is chosen properly as a function of J_4;
(iv) The effective model is expressed in terms of Grassmann integrals which are identical to the ones appearing in certain Quantum Field Theory (QFT) models; we take advantage of the Gauge symmetry and of a property called *anomaly non-renormalization*, see [23], to *exactly compute* the correlations and the critical indices. It turns out that the dependence of the critical indices on λ_∞ is so simple that we can check that the extended scaling relations are verified.

The proof relies on several results derived in detail in previous papers, in particular [22, 23, 6]; here the attention is mostly focused on the new technical points which are required for the proof, and the use of earlier published results is through precise statements proved elsewhere. The proof of (11) is at the end of §2.3, while (12) is proved in §2.4; (10), the main result of this paper, is proved in §4.2.

2 RG analysis of spin models

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In this section we summarize the analysis given in [22] for the 8V or isotropic AT model and in [14] for the anisotropic AT model. The correlations of the AT or 8V models are written in terms of Grassmann integrals and are analyzed using RG methods. The critical indices x_+, x_-, ν and x_T are written, in the small coupling region, as *model independent* convergent series of a single parameter, $\lambda_{-\infty}$, as we shall call the asymptotic limit of the effective coupling on large scale. Notice that $\lambda_{-\infty}$ is in turn a convergent series, depending on all the details of the lattice model, of the coupling J_4. Such expansions allow in principle to compute the indices with arbitrary precision, but the complexity of such expansions makes essentially impossible to see directly from them the extended scaling relations.

2.1 Fermionic representation of the spin models

The partition function $Z(I)$ of the Ising model with external sources $A_{j,\mathbf{x}}$, and periodic conditions at the boundary of Λ is

pf1

$$Z(I) = \sum_\sigma \exp\Big[\sum_{\substack{j=0,1\\ \mathbf{x}\in\Lambda}} I_{j,\mathbf{x}}\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j}\Big] , \tag{14}$$

where $I_{j,\mathbf{x}} = A_{j,\mathbf{x}} + \beta J$. It is well known, see [30] or App. A1 of [14], that $Z(I)$ can be written as sum of *Grassman integrals.* Let $\gamma = (\varepsilon_0, \varepsilon_1)$, with $\varepsilon_0, \varepsilon_1 = \pm$ and let $\{H_{\mathbf{x}},\ \bar H_{\mathbf{x}},\ V_{\mathbf{x}},\ \bar V_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda}$ be a family of Grassmann variables verifying the γ-boundary conditions, namely

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$$\begin{aligned}\bar H_{\mathbf{x}+(L,0)} = \varepsilon_0 \bar H_{\mathbf{x}} \quad &, \quad \bar H_{\mathbf{x}+(0,L)} = \varepsilon_1 \bar H_{\mathbf{x}}\ ,\\ H_{\mathbf{x}+(L,0)} = \varepsilon_0 H_{\mathbf{x}} \quad &, \quad H_{\mathbf{x}+(0,L)} = \varepsilon_1 H_{\mathbf{x}}\ ,\end{aligned} \tag{15}$$

and similar relations for $V, \bar V$ (we are skipping the γ dependence in the H's and V's). Then we consider the *Grassmann functional integral*

2.11

$$Z_\gamma = \int dH dV\ e^{S(t)}\ , \tag{16}$$

where the action $S(t)$ is the following function of the parameters $t = \{t_{j,\mathbf{x}}\}_{\substack{\mathbf{x}\in\Lambda\\ j=0,1}}$ and of the Grassmann variables with $\gamma-$boundary condition:

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$$\begin{aligned}S(t) = &\sum_{\mathbf{x}\in\Lambda}\Big[t_{0,\mathbf{x}}\bar H_{\mathbf{x}} H_{\mathbf{x}+\mathbf{e}_0} + t_{1,\mathbf{x}}\bar V_{\mathbf{x}} V_{\mathbf{x}+\mathbf{e}_1}\Big] +\\ &+\sum_{\mathbf{x}\in\Lambda}\Big[\bar H_{\mathbf{x}} H_{\mathbf{x}} + \bar V_{\mathbf{x}} V_{\mathbf{x}} + \bar V_{\mathbf{x}}\bar H_{\mathbf{x}} + V_{\mathbf{x}} H_{\mathbf{x}} + V_{\mathbf{x}}\bar H_{\mathbf{x}} + H_{\mathbf{x}}\bar V_{\mathbf{x}}\Big]\ .\end{aligned} \tag{17}$$

If we choose $t_{j,\mathbf{x}} = \tanh I_{j,\mathbf{x}}$ and put $c_{j,\mathbf{x}} = \cosh I_{j,\mathbf{x}}$, the partition function (14) can be written in the following way:

17
$$Z(I) = (-1)^{|\Lambda|} 2^{|\Lambda|} \left(\prod_{j,\mathbf{x}} c_{j,x} \right) \sum_{\gamma} \frac{(-1)^{\delta_\gamma}}{2} Z_\gamma \tag{18}$$

where $\delta_\gamma = 1$ for $\gamma = (+,+)$, and $\delta_\gamma = 0$ otherwise.

By using (17), the correlation functions (6) of the spin model (1) can be written as (see §A.1 for details)

$$\langle O^\varepsilon_{\mathbf{x}}; O^\varepsilon_{\mathbf{y}} \rangle^T_\Lambda = \left. \frac{\partial^2 \ln \bar{Z}(\bar{A})}{\partial \bar{A}^\varepsilon_{\mathbf{x}} \partial \bar{A}^\varepsilon_{\mathbf{y}}} \right|_{\bar{A} \equiv 0} + S_L(\mathbf{x}, \mathbf{y}) \,, \tag{19}$$

where $S_L(\mathbf{x}, \mathbf{y})$ is a correction term, vanishing in the thermodynamic limit (see App. 1), and

25
$$\bar{Z}(\bar{A}) = \int dH dV dH' dV' \, e^{S(s) + S(s') + 2\lambda V + B(\bar{A})} \,, \tag{20}$$

s, s' and λ being parameters independent of j and $\mathbf{x}$ (see (170) below), such that $s = \tanh(\beta J) + \mathrm{O}(\beta J_4)$, $s' = \tanh(\beta J) + \mathrm{O}(\beta J_4)$ and $\lambda = \mathrm{O}(\beta J_4)$. Moreover, the (H, V) and (H', V') variables verify antiperiodic boundary conditions and V is a quartic interaction that, in the AT case, is given by

i1
$$V_{AT} = \sum_{\mathbf{x} \in \Lambda} \left[\bar{H}_{\mathbf{x}} H_{\mathbf{x}+\mathbf{e}_0} \bar{H}'_{\mathbf{x}} H'_{\mathbf{x}+\mathbf{e}_0} + \bar{V}_{\mathbf{x}} V_{\mathbf{x}+\mathbf{e}_1} \bar{V}'_{\mathbf{x}} V'_{\mathbf{x}+\mathbf{e}_1} \right] \,, \tag{21}$$

while, in the 8V case, is given by

i2
$$V_{8V} = \sum_{\mathbf{x} \in \Lambda} \left[\bar{H}_{\mathbf{x}} H_{\mathbf{x}+\mathbf{e}_0} \bar{V}'_{\mathbf{x}} V'_{\mathbf{x}+\mathbf{e}_1} + \bar{V}_{\mathbf{x}+\mathbf{e}_0} V_{\mathbf{x}+\mathbf{e}_0+\mathbf{e}_1} \bar{H}'_{\mathbf{x}+\mathbf{e}_1} H'_{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_0} \right] \,. \tag{22}$$

Finally $B(\bar{A})$ is an interaction with external sources $\bar{A}^\varepsilon_{\mathbf{x}}$, given, in the AT case, by

$$\begin{aligned} B(\bar{A}) \;&=\; \sum_{\substack{\mathbf{x} \in \Lambda \\ \varepsilon = \pm}} \bar{A}^\varepsilon_{\mathbf{x}} \left[q_\varepsilon \left(\bar{H}_{\mathbf{x}} H_{\mathbf{x}+\mathbf{e}_0} + \bar{V}_{\mathbf{x}} V_{\mathbf{x}+\mathbf{e}_1} \right) + q'_\varepsilon \left(\bar{H}'_{\mathbf{x}} H'_{\mathbf{x}+\mathbf{e}_0} + \bar{V}'_{\mathbf{x}} V'_{\mathbf{x}+\mathbf{e}_1} \right) \right] + \\ &+ \sum_{\substack{\mathbf{x} \in \Lambda \\ \varepsilon = \pm}} \bar{A}^\varepsilon_{\mathbf{x}} p_\varepsilon \left(\bar{H}_{\mathbf{x}} H_{\mathbf{x}+\mathbf{e}_0} \bar{H}'_{\mathbf{x}} H'_{\mathbf{x}+\mathbf{e}_0} + \bar{V}_{\mathbf{x}} V_{\mathbf{x}+\mathbf{e}_1} \bar{V}'_{\mathbf{x}} V'_{\mathbf{x}+\mathbf{e}_1} \right) \,, \end{aligned} \tag{23}$$

while, in the 8V case, it is given by

b2
$$\begin{aligned} & B(\bar{A}) = \sum_{\mathbf{x} \in \Lambda, \varepsilon = \pm} \bar{A}^\varepsilon_{\mathbf{x}} \left[q_\varepsilon \left(\bar{H}_{\mathbf{x}} H_{\mathbf{x}+\mathbf{e}_0} + \bar{V}_{\mathbf{x}+\mathbf{e}_0} V_{\mathbf{x}+\mathbf{e}_0+\mathbf{e}_1} \right) + \right. \\ & \left. + q'_\varepsilon \left(\bar{H}'_{\mathbf{x}+\mathbf{e}_1} H'_{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_0} + \bar{V}'_{\mathbf{x}} V'_{\mathbf{x}+\mathbf{e}_1} \right) \right] + \\ & + \sum_{\substack{\mathbf{x} \in \Lambda \\ \varepsilon = \pm}} \bar{A}^\varepsilon_{\mathbf{x}} p_\varepsilon \left(\bar{H}_{\mathbf{x}} H_{\mathbf{x}+\mathbf{e}_0} \bar{V}'_{\mathbf{x}} V'_{\mathbf{x}+\mathbf{e}_1} + \bar{V}_{\mathbf{x}+\mathbf{e}_0} V_{\mathbf{x}+\mathbf{e}_0+\mathbf{e}_1} \bar{H}'_{\mathbf{x}+\mathbf{e}_1} H'_{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_0} \right) \,, \end{aligned} \tag{24}$$

with q_ε, q'_ε and p_ε parameters independent of j and $\mathbf{x}$, such that $q_\varepsilon = 1 - \tanh(\beta J) + O(\beta J_4)$, $q'_\varepsilon = \varepsilon[1 - \tanh(\beta J')] + O(\beta J_4)$ and $p_\varepsilon = O(\beta J_4)$. Notice the crucial difference between (14) and (20); in the second case also quartic monomials appears in the exponent, while in the first case only quadratic terms appears. In other words, the Ising model correspond to a *non-interacting* fermionic theory, while the model (1) is mapped into an interacting fermionic system.

2.2 The case $J = J'$

We consider first the case $J = J'$. The Grassmann integral (20) is not very convenient for the RG analysis, but it can be properly rewritten, by a suitable change of variables in the Grassmann algebra, in a much better form (see [22, 14] and App. A for details), such that the analogy of the above functional integral with a fermionic (Euclidean) Quantum Field Model is clearer.

Let $\mathcal{D}$ be the set of $\mathbf{k}$'s such that $k_0 = \frac{2\pi}{L}(n_0 + \frac{1}{2})$ and $k_1 = \frac{2\pi}{L}(n_1 + \frac{1}{2})$, for $n_0, n_1 = -\frac{L}{2}, \dots, \frac{L}{2} - 1$, and L and even integer. Then the functional integral (20) can be written as (see §A.2 for details)

$$\bar{Z}(\bar{A}) = \frac{1}{\mathcal{N}} \int P(d\psi) P_\chi(d\chi)\, e^{\mathcal{Q}(\psi,\chi) + \mathcal{V}(\psi,\chi) + B(\bar{A})} , \tag{25}$$

where $\mathcal{N}$ is a normalization constant, $P(d\psi)$ is the (Grassmannian) Gaussian measure with propagator

$$g(\mathbf{x}) = \frac{1}{L^2} \sum_{\mathbf{k}\in\mathcal{D}} e^{-i\mathbf{k}\mathbf{x}} T^{-1}(\mathbf{k}) , \tag{26}$$

with

TKK
$$T(\mathbf{k}) = u \begin{pmatrix} i\sin k_0 + \sin k_1 & -i\mu(\mathbf{k}) \\ i\mu(\mathbf{k}) & i\sin k_0 - \sin k_1 \end{pmatrix} , \tag{27}$$

$u = \tanh(\beta J) + O(\beta J_4)$,

mukk
$$\mu(\mathbf{k}) = (\cos k_0 + \cos k_1 - 2) + 2\frac{1 - \sqrt{2} + u}{u} , \tag{28}$$

$P_\chi(d\chi)$ is the Gaussian measure with propagator $g_\chi(\mathbf{x})$, which is obtained from $g(\mathbf{x})$ by replacing $T(\mathbf{k})$ with $T^\chi(\mathbf{k})$, $T^\chi(\mathbf{k})$ being the matrix obtained from $T(\mathbf{k})$ by substituting $\mu(\mathbf{k})$ with

sigc
$$\mu^\chi(\mathbf{k}) = (\cos k_0 + \cos k_1 - 2) + 2\frac{1 + \sqrt{2} + u}{u} . \tag{29}$$

The interaction with the external source is given by

BA
$$B(\bar{A}) = i\sum_{\mathbf{x}\in\Lambda} q_+ \bar{A}^+_{\mathbf{x}}[\psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},-} - \psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},+} + \chi^+_{\mathbf{x},+}\chi^-_{\mathbf{x},-} - \chi^+_{\mathbf{x},-}\chi^-_{\mathbf{x},+}] + \\ +i\sum_{\mathbf{x}\in\Lambda} q_- \bar{A}^-_{\mathbf{x}}[\psi^+_{\mathbf{x},+}\psi^+_{\mathbf{x},-} + \psi^-_{\mathbf{x},+}\psi^-_{\mathbf{x},-} + \chi^+_{\mathbf{x},+}\chi^+_{\mathbf{x},-} + \chi^-_{\mathbf{x},+}\chi^-_{\mathbf{x},-}] + R_B(\bar{A})\ , \tag{30}$$

where $R_B(\bar{A})$ contains terms either quartic in the fields or quadratic with derivatives, whose explicit form (not essential for our purposes) can be obtained by performing the changes of variables explained in App. A). Moreover

$$\mathcal{Q}(\psi,\chi) = -\frac{1}{|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}}\sum_{\omega,\omega'}[\widehat{\psi}^+_{\mathbf{k},\omega}\widehat{\chi}^-_{\mathbf{k},\omega'} + \widehat{\chi}^+_{\mathbf{k},\omega}\widehat{\psi}^-_{\mathbf{k},\omega'}]Q_{\omega,\omega'}(\mathbf{k})\ , \tag{31}$$

where $Q(\mathbf{k})$ is a matrix which vanishes at $\mathbf{k}=0$. Finally, the quartic self interaction is given by

int1
$$\mathcal{V}(\psi,\chi) = \lambda\sum_{\mathbf{x}\in\Lambda}\left[\psi^+_{\mathbf{x},+}\psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},+}\psi^-_{\mathbf{x},-} + \chi^+_{\mathbf{x},+}\chi^+_{\mathbf{x},-}\chi^-_{\mathbf{x},+}\chi^-_{\mathbf{x},-}\right] + \\ +v(\psi,\chi) + R_V(\psi,\chi)\ , \tag{32}$$

where $v(\psi,\chi)$ is a quartic interaction depending both on ψ and χ, which has a different expression in the AT and 8V models, and $R_V(\psi,\chi)$ is sum of quartic monomials with at least a derivative; the explicit form for such expressions can again be obtained by performing the changes of variables explained in App. A).

If $J>0$ and J_4 is any real number, u is a strictly increasing function of $\tanh(\beta J)$ and has range $(0,1)$, as one can check by using the definition of s, see (170). On the other hand, $\det T(\mathbf{k})=0$ only if $\mathbf{k}=0$ and $\mu(\mathbf{k})=0$; hence, $g(\mathbf{x})$ has a singularity at $u=u_c=\sqrt{2}-1$, which is an allowed value; moreover, if $\beta|J_4|\ll 1$ (as we shall suppose in the following), $u=\tanh(\beta J)+O(\beta J_4)$. Since we expect that the interaction will move this singularity, it is convenient to modify the interaction by adding a finite *counterterm* $i\nu_1\frac{1}{L^2}\sum_{\omega,\mathbf{k}}\omega\widehat{\psi}^+_{\mathbf{k},\omega}\widehat{\psi}^-_{\mathbf{k},-\omega}$, which is compensated by replacing, in the matrix $T(\mathbf{k})$, $\mu(\mathbf{k})$ with

$$\mu_1(\mathbf{k}) = (\cos k_0 + \cos k_1 - 2) + 2(1-\frac{u^*}{u})\quad,\quad u^* = \sqrt{2}-1-\nu_1\ . \tag{33}$$

Let us call $T_1(\mathbf{k})$ the new matrix and $P_1(d\psi)$ the corresponding measure; we get

2.15
$$\bar{Z}(\bar{A}) = \frac{1}{\mathcal{N}_1}\int P_1(d\psi)P_\chi(d\chi)\ e^{\mathcal{Q}(\psi,\chi)+\mathcal{V}^{(1)}(\psi,\chi)+B(\bar{A})}\ , \tag{34}$$

where

$$\mathcal{V}^{(1)}(\psi,\chi) = i\nu_1\frac{1}{L^2}\sum_{\omega,\mathbf{k}}\omega\widehat{\psi}^+_{\mathbf{k},\omega}\widehat{\psi}^-_{\mathbf{k},-\omega} + \mathcal{V}(\psi,\chi)\ , \tag{35}$$

and ν_1 has to be determined so that the interacting propagator has an infrared singularity at $u = u^*$; the critical temperature is uniquely determined by the value of u^*.

Let us now remark that $\det T^\chi(\mathbf{k})$ is strictly positive for any $\mathbf{k}$, as one can easily see by using the fact that $u \in (0,1)$. Hence, if we define

$$\widetilde{\psi}^+ = \psi^+ Q T_\chi^{-1} \quad , \quad \widetilde{\psi}^- = T_\chi^{-1} Q \psi^- \ , \tag{36}$$

the change of variables $\chi^+ \to \chi^+ + \widetilde{\psi}^+$, $\chi^- \to \chi^- + \widetilde{\psi}^-$, allows us to rewrite (34) in the form

2.15a $$\bar{Z}(\bar{A}) = \frac{1}{\mathcal{N}} \int P_{Z_1,\mu_1}(d\psi) P_\chi(d\chi)\, e^{\mathcal{V}^{(1)}(\psi,\chi-\widetilde{\psi}) + \widetilde{B}(\bar{A})} \ , \tag{37}$$

where $\widetilde{B}(\bar{A})$ is the functional obtained from $B(\bar{A})$ by replacing χ with $\chi - \widetilde{\psi}$ and $P_{Z_1,\mu_1}(d\psi)$ is the Gaussian measure with propagator

lau $$g(\mathbf{x}) = \frac{1}{L^2} \sum_{\mathbf{k}\in\mathcal{D}} e^{-i\mathbf{k}\mathbf{x}} (T^{(1)})^{-1}(\mathbf{k}) \ , \tag{38}$$

where $T^{(1)}(\mathbf{k}) = T(\mathbf{k}) - Q(\mathbf{k}) T_\chi^{-1} Q(\mathbf{k})$. It is also convenient to perform the trivial change of variables

$$\widehat{\psi}^+_{\mathbf{k},\omega} \to -i\omega \widehat{\psi}^+_{\widetilde{\mathbf{k}},\omega} \quad , \quad \widehat{\psi}^-_{\mathbf{k},\omega} \to \widehat{\psi}^-_{\widetilde{\mathbf{k}},\omega} \quad , \quad \mathbf{k} = (k_0, k_1) \quad , \quad \widetilde{\mathbf{k}} = (k_1, k_0) \ . \tag{39}$$

Hence, by an explicit calculation of $Q(\mathbf{k})$ and using the identity $u^*/u = 1 - \mu_1(0)/2$, one can see that $T^{(1)}(\mathbf{k})$ is the matrix

ov1 $$Z_1 C_1(\mathbf{k}) \begin{pmatrix} -i\sin k_0 + \sin k_1 + \mu_{+,+}(\mathbf{k}) & -\mu_1 - \mu_{+,-}(\mathbf{k}) \\ -\mu_1 - \mu_{+,-}(\mathbf{k}) & -i\sin k_0 - \sin k_1 + \mu_{-,-}(\mathbf{k}) \end{pmatrix} \tag{40}$$

with $C_1(\mathbf{k}) = 1$, $\mu_1 = 2\mu_1(0)/(2 - \mu_1(0))$ and $Z_1 = u^*$; moreover $\mu_{+,+}(\mathbf{k}) = -\mu_{-,-}(\mathbf{k})^*$ is an odd function of $\mathbf{k}$ of the form $\mu_{+,+}(\mathbf{k}) = 2\mu_1(0)(-i\sin k_0 + \sin k_1)/(4 - 2\mu_1(0)) + O(|\mathbf{k}|^3)$, while $\mu_{+,-}(\mathbf{k})$ is a real even function, of order $|\mathbf{k}|^2$, which vanishes only at $\mathbf{k} = 0$. Finally, $\det T^{(1)}(\mathbf{k}) \geq C(2 - \cos k_0 - \cos k_1)$, so that $P_{Z_1,\mu_1}(d\psi)$ has the same type of infrared singularity as $P_1(d\psi)$.

The fact that $\det T_\chi(\mathbf{k})$ is strictly positive implies that $g_\chi(\mathbf{x})$ is an exponential decaying function; hence, we can safely perform the integration over the field χ in (37). The result can be written in the following form (see Lemma 1 of [22])

3.1 $$\bar{Z}(\bar{A}) \equiv e^{\mathcal{S}(\bar{A})} = \int P_{Z_1,\mu_1}(d\psi) e^{L^2\mathcal{N}^{(1)} + \bar{\mathcal{V}}^{(1)}(Z_1\psi) + B^{(1)}(\bar{A})} \ , \tag{41}$$

where $\mathcal{N}^{(1)}$ is a constant and the *effective potential* $\bar{\mathcal{V}}^{(1)}(\psi)$ can be represented as

3.2aaa $$\bar{\mathcal{V}}^{(1)} = \sum_{n\geq 1} \sum_{\underline{\alpha},\underline{\omega},\underline{\varepsilon}} \sum_{\mathbf{x}_1,..,\mathbf{x}_n} W_{\underline{\omega},\underline{\alpha},\underline{\varepsilon},2n}(\mathbf{x}_1,..,\mathbf{x}_{2n}) \partial^{\alpha_1}\psi^{\varepsilon_1}_{\mathbf{x}_1,\omega_1} ... \partial^{\alpha_{2n}}\psi^{\varepsilon_{2n}}_{\mathbf{x}_{2n},\omega_{2n}} \ . \tag{42}$$

The kernels $W_{\underline{\omega},\underline{\alpha},\underline{\varepsilon},2n}$ in the previous expansions are analytic functions of λ and ν_1 near the origin; if we suppose that $\nu_1 = O(\lambda)$, their Fourier transforms satisfy, for any $n \geq 1$, the bounds, see [22],

$$|\widehat{W}_{\underline{\alpha},\underline{\omega},\underline{\varepsilon},2n}(\mathbf{k}_1, ...\mathbf{k}_{2n-1})| \leq L^2 C^n |\lambda|^n \; . \tag{43}$$

A similar representation can be written for the functional of the external field $B^{(1)}(\bar{A})$. As explained in detail in [22], the symmetries of the two models we are considering imply that, in the r.h.s. of (42), there are no local terms quadratic in the field, which are relevant or marginal, except those which are already present in the free measure.

2.3 Multiscale analysis

2.3

We briefly recall here the analysis in [22] (see also [7, 8]). The integration in (41) can be done by iteratively integrating the fields with decreasing momentum scale and by moving to the free measure all the marginal terms quadratic in the field. We introduce a scaling parameter $\gamma = 2$, a decomposition of the unity $1 = f_1 + \sum_{h=-\infty}^{0} f_h(\mathbf{k})$, with $f_h(\mathbf{k})$ a smooth function with support $\{\gamma^{h-1}\pi/4 \leq |\mathbf{k}| \leq \gamma^{h+1}\pi/4\}$, and the corresponding decomposition of the field $\psi = \sum_{j=-\infty}^{1} \psi^{(j)}$. If the fields $\psi^{(1)}, .., \psi^{(h+1)}$ are integrated, we get

th1
$$e^{\mathcal{S}(\bar{A})} = e^{S^{(h)}(\bar{A})} \int P_{\bar{Z}_h,\mu_h}(d\psi^{(\leq h)}) e^{\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) + \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)},\bar{A})} \; , \tag{44}$$

where $\psi^{(\leq h)} = \sum_{j=-\infty}^{h} \psi^{(j)}$ and $P_{\bar{Z}_h,\mu_h}(d\psi)$ is the Gaussian fermionic measure with the propagator obtained from (38) by replacing in (40) $C_1(\mathbf{k})$ with $C_h(\mathbf{k}) = [\sum_{k=-\infty}^{h} f_h(\mathbf{k})]^{-1}$, μ_1 with μ_h, Z_1 with the function $\bar{Z}_h(\mathbf{k})$ (to be defined below) and the functions $\mu_{\sigma,\sigma'}(\mathbf{k})$ with similar functions $\mu^{(h)}_{\sigma,\sigma'}(\mathbf{k})$; finally, the constant Z_h, which rescale the field, is given by $Z_h = \bar{Z}_h(0)$.

The *effective interaction* $\mathcal{V}^{(h)}(\psi)$ is a sum over monomials in the Grassmann variables and we define a *localization operator* (see *e.g.* §4 of [22]) as

5.8
$$\mathcal{L}\mathcal{V}^{(h)}(\psi) = (-m_h + \gamma^h n_h) F_\nu^{(h)} + l_h F_\lambda^{(h)} - z_h F_z^{(h)} \; , \tag{45}$$

where m_h, n_h, z_h and l_h are suitable real numbers,

2.112
$$F_\nu^{(h)} = \frac{1}{L^2} \sum_\omega \sum_{\mathbf{k}} \widehat{\psi}^{(\leq h)+}_{\mathbf{k},\omega} \widehat{\psi}^{(\leq h)-}_{\mathbf{k},-\omega} \; , \tag{46}$$

$$F_z^{(h)} = \frac{1}{L^2} \sum_\omega \sum_{\mathbf{k}} (-i \sin k_0 + \omega \sin k_1) \widehat{\psi}^{(\leq h)+}_{\mathbf{k},\omega} \widehat{\psi}^{(\leq h)-}_{\mathbf{k},-\omega} \; , \tag{47}$$

$$F_\lambda^{(\leq h)} = \frac{1}{L^8} \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_4} \widehat{\psi}^{(\leq h)+}_{\mathbf{k}_1,+} \widehat{\psi}^{(\leq h)+}_{\mathbf{k}_3,-} \widehat{\psi}^{(\leq h)-}_{\mathbf{k}_2,+} \widehat{\psi}^{(\leq h)-}_{\mathbf{k}_4,-} \delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4) \; .$$

Moreover we define

$$\mathcal{L}\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\bar A)=\sum_{\varepsilon,\mathbf{x}} Z_h^{(\varepsilon)}\bar A^{\varepsilon}_{\mathbf{x}} O^{(\le h)\varepsilon}_{\mathbf{x}}\;, \tag{48}$$

where

curr

$$\begin{aligned} O^{(\le h)+}_{\mathbf{x}} &= \psi^{(\le h)+}_{\mathbf{x},+}\psi^{(\le h)-}_{\mathbf{x},-}+\psi^{(\le h)+}_{\mathbf{x},-}\psi^{(\le h)-}_{\mathbf{x},+}\;, \\ O^{(\le h)-}_{\mathbf{x}} &= i[\psi^{(\le h)+}_{\mathbf{x},+}\psi^{(\le h)+}_{\mathbf{x},-}+\psi^{(\le h)-}_{\mathbf{x},+}\psi^{(\le h)-}_{\mathbf{x},-}]\;. \end{aligned} \tag{49}$$

We now move to the fermionic measure the terms proportional to m_h and z_h in (45) and we rescale the fields so that

$$\begin{aligned} e^{\mathcal{S}(\bar A)} &= e^{S^{(h)}(\bar A)+L^2 t_h}\int P_{\bar Z_{h-1},\mu_{h-1}}(d\psi^{(\le h)}) \\ & e^{\bar{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})+\bar{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\bar A)}\;, \end{aligned} \tag{50}$$

with t_h a normalization constant and

$$\bar Z_{h-1}(\mathbf{k})=Z_h(1+z_h C_h^{-1}(\mathbf{k}))\;,\;\mu_{h-1}=\frac{Z_h}{\bar Z_{h-1}(\mathbf{k})}[\mu_h(\mathbf{k})+m_h C_h^{-1}(\mathbf{k})]\;. \tag{51}$$

The renormalized potential $\bar{\mathcal{V}}^{(h)}(\psi)$ can be written as

5.8a

$$\bar{\mathcal{V}}^{(h)}(\psi)=\gamma^h\nu_h F^{(h)}_\nu+\lambda_h F^{(h)}_\lambda+R^{(h)}(\psi)\;, \tag{52}$$

with $\nu_h=n_h(Z_h/Z_{h-1})$ and $\lambda_h=l_h(Z_h/Z_{h-1})^2$; $R^{(h)}(\psi)$ is a sum over monomials similar to (42), with $2n+\alpha_1+..+\alpha_{2n}>4$. Finally, $\bar{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\bar A)=\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\bar A)$. The field $\psi^{(h)}$ is integrated and the procedure can be iterated. The above integration procedure is done till the scale h^* defined as the maximal j such that $\gamma^j\le|\mu_j|$, and the integration of the fields $\psi^{(\le h^*)}$ can be done in a single step. Roughly speaking, h^* defines the momentum scale of the mass.

Notice that the propagator of the field $\psi^{(\le h)}$ can be written, for $h\le 0$, as

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$$g^{(\le h)}(\mathbf{x},\mathbf{y})=g_T^{(\le h)}(\mathbf{x},\mathbf{y})+r^{(\le h)}(\mathbf{x},\mathbf{y})\;, \tag{53}$$

where

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$$g_T^{(\le h)}(\mathbf{x},\mathbf{y})=\frac{1}{L^2}\sum_{\mathbf{k}\in\mathcal{D}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{1}{Z_h}T_h^{-1}(\mathbf{k})\;, \tag{54}$$

$$T_h(\mathbf{k})=C_h(\mathbf{k})\begin{pmatrix} -ik_0+k_1 & -\mu_h \\ \mu_h & -ik_0-k_1\end{pmatrix}\;, \tag{55}$$

and, for any positive integer M,

$$|r^{(\le h)}(\mathbf{x},\mathbf{y})|\le C_M\frac{\gamma^{2h}}{1+(\gamma^h|\mathbf{x}-\mathbf{y}|^M)}\;. \tag{56}$$

The propagator $g_T^{(h)}(\mathbf{x},\mathbf{y})$ verifies a similar bound with γ^h replacing γ^{2h}. A similar decomposition can be done for $g^{(h)}(\mathbf{x},\mathbf{y})$.

The definition of the localization operator $\mathcal{L}$ selects in $\mathcal{V}^{(h)}, \mathcal{B}^{(h)}$ the terms with positive or vanishing *scaling dimension*, which is given, for the monomials with n ψ-fields and m A-fields, by

$$\textit{sc} \qquad D = 2 - \frac{n}{2} - m \; . \tag{57}$$

In the RG language, the terms with positive or vanishing dimension are called *relevant* or *marginal* terms, respectively. Notice that *a priori* many other possible local marginal or relevant terms could be generated in the RG integration, with respect to the one listed in (45); however these terms are absent, thanks to the symmetries of the problem, as proved in [22], App. F (see also [14], §A2.2).

The outcome of the above procedure is that the kernels in $\bar{\mathcal{V}}^{(j)}$ and $\bar{\mathcal{B}}^{(j)}$ are *analytic* functions of the *running coupling constants* λ_k, ν_k, $k \geq j$, provided that $\sup_{k\geq j}(|\lambda_k| + |\nu_k|)$ is small enough, see [22] and §3 of [9]. The running couplings λ_j (which, by construction, are the same in the massless $\mu = 0$ or in the massive $\mu \neq 0$ case, see [13]), satisfy a recursive equation of the form

$$\textit{bb} \qquad \lambda_{j-1} = \lambda_j + \beta_\lambda^{(j)}(\lambda_j, ..., \lambda_0) + \bar{\beta}_\lambda^{(j)}(\lambda_j, \nu_j; ...; \lambda_0, \nu_0) \; , \tag{58}$$

where $\beta_\lambda^{(j)}$, $\bar{\beta}_\lambda^{(j)}$ are μ-independent and expressed by a *convergent* expansion in $\lambda_j, \nu_j .., \lambda_0, \nu_0$; moreover $\bar{\beta}_\lambda^{(j)}$ vanishes if at least one of the ν_k is zero. The running coupling λ_j stays close to λ for any j as a consequence of the following property, called *vanishing of the Beta function*, which was proved in Theorem 2 of [11]; for suitable positive constants C and $\vartheta < 1$:

$$\textit{beta} \qquad |\beta_\lambda^{(j)}(\lambda_j, ..., \lambda_j)| \leq C|\lambda_j|^2 \gamma^{\vartheta j} \; . \tag{59}$$

Indeed, it is possible to prove that, for a suitable choice of $\nu_1 = O(\lambda)$, $\nu_j = O(\gamma^{\vartheta j} \bar{\lambda}_j)$, if $\bar{\lambda}_j = \sup_{k \geq j} |\lambda_k|$, and this implies, by the *short memory* property (see for instance A4.6 of [13]), $\bar{\beta}_\lambda^{(j)} = O(\gamma^{\vartheta j} \bar{\lambda}_j^2)$ so that the sequence λ_j converges, as $j \to -\infty$, to a smooth function $\lambda_{-\infty}(\lambda) = \lambda + O(\lambda^2)$, such that

$$\textit{2.42a} \qquad |\lambda_j - \lambda_{-\infty}| \leq C \lambda^2 \gamma^{\vartheta j} \; . \tag{60}$$

Moreover

$$\textit{ffg1} \qquad \frac{Z_{j-1}}{Z_j} = 1 + \beta_z^{(j)}(\lambda_j, ..., \lambda_0) + \bar{\beta}_z^{(j)}(\lambda_j, \nu_j; .., \lambda_0, \nu_0) \; , \tag{61}$$

with $\bar{\beta}_z^{(j)}$ vanishing if at least one of the ν_k is zero so that, by using the bound $\nu_j = O(\gamma^{\vartheta j} \bar{\lambda}_j)$ and the short memory property, $\bar{\beta}_z^{(j)} = O(\lambda_j \gamma^{\vartheta j})$. Finally

$$\textit{lau11} \qquad \beta_z^{(j)}(\lambda_j, ..., \lambda_0) = \beta_z(\lambda_{-\infty}) + O(\lambda \gamma^{\vartheta j}) \; , \tag{62}$$

where the last identity follows from (60) and the *short memory* property. An important point is that the function $\beta_z(\lambda_{-\infty})$ is model independent. Similar equations hold for $Z_h^{(\pm)}, \mu_h$, with leading terms again model independent, that is

lau11a
$$\beta_\pm^{(j)}(\lambda_j, ..., \lambda_0) = \beta_\pm(\lambda_{-\infty}) + O(\lambda\gamma^{\vartheta j}) \; . \tag{63}$$

By an explicit computation and (62), (63), there exist $\eta_+(\lambda_{-\infty}) = c_1\lambda_{-\infty} + O(\lambda_{-\infty}^2)$, $\eta_-(\lambda_{-\infty}) = -c_1\lambda_{-\infty} + O(\lambda_{-\infty}^2)$, $\eta_\mu(\lambda_{-\infty}) = c_1\lambda_{-\infty} + O(\lambda_{-\infty}^2)$ and $\eta_z(\lambda_{-\infty}) = c_2\lambda_{-\infty}^2 + O(\lambda_{-\infty}^3)$, with c_1 and c_2 strictly positive, such that, for any $j \le 0$,

lau12
$$\begin{aligned} &|\log_\gamma(Z_{j-1}/Z_j) - \eta_z(\lambda_{-\infty})| \le C\lambda^2\gamma^{\vartheta j} \; , \\ &|\log_\gamma(\mu_{j-1}/\mu_j) - \eta_\mu(\lambda_{-\infty})| \le C|\lambda|\gamma^{\vartheta j} , \\ &|\log_\gamma(Z_{j-1}^{(\pm)}/Z_j^{(\pm)}) - \eta_\pm(\lambda_{-\infty})| \le C\lambda^2\gamma^{\vartheta j} \; . \end{aligned} \tag{64}$$

The critical indices are functions of $\lambda_{-\infty}$ only, as it is clear from (62); moreover from (6.28) and (5.4) of [22], the indices $x_\pm$ appearing in (8) are such that

pppp3
$$x_\pm = 1 - \eta_\pm + \eta_z \quad , \quad \eta_\mu = \eta_+ - \eta_z = 1 - x_+ \; . \tag{65}$$

When the limit $\mu \to 0$ is taken (after the limit $L \to \infty$, so that all the $Z_{\gamma,\gamma'}$ have the same limit), the multiscale integration procedure implies the power law decay of the correlations given by (8).

If $\mu \ne 0$ (that is, if the temperature is not the critical one), the correlations decay faster than any power with rate proportional to μ_{h^*}, where, if $[x]$ denotes the largest integer $\le x$, h^* is given by

2.45c
$$h^* = \left[\frac{\log_\gamma|\mu|}{1+\eta_\mu}\right] \; , \tag{66}$$

which implies, together with (65), the identity (11) of Theorem 1.1.

2.4 The anisotropic Ashkin-Teller model

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In order to derive (12), we briefly recall the analysis of the anisotropic Ashkin-Teller model in [13, 14] with $J \ne J'$. We still obtain an expression similar to (34), the main difference being that (see (182) below) $P_1(d\psi)$ contains in the exponents also terms of the form $\psi_{\mathbf{x},\omega}^{\varepsilon(\le h)}\psi_{\mathbf{x},-\omega}^{\varepsilon(\le h)}$, and the same is true for $P_\chi(d\chi)$. The integration procedure is similar to the one in §2.3, but we have to substitute the Grassmann integration $P_{Z_h,\mu_h}(d\psi^{(\le h)})$ in (44) with a new measure $P_{Z_h,\mu_h,\sigma_h}(d\psi^{(\le h)})$, where μ_h and σ_h are the constants multiplying, respectively, the quadratic *mass terms*

$$2\sum_{\omega=\pm}\psi_{\mathbf{x},\omega}^{(\le h)+}\psi_{\mathbf{x},-\omega}^{(\le h)-} \qquad \text{and} \qquad -2i\sum_{\varepsilon=\pm}\psi_{\mathbf{x},+}^{(\le h)\varepsilon}\psi_{\mathbf{x},-}^{(\le h)\varepsilon} \; . \tag{67}$$

One can see that

$$|\log_\gamma(\mu_{j-1}/\mu_j) - \eta_\mu(\lambda_{-\infty})| \leq C\lambda^2\gamma^{\vartheta j} ,$$

$$|\log_\gamma(\sigma_{j-1}/\sigma_j) - \eta_\sigma(\lambda_{-\infty})| \leq C\lambda^2\gamma^{\vartheta j} . \tag{68}$$

Hence, since the two mass terms are clearly proportional, respectively, to the operators O^+ and O^-, we find that

$$\eta_\mu = \eta_+ - \eta_z , \qquad \eta_\sigma = \eta_- - \eta_z . \tag{69}$$

It turns out that the difference of the critical temperatures scales as $|J - J'|^{x_T}$ where x_T, see (5.26) of [13] (where the indices are defined with a different sign and the definitions of μ_h and σ_h are exchanged), is given by

$$x_T = \frac{1+\eta_\mu}{1+\eta_\sigma} , \tag{70}$$

which implies (12), since $\eta_\mu = 1 - x_+$ and $\eta_\sigma = 1 - x_-$.

3 Equivalence with a continuum model

3

In this section we show that the spin model (1) is *equivalent*, for the purpose of computing the long distance behavior of the correlations we are considering, to a fermionic theory defined as the formal scaling limit of the original one plus an ultraviolet regularization; more exactly, we prove that the critical indices x_+, x_-, ν and x_T of the spin model (1) are *equal* to the indices of a fermionic theory provided that the bare coupling λ_∞ of the new theory is properly chosen as a suitable function of the parameters of the 8V or AT models. The new fermionic theory has correlations expressed by Grassmann integrals which are identical to the ones appearing in certain Quantum Field Theory models; in particular it verifies extra *Gauge symmetries* with respect to the original spin Hamiltonian (1).

3.1 The model

3.1

The continuum (or QFT) model is defined as the limit $N \to \infty$, followed by the limit $-l \to \infty$, to be called *the removed cutoff limit*, of a model with an infrared γ^l and an ultraviolet γ^N momentum cut-off, $-l, N \gg 0$. This model is expressed in terms of the following Grassmann integral

*th*1111

$$e^{\mathcal{W}_N(A,J,\eta)} = \int P(d\psi^{[l,N]}) \exp\Big\{\mathcal{V}^{(N)}(\psi^{[l,N]}) + \sum_\varepsilon \int d\mathbf{x} A^\varepsilon_\mathbf{x} O_{\varepsilon,\mathbf{x}} + \tag{71}$$
$$+ \sum_\omega \int d\mathbf{x}\ [J_{\mathbf{x},\omega}\psi^{[l,N]+}_{\mathbf{x},\omega}\psi^{[l,N]-}_{\mathbf{x},\omega} + \psi^{+[l,N]}_{\mathbf{x},\omega}\eta^-_{\mathbf{x},\omega} + \eta^+_{\mathbf{x},\omega}\psi^{[l,N]-}_{\mathbf{x},\omega}]\Big\} ,$$

where $\mathbf{x} \in \widetilde{\Lambda}$, a square subset of $\mathbb{R}^2$ of size $|\widetilde{\Lambda}| \le \gamma^{-2l}$, $O^+_{\mathbf{x}}$ and $O^-_{\mathbf{x}}$ are defined in (49) and $P(d\psi^{[l,N]})$ is a Gaussian measure with propagator $g_T^{[l,N]}(\mathbf{x},\mathbf{y})$ given by (54) with $\mu_h = 0, Z_h = 1$ and $C_h^{-1}(\mathbf{k})$ replaced by $C_{l,N}^{-1}(\mathbf{k}) = \sum_{k=l}^N f_k(\mathbf{k})$; moreover $\eta^\pm_{\mathbf{x}}$ are external fermionic fields and $A^\varepsilon_{\mathbf{x}}, J_{\mathbf{x},\omega}$ are external bosonic fields. The interaction is

gjhfk
$$\mathcal{V}^{(N)}(\psi) = \frac{\lambda_\infty}{2} \sum_\omega \int d\mathbf{x} \int d\mathbf{y} v_K(\mathbf{x}-\mathbf{y}) \psi^+_{\mathbf{x},\omega} \psi^+_{\mathbf{y},-\omega} \psi^-_{\mathbf{x},\omega} \psi^-_{\mathbf{y},-\omega} \;, \tag{72}$$

where $K < N$ and $v_K(\mathbf{x}-\mathbf{y})$ is given by

$$v_K(\mathbf{x}-\mathbf{y}) = \frac{1}{L^2} \sum_{\mathbf{p}} \chi_0(\gamma^{-K}\mathbf{p}) e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \;, \tag{73}$$

$\chi_0(\mathbf{p})$ being a smooth function with support in $\{|\mathbf{p}| \le 2\}$ and equal to 1 for $\{|\mathbf{p}| \le 1\}$. The correlation functions are found by making suitable derivatives with respect to the external fields $A_{\mathbf{x}}$, $J_{\mathbf{x}}$, $\eta_{\mathbf{x}}$ and setting them equal to zero. We consider K fixed, for example $K = 0$, so that no ultraviolet regularization is needed, as we shall see, when we take the limit $N \to \infty$.

We shall study the functional $\mathcal{W}_N(A, J, \eta)$ by performing a multiscale integration of (71); we have to distinguish two different regimes: the first regime, called *ultraviolet*, contains the scales $h \in [K+1, N]$, while the second one contains the scales $h \le K$, and is called *infrared*.

3.2 The ultraviolet integration

3.2

We describe how to control the integration of the ultraviolet scales, in order to remove the ultraviolet cut-off $N \to \infty$. For simplicity, we shall only consider the case $A = \eta = 0$, but the result is valid for the full problem (see also the remark at the end of this section).

Assume that the fields $\psi^{(N)}, \psi^{(N-1)}, ..., \psi^{(h+1)}$ are integrated so that

th1111a
$$e^{\mathcal{W}_N(0,J,0)} = e^{\mathcal{S}^{(h)}(J)} \int P(d\psi^{[l,h]}) \exp\left\{ \mathcal{V}^{(h)}(\psi^{[l,h]}) + \mathcal{B}^{(h)}(\psi^{[l,h]}, J) \right\} \tag{74}$$

where $\mathcal{V}^{(h)} + \mathcal{B}^{(h)}$ is sum of integrated monomials in m $\psi^+_{\mathbf{x}_i,\omega_i}$ variables, $i = 1, \dots, m$, m $\psi^-_{\mathbf{y}_i,\omega_i}$ variables and n $J_{\mathbf{z}_j,\omega'_j}$ external fields, $j = 1, \dots, n$, multiplied by suitable kernels $W^{(n;2m)(h)}_{\underline{\omega}';\underline{\omega}}(\underline{\mathbf{z}}; \underline{\mathbf{x}}, \underline{\mathbf{y}})$. The scaling dimension is again (57), and, as in §2.3, we define a localization operator on the terms with positive or vanishing scaling dimensions which, as in the previous case, are the terms with $(2m, n) = (2, 0)$ or $(4, 0)$ or $(2, 1)$. Notice however that in this case the localization operation is defined as the identity on the relevant or marginal terms, that is $W^{(0;2)(h)}_\omega$, $W^{(1;2)(h)}_{\omega';\omega}$ and $W^{(0;4)(h)}_{\omega,\omega'}$, while it annihilates, as always, all the other contributions to the effective potential.

These kernels $W^{(n;2m)(h)}_{\underline{\omega}';\underline{\omega}}(\underline{\mathbf{z}};\underline{\mathbf{x}},\underline{\mathbf{y}})$ are represented as power expansions in the *running coupling functions* $W^{(0;2)(k)}_{\omega}$, $W^{(1;2)(k)}_{\omega';\omega}$ and $W^{(0;4)(k)}_{\omega,\omega'}$, $k\ge h$, whose size is estimated by the L^1 norm, as well as the kernels themselves. Of course, since the kernels may contain delta functions, we extend, as usual, the definition of L^1 norm, by treating the delta as a positive funcion. Hence, we define

norm
$$\|W^{(n;2m)(k)}_{\underline{\omega}';\underline{\omega}}\| \stackrel{def}{=} \frac{1}{|\widetilde{\Lambda}|}\int d\underline{\mathbf{z}}d\underline{\mathbf{x}}d\underline{\mathbf{y}}\;\left|W^{(n;2m)(k)}_{\underline{\omega}';\underline{\omega}}(\underline{\mathbf{z}};\underline{\mathbf{x}},\underline{\mathbf{y}})\right| \tag{75}$$

and we prove the following theorem.

Theorem 3.1 *If λ_∞ is small enough, there exist two constants $C_1>1$ and C_2, such that, if $K\le h\le N$, the relevant or marginal contributions to the effective potential satisfy the bounds:*

hb1
$$\|W^{(0;2)(h)}_{\omega}\| \le C_1|\lambda_\infty|\gamma^h\gamma^{-2(h-K)}\;, \tag{76}$$
hb2
$$\|W^{(1;2)(h)}_{\omega';\omega} - \delta_2\delta_{\omega,\omega'}\| \le C_2|\lambda_\infty|\gamma^{-(h-K)}\;, \tag{77}$$
hb3
$$\|W^{(0;4)(h)}_{\omega,\omega'} - \lambda_\infty v\delta_4\delta_{\omega,-\omega'}\| \le C_2|\lambda_\infty|^2\gamma^{-(h-K)}\;, \tag{78}$$

where $\delta_2(\mathbf{z};\mathbf{x},\mathbf{y})\equiv\delta(\mathbf{z}-\mathbf{x})\delta_(\mathbf{z}-\mathbf{y})$ and $v\delta_4(\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}_1,\mathbf{y}_2)\equiv\delta(\mathbf{x}_1-\mathbf{y}_1)v_K(\mathbf{x}_1-\mathbf{x}_2)\delta(\mathbf{x}_2-\mathbf{y}_2)$.

Before proving the theorem, notice that, as for the multiscale analysis in §2.3, the fact that the running coupling functions are small for λ_∞ small enough (as it follows by the bounds (76), (77), (78)) implies the following standard "dimensional" bound for all other kernels with negative scaling dimension, for λ_∞ small enough, see *e.g.* App. A of [23]:

pc1
$$\|W^{(n;2m)(k)}_{\underline{\omega}';\underline{\omega}}\| \le C^{n+d_{n,m}}|C_1\lambda_\infty|^{d_{n,m}}\gamma^{k(2-n-m)}\;, \tag{79}$$

where $d_{n,m}=\max\{m-1,0\}$, if $n>0$, and $d_{n,m}=\max\{m-1,1\}$, if $n=0$, and C is a suitable constant larger, at least, than γ. Indeed, in the tree expansion for $W^{(n;2m)(k)}_{\underline{\omega}';\underline{\omega}}$ defined in [23], all the vertices of the tree have negative scaling dimension and there are three types of endpoints (see [23]), associated to $W^{(0;2)(h)}_{\omega}$, $W^{(1;2)(h)}_{\omega';\omega}$, $W^{(0;4)(h)}_{\omega,\omega'}$, which contribute (up to dimensional factors and for λ_∞ small enough) a factor $C_1|\lambda_\infty|$, $1+C_2|\lambda_\infty|\le C$ and $|\lambda_\infty|[1+C_2|\lambda_\infty|]\le C_1|\lambda_\infty|$, respectively. Notice that the condition $C>\gamma$ comes from the bound of the trivial tree (that with only one endpoint) contributing to the tree expansion of $W^{(0;2)(k)}_{\omega}$. The bounds (76), (77), (78) follow from a "power counting improvement", similar to the one used in [19] for the Yukawa model, in which the non-locality of the interaction plays an essential role.

Proof of Theorem 3.1 The proof is by induction: we assume that the bounds (76)-(78) hold for $h : k+1 \leq h \leq N$ (for $h = N$ they are true with $C_1 = C_2 = 0$) and we prove them for $h = k$.

The inductive assumption implies the validity of (79) and we need to improve such bound when $2 - n - m \geq 0$. We can write, by using the properties of the fermionic truncated expectations and the fact that, by the oddness of the free propagator, $W_\omega^{(1;0)}(\mathbf{k}) = 0$,

111b
$$W_\omega^{(0;2)(k)}(\mathbf{x}, \mathbf{y}) = \tag{80}$$
$$= \lambda_\infty \int d\mathbf{w} d\mathbf{w}' \; v_K(\mathbf{x} - \mathbf{w}) g_\omega^{[k+1,N]}(\mathbf{x} - \mathbf{w}') W_{-\omega;\omega}^{(1;2)(k)}(\mathbf{w}; \mathbf{w}', \mathbf{y}) \; ,$$

which can be bounded, by using (79), as

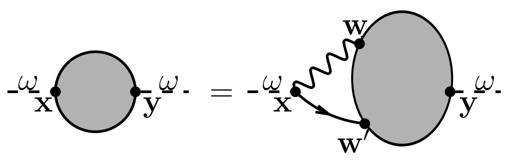

Figure 2: : Graphical representation of (80); the gray blobs represent the kernels $W_\omega^{(0;2)(k)}$ and $W_{-\omega;\omega}^{(1;2)(k)}$, the dotted lines the external fermionic lines, the paired line is the fermionic propagator $g_\omega^{[k+1,N]}$ and the wiggly line is the interaction v_K.

111c
$$\|W_\omega^{(0;2)(k)}\| \leq |\lambda_\infty| \|v_K\|_{L^\infty} \|W_{-\omega;\omega}^{(1;2)(k)}\| \sum_{j=k+1}^{N} \|g_\omega^{(j)}\|_{L^1} \leq$$
$$\leq \frac{c_1}{1-\gamma^{-1}} \gamma^{2K} C |\lambda_\infty| \gamma^{-k} \leq C_1 |\lambda_\infty| \gamma^{k} \gamma^{-2(k-K)} \; , \tag{81}$$

where, for example, $C_1 = \max\{2, \frac{c_1}{1-\gamma^{-1}} C\}$; hence (76) is proved. Notice that the condition $C_1 \geq 2$ is introduced only because C_1 is the same constant appearing in (79).

Let us now consider $W_{\omega';\omega}^{(1;2)(k)}(\mathbf{z}; \mathbf{x}, \mathbf{y})$ and notice that it can be decomposed as the sum of the five terms in Fig.3. The term denoted by (a) in Fig.3 can be bounded as

$$\|W_{(a);\omega';\omega}^{(1;2)(k)}\| \leq |\lambda_\infty| \|v_K\|_{L^\infty} \|W_{\omega',-\omega;\omega}^{(2;2)(k)}\| \sum_{j=k+1}^{N} \|g_\omega^{(j)}\|_{L^1} \leq C C_1 |\lambda_\infty| \gamma^{-2(k-K)} \; . \tag{82}$$

The bounds for the graphs (c) and (d) are an easy consequence of the bound for $W_\omega^{(0;2)(k)}$.

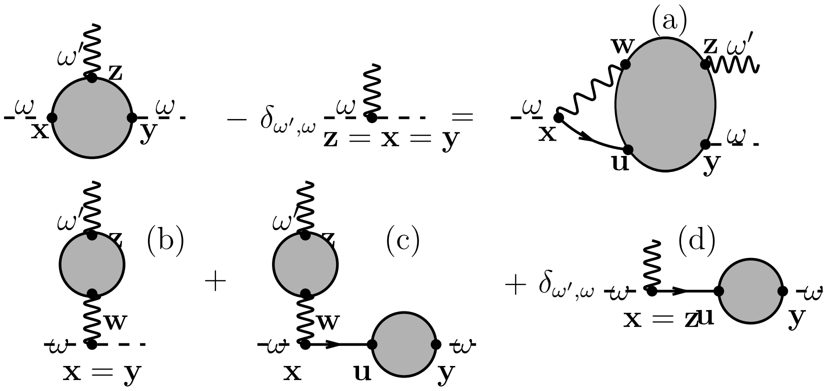

Figure 3: : Graphical representation of $W^{(1;2)(k)}_{\omega';\omega}(\mathbf{z};\mathbf{x},\mathbf{y})$; the external wiggly line represent the external field J, while the internal wiggly line is the interaction v_K, as before.

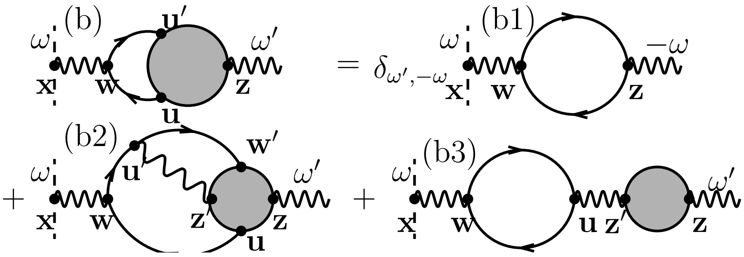

Figure 4: : Graphical representation of the term (b) in Fig.3

In order to obtain an improved bound also for the graph (b) of Fig. 3, we need to further expand $W^{(2;0)(k)}_{\omega,\omega'}$ as done in Fig 4, if we define the graph (b2) so that the vertex u' can be either on the fermion line joining w with w' (as in the figure) or on the other fermion line ending in w.

The bound for the graph $(b2)$ can be done by using the previous arguments. We can write

$$W^{(1;2)(k)}_{(b2)\omega';\omega}(\mathbf{z};\mathbf{x},\mathbf{y}) = \lambda_\infty^2 \delta(\mathbf{x}-\mathbf{y}) \int d\mathbf{w} d\mathbf{u}' d\mathbf{z}' \; v_K(\mathbf{x}-\mathbf{w}) v_K(\mathbf{u}'-\mathbf{z}') \cdot$$
$$\cdot \int d\mathbf{u} d\mathbf{w}' \; g^{[k+1,N]}_\omega(\mathbf{w}-\mathbf{u}) g^{[k+1,N]}_\omega(\mathbf{u}'-\mathbf{w}) g^{[k+1,N]}_\omega(\mathbf{w}'-\mathbf{u}') \cdot$$
$$\cdot W^{(2;2)(k)}_{\omega',\omega;-\omega}(\mathbf{z},\mathbf{z}';\mathbf{w}',\mathbf{u}) \; . \tag{83}$$

In order to get the right bound, it is convenient to decompose the three propagators g_ω into scales and then bound by the L^∞ norm the propagator of lowest scale, while the two others are used to control the integration over the inner space variables through their L^1 norm. Hence we get:

27 $$\|W^{(1;2)(k)}_{(b2)\omega';\omega}\| \le |\lambda_\infty|^2 \|v_K\|_{L^\infty} \|v_K\|_{L^1} \|W^{(2;2)(k)}_{\omega',-\omega;\omega}\| \cdot \tag{84}$$

$$\cdot 3! \sum_{k+1\le i'\le j\le i\le N} \|g^{(j)}_\omega\|_{L^1}\|g^{(i)}_\omega\|_{L^1}\|g^{(i')}_\omega\|_{L^\infty} \le C_3|\lambda_\infty|^2\gamma^{-2(k-K)} \ . \quad (85)$$

for some constant C_3.

The bound of $(b1)$ and $(b3)$ requires a new argument, based on a cancelation following from the particular form of the free propagator. Let us consider, for instance, $(b1)$:

$$W^{(1;2)(k)}_{(b1)\omega';\omega}(\mathbf{z};\mathbf{x},\mathbf{y}) = \\ = \lambda_\infty \delta_{\omega',-\omega}\delta(\mathbf{x}-\mathbf{y}) \int d\mathbf{w}\ v_K(\mathbf{x}-\mathbf{w}) \left[g^{[k+1,N]}_{-\omega}(\mathbf{w}-\mathbf{z})\right]^2 \ . \quad (86)$$

Since the cutoff function $C_{k,N}(\mathbf{k})$ is symmetric in the exchange between k_0 and k_1, it is easy to see that $g^{[k,N]}_\omega(x_0,x_1) = -i\omega g^{[k,N]}_\omega(x_1,-x_0)$; hence

$$\int d\mathbf{u}\ \left[g^{[k+1,N]}_{-\omega}(\mathbf{u})\right]^2 = 0 \ . \quad (87)$$

mas3

It follows, by using (87) and the identity

$$v_K(\mathbf{x}-\mathbf{w}) = v_K(\mathbf{x}-\mathbf{z}) + \sum_{j=0,1}(z_j-w_j)\int_0^1 ds\ (\partial_j v_K)(\mathbf{x}-\mathbf{z}+s(\mathbf{z}-\mathbf{w}))\ , \quad (88)$$

b

that we can write

$$W^{(1;2)(k)}_{(b1)\omega';\omega}(\mathbf{z};\mathbf{x},\mathbf{y}) = \lambda_\infty \delta_{\omega',-\omega}\delta(\mathbf{x}-\mathbf{y}) \cdot \quad (89) \\ \cdot \sum_{j=0,1}\int_0^1 ds \int d\mathbf{w}\ (\partial_j v_K)(\mathbf{x}-\mathbf{z}+s(\mathbf{z}-\mathbf{w}))(z_j-w_j) \left[g^{[k+1,N]}_{-\omega}(\mathbf{w}-\mathbf{z})\right]^2 \ .$$

Hence,

$$\|W^{(1;2)(k)}_{(b1)\omega';\omega}\| \le 4|\lambda_\infty| \sum_{i=k}^N\sum_{j=k}^i \|g^{(j)}_{-\omega}\|_{L^\infty} \int d\mathbf{x}\ |(\partial_j v_K)(\mathbf{x})| \cdot \quad (90)$$

$$\cdot \int d\mathbf{w}\ |w_j||g^{(i)}_{-\omega}(\mathbf{w})| \le C_4|\lambda_\infty|\gamma^{-(k-K)} \ . \quad (91)$$

By summing all the bounds, we see that there is a constant C_2 such that

$$\|W^{(1;2)(k)}_{\omega';\omega} - \delta_{\omega,\omega'}\delta_2\| \le C_2|\lambda_\infty|\gamma^{-(k-K)} \ , \quad (92)$$

ab

which proves (77). The bound (78) for $W^{(0;4)(k)}$ follows from similar arguments. ∎

Remark In presence of the A fields the above analysis can be repeated, with the only difference that in the analogue of Fig. (3.3) the (b1) and (b3) terms are missing.

3.3 Equivalence of the spin and the QFT models

As a consequence of the integration of the ultraviolet scales discussed in the previous section, we can write the removed cutoffs limit of (71), with $\eta = 0$ and with the choice $K = 0$, as

221 $$\lim_{l\to-\infty}\lim_{N\to\infty}\int P_{\mu_0,Z_0}(d\psi^{(\le 0)})e^{\mathcal{V}^{(0)}(\psi^{(\le 0)})+\mathcal{B}^{(0)}(\psi^{(\le 0)},A,J)}\;, \tag{93}$$

where the propagator of the integration measure in (93) coincides with $g_T^{(\le 0)}(\mathbf{x},\mathbf{y})$, defined in (54), $\mathcal{L}\mathcal{V}^{(0)} = \lambda_0 F_\lambda^{(0)}$ and $\mathcal{L}\mathcal{B}^{(0)}$, when $J = 0$, defined as in (2.37); from the analysis of the previous section it follows that λ_0 is a smooth function of λ_∞, such that $\lambda_0 = \lambda_\infty + O(\lambda_\infty^2)$.

The multiscale integration for the negative scales can be done exactly as described in §2.3, with the only difference that, by the oddness of the free propagator, $\nu_j = 0$ and

$$\lambda_{j-1} = \lambda_j + \widehat\beta_\lambda^{(j)}(\lambda_j, ...\lambda_0)\;, \tag{94}$$

where, by (53) and the short memory property,

$$\widehat\beta_\lambda^{(j)}(\lambda_j, ...\lambda_0) = \beta_\lambda^{(j)}(\lambda_j, ...\lambda_0) + O(\bar\lambda_j^2\gamma^{\vartheta j})\;, \tag{95}$$

$\beta_\lambda^{(j)}(\lambda_j, ...\lambda_j)$ being the function appearing in the bound (59), so that we can prove in the usual way that $\lambda_{-\infty} = \lambda_0 + O(\lambda_0^2)$; since $\lambda_0 = \lambda_\infty + O(\lambda_\infty^2)$, we have

$$\lambda_{-\infty} = h(\lambda_\infty) = \lambda_\infty + O(\lambda_\infty^2)\;, \tag{96}$$

for some analytic function $h(\lambda_\infty)$, invertible for λ_∞ small enough. Moreover, by using (53)

ffg2 $$\frac{Z_{j-1}^\pm}{Z_j^\pm} = 1 + \widehat\beta_\pm^{(j)}(\lambda_j, ..., \lambda_0)\;, \tag{97}$$

with

dx1 $$\widehat\beta_\pm^{(j)}(\lambda_j, ..., \lambda_0) = \beta_\pm^{(j)}(\lambda_j, ...\lambda_0) + O(\bar\lambda_j^2\gamma^{\vartheta j})\;, \tag{98}$$

$\beta_\pm^{(j)}$ being the functions appearing in (63). This implies that

dx2 $$\eta_\pm = \log_\gamma[1 + \beta_\pm(\lambda_{-\infty})]\;, \tag{99}$$

that is *the critical indices in the AT or 8V or in the model (71) are the same as functions of* $\lambda_{-\infty}$.

Of course $\lambda_{-\infty}$ is a rather complex function of all the details of the models. However, if we call $\lambda'_j(\lambda)$ the effective couplings of the lattice model of the previous sections, the invertibility of $h(\lambda_\infty)$ implies that we can choose λ_∞ so that

dx3 $$h(\lambda_\infty) = \lambda'_{-\infty}(\lambda)\;. \tag{100}$$

With this choice of $\lambda_\infty(\lambda)$, the critical indices are the same, as they depend only on $\lambda_{-\infty}$; the rest of this chapter is devoted to the proof that the critical indices have, as functions of λ_∞, simple expressions, which imply the scaling relations in the main theorem.

4 Ward Identities and Schwinger-Dyson equation

In this section we prove that the Gauge symmetry of the equivalent QFT model implies closed equations for the correlations, from which an explicit expression of the correlations and the indices as a function of λ_∞ can be derived; such expressions are so simple that the validity of the extended scaling relation can be checked. Such a simplicity follows from the fact that the Ward Identities for the equivalent QFT model, from which the closed equations are derived, verify a property called *anomaly non-renormalization.*

4.1 Schwinger-Dyson equations and Ward Identities

3.4

The Schwinger-Dyson equations for the model (71) are generated by the identity, see [6],

SDE

$$D_\omega(\mathbf{k})\frac{\partial e^{\mathcal{W}_N}}{\partial \widehat{\eta}^+_{\mathbf{k},\omega}}(0,\eta) = \chi_{l,N}(\mathbf{k})\Bigg[\widehat{\eta}^-_{\mathbf{k},\omega}e^{\mathcal{W}_N(0,\eta)} - \\ -\lambda_\infty\int\frac{d\mathbf{p}}{(2\pi)^2}\;\widehat{v}_K(\mathbf{p})\frac{\partial^2 e^{\mathcal{W}_N}}{\partial \widehat{J}_{\mathbf{p},-\omega}\partial\widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}}(0,\eta)\Bigg]\;, \qquad (101)$$

where $D_\omega(\mathbf{k}) = -ik_0 + \omega k$ and we have shortened the notation of $\mathcal{W}_N(0,J,\eta)$ into $\mathcal{W}_N(J,\eta)$.

On the other hand, by the change of variables $\psi^\pm_{\mathbf{x},\omega}\to e^{\pm i\alpha_{\mathbf{x},\omega}}\psi^\pm_{\mathbf{x},\omega}$, we obtain another identity:

grez

$$D_\omega(\mathbf{p})\frac{\partial\mathcal{W}_N}{\partial\widehat{J}_{\mathbf{p},\omega}}(0,\eta) - \tau\;\widehat{v}_K(\mathbf{p})D_{-\omega}(\mathbf{p})\frac{\partial\mathcal{W}_N}{\partial\widehat{J}_{\mathbf{p},-\omega}}(0,\eta) = \qquad (102) \\ = \int\frac{d\mathbf{k}}{(2\pi)^2}\left[\widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}\frac{\partial\mathcal{W}_N}{\partial\widehat{\eta}^+_{\mathbf{k},\omega}}(0,\eta) - \frac{\partial\mathcal{W}_N}{\partial\widehat{\eta}^-_{\mathbf{k}+\mathbf{p},\omega}}(0,\eta)\widehat{\eta}^-_{\mathbf{k},\omega}\right] + \frac{\partial\mathcal{W}_\mathcal{A}}{\partial\widehat{\alpha}_{\mathbf{p},\omega}}(0,0,\eta)\;,$$

where τ is a constant to be chosen later,

h11

$$e^{\mathcal{W}_\mathcal{A}(\alpha,J,\eta)} = \int P(d\psi^{[l,N]})e^{\mathcal{V}^{(N)}(\psi^{[l,N]})+\sum_\omega\int d\mathbf{x}\;J_{\mathbf{x},\omega}\psi^{[l,N]+}_{\mathbf{x},\omega}\psi^{[l,N]-}_{\mathbf{x},\omega}} \\ \cdot e^{\sum_\omega\int d\mathbf{x}[\psi^{[l,N]+}_{\mathbf{x},\omega}\eta^-_{\mathbf{x},\omega}+\eta^+_{\mathbf{x},\omega}\psi^{[l,N]-}_{\mathbf{x},\omega}]}e^{[\mathcal{A}_0-\tau\mathcal{A}_-]\left(\alpha,\psi^{[l,N]}\right)}\;, \qquad (103)$$

$$\mathcal{A}_0(\alpha,\psi) \stackrel{def}{=} \sum_{\omega=\pm} \int \frac{d\mathbf{q}\, d\mathbf{p}}{(2\pi)^4}\, C_\omega(\mathbf{q},\mathbf{p}) \widehat{\alpha}_{\mathbf{q}-\mathbf{p},\omega} \widehat{\psi}^+_{\mathbf{q},\omega} \widehat{\psi}^-_{\mathbf{p},\omega}\ , \tag{104}$$

$$\mathcal{A}_-(\alpha,\psi) \stackrel{def}{=} \sum_{\omega=\pm} \int \frac{d\mathbf{q}\, d\mathbf{p}}{(2\pi)^4}\, D_{-\omega}(\mathbf{p}-\mathbf{q}) \widehat{v}_K(\mathbf{p}-\mathbf{q}) \widehat{\alpha}_{\mathbf{q}-\mathbf{p},\omega} \widehat{\psi}^+_{\mathbf{q},-\omega} \widehat{\psi}^-_{\mathbf{p},-\omega}, \tag{105}$$

$$C_\omega(\mathbf{q},\mathbf{p}) = [\chi^{-1}_{l,N}(\mathbf{p}) - 1] D_\omega(\mathbf{p}) - [\chi^{-1}_{l,N}(\mathbf{q}) - 1] D_\omega(\mathbf{q})\ , \tag{106}$$

and $\chi_{l,N}(\mathbf{k}) = \sum_{k=l}^N f_k(\mathbf{k})$.

An explicit derivation of (102) can be found in §2.2 of [10]; (102) is obtained by introducing a cut-off function $\chi^\varepsilon_{l,N}(\mathbf{k})$ never vanishing for all values of $\mathbf{k} \neq 0$ and equivalent to $\chi_{l,N}(\mathbf{k})$ as far as the scaling properties of the theory are concerned; ε is a small parameter and $\lim_{\varepsilon\to 0^+} \chi^\varepsilon_{l,N}(\mathbf{k}) = \chi_{l,N}(\mathbf{k})$. This further regularization (to be removed before taking the removed cutoffs limit) ensures that the identity $[(\chi^\varepsilon_{l,N})^{-1}(\mathbf{k}) - 1]\chi^\varepsilon_{l,N}(\mathbf{k}) = 1 - \chi^\varepsilon_{l,N}(\mathbf{k})$ is satisfied for all $\mathbf{k} \neq 0$. When this further regularization is removed, all the quantities we shall study have a well defined expression and, by the change of variables $\psi^\pm_{\mathbf{x},\omega} \to e^{\pm i\alpha_{\mathbf{x},\omega}} \psi^\pm_{\mathbf{x},\omega}$, we get the Ward identity (WI):

ez1

$$D_\omega(\mathbf{p}) \frac{\partial \mathcal{W}_N}{\partial \widehat{J}_{\mathbf{p},\omega}}(0,\eta) = \tag{107}$$
$$= \int \frac{d\mathbf{k}}{(2\pi)^2} \left[\widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega} \frac{\partial \mathcal{W}_N}{\partial \widehat{\eta}^+_{\mathbf{k},\omega}}(0,\eta) - \frac{\partial \mathcal{W}_N}{\partial \widehat{\eta}^-_{\mathbf{k}+\mathbf{p},\omega}}(0,\eta) \widehat{\eta}^-_{\mathbf{k},\omega} \right] + \frac{\partial \mathcal{W}_\mathcal{A}}{\partial \widehat{\alpha}_{\mathbf{p},\omega}}(0,0,\eta)\Big|_{\tau=0}\ .$$

(102) is obtained by just adding and subtracting the second term in the first line of (102). The last term in (107) is a correction to the formal WI, due to presence of the ultraviolet cut-off function $\chi_{l,N}$.

The two equations obtained from (102) by putting $\omega = \pm 1$ can be solved w.r.t. $\partial e^{\mathcal{W}_N}/\partial \widehat{J}_{\mathbf{p},\omega}$ and, if we define

$$a(\mathbf{p}) = \frac{1}{1 - \tau\, \widehat{v}_K(\mathbf{p})}\quad ,\quad \bar{a}(\mathbf{p}) = \frac{1}{1 + \tau\, \widehat{v}_K(\mathbf{p})}\ ,$$

Aeps

$$A_\varepsilon(\mathbf{p}) = \frac{a(\mathbf{p}) + \varepsilon \bar{a}(\mathbf{p})}{2}\ , \tag{108}$$

we obtain the identity

WT1

$$\frac{\partial e^{\mathcal{W}_N}}{\partial \widehat{J}_{\mathbf{p},\omega}}(0,\eta) - \sum_{\omega'} \frac{A_{\omega\omega'}(\mathbf{p})}{D_\omega(\mathbf{p})} \frac{\partial e^{\mathcal{W}_\mathcal{A}}}{\partial \widehat{\alpha}_{\mathbf{p},\omega'}}(0,0,\eta) = \tag{109}$$
$$= \sum_{\omega'} \frac{A_{\omega\omega'}(\mathbf{p})}{D_\omega(\mathbf{p})} \int \frac{d\mathbf{k}}{(2\pi)^2} \left[\widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega'} \frac{\partial e^{\mathcal{W}_N}}{\partial \widehat{\eta}^+_{\mathbf{k},\omega'}}(0,\eta) - \frac{\partial e^{\mathcal{W}_N}}{\partial \widehat{\eta}^-_{\mathbf{k}+\mathbf{p},\omega'}}(0,\eta) \widehat{\eta}^-_{\mathbf{k},\omega'} \right]\ .$$

By using (109) and (101), we easily get:

e

$$D_\omega(\mathbf{k}) \frac{\partial e^{\mathcal{W}_N}}{\partial \widehat{\eta}^+_{\mathbf{k},\omega}}(0,\eta) = \chi_{l,N}(\mathbf{k}) \Big\{ \widehat{\eta}^-_{\mathbf{k},\omega} e^{\mathcal{W}_N(0,\eta)} -$$

$$
\begin{aligned}
&-\lambda_\infty \sum_{\omega'} \int \frac{d\mathbf{p}}{(2\pi)^2}\, v_K(\mathbf{p}) \frac{A_{-\omega\omega'}(\mathbf{p})}{D_{-\omega}(\mathbf{p})} \cdot \\
&\cdot \int \frac{d\mathbf{q}}{(2\pi)^2} \left[\widehat{\eta}^+_{\mathbf{q}+\mathbf{p},\omega'} \frac{\partial^2 e^{\mathcal{W}_N}}{\partial \widehat{\eta}^+_{\mathbf{q},\omega'} \partial \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}}(0,\eta) - \frac{\partial^2 e^{\mathcal{W}_N}}{\partial \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega} \partial \widehat{\eta}^-_{\mathbf{q}+\mathbf{p},\omega'}}(0,\eta) \widehat{\eta}^-_{\mathbf{q},\omega'} \right] - \\
&-\lambda_\infty \sum_{\omega'} \int \frac{d\mathbf{p}}{(2\pi)^2}\, v_K(\mathbf{p}) \frac{A_{-\omega\omega'}(\mathbf{p})}{D_{-\omega}(\mathbf{p})} \frac{\partial^2 e^{\mathcal{W}_A}}{\partial \widehat{\alpha}_{\mathbf{p},\omega'} \partial \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}}(0,0,\eta) \Bigg\} \;,
\end{aligned}
\tag{110}
$$

where we have used that, by simple parity arguments,

$$
\lambda_\infty \int \frac{d\mathbf{p}}{(2\pi)^2}\, v_K(\mathbf{p}) \frac{A_{-\omega\omega}(\mathbf{p})}{D_{-\omega}(\mathbf{p})} \frac{\partial e^{\mathcal{W}_N}}{\partial \widehat{\eta}^+_{\mathbf{k},\omega}}(0,\eta) = 0 \;. \tag{111}
$$

4.2 Closed equation

3.4a

If we make an arbitrary number of functional derivatives with respect to the η external fields in (110), then we set $\eta = 0$ and perform the Fourier transform, we obtain a set of differential equations. We will prove in the last section the following crucial result

.1

Theorem 4.1 *If λ_∞ is small enough and we put*

61bb
$$
\tau = \frac{\lambda_\infty}{4\pi} \;, \tag{112}
$$

then the Fourier transforms of the correlation functions generated by setting $\eta = 0$ after deriving w.r.t. η the functional

ff
$$
\sum_{\omega'} \int \frac{d\mathbf{p}}{(2\pi)^2}\, v_K(\mathbf{p}) \frac{A_{-\omega\omega'}(\mathbf{p})}{D_{-\omega}(\mathbf{p})} \frac{\partial^2 e^{\mathcal{W}_A(0,0,\eta)}}{\partial \widehat{\alpha}_{\mathbf{p},\omega'} \partial \widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}} \;, \tag{113}
$$

vanish at distinct points in the removed cutoff limit (defined at the beginning of §*3.1).*

This theorem will be proved in §4.3. We want now to show how to use it to prove the identity (10), so completing the proof of Theorem 1.1.

By using Theorem 4.1 and some regularity property of the Schwinger functions (for details, see §A.1 in [6]) we get, in the removed cutoff limit, a set of *closed equation* for the Schwinger functions. In particular, if we define

$$
\langle \psi^-_{\mathbf{x},\omega} \psi^+_{\mathbf{y},\omega} \rangle \stackrel{def}{=} S_\omega(\mathbf{x}-\mathbf{y}) \;, \tag{114}
$$

Gom
$$
\langle \psi^-_{\mathbf{x},\omega} \psi^-_{\mathbf{y},-\omega} \psi^+_{\mathbf{u},-\omega} \psi^+_{\mathbf{v},\omega} \rangle \stackrel{def}{=} G_\omega(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}) \;, \tag{115}
$$

we get

eqx
$$\left(\partial_\omega S_\omega\right)(\mathbf{x}) - \lambda_\infty F_{K,-}(\mathbf{x}) S_\omega(\mathbf{x}) = \delta(\mathbf{x}) \;, \tag{116}$$

where $\partial_\omega = \partial_{x_0} + i\omega\partial_{x_1}$ and $F_{K,-}(\mathbf{x}) = \int d\mathbf{p}/(2\pi)^2 e^{-i\mathbf{p}\mathbf{x}} \widehat{F}_{K,-}(-\mathbf{p})$, with

$$\widehat{F}_{K,\varepsilon}(\mathbf{p}) \stackrel{def}{=} \frac{v_K(\mathbf{p}) A_\varepsilon(\mathbf{p})}{D_{-\omega}(\mathbf{p})} \;. \tag{117}$$

The solution of (116) is:

Som
$$S_\omega(\mathbf{x}) = e^{\lambda_\infty \Delta_-(\mathbf{x},0)} g_\omega(\mathbf{x}) \;, \tag{118}$$

having defined

$$\Delta_\varepsilon(\mathbf{x},\mathbf{z}) = \int \frac{d\mathbf{k}}{(2\pi)^2} \, \frac{e^{-i\mathbf{k}\mathbf{x}} - e^{-i\mathbf{k}\mathbf{z}}}{D_\omega(\mathbf{k})} \widehat{F}_{K,\varepsilon}(-\mathbf{k}) \;. \tag{119}$$

Notice that, for large $|\mathbf{x}|$, thanks to (108),

delta
$$\Delta_\varepsilon(\mathbf{x},0) \sim -\frac{A_\varepsilon(0)}{2\pi} \ln|\mathbf{x}| = -\frac{a(0)+\varepsilon\bar{a}(0)}{4\pi} \ln|\mathbf{x}| \;, \tag{120}$$

which implies, in particular, that the critical index η_z, defined in (64) is given by

$$\eta_z = \frac{\lambda_\infty}{4\pi}[a(0) - \bar{a}(0)] = \frac{2\tau^2}{1-\tau^2} \;. \tag{121}$$

Moreover, if we take in (110) three derivatives w.r.t. $\widehat{\eta}^+_{\mathbf{q},-\omega}$, $\widehat{\eta}^-_{\mathbf{k}+\mathbf{q}-\mathbf{s},\omega}$ and $\widehat{\eta}^-_{\mathbf{s},-\omega}$, we find, after Fourier transforming and in the cutoffs limit,

$$\begin{aligned} &\left(\partial^{\mathbf{x}}_\omega G_\omega\right)(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}) = \delta(\mathbf{x}-\mathbf{v}) S_{-\omega}(\mathbf{y}-\mathbf{u}) + \\ &+\lambda_\infty \Big[F_{K,+}(\mathbf{x}-\mathbf{y}) - F_{K,+}(\mathbf{x}-\mathbf{u}) - F_{K,-}(\mathbf{x}-\mathbf{v})\Big] G_\omega(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}) \;, \end{aligned} \tag{122}$$

which is solved, by using (118), by

eqG
$$\begin{aligned} G_\omega(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}) \;&=\; e^{-\lambda_\infty \left[\Delta_+(\mathbf{x}-\mathbf{y},\mathbf{v}-\mathbf{y}) - \Delta_+(\mathbf{x}-\mathbf{u},\mathbf{v}-\mathbf{u})\right]} \cdot \\ &\cdot\; S_\omega(\mathbf{x}-\mathbf{v}) S_{-\omega}(\mathbf{y}-\mathbf{u}) \;. \end{aligned} \tag{123}$$

The r.h.s. of (123) is well defined for $\mathbf{x} = \mathbf{u}$ and $\mathbf{y} = \mathbf{v}$, if $\mathbf{x} \neq \mathbf{y}$, or $\mathbf{x} = \mathbf{y}$ and $\mathbf{u} = \mathbf{v}$, if $\mathbf{x} \neq \mathbf{u}$. This is a consequence of the fact that the operators $\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega}$ and $\psi^+_{\mathbf{x},\omega}\psi^+_{\mathbf{x},-\omega}$ are well defined even in the limit $N \to \infty$, thanks to the non locality of the interaction. Hence, one expects that $\frac{\partial^2 \mathcal{W}_N}{\partial A^+_{\mathbf{x}} \partial A^+_{\mathbf{y}}}$ and $\frac{\partial^2 \mathcal{W}_N}{\partial A^+_{\mathbf{x}} \partial A^+_{\mathbf{y}}}$ can be calculated by simply using equations (123), (118). A rigorous proof of this statement could be done by a simple extension of Lemma 4.1

of [6] (with $\bar{Z}_N^{(1)} = c_1 = 1 + O(\lambda_\infty)$), where a similar (more difficult) problem is considered.

If we put (123) $\mathbf{x} = \mathbf{u}$ and $\mathbf{y} = \mathbf{v}$, we find, using also (115) and (120), that

$$\langle \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega}\psi^+_{\mathbf{y},-\omega}\psi^-_{\mathbf{y},\omega}\rangle = \langle \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega}\psi^+_{\mathbf{y},-\omega}\psi^-_{\mathbf{y},\omega}\rangle_0 e^{-2\lambda_\infty[\Delta_+(\mathbf{x}-\mathbf{y},0)-\Delta_-(\mathbf{x}-\mathbf{y},0)]}$$
$$\underset{|\mathbf{x}-\mathbf{y}|\to\infty}{\sim} \quad \frac{C}{|\mathbf{x}-\mathbf{y}|^{2[1-\bar{a}(0)(\lambda_\infty/2\pi)]}} \; . \tag{124}$$

If we put instead $\mathbf{x} = \mathbf{y}$ and $\mathbf{u} = \mathbf{v}$, we get

$$\langle \psi^+_{\mathbf{x},\omega}\psi^+_{\mathbf{x},-\omega}\psi^-_{\mathbf{u},-\omega}\psi^-_{\mathbf{u},\omega}\rangle = \langle \psi^+_{\mathbf{x},\omega}\psi^+_{\mathbf{x},-\omega}\psi^-_{\mathbf{u},-\omega}\psi^-_{\mathbf{u},\omega}\rangle_0 e^{2\lambda_\infty[\Delta_+(\mathbf{x}-\mathbf{u},0)+\Delta_-(\mathbf{x}-\mathbf{u},0)]}$$
$$\underset{|\mathbf{x}-\mathbf{u}|\to\infty}{\sim} \quad \frac{C}{|\mathbf{x}-\mathbf{u}|^{2[1+a(0)(\lambda_\infty/2\pi)]}} \; . \tag{125}$$

Let us now choose λ_∞, so that (100) is satisfied. Then, by using (49), (108), (112) and the definition (8) of $x_\pm$ we get the identities

$$x_+ = \frac{1-\tau}{1+\tau} \quad , \quad x_- = \frac{1+\tau}{1-\tau} \, , \tag{126}$$

which imply the identity (10).

Remark The proof of the relation $x_- x_+ = 1$ follows from two main ingredients, namely the linearity of τ as a function of λ_∞, see (112), and the vanishing of the last term in (110). The validity of such properties is due to our choice of the equivalent continuum model; it is indeed known, as proved in [5], that in other QFT models, still equivalent to the spin model, such properties are not true so that they do not allow to derive the relation $x_- x_+ = 1$. The linearity of τ as a function of λ_∞ corresponds to a property called in the physical literature non renormalization of the anomaly or Adler-Bardeen theorem, see [23, 24, 25].

4.3 Proof of Theorem 4.1

We start with the multiscale integration of the Grassman integral $\mathcal{W}_\mathcal{A}(\alpha, 0, \eta)$ (103) appearing on the WI (4.2). Notice that $\mathcal{W}_\mathcal{A}(\alpha, 0, \eta)$ (103) is very similar to $\mathcal{W}_N(J, \eta)$, see (71), the difference being that $\int J_{\mathbf{x},\omega}\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}$ is replaced by $\mathcal{A}_0 - \tau\mathcal{A}_-$. A crucial role in the analysis is played by the function $C_\omega(\mathbf{p}, \mathbf{q})$ appearing in the definition of $\mathcal{A}_0$; this function is very singular, but it appears in the various equations relating the correlation functions only through the regular function

mjmj $$\widehat{U}^{(i,j)}_\omega(\mathbf{q}+\mathbf{p},\mathbf{q}) \stackrel{def}{=} \widetilde{\chi}_N(\mathbf{p}) C_\omega(\mathbf{q}+\mathbf{p},\mathbf{q}) \widehat{g}^{(i)}_\omega(\mathbf{q}+\mathbf{p}) \widehat{g}^{(j)}_\omega(\mathbf{q}) \; , \tag{127}$$

where $\widetilde{\chi}_N(\mathbf{p})$ is a smooth function, with support in the set $\{|\mathbf{p}| \le 3\gamma^{N+1}\}$ and equal to 1 in the set $\{|\mathbf{p}| \le 2\gamma^{N+1}\}$; we can add freely this factor in the definition, since $\widehat{U}^{(i,j)}_\omega(\mathbf{q}+\mathbf{p},\mathbf{q})$ will only be used for values of $\mathbf{p}$ such that $\widetilde{\chi}_N(\mathbf{p}) = 1$, thanks to the support properties of the propagator. It is easy to see that $\widehat{U}^{(i,j)}_\omega$ vanishes if neither j nor i equals N or l; this has the effect that at least one of the fields in $\mathcal{A}_0$ has to be integrated at the N or l scale.

As a matter of fact, the terms in which at least one field is integrated at scale l are much easier to analyze, see the considerations after (161) below. In order to study the others, it is convenient to introduce the function $\widehat{S}^{(i,j)}_{\bar\omega,\omega}$ such that

91 $$\widehat{U}^{(i,j)}_\omega(\mathbf{q}+\mathbf{p},\mathbf{q}) = \sum_{\bar\omega} D_{\bar\omega}(\mathbf{p}) \widehat{S}^{(i,j)}_{\bar\omega,\omega}(\mathbf{q}+\mathbf{p},\mathbf{q}) \; . \tag{128}$$

One can show that, if we define

$$S^{(i,j)}_{\bar\omega,\omega}(\mathbf{z};\mathbf{x},\mathbf{y}) = \int \frac{d\mathbf{p}\, d\mathbf{q}}{(2\pi)^4} \, e^{-i\mathbf{p}(\mathbf{x}-\mathbf{z})} e^{i\mathbf{q}(\mathbf{y}-\mathbf{z})} \widehat{S}^{(i,j)}_{\bar\omega,\omega}(\mathbf{p},\mathbf{q}) \; , \tag{129}$$

then, given any positive integer M, there exists a constant C_M such that, if $j > l$,

61 $$|S^{(N,j)}_{\bar\omega,\omega}(\mathbf{z};\mathbf{x},\mathbf{y})| \le C_M \frac{\gamma^N}{1+[\gamma^N|\mathbf{x}-\mathbf{z}|]^M} \frac{\gamma^j}{1+[\gamma^j|\mathbf{y}-\mathbf{z}|]^M} \; , \tag{130}$$

a bound which is used to control the renormalization of the marginal terms containing a vertex of type $\mathcal{A}_0$. We choose τ as given by

61b $$\tau = \lambda_\infty \sum_{i,j=l+1}^{N} \int \frac{d\mathbf{q}}{(2\pi)^2} \; \widehat{S}^{(i,j)}_{-\omega,\omega}(\mathbf{q},\mathbf{q}) \; ; \tag{131}$$

by an explicit calculation one can see that, for any $l < 0$ and $N > 0$, τ satisfies (112). We remark that, to get this result, it is important to exclude from the sum in the r.h.s. of (131) the couples (i,j) with one of the indices equal to l; without this restriction, τ would be equal to 0, for any $N > 0$.

We will proceed as in the analysis of $\mathcal{W}_N(J,\eta)$, by integrating first the ultraviolet scales $N, N-1, \ldots, h+1$, $h \ge K$, and following a procedure very similar to the one described in §3.2; we have new marginal terms with one α field and two ψ fields and we have to prove the analogue of (77) for them. The marginal terms such that only one of these two fields is contracted are proportional to $W^{(0;2)(k)}$, so that one can use (76) to bound them. Hence, we shall consider in detail only the terms such that both fields of $\mathcal{A}_0$ or $\mathcal{A}_1$ are contracted and we shall call $\widehat{K}^{(n;2m)(k)}_{\Delta;\omega;\underline{\omega}'}$ the corresponding kernels of the monomials with $2m$ ψ-fields and n α-fields. In the case $n=1$, we decompose them as follows:

$$\widehat{K}^{(1;2m)(k)}_{\Delta;\omega;\underline{\omega}'}(\mathbf{p};\underline{\mathbf{k}}) = \sum_\sigma D_{\sigma\omega}(\mathbf{p})\widehat{W}^{(1;2m)(k)}_{\Delta;\sigma,\omega;\underline{\omega}'}(\mathbf{p};\underline{\mathbf{k}}) \;, \tag{132}$$

where $\mathbf{p}$ is the momentum flowing along the external α-field. As in §3.2, we have to improve the dimensional bound of $W^{(1;2)(k)}_{\Delta;\sigma,\omega;\omega'}$. We can write the following identity, which is represented by the first line of Fig.5 in the case $\sigma=-1$:

71ter

$$W^{(1;2)(k)}_{\Delta;\sigma,\omega;\omega'}(\mathbf{z};\mathbf{x},\mathbf{y}) = \sum_{i,j=k}^N \int d\mathbf{u}d\mathbf{w}\; S^{(i,j)}_{\sigma\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{w})W^{(0;4)(k)}_{\omega,\omega'}(\mathbf{u},\mathbf{w},\mathbf{x},\mathbf{y}) -$$
$$-\tau\,\delta_{-1,\sigma}\int d\mathbf{w}\; v_K(\mathbf{z}-\mathbf{w})W^{(1;2)(k)}_{-\omega;\omega'}(\mathbf{w};\mathbf{x},\mathbf{y})\;. \tag{133}$$

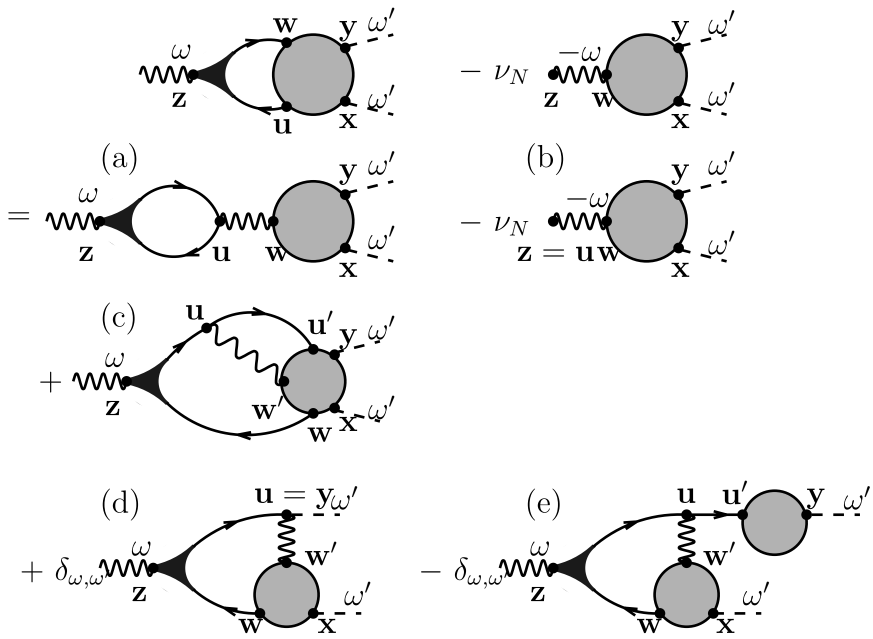

Figure 5: : Graphical representation of $W^{(1;2)(k)}_{\Delta;-1,\omega;\omega'}$; the triangular vertex represent $C_\omega(\mathbf{q},\mathbf{p})$ given by (4.5)

We can further decompose $W^{(1;2)(k)}_{\Delta;-1,\omega;\omega'}$ as in the last three lines of Fig.5. The term (c) can be written as

$$\lambda_\infty \sum_{i,j=k}^N \int d\mathbf{u}d\mathbf{u}'d\mathbf{w}d\mathbf{w}'\; S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{w})g^{[k,N]}_\omega(\mathbf{u}-\mathbf{u}')v_K(\mathbf{u}-\mathbf{w}')\cdot$$

$$\cdot W^{(1;4)(k)}_{-\omega;\omega,\omega'}(\mathbf{w}';\mathbf{u}',\mathbf{w},\mathbf{x},\mathbf{y}) \; . \tag{134}$$

Hence, if we put $b_j(\mathbf{x}) \stackrel{def}{=} \gamma^j/(1+[\gamma^j|\mathbf{x}|]^3)$, we recall that $S^{(i,j)}_{-\omega,\omega}$ is different from 0 only if either i or j is equal to N, and we use the bound (130), we see that the norm of (c) is bounded by

$$C_3|\lambda_\infty|\|v_K\|_{L^\infty} \sum_{i,j,m=k}^{N\,*} \int d\mathbf{x}d\mathbf{u}'d\mathbf{w}d\mathbf{w}' \; |W^{(1;4)(k)}_{-\omega;\omega,\omega'}(\mathbf{w}';\mathbf{u}',\mathbf{w},\mathbf{x},\mathbf{y})| \cdot$$
$$\cdot \int d\mathbf{z}d\mathbf{u} \; b_i(\mathbf{z}-\mathbf{w})b_j(\mathbf{z}-\mathbf{u})|g^{(m)}_\omega(\mathbf{u}-\mathbf{u}')| \; , \tag{135}$$

where $*$ reminds that $\max\{i,j\}=N$. Since the L^1 and the L^∞ norm of b_j satisfy a bound similar to analogous bounds of $g^{(j)}_\omega$, we can proceed as in the previous section to bound $\int d\mathbf{z}d\mathbf{u} \; b_i(\mathbf{z}-\mathbf{w})b_j(\mathbf{z}-\mathbf{u})|g^{(m)}_\omega(\mathbf{u}-\mathbf{u}')|$, by taking the L^∞ norm for the factor with the smaller index and the L^1 norm for the other two. By also using (79), we get the bound

$$C_\vartheta|\lambda_\infty|^2\gamma^{-2(k-K)}\gamma^{-\vartheta(N-k)} \; , \tag{136}$$

for any $0<\vartheta<1$ (C_ϑ is divergent for $\vartheta\to 1$). With respect to analogous bound in §3.2 ((b2) in Fig.4), there is an improvement of a factor $\gamma^{-\vartheta(N-k)}$. The term (d) can be bounded by

$$C|\lambda_\infty|\|v_K\|_{L^\infty} \sum_{i,j=k}^{N\,*} \|b_i\|_{L^1} \; \|b_j\|_{L^1} \le C|\lambda_\infty|\gamma^{-(k-K)}\gamma^{-(N-k)} \; ;$$

for the term (e) we get the bound $C|\lambda_\infty|^2\gamma^{-3(k-K)}\gamma^{-(N-k)}$. By putting together all the previous bounds, we get

222
$$\|(c)+(d)+(e)\| \le C_\vartheta|\lambda_\infty|\gamma^{-(k-K)}\gamma^{-\vartheta(N-k)} \; . \tag{137}$$

We consider now the terms (a) and (b), whose sum can be written as

75
$$\int d\mathbf{u} \; \left[\lambda_\infty \sum_{i,j=k}^{N} S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u}) - \tau\delta(\mathbf{z}-\mathbf{u})\right] \cdot$$
$$\cdot \int d\mathbf{w} \; v_K(\mathbf{u}-\mathbf{w})W^{(1;2)(k)}_{-\omega;\omega'}(\mathbf{w};\mathbf{x},\mathbf{y}) \; . \tag{138}$$

By using the identity (88), (138) can be written also as

5b
$$\left[\lambda_\infty \sum_{i,j=k}^{N} \int d\mathbf{u} \; S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u}) - \tau\right] \int d\mathbf{w} \; v_K(\mathbf{z}-\mathbf{w})W^{(1;2),(k)}_{-\omega;\omega'}(\mathbf{w};\mathbf{x},\mathbf{y}) +$$
$$+\lambda_\infty \sum_{p=0,1} \sum_{i,j=k}^{N} \int d\mathbf{u} \; S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u})(u_p-z_p) \cdot \tag{139}$$
$$\cdot \int_0^1 ds \; \int d\mathbf{w} \; (\partial_p v_K)(\mathbf{z}-\mathbf{w}+s(\mathbf{u}-\mathbf{z}))W^{(1;2),(k)}_{-\omega;\omega'}(\mathbf{w};\mathbf{x},\mathbf{y}) \; .$$

The latter term is again irrelevant and vanishing for $N-k\to+\infty$; in fact, its norm can be bounded by

$$2|\lambda_\infty|\|W^{(1;2),(k)}_{-\omega;\omega'}\|\,\|\partial v_K\|_{L^1}\sum_{i,j=k}^{N\ *}\int d\mathbf{z}\; b_i(\mathbf{z}-\mathbf{u})b_j(\mathbf{z}-\mathbf{u})|\mathbf{u}-\mathbf{z}_p|\le$$
$$\le C|\lambda_\infty|\gamma^{-(k-K)}\gamma^{-(N-k)}\;. \tag{140}$$

Contrary to what happened for the graph (b1) of Fig4, the contribution of the graph (a) to the first term in the r.h. side of (139) is not zero (that is, *the fermionic bubble is not vanishing*); however, in this case its value is compensated by the graph (b), thanks to the explicit choice we made for τ. Indeed we have

78bis
$$\lambda_\infty\sum_{i,j=k}^{N}\int d\mathbf{u}\; S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u})-\tau=-2\lambda_\infty\sum_{j=l+1}^{k-1}\int d\mathbf{u}\; S^{(N,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u})\;, \tag{141}$$

that easily implies that the norm of the first term in the r.h. side of (139) is bounded by $C|\lambda_\infty|\gamma^{-(N-k)}$.

Let us finally consider $W^{(1;2)(k)}_{\Delta;+1,\omega;\omega'}$, for which we can use a graph expansion similar to that of Fig.5, the only differences being that τ is replaced by 0 and the indices $-\omega$ are replaced by ω. Hence a bound can be obtained with the same arguments used above, with only one important difference: the contribution that in the previous analysis was compensated by the graph (b) now is zero by symmetry reasons. Indeed, if we call $\mathbf{k}^*$ the vector $\mathbf{k}$ rotated by $\pi/2$, it is easy to see that $\widehat{S}^{(i,j)}_{\bar\omega,\omega}(\mathbf{k}^*,\mathbf{p}^*)=-\omega\bar\omega\widehat{S}^{(i,j)}_{\bar\omega,\omega}(\mathbf{k},\mathbf{p})$, which implies that

79
$$\sum_{i,j=k}^{N}\int d\mathbf{u}\; S^{(i,j)}_{\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u})=\sum_{i,j=k}^{N}\int\frac{d\mathbf{k}}{(2\pi)^2}\;\widehat{S}^{(i,j)}_{\omega,\omega}(\mathbf{k},-\mathbf{k})=0\;. \tag{142}$$

We have then proved that

de
$$\|W^{(1;2)(k)}_{\Delta;\sigma,\omega;\underline{\omega}'}\|\le C|\lambda_\infty|\gamma^{-\vartheta(N-k)}\;, \tag{143}$$

which implies, as in the proof of Theorem 3.1, that, for $K\le k\le N$,

de1
$$\|W^{(1;2m)(k)}_{\Delta;\sigma,\omega;\underline{\omega}'}\|\le (C|\lambda_\infty|)^m\gamma^{(1-m)k}\gamma^{-\vartheta(N-k)}\;. \tag{144}$$

With respect to the bounds appearing in Theorem 3.1, there is an extra factor $\gamma^{-\vartheta(N-k)}$, implying that such kernels vanish at fixed k in the $N\to\infty$ limit. However, this is not sufficient to prove Theorem 4.1, as in (113) the derivatives of $\mathcal{W}_{\mathcal{A}}(\alpha,0,\eta)$ with respect to the external fields are integrated over $\mathbf{p}$.

It is convenient to write (113) as

$$\sum_{\omega'}\int \frac{d\mathbf{p}}{(2\pi)^2}\, v_K(\mathbf{p})\frac{A_{-\omega\omega'}(\mathbf{p})}{D_{-\omega}(\mathbf{p})}\frac{\partial^2 e^{\mathcal{W}_{\mathcal{A}}(0,0,\eta)}}{\partial\widehat{\alpha}_{\mathbf{p},\omega'}\partial\widehat{\eta}^+_{\mathbf{k}+\mathbf{p},\omega}} = \sum_{\varepsilon=\pm}\frac{\partial \mathcal{W}_{T,\varepsilon}}{\partial\widehat{\beta}_{\mathbf{k},\omega}}(0,\eta) \tag{145}$$

where

$$\begin{aligned} e^{\mathcal{W}_{T,\varepsilon}(\beta,\eta)} &= \int P(d\psi^{[l,N]})e^{\mathcal{V}^{(N)}(\psi^{[l,N]})+\sum_\omega\int d\mathbf{x}[\psi^{[l,N]+}_{\mathbf{x},\omega}\eta^-_{\mathbf{x},\omega}+\eta^+_{\mathbf{x},\omega}\psi^{[l,N]-}_{\mathbf{x},\omega}]}\,\cdot \\ &\cdot\ e^{\left[T_1^{(\varepsilon)}-\tau T_-^{(\varepsilon)}\right]\left(\psi^{l,N},\beta\right)} \end{aligned} \tag{146}$$

and

80
$$\begin{aligned} T_1^{(\varepsilon)}(\psi,\beta) &= \sum_\omega\int\frac{d\mathbf{k}\,d\mathbf{p}\,d\mathbf{q}}{(2\pi)^4}\,\widehat{v}_K^{(\varepsilon)}(\mathbf{p})\frac{C_{-\varepsilon\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})}{D_{-\omega}(\mathbf{p})}\,\cdot \\ &\cdot\ \widehat{\beta}_{\mathbf{k},\omega}\widehat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega}\widehat{\psi}^+_{\mathbf{q}+\mathbf{p},-\varepsilon\omega}\widehat{\psi}^-_{\mathbf{q},-\varepsilon\omega}\ , \end{aligned} \tag{147}$$

80a
$$T_-^{(\varepsilon)}(\psi,\beta) = \sum_\omega\int\frac{d\mathbf{k}\,d\mathbf{p}\,d\mathbf{q}}{(2\pi)^4}\,\widehat{u}_K^{(\varepsilon)}(\mathbf{p})\widehat{\beta}_{\mathbf{k},\omega}\widehat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega}\widehat{\psi}^+_{\mathbf{q}+\mathbf{p},\varepsilon\omega}\widehat{\psi}^-_{\mathbf{q},\varepsilon\omega}\ , \tag{148}$$

where

$$\widehat{v}_K^{(\varepsilon)}(\mathbf{p}) \stackrel{def}{=} v_K(\mathbf{p})\widehat{A}_\varepsilon(\mathbf{p}) \quad , \quad \widehat{u}_K^{(\varepsilon)}(\mathbf{p}) = \widehat{v}_K^{(\varepsilon)}(\mathbf{p})\widehat{v}_K(\mathbf{p})\frac{D_{\varepsilon\omega}(\mathbf{p})}{D_{-\omega}(\mathbf{p})}\ . \tag{149}$$

Notice that $v_K^{(\pm)}(\mathbf{x})$ and $u_K^{(-)}(\mathbf{x})$ are smooth functions of fast decay, hence they are equivalent to $v_K(\mathbf{x})$ in the bounds. This is not true for $u_K^{(+)}(\mathbf{x})$, whose Fourier transform is bounded but discontinuous in $\mathbf{p}=0$. However, in the following we shall only need to know that $\|u_K^{(+)}\|_{L^\infty}\le C\gamma^{2K}$ and that $|\widehat{u}_K^{(+)}(\mathbf{p})|\le|\widehat{v}_K^{(+)}(\mathbf{p})\widehat{v}_K(\mathbf{p})|$, which are easy to prove.

As in §4.1, we now perform a multiscale integration for the ultraviolet scales $N, N-1,\ldots,k+1$, $k\ge K$, very similar to the one described in §3.2, the main difference being that that there appear in the effective potential new monomials in the external field β and in ψ. Again, as in Theorem 3.1, one has to produce an improved bound only on the terms with positive or vanishing dimension, so that one has to analyze the kernels of the monomials with a β field and one or three ψ fields.

We consider first the terms contributing to $\mathcal{W}_{T,\varepsilon}(\beta,\eta)$, in which at least one of the two ψ-fields in $T_1^{(\varepsilon)}(\psi,\beta)$ with momentum $\mathbf{q}+\mathbf{p}$ or $\mathbf{q}$ is contracted at scale N. We shall call $W^{(1;2m-1)}_{T,\varepsilon;\omega;\underline{\omega}'}$ the corresponding kernels of the monomials with $2m-1$ ψ-fields and 1 α-field.

We can write

$$W^{(1;1)(k)}_{T,\varepsilon;\omega,\omega} = W^{(1;1)(k)}_{(a)T,\varepsilon;\omega,\omega} + W^{(1;1)(k)}_{(b)T,\varepsilon;\omega,\omega} \tag{150}$$

where

a) $W^{(1;1)(k)}_{(a)T,\varepsilon;\omega,\omega}$ is the sum over the terms such that the field β belongs only to a $T_1^{(\varepsilon)}$-vertex, whose ψ-field $\widehat\psi^+_{\mathbf{q}+\mathbf{p},-\varepsilon\omega}$ either is contracted with $\widehat\psi^-_{\mathbf{k}+\mathbf{p},\omega}$ (this can happen only for $\varepsilon=-1$) or is connected to it through a kernel $\widehat W^{(0;2)(k)}_\omega(\mathbf{q}+\mathbf{p})$.

b) $W^{(1;1)(k)}_{(b)T,\varepsilon;\omega,\omega}$ is the sum over the remaining terms.

Let us consider the first term. Given $\mathbf{k}$, for N large enough, $\chi^{-1}_{l,N}(\mathbf{k})-1=0$; hence we can write:

95
$$\widehat W^{(1;1)(k)}_{(a)T,\varepsilon;\omega,\omega}(\mathbf{k})=\delta_{\varepsilon,-1}\int\frac{d\mathbf{p}}{(2\pi)^2}\,\frac{\widehat v_K^{(-1)}(\mathbf{p})}{D_{-\omega}(\mathbf{p})}[\chi_{-\infty,N}(\mathbf{p}+\mathbf{k})-1]\cdot \tag{151}$$
$$\cdot\left[1+\widehat g^{[k+1,N]}_\omega(\mathbf{p}+\mathbf{k})\widehat W^{(0;2)(k)}_\omega(\mathbf{p}+\mathbf{k})\right]\left[1+\widehat g^{[k+1,N]}_\omega(\mathbf{k})\widehat W^{(0;2)(k)}_\omega(\mathbf{k})\right]\ .$$

Moreover, since $\widehat v_K^{(-1)}(\mathbf{p})=0$ for $|\mathbf{p}|\ge 2\gamma^K$, then $\chi_{-\infty,N}(\mathbf{p}+\mathbf{k})-1=0$, if $\widehat v_K^{(-1)}(\mathbf{p})\neq 0$ and N is large enough. It follows that, given a fixed $\mathbf{k}$, for N large enough,

$$\widehat W^{(1;1)(k)}_{(a)T,\varepsilon;\omega,\omega}(\mathbf{k})=0\ . \tag{152}$$

Let us now consider $W^{(1;1)(k)}_{(b)T,\varepsilon;\omega,\omega}(\mathbf{x}-\mathbf{y})$, which can be decomposed as in Fig. 6.

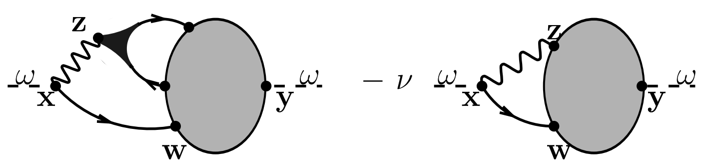

Figure 6: : Graphical representation of $W^{(1;1)(k)}_{(b)T,\varepsilon;\omega,\omega}$; the dotted line coming from $\mathbf{x}$ represent the external field β.

By using (133), it can be written as

$$\sum_\sigma\int d\mathbf{z}\ u_K^{(\varepsilon)}(\mathbf{x}-\mathbf{z})g^{[k,N]}_\omega(\mathbf{x}-\mathbf{w})W^{(1;2)(k)}_{\Delta;\sigma,-\varepsilon\omega;\omega}(\mathbf{z};\mathbf{y},\mathbf{w})\ , \tag{153}$$

hence its norm, by using (143), can be bounded by

$$\|u_K^{(\varepsilon)}\|_{L^\infty}\sum_{j=k}^N|g^{(j)}_\omega|_{L^1}\|W^{(1;2)(k)}_{\Delta;\sigma,-\varepsilon\omega;\omega}\|\le C|\lambda_\infty|\gamma^k\gamma^{-2(k-K)}\gamma^{-\vartheta(N-k)}\ . \tag{154}$$

so that

t1
$$\|W^{(1;1)(k)}_{T,\varepsilon;\omega,\omega}\|\le C|\lambda_\infty|\gamma^k\gamma^{-\vartheta(N-k)}\gamma^{-2(k-K)}\ . \tag{155}$$

Moreover

fffg
$$W^{(1;3)(k)}_{T,\varepsilon;\omega;\underline{\omega}'}=W^{(1;3)(k)}_{(a)T,\varepsilon;\omega;\underline{\omega}'}+W^{(1;3)(k)}_{(b)T,\varepsilon;\omega;\underline{\omega}'}\ , \tag{156}$$

where $W^{(1;3)(k)}_{(a)T,\varepsilon;\omega;\underline{\omega}'}$ contains the terms in which the field $\widehat{\psi}_{\mathbf{k}+\mathbf{p},\omega}$ of T_1 and T_- is not contracted or is linked to a kernel $\widehat{W}^{(0;2)(k)}_{\omega}$, while the other terms are collected in $W^{(1;3)(k)}_{(b)T,\varepsilon;\omega;\underline{\omega}'}$.

Let us consider first $W^{(1;3)(k)}_{(b)T,\varepsilon;\omega;\underline{\omega}'}$, which can be represented as in Fig.7.

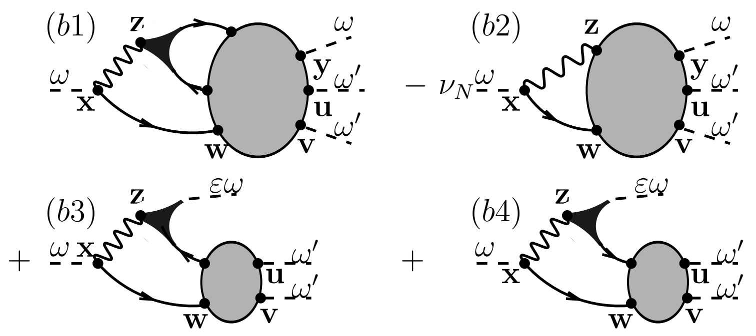

Figure 7: : Graphical representation of $W^{(1;3)(k)}_{(b)T,\varepsilon;\omega;\omega'}$

We can write

$$W^{(1;3)(k)}_{(b)T,\varepsilon;\omega,\underline{\omega}'}(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}) = \tag{157}$$
$$= \int d\mathbf{z}d\mathbf{w}\; u^{(\varepsilon)}_K(\mathbf{x}-\mathbf{z})g^{[k,N]}_{\omega}(\mathbf{x}-\mathbf{w})W^{(1;4)(k)}_{\Delta,\varepsilon;\omega,\omega'}(\mathbf{z};\mathbf{w},\mathbf{y},\mathbf{u},\mathbf{v}) \;,$$

so that, by the bounds (143), $\|W^{(1;4)(k)}_{\Delta,\varepsilon;\omega,\omega'}\| \le C|\lambda_\infty|\gamma^{-k}\gamma^{-\vartheta(N-k)}$ and $\|u^{(\varepsilon)}_K\|_{L^\infty} \le C\gamma^{2K}$, we get:

$$\|W^{(1;3)(k)}_{(b)T,\varepsilon;\omega;\omega'}\| \le C|\lambda_\infty|\gamma^{-2(k-K)}\gamma^{-\vartheta(N-k)} \;. \tag{158}$$

Let us now consider $W^{(1;3)(k)}_{(a)T,\varepsilon;\omega;\underline{\omega}'}$; its Fourier transform, if we call $\mathbf{k}^+$ and $\mathbf{k}^-$ the momenta of the two fields connected to the line $u^{(\varepsilon)}_K$, can be written as (notice that $\underline{\omega}'$ is of the form (ω,ω',ω')):

$$\widehat{W}^{(1;3)(k)}_{(a)T,\varepsilon;\omega;\underline{\omega}'}(\mathbf{k};\mathbf{k}^+,\mathbf{k}^-) = \Big[1+\widehat{g}^{[k+1,N]}_{\omega}(\mathbf{k}+\mathbf{k}^+-\mathbf{k}^-)\widehat{W}^{(0;2)(k)}_{\omega}(\mathbf{k}+\mathbf{k}^+-\mathbf{k}^-)\Big]\cdot$$
$$\cdot\widehat{u}^{(\varepsilon)}_K(\mathbf{k}^+-\mathbf{k}^-)\sum_\sigma \widehat{W}^{(1;2)(k)}_{\Delta;\sigma,-\varepsilon\omega,\omega'}(\mathbf{k}^-+\mathbf{k}^+-\mathbf{k}^-,\mathbf{k}^-)\;. \tag{159}$$

Then, if $\varepsilon=-1$, since $\|v^{(-1)}_K\|_{L^1}\le C$, by using the bounds (143) and (76), we find

$$w13 \qquad \|W^{(1;3)(k)}_{(a)T,-1;\omega;\underline{\omega}'}\| \le C|\lambda_\infty|\gamma^{-\vartheta(N-k)} \tag{160}$$

Using the general bound (2.58) of [5], we get that the contributions to the derivatives of $\mathcal{W}_{T,\varepsilon}$ with respect to η at distinct space points, coming from trees containing one endpoint associated with one of the kernels $W^{(1;1)(k)}_{T,-1;\omega;\underline{\omega}'}$,

$W^{(1;3)(k)}_{(a)T,-1;\omega;\underline{\omega}'}$, $W^{(1;3)(k)}_{(a)T,-1;\omega;\underline{\omega}'}$, are bounded by $C^k k!^4 \lambda_\infty^k \delta^{-2k} (\frac{\gamma^{-N}}{\delta})^\vartheta$, with $0 < \vartheta < 1$ and δ the minimal distance between the external points; hence they are vanishing in the removed cutoff limit.

A similar conclusion is true for the contributions to the derivatives of $\mathcal{W}_{T,\varepsilon}$ with respect to η at distinct points, coming from trees containing one endpoint associated with $W^{(1;3)(k)}_{(a)T,+;\omega;\underline{\omega}'}$, even if $u^{(+)}_K(\mathbf{x})$ is not integrable. In fact, since $|\widetilde{\Lambda}| \le \gamma^{-2l}$, it is easy to show that $\int_{\widetilde{\Lambda}} d\mathbf{x} |v^{(+1)}_K(\mathbf{x})| \le C\gamma^{-l}$, so that the previous bound has to be multiplied by γ^{-l}; however, we take the limit $-l \to \infty$ after the limit $N \to \infty$, hence the conclusion is the same.

Finally we have to consider the contributions to the correlation functions such that one of the ψ-fields in $T_1^{(\varepsilon)}(\psi, \beta)$ with momentum $\mathbf{q}+\mathbf{p}$ or $\mathbf{q}$, see (147), is contracted at scale l. In such a case we can use the bound

61a
$$\left| \frac{\widehat{U}^{(i,l)}_{\omega'}(\mathbf{q}+\mathbf{p},\mathbf{q})}{D_\omega(\mathbf{p})} \right| \le C\gamma^{-(i-l)} \frac{\gamma^{-l-i}}{Z_{i-1}} \quad , \quad \text{if } |\mathbf{p}| \ge 2\gamma^{l+1} \ , \tag{161}$$

and the factor $\gamma^{-(i-l)}$ in the r.h.s. of this bound makes negative the scaling dimension of marginal terms with an external β line (there are no relevant terms), see §4.8 of [11] for details. Again by the bound (2.58) of [5] we get, for this kind of contributions to the the derivatives of $\mathcal{W}_{T,\varepsilon}$ with respect to η at distinct points, the bound $C^k k!^4 \lambda_\infty^k \delta^{-2k} (\gamma^l/\delta)^\vartheta$, with $0 < \vartheta < 1$, so they are vanishing as $l \to -\infty$. This completes the proof of Theorem 4.1.

A Fermionic representation of the partition function

A.1 Proof of (19)

5.1

Since $\sigma_\mathbf{x}, \sigma'_\mathbf{x} = \pm 1$,

$$\exp\left(\alpha \sigma_\mathbf{x} \sigma_{\mathbf{x}+\mathbf{e}_j} \sigma'_\mathbf{y} \sigma'_{\mathbf{y}+\mathbf{e}_{j'}}\right) = \cosh(\alpha) + \sigma_\mathbf{x} \sigma_{\mathbf{x}+\mathbf{e}_j} \sigma'_\mathbf{y} \sigma'_{\mathbf{y}+\mathbf{e}_{j'}} \sinh(\alpha) \ ,$$

so that the partition function for the AT or 8V model with external fields is given by:

int
$$Z(I, I') = [\cosh(\beta J_4)]^{2|\Lambda|} \cdot \cdot \prod_{\substack{j=0,1 \\ \mathbf{x}\in\Lambda}} \left[1 + \tanh(\beta J_4) \frac{\partial^2}{\partial \widetilde{A}_{j,\mathbf{x}} \widetilde{A}'_{j,\mathbf{x}}} \right] Z(I) Z(I') \ , \tag{162}$$

where $I'_{j,\mathbf{x}} = A'_{j,\mathbf{x}} + \beta J'$ and, in the AT case, $\widetilde{A}_{j,\mathbf{x}} = A_{j,\mathbf{x}}$ and $\widetilde{A}'_{j,\mathbf{x}} = A'_{j,\mathbf{x}}$, while, in the 8V case, $\widetilde{A}_{0,\mathbf{x}} = A_{0,\mathbf{x}}$, $\widetilde{A}'_{0,\mathbf{x}} = A'_{1,\mathbf{x}}$, $\widetilde{A}_{1,\mathbf{x}} = A_{1,\mathbf{x}+\mathbf{e}_0}$, $\widetilde{A}'_{1,\mathbf{x}} = A'_{0,\mathbf{x}+\mathbf{e}_1}$.

Let us call $\widetilde{t}_{j,\mathbf{x}}$, $\widetilde{c}_{j,\mathbf{x}}$ the expressions obtained from $t_{j,\mathbf{x}}$, $c_{j,\mathbf{x}}$ by substituting $A_{j,\mathbf{x}}$ with $\widetilde{A}_{j,\mathbf{x}}$; in a similar way we define $\widetilde{t}'_{j,\mathbf{x}}$, $\widetilde{c}'_{j,\mathbf{x}}$. Let us now define:

$$\begin{aligned} f_{j,\mathbf{x}} &= 1 + \tanh(\beta J_4)\widetilde{t}_{j,\mathbf{x}}\widetilde{t}'_{j,\mathbf{x}} \;, \\ g_{j,\mathbf{x}} &= \frac{\widetilde{t}'_{j,\mathbf{x}}}{(\widetilde{c}_{j,\mathbf{x}})^2}\frac{\tanh(\beta J_4)}{f_{j,\mathbf{x}}} \quad , \quad g'_{j,\mathbf{x}} = \frac{\widetilde{t}_{j,\mathbf{x}}}{(\widetilde{c}'_{j,\mathbf{x}})^2}\frac{\tanh(\beta J_4)}{f_{j,\mathbf{x}}} \;, \\ h_{j,\mathbf{x}} &= \frac{1}{(\widetilde{c}'_{j,\mathbf{x}})^2(\widetilde{c}_{j,\mathbf{x}})^2}\frac{\tanh(\beta J_4)}{f_{j,\mathbf{x}}} - g_{j,\mathbf{x}}g'_{j,\mathbf{x}} \;. \end{aligned} \tag{163}$$

By explicitly taking the derivatives w.r.t. $\widetilde{A}_{j,\mathbf{x}}$ and $\widetilde{A}'_{j,\mathbf{x}}$ we can write the partition function (162) as

2.10
$$\begin{aligned} Z(I,I') = 4^{|\Lambda|}\left[\cosh(\beta J_4)\right]^{2|\Lambda|}\left(\prod_{j,\mathbf{x}} f_{j,\mathbf{x}}c_{j,\mathbf{x}}c'_{j,\mathbf{x}}\right)\cdot \\ \cdot\sum_{\gamma,\gamma'}\frac{(-1)^{\delta_\gamma+\delta_{\gamma'}}}{4}Z_{\gamma,\gamma'}(I,I') \;, \end{aligned} \tag{164}$$

where $Z_{\gamma,\gamma'}(I,I')$ is the Grassmannian functional integral

2.111a
$$Z_{\gamma,\gamma'}(I,I') = \int dH dV dH' dV' \; e^{\widetilde{S}(\widetilde{t}+g)+\widetilde{S}'(\widetilde{t}'+g')+V(h)} \;, \tag{165}$$

with boundary conditions $\gamma = (\varepsilon_0, \varepsilon_1)$ and $\gamma' = (\varepsilon'_0, \varepsilon'_1)$ on the variables H, V and H', V', respectively. Moreover $\widetilde{S}(t)$ and $\widetilde{S}'(t)$ have a definition which depends on the model. $\widetilde{S}(t)$ is equal to $S(t)$ in the AT model, while, in the 8V model, it is the function which is obtained from $S(t)$, by substituting, in the first line of (17), $\bar{V}_\mathbf{x}V_{\mathbf{x}+\mathbf{e}_1}$ with $\bar{V}_{\mathbf{x}+\mathbf{e}_0}V_{\mathbf{x}+\mathbf{e}_0+\mathbf{e}_1}$. $\widetilde{S}'(t)$, in the AT case, is obtained from $S(t)$, by simply replacing H, V with H', V', while, in the 8V case, we also have to substitute $\bar{H}'_\mathbf{x}H'_{\mathbf{x}+\mathbf{e}_0}$ with $\bar{V}'_\mathbf{x}V'_{\mathbf{x}+\mathbf{e}_1}$ and $\bar{V}'_\mathbf{x}V'_{\mathbf{x}+\mathbf{e}_1}$ with $\bar{H}'_{\mathbf{x}+\mathbf{e}_1}H'_{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_0}$. $V(h)$ is a quartic interaction that, in the AT case, is given by

$$V_{AT}(h) = \sum_{\mathbf{x}\in\Lambda}\left[h_{0,\mathbf{x}}\bar{H}_\mathbf{x}H_{\mathbf{x}+\mathbf{e}_0}\bar{H}'_\mathbf{x}H'_{\mathbf{x}+\mathbf{e}_0} + h_{1,\mathbf{x}}\bar{V}_\mathbf{x}V_{\mathbf{x}+\mathbf{e}_1}\bar{V}'_\mathbf{x}V'_{\mathbf{x}+\mathbf{e}_1}\right] \;, \tag{166}$$

while, in the 8V case, is given by

$$V_{8V}(h) = \sum_{\mathbf{x}\in\Lambda}\left[h_{0,\mathbf{x}}\bar{H}_\mathbf{x}H_{\mathbf{x}+\mathbf{e}_0}\bar{V}'_\mathbf{x}V'_{\mathbf{x}+\mathbf{e}_1} + h_{1,\mathbf{x}}\bar{V}_{\mathbf{x}+\mathbf{e}_0}V_{\mathbf{x}+\mathbf{e}_0+\mathbf{e}_1}\bar{H}'_{\mathbf{x}+\mathbf{e}_1}H'_{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_0}\right] \;. \tag{167}$$

We remark that $g_{j,\mathbf{x}}, g'_{j,\mathbf{x}}, h_{j,\mathbf{x}} = \mathrm{O}(\beta J_4)$.

The truncated correlations of the quadratic observables are obtained by taking two derivatives of $\ln Z(I,I')$ w.r.t. the external sources in two different points, and putting such external sources to zero. The addends $2|\Lambda|\ln[2\cosh(\beta J_4)]$ and $\sum_{j,\mathbf{x}}(\ln f_{j,\mathbf{x}}+\ln c_{j,\mathbf{x}}+\ln c'_{j,\mathbf{x}})$ do not contribute when we take two derivatives in the A variables of two different points. If we define $\partial^\varepsilon_{j,\mathbf{x}}=\partial/\partial A_{j,\mathbf{x}}+\varepsilon\partial/\partial A'_{j,\mathbf{x}}$, we get:

$$\langle O^\varepsilon_{\mathbf{x}};O^\varepsilon_{\mathbf{y}}\rangle^T_\Lambda=\sum_{i,j}\partial^\varepsilon_{i,\mathbf{x}}\partial^\varepsilon_{j,\mathbf{y}}\ln\left[\sum_{\gamma,\gamma'}(-1)^{\delta_\gamma+\delta_{\gamma'}}Z_{\gamma,\gamma'}(I,I')\right]\Bigg|_{A\equiv 0}\;, \tag{168}$$

so that we get, by some simple algebraic calculations:

$$\begin{aligned}\langle O^\varepsilon_{\mathbf{x}};O^\varepsilon_{\mathbf{y}}\rangle^T_\Lambda&=\frac{1}{\widehat{Z}}\sum_{\gamma,\gamma'}(-1)^{\delta_\gamma+\delta_{\gamma'}}\frac{\partial^2\bar{Z}_{\gamma,\gamma'}(A)}{\partial\bar{A}^\varepsilon_{\mathbf{x}}\partial\bar{A}^\varepsilon_{\mathbf{y}}}\Bigg|_{\bar{A}=0}-\\&-\frac{1}{(\widehat{Z})^2}\sum_{\gamma,\gamma'}(-1)^{\delta_\gamma+\delta_{\gamma'}}\frac{\partial\bar{Z}_{\gamma,\gamma'}(A)}{\partial\bar{A}^\varepsilon_{\mathbf{x}}}\Bigg|_{\bar{A}=0}\sum_{\gamma,\gamma'}(-1)^{\delta_\gamma+\delta_{\gamma'}}\frac{\partial\bar{Z}_{\gamma,\gamma'}(A)}{\partial\bar{A}^\varepsilon_{\mathbf{y}}}\Bigg|_{\bar{A}=0},\end{aligned} \tag{169}$$

with $Z_{\gamma,\gamma'}$ defined as in (20), with (γ,γ')-boundary conditions (instead of antiperiodic in all variables) and $\widehat{Z}=\sum_{\gamma,\gamma'}(-1)^{\delta_\gamma+\delta_{\gamma'}}\bar{Z}_{\gamma,\gamma'}(0,0)$; the parameters s, s' and λ are given by

$$\begin{aligned}s&=t_{j,\mathbf{x}}+g_{j,\mathbf{x}}|_{A\equiv 0}=\tanh(\beta J)+\mathrm{O}(\beta J_4)\\s'&=t'_{j,\mathbf{x}}+g'_{j,\mathbf{x}}\big|_{A\equiv 0}=\tanh(\beta J')+\mathrm{O}(\beta J_4)\\2\lambda&=h_{j,\mathbf{x}}|_{A\equiv 0}=\mathrm{O}(\beta J_4)\;,\end{aligned} \tag{170}$$

and the parameters appearing in (23) and (24) are given by

$$\begin{aligned}q_\varepsilon&=\sum_i\left(\frac{\partial}{\partial A_{j,\mathbf{x}}}+\varepsilon\frac{\partial}{\partial A'_{j,\mathbf{x}}}\right)(\tilde{t}_{i,\mathbf{x}}+g_{i,\mathbf{x}})\Bigg|_{A\equiv 0}\quad,\quad q'_\varepsilon=\{\tilde{t},g\to\tilde{t}',g'\}\;,\\p_\varepsilon&=\sum_i\left(\frac{\partial h_{i,\mathbf{x}}}{\partial A_{j,\mathbf{x}}}+\varepsilon\frac{\partial h_{j,\mathbf{x}}}{\partial A'_{j,\mathbf{x}}}\right)\Bigg|_{A\equiv 0}\;.\end{aligned} \tag{171}$$

In order to prove (19) we note that, as proved in App. G of [22], if we put $\bar{Z}_\gamma=Z_\gamma|_{A=0}$, the quantities $\bar{Z}_{\gamma,\gamma'}(0)/\bar{Z}_\gamma\bar{Z}_{\gamma'}$ are exponentially insensitive to boundary conditions in the thermodynamic limit, away from the critical temperature; this implies that $\widehat{Z}$ coincides, in the thermodynamic limit, with $(\bar{Z}_{\bar{\gamma},\bar{\gamma}}(0)/\bar{Z}_{\bar{\gamma}}\bar{Z}_{\bar{\gamma}})(\bar{Z})^2$ with $\bar{\gamma}=(-,-)$ and $\bar{Z}=\sum_\gamma(-1)^{\delta_\gamma}\bar{Z}_\gamma$. Notice that $\bar{Z}$ is non vanishing; indeed, as proved in §4 of [26], away from the critical temperature $|\bar{Z}_\gamma|$ is exponentially insensitive to boundary conditions and below the critical temperature Z_γ is positive for any γ while above is negative if $\gamma = (+,+)$ and positive in all other cases. Moreover, as proved in App. G of [22],

$$\frac{1}{\bar{Z}_{\gamma,\gamma'}(0)} \frac{\partial \bar{Z}_{\gamma,\gamma'}(A)}{\partial \bar{A}^{\varepsilon}_{\mathbf{x}}}\Big|_{\bar{A}=0} \quad \text{and} \quad \frac{1}{\bar{Z}_{\gamma,\gamma'}(0)} \frac{\partial^2 \bar{Z}_{\gamma,\gamma'}(A)}{\partial^2 \bar{A}^{\varepsilon}_{\mathbf{x}} \partial \bar{A}^{\varepsilon}_{\mathbf{y}}}\Big|_{\bar{A}=0} \tag{172}$$

are exponentially insensitive to boundary conditions, so that the r.h.s. of (169) coincides, in the thermodynamic limit, with $\frac{\partial^2 \log \bar{Z}_{\bar{\gamma},\bar{\gamma}}(\bar{A})}{\partial^2 \bar{A}^{\varepsilon}_{\mathbf{x}} \partial \bar{A}^{\varepsilon}_{\mathbf{y}}}|_{\bar{A}=0}$.

A.2 Proof of (25)

5.2

In order to make more evident the analogy of the above functional integral with the action of a fermionic (Euclidean) Quantum Field Model, it is convenient to make a change of variables in the Grassmann algebra. The new Grassmannian variables will be denoted by $\psi_{\mathbf{x}}$, $\bar{\psi}_{\mathbf{x}}$, $\chi_{\mathbf{x}}$ and $\bar{\chi}_{\mathbf{x}}$ and are related to the old ones by the equations:

2.12

$$\begin{aligned} \bar{H}_{\mathbf{x}} + iH_{\mathbf{x}} &= e^{i\frac{\pi}{4}} (\psi_{\mathbf{x}} - \chi_{\mathbf{x}}) \quad , \quad \bar{V}_{\mathbf{x}} + iV_{\mathbf{x}} = \psi_{\mathbf{x}} + \chi_{\mathbf{x}} \ , \\ \bar{H}_{\mathbf{x}} - iH_{\mathbf{x}} &= e^{-i\frac{\pi}{4}} \left(\bar{\psi}_{\mathbf{x}} - \bar{\chi}_{\mathbf{x}}\right) \quad , \quad \bar{V}_{\mathbf{x}} - iV_{\mathbf{x}} = \bar{\psi}_{\mathbf{x}} + \bar{\chi}_{\mathbf{x}} \ . \end{aligned} \tag{173}$$

A similar transformation is done for the primed variables. After a straightforward computation, we see that the action (17), calculated at $t_{j,\mathbf{x}} = s$, $\forall j, \mathbf{x}$, can be written in terms of the Majorana fields as

5.9aa

$$S(s) = A(\psi, m_s) + A(\chi, M_s) + Q(\psi, \chi) \ , \tag{174}$$

where $m_s = 1 - \sqrt{2} + s$, $M_s = 1 + \sqrt{2} + s$ and, if we define $\partial^i \psi_{\mathbf{x}} = \psi_{\mathbf{x}+\mathbf{e}_i} - \psi_{\mathbf{x}}$,

$$\begin{aligned} A(\psi, m) \;=\; & \frac{s}{4} \sum_{\mathbf{x}\in\Lambda} \left[\psi_{\mathbf{x}} \left(\partial^0 - i\partial^1\right) \psi_{\mathbf{x}} + \text{c.c.}\right] - im \sum_{\mathbf{x}\in\Lambda} \bar{\psi}_{\mathbf{x}} \psi_{\mathbf{x}} + \\ + \; & \frac{s}{4} \sum_{\mathbf{x}\in\Lambda} \left[\bar{\psi}_{\mathbf{x}} \left(-i\partial^0 - i\partial^1\right) \psi_{\mathbf{x}} + \text{c.c.}\right] \ , \end{aligned} \tag{175}$$

$$\begin{aligned} Q(\psi,\chi) = \; - \; & \frac{s}{4} \sum_{\mathbf{x}\in\Lambda} \left[\psi_{\mathbf{x}} \left(\partial^0 + i\partial^1\right) \chi_{\mathbf{x}} + \left\{\psi \leftrightarrow \chi\right\} + \text{c.c.}\right] - \\ - \; & \frac{s}{4} \sum_{\mathbf{x}\in\Lambda} \left[\bar{\chi}_{\mathbf{x}} \left(-i\partial^0 + i\partial^1\right) \psi_{\mathbf{x}} + \left\{\psi \leftrightarrow \chi\right\} + \text{c.c.}\right] \ , \end{aligned} \tag{176}$$

where, in agreement with (173), we are calling complex conjugation (c.c.) the operation on the Grassmann algebra which amounts to exchange $\psi_{\mathbf{x}}$ with $\bar{\psi}_{\mathbf{x}}$, $\chi_{\mathbf{x}}$ with $\bar{\chi}_{\mathbf{x}}$ and i with $-i$.

The quartic interaction of the AT model becomes:

$$V_{AT} = -\lambda \sum_{\mathbf{x}\in\Lambda} \left[\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}}\bar\psi'_{\mathbf{x}}\psi'_{\mathbf{x}} + \bar\psi_{\mathbf{x}}\psi_{\mathbf{x}}\bar\chi'_{\mathbf{x}}\chi'_{\mathbf{x}} + \{\psi \leftrightarrow \chi\}\right] - \tag{177}$$
$$-\lambda \sum_{\mathbf{x}\in\Lambda} \left[\bar\chi_{\mathbf{x}}\psi_{\mathbf{x}}\bar\chi'_{\mathbf{x}}\psi'_{\mathbf{x}} + \bar\chi_{\mathbf{x}}\psi_{\mathbf{x}}\bar\psi'_{\mathbf{x}}\chi'_{\mathbf{x}} + \{\psi \leftrightarrow \chi\}\right] + R_V \;,$$

where R_V is sum of quartic terms with at least one (discrete) derivative. In the case of the 8V model, the second square bracket has $+\lambda$ in front, rather than $-\lambda$.

If we set $b_\varepsilon = (q_\varepsilon + \varepsilon q'_\varepsilon)/2$ and $d_\varepsilon = (q_\varepsilon - \varepsilon q'_\varepsilon)/2$, the interaction with the external field is given by

$$B(\bar A) = -i \sum_{\substack{\mathbf{x}\in\Lambda\\ \varepsilon=\pm}} b_\varepsilon \bar A^\varepsilon_{\mathbf{x}} \left[\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}} + \varepsilon\bar\psi'_{\mathbf{x}}\psi'_{\mathbf{x}} + \bar\chi_{\mathbf{x}}\chi_{\mathbf{x}} + \varepsilon\bar\chi'_{\mathbf{x}}\chi'_{\mathbf{x}}\right] -$$
$$-i \sum_{\substack{\mathbf{x}\in\Lambda\\ \varepsilon=\pm}} d_\varepsilon \bar A^\varepsilon_{\mathbf{x}} \left[\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}} - \varepsilon\bar\psi'_{\mathbf{x}}\psi'_{\mathbf{x}} + \bar\chi_{\mathbf{x}}\chi_{\mathbf{x}} - \varepsilon\bar\chi'_{\mathbf{x}}\chi'_{\mathbf{x}}\right] + R_B,$$

where R_B is sum of monomials quartic in the fields or quadratic with derivatives. We remark that, if $J = J'$, then $d_\varepsilon = 0$, while $b_\varepsilon = 1 - \tanh(\beta J) + O(\beta J_4)$.

We now make another change of variables, defined by the relations

2.12a
$$\psi^\varepsilon_{\mathbf{x},+} = \frac{\psi_{\mathbf{x}} - \varepsilon i \psi'_{\mathbf{x}}}{\sqrt{2}} \quad , \quad \psi^\varepsilon_{\mathbf{x},-} = \frac{\bar\psi_{\mathbf{x}} - \varepsilon i \bar\psi'_{\mathbf{x}}}{\sqrt{2}} \quad , \quad \varepsilon = \pm \;, \tag{178}$$

and the similar ones for the χ-variables. This change of variables is the analogous in the euclidean theories of the transformation from *Majorana fermions* to *Dirac fermions* in real time QFT.

If we put $u = (s + s')/2$, $v = (s - s')/2$ (s, s' defined in (170)) and $m_\varepsilon = (m_s + \varepsilon m_{s'})/2$ (m_s and $m_{s'}$ defined after (174)), we get

36
$$A(\psi, m_s) + A(\psi', m_{s'}) = \tag{179}$$
$$= \sum_{\mathbf{x}\in\Lambda} \Big\{ \frac{u}{4} \left[\psi^+_{\mathbf{x},+}\left(\partial^0 - i\partial^1\right)\psi^-_{\mathbf{x},+} + \psi^-_{\mathbf{x},+}\left(\partial^0 - i\partial^1\right)\psi^+_{\mathbf{x},+} + \text{c.c.}\right] +$$
$$+\frac{u}{4} \left[\psi^-_{\mathbf{x},+}\left(i\partial^0 + i\partial^1\right)\psi^+_{\mathbf{x},-} + \psi^+_{\mathbf{x},+}\left(i\partial^0 + i\partial^1\right)\psi^-_{\mathbf{x},-} + \text{c.c.}\right] +$$
$$+\frac{v}{4} \left[\psi^+_{\mathbf{x},+}\left(\partial^0 - i\partial^1\right)\psi^+_{\mathbf{x},+} + \psi^-_{\mathbf{x},+}\left(\partial^0 - i\partial^1\right)\psi^-_{\mathbf{x},+} + \text{c.c.}\right] +$$
$$+\frac{v}{4} \left[\psi^-_{\mathbf{x},+}\left(i\partial^0 + i\partial^1\right)\psi^-_{\mathbf{x},-} + \psi^+_{\mathbf{x},+}\left(i\partial^0 + i\partial^1\right)\psi^+_{\mathbf{x},-} + \text{c.c.}\right] -$$
$$-im_+ \left[\psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},+} - \psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},-}\right] + im_- \left[\psi^-_{\mathbf{x},+}\psi^-_{\mathbf{x},-} + \psi^+_{\mathbf{x},+}\psi^+_{\mathbf{x},-}\right]\Big\} \;,$$

where now the c.c. operation amounts to exchange $\psi^\varepsilon_{\mathbf{x},\omega}$ with $\psi^{-\varepsilon}_{\mathbf{x},-\omega}$ and i with $-i$. In the new variables the interaction with the external source is given by

$BAapp$

$$B(\bar{A}) = i\sum_{\mathbf{x}\in\Lambda}(b_+\bar{A}^+_{\mathbf{x}} + d_-\bar{A}^-_{\mathbf{x}})[\psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},-} - \psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},+} + \tag{180}$$
$$+\chi^+_{\mathbf{x},+}\chi^-_{\mathbf{x},-} - \chi^+_{\mathbf{x},-}\chi^-_{\mathbf{x},+}] + i\sum_{\mathbf{x}\in\Lambda}(b_-\bar{A}^-_{\mathbf{x}} + d_+\bar{A}^+_{\mathbf{x}})\cdot$$
$$\cdot[\psi^+_{\mathbf{x},+}\psi^+_{\mathbf{x},-} + \psi^-_{\mathbf{x},+}\psi^-_{\mathbf{x},-} + \chi^+_{\mathbf{x},+}\chi^+_{\mathbf{x},-} + \chi^-_{\mathbf{x},+}\chi^-_{\mathbf{x},-}] + R_B(\bar{A})\ ,$$

and $\mathcal{V}(\psi,\chi)$ is given by (32).

Let $\mathcal{D}$ be the set of $\mathbf{k}$'s such that $k_0 = \frac{2\pi}{L}(n_0+\frac{1}{2})$ and $k_1 = \frac{2\pi}{L}(n_1+\frac{1}{2})$, for $n_0, n_1 = -\frac{L}{2},\ldots,\frac{L}{2}-1$, and L and even integer. Then, the Fourier transform for the fermions with antiperiodic boundary condition is defined by

$$\psi^\varepsilon_{\mathbf{x},\omega} \stackrel{def}{=} \frac{1}{|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}} e^{i\varepsilon\mathbf{k}\mathbf{x}}\widehat{\psi}^\varepsilon_{\mathbf{k},\omega}\ . \tag{181}$$

Therefore (179) can be written as

2.29

$$A(\psi, m_s) + A(\psi', m_{s'}) = \frac{u}{2|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}}\Phi^+_{\mathbf{k}}S(\mathbf{k})\Phi_{\mathbf{k}}\ , \tag{182}$$

where

$$\begin{aligned}\Phi_{\mathbf{k}} &= (\widehat{\psi}^-_{\mathbf{k},+}, \widehat{\psi}^-_{\mathbf{k},-}, \widehat{\psi}^+_{-\mathbf{k},+}, \widehat{\psi}^+_{-\mathbf{k},-})\ ,\\ \Phi^+_{\mathbf{k}} &= (\widehat{\psi}^+_{\mathbf{k},+}, \widehat{\psi}^+_{\mathbf{k},-}, \widehat{\psi}^-_{-\mathbf{k},+}, \widehat{\psi}^-_{-\mathbf{k},-})\ ,\end{aligned} \tag{183}$$

and , if we define $\mu(\mathbf{k})$ as in (28) and

$$\begin{aligned}\widehat{D}_\omega(\mathbf{k}) &= -i\sin k_0 + \omega\sin k_1\ ,\\ \sigma(\mathbf{k}) &= \frac{v}{u}(\cos k_0 + \cos k_1 - 2) + 2\frac{v}{u}\ ,\end{aligned}$$

the matrix $S(\mathbf{k})$ is given by

sds

$$S(\mathbf{k}) = \begin{pmatrix} \widehat{D}_-(\mathbf{k}) & i\mu(\mathbf{k}) & \frac{v}{u}\widehat{D}_-(\mathbf{k}) & i\sigma(\mathbf{k}) \\ -i\mu(\mathbf{k}) & \widehat{D}_+(\mathbf{k}) & -i\sigma(\mathbf{k}) & \frac{v}{u}\widehat{D}_+(\mathbf{k}) \\ \frac{v}{u}\widehat{D}_-(\mathbf{k}) & +i\mu(\mathbf{k}) & \widehat{D}_-(\mathbf{k}) & i\sigma(\mathbf{k}) \\ -i\mu(\mathbf{k}) & \frac{v}{u}\widehat{D}_+(\mathbf{k}) & -i\sigma(\mathbf{k}) & \widehat{D}_+(\mathbf{k}) \end{pmatrix}\ . \tag{184}$$

In the case $J = J'$ we have $v = 0$ and $\sigma(\mathbf{k}) \equiv 0$, so that we get the much simpler equation

$$A(\psi, m_s) + A(\psi', m_{s'}) = -\frac{1}{|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}}\sum_{\omega,\omega'}\widehat{\psi}^+_{\mathbf{k},\omega}\widehat{\psi}^-_{\mathbf{k},\omega'}T_{\omega,\omega'}(\mathbf{k})\ , \tag{185}$$

with $T_{\omega,\omega'}(\mathbf{k})$ given by (27) and

$$A(\chi, M_s) + A(\chi', M_{s'}) = -\frac{1}{|\Lambda|} \sum_{\mathbf{k}\in\mathcal{D}} \sum_{\omega,\omega'} \widehat{\chi}^+_{\mathbf{k},\omega} \widehat{\chi}^-_{\mathbf{k},\omega'} T^\chi_{\omega,\omega'}(\mathbf{k}) \ , \tag{186}$$

$T^\chi(\mathbf{k})$ being the matrix obtained from $T(\mathbf{k})$ by substituting $\mu(\mathbf{k})$ with (29). Moreover, $\varepsilon q'_\varepsilon = q_\varepsilon$, so that $d_\varepsilon = 0$ and $b_\varepsilon = q_\varepsilon$; it follows that $B(\bar{A})$ can be written as in (30). This completes the proof of (25).

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