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A new dual approach for a class of phase transitions with memory: existence and long-time behaviour of solutions

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Abstract

In this paper we describe a class of phase transitions with thermal memory using a dual approach with respect to the energy functionals. More precisely, we use as state variables the phase parameter, the entropy (in place of the absolute temperature), and the history contribution of the entropy flux. The equations are recovered from a generalization of the principle of virtual power (to describe the evolution of the phases), including the effects of micro-motions responsible for the phase transition, and a rescalation of the internal energy balance (to describe the evolution of the entropy). We discuss thermodynamical consistency in terms of the properties of the involved energy functionals: the internal energy (in place of the free energy) and the pseudo-potential of dissipation. Hence, we prove existence of a solution (in a proper functional framework) for the resulting nonlinear integrodifferential PDE system. Finally, we discuss the long-time behaviour of solutions holding on $(0,+\infty)$ characterizing the ω-limit of trajectories.

Résumé

Des changements de phase avec mémoire thermique sont étudiés en utilisant l'entropie, variable duale de la température en mécanique des milieux continus, ainsi que l'histoire du flux d'entropie. Les équations sont la conservation de l'entropie, équivalente à la conservation de l'énergie, et une équation du mouvement obtenue avec le principe des puissances virtuelles prenant en compte les mouvements microscopiques qui apparaissent lors des changements de phase. On étudie la thermodynamique du système avec l'énergie interne, fonction duale de l'énergie libre, et du pseudo-potentiel de dissipation. Nous démontrons l'existence de solutions dans un cadre fonctionnel convenable pour le système intégro-différentiel et non linéaire d'équations aux dérives partielles. Nous étudions enfin le comportement à long terme des solutions dont on caractérise l'ensemble ω-limite.

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1. Introduction

Phase transitions phenomena have been deeply investigated in the literature both from the modelling and analytical viewpoints. Different physical situations have been analyzed, also including dissipative effects, thermal memory, and hysteresis (cf., e.g., [11]). Results of existence, uniqueness, regularity, and approximation of solutions have been obtained (cf. among the others [30] and references therein), mainly as generalization and/or a regularization of the classical Stefan problem (cf., e.g., [15]). Hence, the theory has been generalized accounting for possible thermal memory effects (cf. [12,22]) influencing the phase transitions, i.e. assuming that the thermodynamical equilibrium of the system depends also on the past history of the gradient of the temperature (see, e.g., [24]).

In the last years, a new approach for modelling phase transitions has been proposed by Frémond (cf. [17]): it is based on a generalization of the principle of virtual power, including microscopic motions responsible for the phase transition. In particular, the equation governing the evolution of the phase parameter is recovered as a balance equation for micro-movements. The resulting system turns out to be thermodynamically consistent and it has been applied to describe (reversible and irreversible) phase transitions in binary systems and also to more complex thermomechanical situations, as solid–solid phase transitions in shape memory alloys and damage of elastic and viscoelastic bodies.

Recently, this issue has been combined with a new idea to describe the thermal evolution of the system by means of the entropy variable. This model has been first introduced in [7] for a simplified situation and then in [6] for a general setting, also including thermal memory. The peculiar novelty consists in describing the evolution of the temperature through an equation written in terms of the entropy of the system (cf. [16] for a similar approach). Actually, this "entropy balance" is recovered rescaling the internal energy balance (and neglecting some higher order dissipative contributions, which is physically reasonable due to the small perturbations assumption). The main advantage of this approach is that, once the problem is solved in a suitable sense, one can obtain directly the positivity of the temperature, due to the presence of a logarithmic nonlinearity in the PDE system. This fact, which turns out to be essential in order to prove the thermodynamical consistency of the model, is of particular interest in the investigation of phenomena including thermal memory effects, as it avoids to apply any maximum principle technique. This fact is put in evidence in [6] where this model has been introduced and a global existence result is proved for a weak formulation of the related initial and boundary value problem. This result has been recently extended in [8] to the case of more general phase potentials (also including non-convex contributions, see the form of the energy functional (2.11) in the next section). However, the singularity of the system as well as the presence of strong nonlinearities prevent to prove uniqueness of the solution—which is still an open problem—mainly due to the lack of regularity of the temperature-variable. A well-posedness result is given in [7], for a simplified situation in which no thermal memory effects are considered and the phase dynamics is not diffusive. The problem of characterizing the ω-limit of solutions trajectories is faced in [8] for the general system and in [3] for the problem studied in [7]. In [4,5] a different choice for the present contribution of the heat flux, leading to a linear space operator acting on the temperature, allows the authors to prove existence, uniqueness, continuous dependence on the data, and regularity of the solutions as well as to investigate their long-time behaviour. Finally, for the sake of completeness, let us quote [18], in which the "entropy approach" is exploited to describe phase transitions processes with the possibility to observe macroscopically the presence of some voids between the phases.

In this paper, we refer to the model introduced in [6] (cf. also [8]) considering a more general relation between the entropy and the absolute temperature. We are interested both in the derivation of the model and in the analytical investigation of the initial-boundary value problem for the corresponding PDE system. As far as the modelling aspects are concerned, a novelty of our contribution also consists in writing the first principle of thermodynamics (corresponding to the energy balance in the classical approach) in a dual formulation (in the sense of convex analysis). More precisely, we choose as state variables the phase parameter, its gradient, the entropy (in place of the temperature), and the (non-dissipative) entropy flux. Hence, the equilibrium of the system turns out to be described by the internal energy functional (in place of the free energy). The relation between the internal energy and the free energy is established by means of the Legendre transformation (cf., e.g., [20,26]). For the sake of completeness, let us recall that a fairly similar approach has been recently used in a different framework to describe solidification processes in the alloys (cf. [29]). Moreover, concerning the dissipative variables, we use the dissipative entropy flux in place of the past history of the temperature gradient (related to the entropy flux by duality with respect to the convex dissipative functional). The main advantage of this approach is concerned with the proof of the second law of thermodynamics, which directly follows from the resulting equations and the properties of the involved functionals.

Here is the plan of the paper. After introducing in Section 2 the model and the derivation of the PDE system we deal with, we state our main existence result in Section 3. The proof is performed in Section 4 regularizing the system and passing to the limit by means of a priori estimates combined with compactness and lower-semicontinuity tools. Unfortunately, uniqueness of solutions for such a problem is still an open question in the general case: only for particular choices of the involved nonlinearities we can prove it. The main difficulty encountered at this step relies on the doubly nonlinear (and possibly singular) structure of the system (cf. also Remark 5.2 in [6] for further considerations on this topic). Finally, in the last Section 5 we investigate the long-time behaviour of the system, characterizing the ω-limit set of trajectories of the solutions (which are obtained through an approximation-passage to the limit technique) as the set of solutions of the associated stationary problem.

2. The model

Let us consider a phase transition phenomenon for a binary system, located in a bounded connected domain $\Omega \subset \mathbb{R}^3$ (with Lipschitzian boundary $\Gamma := \partial\Omega$). For the sake of completeness, we first recall the original modelling approach to phase transitions phenomena proposed by Frémond (e.g. in [17]). Then, we introduce our new modelling approach and point out the main differences.

In the usual phase-field theory, the unknowns are the absolute temperature $\vartheta (> 0)$ and a phase parameter χ, representing, e.g., the fraction of one phase with respect to the other. In the classical literature for phase transitions, the phase parameter is forced to satisfy some internal constraint (e.g., $\chi \in [0, 1]$). Actually, this is not the case of our problem in which we cannot introduce a constraint on χ but only some "penalty" function. Assuming in addition that the materials possibly present some thermal memory effects (cf. [22]), the thermodynamical equilibrium of the system at time t is defined in terms of the free energy functional Ψ, depending on the set of state variables $\mathcal{E} = \mathcal{E}_P \cup \mathcal{E}_H$, given by $\mathcal{E}_P = (\vartheta(t), \chi(t), \nabla\chi(t))$ and the summed past history of the gradient $\nabla\vartheta$, i.e. (cf. [6]) $\mathcal{E}_H = \widetilde{\nabla\vartheta}^t(\tau)$, where

$$\widetilde{\nabla\vartheta}^t(\tau) := \int_0^\tau \nabla\vartheta^t(\iota)\,\mathrm{d}\iota \quad \text{where } \vartheta^t(\iota) := \vartheta(t-\iota). \tag{2.1}$$

More precisely, the free energy Ψ is given by the sum of a contribution Ψ_P, depending on the present variables $\mathcal{E}_P$, and Ψ_H, depending on the histories $\mathcal{E}_H$. Concerning the dependence of Ψ_P on ϑ and of Ψ_H on $\widetilde{\nabla\vartheta}^t$, we point out that from thermodynamics it follows that Ψ_P is concave w.r.t. ϑ, while Ψ_H is assumed to be convex w.r.t. the history variable $\widetilde{\nabla\vartheta}^t$. Hence, following the approach of [20,26], the evolution is described by a so-called pseudo-potential of dissipation Φ depending on dissipative variables, as $\delta\mathcal{E} = (\chi_t, \nabla\chi_t, \nabla\vartheta)$. It is a proper, convex, lower semicontinuous functional attaining its minimum 0 in $\delta\mathcal{E} = \mathbf{0}$, so that, denoting by $\partial\Phi$ its subdifferential (in the sense of convex analysis), we have:

$$\partial\Phi(\delta\mathcal{E}) \cdot \delta\mathcal{E} \geqslant 0. \tag{2.2}$$

This fact is used in the sequel to show that Clausius–Duhem inequality holds true (cf. (2.39)).

The equations of the system are recovered from the first principle of thermodynamics (the energy balance describing the evolution of ϑ) and a generalization of the principle of virtual powers (cf. [17]) including the effects of micro-motions responsible for the phase transition (describing the evolution of χ). A consequence is that the internal energy balance has to account for heat contributions induced by microscopic mechanical actions. Thus, the first principle can be written as (see [17, p. 7, Section 3.2]),

$$e_t + \operatorname{div}\mathbf{q} = r + B\chi_t + \mathbf{H}\cdot\nabla\chi_t, \tag{2.3}$$

where B and $\mathbf{H}$ are new interior forces (responsible for the phase transition) and χ_t, $\nabla\chi_t$ represent microscopic velocities, e is the internal energy, $\mathbf{q}$ the heat flux, and r an external heat source. Hence, the balance equations for micro-motions is given by (see [17, Section 2.4, p. 53]):

$$B - \operatorname{div}\mathbf{H} = 0, \tag{2.4}$$

where we are assuming that no external mechanical actions are present at the microscopic level. Finally, the above equations are combined with suitable boundary conditions on Γ,

$$-\mathbf{q}\cdot\mathbf{n} = g, \qquad \mathbf{H}\cdot\mathbf{n} = 0,$$

g being a known boundary source. Then, the constitutive equations for the involved thermomechanical quantities are recovered from a free energy and a pseudo-potential of dissipation.

Thus, the internal energy e is written (by means of the in terms of Helmholtz relation) in terms of Ψ, ϑ, and of the entropy s as follows,

$$e(s,\cdot)=\Psi(\vartheta,\cdot)+\vartheta s, \tag{2.5}$$

where we have:

$$s=-\frac{\partial\Psi}{\partial\vartheta}.$$

Note that, as $-\Psi$ is assumed to be convex w.r.t. ϑ (in agreement with thermodynamics) and sufficiently regular, from the above relation we read that s and ϑ are convex conjugate functions. In particular, we are allowed to introduce the dual function of $-\Psi$ w.r.t. ϑ ($-\Psi$ is assumed to be convex and proper),

$$(-\Psi)^*(s,\cdot):=\sup_{\vartheta}\big\{s\vartheta+\Psi(\vartheta,\cdot)\big\}, \tag{2.6}$$

there holds

$$\vartheta=\frac{\partial(-\Psi)^*}{\partial s},$$

with $(-\Psi)^*(s,\cdot)=e(s,\cdot)$. For the heat flux $\mathbf{q}$ we have $\mathbf{q}=\vartheta\mathbf{Q}$, where $\mathbf{Q}$ represents an entropy flux. It is given by the sum of a dissipative and a non-dissipative contribution. The dissipative part of the entropy flux $-\mathbf{Q}^d$ may be introduced as the conjugate variable of $\nabla\vartheta$ w.r.t. the convex function Φ (which is the pseudo-potential of dissipation), i.e.

$$-\mathbf{Q}^d=\frac{\partial\Phi}{\partial\nabla\vartheta}. \tag{2.7}$$

Moreover, in the standard theory for materials with thermal memory (see, e.g. [22]), the heat flux $\mathbf{q}$ contains a non-dissipative contribution involving the summed past history of the gradient of the temperature $\widetilde{\nabla\vartheta}^t$ (cf. (2.1)). For the sake of simplicity we do not enter the details (cf. [6]) and just point out some main ideas. We recall that $\widetilde{\nabla\vartheta}^t$ is required to be an element of the space of the past histories,

$$S:=\left\{\mathbf{f}:(0,+\infty)\to\mathbb{R}^3 \text{ measurable such that } \int_0^{+\infty}h(\tau)\big|\mathbf{f}(\tau)\big|^2\,\mathrm{d}\tau<+\infty\right\},$$

where

$$h:(0,+\infty)\to(0,+\infty) \text{ is a continuous, decreasing function such that } \int_0^{+\infty}\tau^2h(\tau)\,\mathrm{d}\tau<\infty, \tag{2.8}$$

and S is endowed with the (natural) norm $|\mathbf{f}|_S^2:=\int_0^{+\infty}h(\tau)|\mathbf{f}(\tau)|^2\,\mathrm{d}\tau$ and the related scalar product $(\mathbf{v},\mathbf{u})_S=\int_0^{+\infty}h(\tau)\mathbf{v}(\tau)\cdot\mathbf{u}(\tau)\,\mathrm{d}\tau$. Then, letting the history contribution to the free energy $\Psi_H:S\to\mathbb{R}$ be a convex functional (actually it is a quadratic form, cf. (2.18)), one may consider its Fréchet derivative $\delta\Psi_H$ and define,

$$-\mathbf{Q}^{nd}:=\delta\Psi_H\big[\widetilde{\nabla\vartheta}^t\big], \tag{2.9}$$

as a linear and bounded operator from S to $\mathbb{R}$. Letting Ψ_H be sufficiently regular w.r.t. the topology of S, we may identify $-\mathbf{Q}^{nd}$ with the convex conjugate variable of the history $\widetilde{\nabla\vartheta}^t$, that is

$$-\mathbf{Q}^{nd}=\frac{\partial\Psi_H}{\partial\widetilde{\nabla\vartheta}^t}, \tag{2.10}$$

the relation holding in S. Finally, the new forces $\mathbf{B}$ and $\mathbf{H}$ are introduced in terms of Ψ and Φ (cf. [17]):

$$\mathbf{B}=\frac{\partial\Psi}{\partial\chi}+\frac{\partial\Phi}{\partial\chi_t},\qquad \mathbf{H}=\frac{\partial\Psi}{\partial\nabla\chi}+\frac{\partial\Phi}{\partial\nabla\chi_t}.$$

2.1. Our new approach

In this paper, we assume a different point of view and consider the entropy s as state variable, in place of the temperature ϑ, and the history contribution to the entropy flux $-\mathbf{Q}^{nd}$ in place of the summed past history of the temperature gradient $\widetilde{\nabla\vartheta}^t$ (cf. (2.1)). Consequently, the thermodynamical equilibrium of the system is defined in terms of the internal energy functional e in place of the free energy Ψ. The relation between e and Ψ is given exploiting convex analysis arguments (roughly speaking it is $e = -\Psi^*$, where Ψ^* is the dual function of Ψ with respect to temperature ϑ).

As far as concern the evolution of the system, we introduce a dissipative functional p, which we prescribe to be a pseudo-potential of dissipation related to $\Phi(\nabla\vartheta)$ through the Legendre transformation (i.e. $p = \Phi^*$, cf. the following (2.30)). Thus, we choose as dissipative variables the time derivative of the phase parameter χ_t and the dual conjugate variable of $\nabla\vartheta$, i.e. the dissipative contribution to the entropy flux $-\mathbf{Q}^d$ (cf. (2.7)),

$$\overline{\delta\mathcal{E}} = \big(\chi_t, -\mathbf{Q}^d\big).$$

Hence, we introduce the set of the state variables of our system including the entropy s (the conjugate function of the absolute temperature ϑ), the phase variable χ, its gradient $\nabla\chi$, and $-\mathbf{Q}^{nd}$ (the conjugate variable of $\widetilde{\nabla\vartheta}^t$, which is our history variable), i.e.

$$\widetilde{\mathcal{E}}_P(t) = \big(s(t), \chi(t), \nabla\chi(t)\big), \qquad \widetilde{\mathcal{E}}_H = \big(-\mathbf{Q}^{nd}\big).$$

Thus, the internal energy functional is assumed to be given by the sum of an internal energy contribution $\tilde{e}_P(\widetilde{\mathcal{E}}_P)$ (related to $\Psi_P(\mathcal{E}_P)$), depending on the "present" state variables—at time t—$\widetilde{\mathcal{E}}_P(t)$, and the history internal energy $\tilde{e}_H(\widetilde{\mathcal{E}}_H)$ (related to $\Psi_H(\mathcal{E}_H)$), depending on $-\mathbf{Q}^{nd}$.

It could be possible to give directly the three potentials $\tilde{e}_P(\widetilde{\mathcal{E}}_P)$, $\tilde{e}_H(\widetilde{\mathcal{E}}_H)$, and $p(\overline{\delta\mathcal{E}})$, but, in order to explain our point of view, we prefer to relate each element of the new approach to the previous ones mainly using convex analysis tools. Three dualities are involved in this argument. The first one is the classical entropy–temperature duality, obtained introducing as convex conjugate functionals the internal energy depending on the entropy and the free energy depending on the temperature. The two others are unusual dualities between thermal quantities. The first thermal duality concerns the gradient of temperature and the dissipative heat flux. The dual functions are both pseudo-potentials of dissipation. The second thermal duality is even more unusual because it is the duality between the temperature gradient history and the non-dissipative heat flux vector. These history functionals may be called either free energies or internal energies because both of them do depend neither on the temperature nor on the entropy.

Let us note that we use the freedom we have in mechanics to choose the state quantities and the quantities which describe the evolution. The main advantage in replacing the absolute temperature, as unknown, by the entropy is given by the fact that this approach leads to the positivity of the temperature as a direct consequence of the resulting equations.

2.1.1. The internal energy and the duality between the entropy s and the absolute temperature ϑ

We start discussing the choice of the "internal energy functional" $\tilde{e}_P$. We let:

$$\tilde{e}_P(\widetilde{\mathcal{E}}_P) = \tilde{e}_P = \hat{\alpha}\big(s - \lambda(\chi)\big) + \sigma(\chi) + \hat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2, \tag{2.11}$$

where

- σ and λ are smooth functions, with $\lambda'(\chi)$ denoting the latent heat of the phase transition and $\sigma(\chi)$ representing a possibly non-convex term;
- $\hat{\beta}:\mathbb{R}\to\mathbb{R}$ is a proper, convex, and lower semicontinuous function (the sum $\hat{\beta}(\chi) + \sigma(\chi)$ may be regarded as, e.g., a double-well potential as $\chi^2(\chi^2 - 1)$);
- $\hat{\alpha}:\mathbb{R}\to\mathbb{R}$ is a sufficiently regular, increasing, and convex function depending on the purely caloric part of the entropy $(s - \lambda(\chi))$ (actually in the sequel we will generalize these assumptions on $\hat{\alpha}$).

The term $\nu/2|\nabla\chi|^2$ stands for a local interaction (between phases) potential, ν representing the non-negative interfacial energy coefficient. As $\tilde{e}_P(\widetilde{\mathcal{E}}_P)$ is convex with respect to the entropy s, we can define its dual function, w.r.t. s, as follows,

$$\tilde{e}_P^*(\vartheta,\cdot)=\sup_s\bigl\{\vartheta s-\tilde{e}_P(s,\cdot)\bigr\}, \tag{2.12}$$

being

$$\vartheta=\frac{\partial\tilde{e}_P}{\partial s}=\hat{\alpha}'\bigl(s-\lambda(\chi)\bigr)=:\alpha\bigl(s-\lambda(\chi)\bigr)>0, \tag{2.13}$$

as $\hat{\alpha}$ is increasing. In particular, we point out that, on account of (2.11), (2.12), and just setting

$$\Psi_P(\vartheta)=-\tilde{e}_P^*(\vartheta), \tag{2.14}$$

one may recover the following form for the present part of the free energy functional,

$$\Psi_P(\vartheta,\chi,\nabla\chi)=-\hat{\alpha}^*(\vartheta)-\lambda(\chi)\vartheta+\sigma(\chi)+\hat{\beta}(\chi)+\frac{1}{2}|\nabla\chi|^2, \tag{2.15}$$

where $\hat{\alpha}^*$ denotes the convex conjugate of $\hat{\alpha}$. Note that by the above argument it follows that (2.5) holds true with $\tilde{e}_P$ in place of e. Moreover, as $\tilde{e}_P^*(\vartheta,\chi,\nabla\chi)$ is convex w.r.t. to ϑ, $\Psi_P(\vartheta,\chi,\nabla\chi)$ is concave with respect to the absolute temperature, which is expected by classical thermodynamical arguments.

Remark 2.1. Observe that the usual form for the free energy (the so-called Ginzburg–Landau potential),

$$\Psi_P(\vartheta,\chi,\nabla\chi)=c_V\vartheta(1-\log\vartheta)-\lambda(\chi)\vartheta+\sigma(\chi)+\hat{\beta}(\chi)+\frac{1}{2}|\nabla\chi|^2, \tag{2.16}$$

with a constant specific heat c_V, corresponds to the choice:

$$\hat{\alpha}\bigl(s-\lambda(\chi)\bigr)=-c_V\exp\left(-\frac{s-\lambda(\chi)}{c_V}\right).$$

Actually, our analysis accounts for more general form of the specific heat c_V in (2.16), which can be seen as a function of ϑ such that the resulting $\hat{\alpha}$ satisfies the assumptions we are requiring (i.e., $\hat{\alpha}(s-\lambda(\chi))$ is convex and increasing with respect to s). For instance, it is known from physics that the specific heat c_V can be taken of the type $c_V(\vartheta)=\vartheta^\gamma$, for ϑ small, with $\gamma\geqslant 0$. In this case, using the standard thermodynamic relation linking Ψ_P and c_V, that is

$$c_V=-\vartheta\frac{\partial^2\Psi_P}{\partial\vartheta^2},$$

we get:

$$\hat{\alpha}^*(\vartheta)=\frac{\vartheta^{\gamma+1}}{\gamma(\gamma+1)},$$

which is indeed convex and such that the function $\hat{\alpha}$ turns out to be convex and increasing, i.e.

$$\hat{\alpha}\bigl(s-\lambda(\chi)\bigr)=\frac{(s-\lambda(\chi))^{\frac{\gamma+1}{\gamma}}}{\gamma+1}.$$

Let us here introduce a useful notation for the sequel: denote by u the purely caloric part of the entropy, i.e.

$$u:=s-\lambda(\chi). \tag{2.17}$$

2.1.2. The duality between $-\mathbf{Q}^{nd}$ and $\widetilde{\nabla\vartheta}^t$

Now, we discuss with more details the presence of the thermal memory justifying the choice of the state variable $-\mathbf{Q}^{nd}$ and making precise the history energy functional $\tilde{e}_H$. We are mainly referring to the theory by Gurtin and Pipkin (cf. [22]). Before proceeding, let us recall that usually in Gurtin–Pipkin's theory (cf. also [6]) the history variable is defined as in (2.1).

Let Ψ_H be the history contribution to the free energy functional defined by (cf. [22]):

$$\Psi_H\big(\widetilde{\nabla\vartheta}^t\big) = \frac{1}{2}\int_0^{+\infty} h(\tau)\widetilde{\nabla\vartheta}^t(\tau)\cdot\widetilde{\nabla\vartheta}^t(\tau)\,\mathrm{d}\tau = \frac{1}{2}\big|\widetilde{\nabla\vartheta}^t\big|_S^2. \tag{2.18}$$

As Ψ_H is a convex functional in S, we are allowed to consider its Legendre transformation depending on the history variable $-\mathbf{Q}^{nd}$, which is the conjugate dual variable of $\widetilde{\nabla\vartheta}^t(\tau)$ (cf. (2.9)) in S,

$$\Psi_H^*\big(-\mathbf{Q}^{nd}\big) = \sup_{\widetilde{\nabla\vartheta}^t}\big\{\big(\widetilde{\nabla\vartheta}^t, -\mathbf{Q}^{nd}\big)_S - \Psi_H\big(\widetilde{\nabla\vartheta}^t\big)\big\} = \frac{1}{2}\big|-\mathbf{Q}^{nd}\big|_S^2, \tag{2.19}$$

from which we deduce (cf. also (2.18)),

$$\Psi_H^*\big(-\mathbf{Q}^{nd}\big) = \Psi_H\big(\widetilde{\nabla\vartheta}^t\big),$$

with

$$\widetilde{\nabla\vartheta}^t = \frac{\partial\Psi_H^*(-\mathbf{Q}^{nd})}{\partial(-\mathbf{Q}^{nd})} = -\mathbf{Q}^{nd}, \tag{2.20}$$

the relation holding in S. With our choice for Ψ_H, the constitutive law turns eventually out to be (cf. also (2.13)),

$$-\mathbf{Q}^{nd} = \widetilde{\nabla\vartheta}^t = \widetilde{\nabla\alpha(u)}^t. \tag{2.21}$$

Since the internal energy is the dual function with respect to the temperature of the opposite of the free energy and the free energy Ψ_H does not depend on ϑ, we have either,

$$\tilde{e}_H(\widetilde{\mathcal{E}}_H) = \tilde{e}_H\big(-\mathbf{Q}^{nd}\big) = \Psi_H^*\big(-\mathbf{Q}^{nd}\big) = \frac{1}{2}\big|-\mathbf{Q}^{nd}\big|_S^2, \tag{2.22}$$

when choosing $\widetilde{\mathcal{E}}_H$ as state quantity, or

$$e_H(\mathcal{E}_H) = e_H\big(\widetilde{\nabla\vartheta}^t\big) = \Psi_H\big(\widetilde{\nabla\vartheta}^t\big) = \frac{1}{2}\big|\widetilde{\nabla\vartheta}^t\big|_S^2,$$

when choosing $\mathcal{E}_H$ as state quantity. Of course, both functions $\tilde{e}_H(\widetilde{\mathcal{E}}_H)$ and $e_H(\mathcal{E}_H)$ are equal when constitutive laws (2.10) and (2.20) are satisfied in S. Let us note that, using the chain rule in S, we have:

$$\frac{\mathrm{d}\tilde{e}_H(\widetilde{\mathcal{E}}_H)}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|-\mathbf{Q}^{nd}\big|_S^2 = \left(-\mathbf{Q}^{nd}, \frac{\mathrm{d}(-\mathbf{Q}^{nd})}{\mathrm{d}t}\right)_S.$$

Thus, by constitutive law (2.21) and using the fact that $\nabla\vartheta(t)$ is constant in S, we get:

$$\begin{aligned}\frac{\mathrm{d}\tilde{e}_H(\widetilde{\mathcal{E}}_H)}{\mathrm{d}t} &= \left(-\mathbf{Q}^{nd}, \frac{\mathrm{d}\widetilde{\nabla\vartheta}^t}{\mathrm{d}t}\right)_S = \big(-\mathbf{Q}^{nd}, \nabla\vartheta(t) - \nabla\vartheta^t(\tau)\big)_S \\ &= -\nabla\vartheta(t)\cdot\int_0^{+\infty} h(\tau)\mathbf{Q}^{nd}(\tau)\,\mathrm{d}\tau + \big(\mathbf{Q}^{nd}, \nabla\vartheta^t(\tau)\big)_S.\end{aligned} \tag{2.23}$$

We point out that in materials without thermal memory the internal energy e in (2.3) does not depend on $-\mathbf{Q}^{nd}$. Hence, to prove the Clausius–Duhem inequality:

$$s_t + \operatorname{div}\mathbf{Q} - \frac{r}{\vartheta} \geqslant 0, \tag{2.24}$$

in case of materials without thermal memory, one has just to ensure that $-\mathbf{Q}\cdot\nabla\vartheta = -\mathbf{Q}^d\cdot\nabla\vartheta$ is non-negative. On the contrary, in our case we have to discuss in particular the sign of an additional term (see (2.23)) given by $(\mathbf{Q}^{nd}, \nabla\vartheta^t(\tau))_S$. As we will detail later, this term actually accounts for an internal heat source r^{int} (cf. the following (2.37)). Thus we prescribe that

$$\mathbf{Q}^{nd} = \mathbf{Q} - \mathbf{Q}^d := -\int_0^{+\infty} h(\tau)\mathbf{Q}^{nd}\,\mathrm{d}\tau = \delta\Psi_H^*\big[-\mathbf{Q}^{nd}\big] = \frac{\partial(-\Psi_H)^*}{\partial(-\mathbf{Q}^{nd})}, \tag{2.25}$$

the last equality holding in S. This choice leads to the non-negativity of r^{int} in (2.37), which immediately proves the Clausius–Duhem inequality (cf. (2.39)).

Remark 2.2. Note that the history energy functional $\tilde{e}_H(\widetilde{\mathcal{E}}_H)$ accounts for the dissipative thermal memory effects which, indeed, avoid an immediate response of the material to a disturbance at a distant point. This turns to be a relevant property for some classes of materials (cf., e.g., [24]).

2.1.3. The duality between $-\mathbf{Q}^d$ and $\nabla\vartheta$

Now, let us introduce our dissipative functional:

$$p(\overline{\delta\mathcal{E}}) = p(\chi_t, -\mathbf{Q}^d) = \frac{1}{2}|\chi_t|^2 + \frac{1}{2}\alpha'(u)\left|-\mathbf{Q}^d\right|^2. \tag{2.26}$$

After observing that—since α is increasing—p is convex with respect to $-\mathbf{Q}^d$, we can compute its conjugate function w.r.t. $-\mathbf{Q}^d$, finding

$$p^*(\delta\mathcal{E}) = p^*(\chi_t, \nabla\vartheta) = \sup_{-\mathbf{Q}^d}\left\{-\nabla\vartheta\cdot\mathbf{Q}^d - p(\chi_t, -\mathbf{Q}^d)\right\}, \tag{2.27}$$

from which it follows that

$$\nabla\vartheta = -\alpha'(u)\mathbf{Q}^d. \tag{2.28}$$

Then, using (2.13) (from which we have $\nabla\vartheta = \alpha'(u)\nabla u$) and (2.28) we eventually deduce,

$$-\mathbf{Q}^d = \nabla u. \tag{2.29}$$

We also point out that (2.26)–(2.28) lead to (cf. Remark 2.3 below)

$$\Phi(\chi_t, \nabla\vartheta) = p^*(\chi_t, \nabla\vartheta) = \frac{1}{2}|\chi_t|^2 + \frac{1}{2\alpha'(\alpha^{-1}(\vartheta))}|\nabla\vartheta|^2. \tag{2.30}$$

Observe that the properties of p and Φ imply in particular that (see the Fenchel–Moreau theorem, e.g., in [26]) $\Phi^* = (p^*)^* = p$.

Remark 2.3. The more usual form for the pseudo-potential of dissipation, leading to the Fourier law for the heat flux, (cf., e.g., [20]) is,

$$\Phi(\chi_t, \nabla\vartheta) = \frac{1}{2}|\chi_t|^2 + \frac{|\nabla\vartheta|^2}{2\vartheta}, \tag{2.31}$$

which corresponds to the choice of a function of α in (2.30) such that

$$\alpha'\circ\alpha^{-1}(\vartheta) = \vartheta. \tag{2.32}$$

Note that all functions α satisfying (2.32) have the form $\alpha = c\exp$ for some $c \in \mathbb{R}$. Thus, we recover in this case, the model introduced in [6]. Moreover, the presence of the factor $1/\vartheta$ in (2.31) entails that the thermal dissipation becomes more relevant at low temperature and this is actually physically reasonable.

2.1.4. The constitutive laws and the model equations

Finally, we make precise the constitutive relations for the remaining involved physical quantities B and $\mathbf{H}$ in (2.4). We first recall that (2.13) holds. Then, we set (cf. [17, p. 9, Section 3.3]):

$$B := B^{nd} + B^d = \frac{\partial\tilde{e}_P}{\partial\chi} + \frac{\partial p}{\partial\chi_t}, \qquad \mathbf{H} := \mathbf{H}^{nd} = \frac{\partial\tilde{e}_P}{\partial\nabla\chi}. \tag{2.33}$$

Now, we are in the position to recover our PDE system from the basic thermo-mechanical laws (2.3) and (2.4) with $\tilde{e} = \tilde{e}_P + \tilde{e}_H$ in place of e. To this aim, we first observe that, using the chain rule, we get for the time derivative of $\tilde{e}_P(\widetilde{\mathcal{E}}_P) = \tilde{e}_P(s, \chi, \nabla\chi)$ (cf. (2.11)) the following formula:

$$\frac{\mathrm{d}\tilde{e}_P(\widetilde{\mathcal{E}}_P)}{\mathrm{d}t} = \alpha(s - \lambda(\chi))(s_t - \lambda'(\chi)\chi_t) + (\sigma'(\chi) + \beta(\chi))\chi_t + \nabla\chi\cdot\nabla\chi_t. \tag{2.34}$$

Analogously, using the notation $u = s - \lambda(\chi)$, from (2.23) and (2.13) we get (cf. [6]),

$$\frac{\mathrm{d}\tilde{e}_H(\widetilde{\mathcal{E}}_H)}{\mathrm{d}t} = \nabla\alpha(u)\int_0^{+\infty} h(\tau)\widetilde{\nabla\alpha(u)}^t(\tau)\,\mathrm{d}\tau - \int_0^{+\infty} h(\tau)\widetilde{\nabla\alpha(u)}^t(\tau)\cdot\nabla\alpha(u)^t(\tau)\,\mathrm{d}\tau, \tag{2.35}$$

where $\nabla\alpha(u)^t(\tau) = \nabla\alpha(u)(t-\tau)$. Thus, combining (2.34) and (2.35) in (2.3), using (2.25) (note that the heat flux is $\mathbf{q} = \vartheta\mathbf{Q}$) and formulas (2.33), some terms cancel out and we actually get:

$$\alpha(u)(s_t + \operatorname{div}\mathbf{Q} - R) = r^{\mathrm{int}} + \frac{\partial p}{\partial\chi_t}\chi_t + \left(-\mathbf{Q}^d\right)\frac{\partial p}{\partial(-\mathbf{Q}^d)}, \tag{2.36}$$

where R is the entropy source $R := r/\alpha(u)$ (which is supposed to be known) and r^{int} represents an internal heat production resulting from the thermal history of the material (cf. (2.35)), i.e.

$$r^{\mathrm{int}} := \int_0^{+\infty} h(\tau)\widetilde{\nabla\alpha(u)}^t(\tau)\cdot\nabla\alpha(u)^t(\tau)\,\mathrm{d}\tau. \tag{2.37}$$

The case of a source $R = R(\alpha(u))$—possibly singular w.r.t. the absolute temperature $\alpha(u)$—is still an open problem. Then, substituting (2.33) in (2.4) we have (cf. also (2.11) and (2.26)):

$$\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \alpha\big(s - \lambda(\chi)\big)\lambda'(\chi) \ni 0, \tag{2.38}$$

where $\beta = \partial\hat{\beta}$ represents here the subdifferential of $\hat{\beta}$ in the sense of convex analysis (cf. [9,26]). We put in (2.38) an inclusion symbol as β may also be a multivalued operator.

As far as concern the thermodynamical consistency of the model, we can immediately deduce that

$$\partial p\big(\chi_t, -\mathbf{Q}^d\big)\cdot\big(\chi_t, -\mathbf{Q}^d\big) \geqslant 0,$$

owing to the fact that p is a pseudo-potential and by standard monotonicity arguments (cf. (2.26)). Thus, by (2.36) and since $\hat{\alpha}$ is increasing—hence $\hat{\alpha}'(u) = \alpha(u) = \vartheta > 0$ (cf. (2.13))—we immediately deduce that then the Second Law of Thermodynamics (in the form of the Clausius–Duhem inequality (2.24)) is satisfied only in case of a non-negative r^{int}, i.e.

$$s_t + \operatorname{div}\mathbf{Q} - R \geqslant 0 \quad\Longleftrightarrow\quad r^{\mathrm{int}} \geqslant 0. \tag{2.39}$$

Remark 2.4. Actually, to get a solution to our problem, we do not need to require α to be positive, hence we will make our analysis in the next sections without this assumption. The positivity of the temperature results from the mechanical assumption that function α is positive.

Now, it remains to establish (2.39) in terms of the properties of h (cf. (2.37)). Before proceeding, let us introduce the standard notation for convolution product $(a * b)(t) := \int_0^t a(t-s)b(s)\,\mathrm{d}s$. To this aim we introduce the auxiliary function $k : (0, +\infty) \to \mathbb{R}$ such that

$$h(\tau) =: -k'(\tau) \quad \forall\tau \in (0, +\infty)$$

and require that

$$k, k', k'' \in L^1(0, +\infty), \qquad \lim_{\tau\to+\infty} k(\tau) = 0. \tag{2.40}$$

Then, recalling (2.8), we also have:

$$k' \leqslant 0, \qquad k'' \geqslant 0. \tag{2.41}$$

Note that we prescribe these assumptions on the kernel k in order to ensure thermodynamical consistency. Indeed, from (2.40)–(2.41) it follows that k is of positive type, i.e. $\int_0^t (k * v)v \geqslant 0$ for any $v \in L^2(0, +\infty)$ and for all t (cf. [1, Prop. 4.1, p. 237]).

This is a sufficient condition for thermodynamical consistency (cf. [19]). Following the ideas contained in [6], i.e. integrating by parts in time in (2.37) and using assumptions (2.8) and (2.40), we can easily deduce that $r^{\mathrm{int}} \geqslant 0$.

Hence, we can write $\mathbf{Q}$ in terms of k and u (cf. (2.25) and (2.29)). Integrating by parts in time and exploiting (2.40) yields:

$$\mathbf{Q} = -\int_{-\infty}^{t} k(t-\tau)\nabla\alpha\big(u(\tau)\big)\,\mathrm{d}\tau - \nabla u.$$

Now, we are in the position of writing the complete PDE system we are dealing with. We assume that the past history till $t = 0$ of $\nabla\alpha(u)$, i.e. $\int_{-\infty}^{0} k(t-\tau)\nabla\alpha(u(\tau))\,\mathrm{d}\tau$ is known and included in a generalized source, still denoted by R to simplify notation. Then, using the small perturbations assumption (cf. [20]), we are allowed to neglect some higher order dissipative contributions in (2.36) (which are quadratic nonlinearities on the time derivatives, cf. also (2.26)). Notice that, as far as some applications are concerned, we should note that the quantity,

$$\frac{\partial p}{\partial \chi_t}\chi_t + \big(-\mathbf{Q}^d\big)\frac{\partial p}{\partial(-\mathbf{Q}^d)},$$

on the right-hand side in (2.36), which may be also called *intrinsic dissipation*, can be computed with a predictive theory and indirectly measured using infrared cameras and heat analysis (cf., e.g., [23] for a reference on this subject).

Finally, we rescale (2.36) by the absolute temperature $\alpha(u) > 0$ and couple it to (2.38), getting (now u is given as unknown in place of s, see (2.17)) the following PDE system:

$$\partial_t\big(u + \lambda(\chi)\big) - \Delta u - k * \Delta\alpha(u) = R \quad \text{in } \Omega, \tag{2.42}$$

$$\partial_t\chi - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \alpha(u)\lambda'(\chi) \ni 0 \quad \text{in } \Omega. \tag{2.43}$$

In the next sections, we study the initial-boundary value problem obtained coupling the system (2.42)–(2.43) with the following Neumann boundary conditions:

$$\partial_{\mathbf{n}}\big(u + k * \alpha(u)\big) = g, \quad \partial_{\mathbf{n}}\chi = 0 \quad \text{on } \Gamma, \tag{2.44}$$

when g represents a known boundary source, and with the initial conditions for the unknowns u and χ,

$$u(0) = u_0, \quad \chi(0) = \chi_0 \quad \text{in } \Omega. \tag{2.45}$$

Remark 2.5. Note that R in (2.42) is assumed to be known and not depending explicitly on the temperature. The case of a source $R(\cdot, \alpha(u))$—possibly singular in $\alpha(u)$—is still an open problem in this framework.

In the following Section 3, we introduce a weak formulation of (2.42)–(2.43), generalizing in particular the assumptions on α.

3. Main results

In this section we introduce the abstract formulation of our problem (2.42)–(2.45) and state the main existence (and uniqueness) results, holding under suitable assumptions on the data.

Let $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) be a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$, and $(0, T)$ a finite time interval. We use the notation $Q := \Omega \times (0, T)$ and, for every fixed time $t \in (0, T)$, $Q_t := \Omega \times (0, t)$. Moreover, we introduce the Hilbert triplet:

$$V := H^1(\Omega) \hookrightarrow H := L^2(\Omega) \equiv H' \hookrightarrow V',$$

where H is identified as usual with its dual space and the embeddings are continuous and compact. Moreover, we let $|\cdot|_X$ denote the norm in the Banach space X and we use the same notation $|\cdot|_H$ for the usual norm both in H and in $(L^2(\Omega))^3$. By $\langle\cdot,\cdot\rangle$ we denote the duality pairing between V' and V. Finally, we introduce the realization of the Laplace operator associated with homogeneous Neumann boundary conditions in the duality between V' and V:

$$A : V \to V', \quad \langle Av, w\rangle = \int_{\Omega} \nabla v \cdot \nabla w, \ \forall v, w \in V. \tag{3.1}$$

We collect the contribution of the external source R and the flux through the boundary g (cf. (2.42) and (2.44)) in the datum $F(t) \in V'$ defined, for a.e. $t \in (0, T)$, as

$$\langle F(t), v\rangle = \int_\Omega R(\cdot, t)v + \int_\Gamma g(\cdot, v)v_{|\Gamma}, \quad v \in V. \tag{3.2}$$

Then, we point out that actually we generalize the assumptions for $\hat{\alpha}$, as we can consider α in (2.42)–(2.43) as a multivalued monotone operator. On the contrary, we have to prescribe some growth conditions on $\hat{\beta}$. Thus, our main assumptions on the data are specified by:

$$\hat{\alpha} : \mathbb{R} \to [0, +\infty] \text{ is a proper, convex, lower-semicontinuous function, } \alpha := \partial\hat{\alpha}, \tag{3.3}$$

$$\hat{\beta} : \mathbb{R} \to [0, +\infty) \text{ is a convex, lower-semicontinuous function, } \beta := \partial\hat{\beta}, \tag{3.4}$$

$$|\xi| \leqslant c_\beta + c'_\beta \min\{|r|^{5-\eta}, |\hat{\beta}(r)|\} \quad \forall r \in \mathbb{R}, \xi \in \beta(r), \text{ and for some } \eta > 0, \tag{3.5}$$

$$\sigma \in C^2(\mathbb{R}), \quad \sigma'' \in L^\infty(\mathbb{R}), \tag{3.6}$$

$$k \in W^{2,1}(\mathbb{R}), \quad k(0) > 0, \quad \nu > 0, \tag{3.7}$$

$$F \in W^{1,1}(0, T; V'), \tag{3.8}$$

$$u_0 \in H, \quad \hat{\alpha}(u_0) \in L^1(\Omega), \quad \chi_0 \in V, \tag{3.9}$$

being c_β and c'_β two positive constants depending only on β. Let us point out that due to (3.3) and (3.4), α and β are maximal monotone operators $\mathbb{R} \to 2^{\mathbb{R}}$.

Remark 3.1. Roughly speaking, we may say that hypothesis (3.4) regards convex functions $\hat{\beta}(r)$ that grow at most like a power $(6 - \eta)$ $(\eta > 0)$ as $|r| \nearrow \infty$. This exponent—needed in the 3D (in space) case—is justified in our estimates (cf. (4.10) and Remark 4.1 below). Let us point out that our analysis does not include the choice $\widehat{\beta} = I_{[0,1]}$. On the contrary, we allow $\hat{\alpha}$, and thus α, to be fairly general.

Remark 3.2. Let us observe that (3.7) turns out to be weaker than the usual assumptions one has to require to get thermodynamical consistency of the model. Indeed, we do not consider any restrictions on the long time behaviour of k nor on the sign of its derivatives (usually it is $k' \leqslant 0$, $k'' \geqslant 0$, and $\lim_{s\to+\infty} k(s) = 0$, cf. [6]). In fact, our first result (cf. Thm. 3.6) deals with the finite time interval $(0, T)$ and not the whole $(0, +\infty)$, so that we can control possible perturbations of a kernel like k within a finite range of times. Note that using (3.7) and prescribing the above sign on the derivatives of k imply that k is of positive type, i.e. $\int_0^t \int_\Omega (k * v)v \geqslant 0$ for any admissible test function v. Actually, to deal with the long-time behaviour of the solutions we should require further hypotheses on k (cf. assumption (3.40) and Theorems 3.15 and 3.17 below). Note that (2.40) and (2.41) ensures that (3.40) holds (cf. [1, Prop. 4.1, p. 237]).

Now, we refer to [2] and collect some properties of the operators involved in our problem and generalizing (2.42) and (2.43). We first associate to $\hat{\alpha}$ defined in (3.3) the following functionals $\hat{\alpha}_H$ and $\hat{\alpha}_V$ (defined on H and V respectively):

$$\hat{\alpha}_H(v) = \int_\Omega \hat{\alpha}(v(x))\,dx \quad \text{if } v \in H \text{ and } \hat{\alpha}(v) \in L^1(\Omega), \tag{3.10}$$

$$\hat{\alpha}_H(v) = +\infty \quad \text{if } v \in H \text{ and } \hat{\alpha}(v) \notin L^1(\Omega), \tag{3.11}$$

$$\hat{\alpha}_V(v) = \hat{\alpha}_H(v) \quad \text{if } v \in V. \tag{3.12}$$

Analogously, we could introduce $\hat{\beta}_H$. Since, both $\hat{\alpha}_V$ and $\hat{\alpha}_H$ are proper, convex and lower semicontinuous functionals (on V and H respectively), their subdifferentials (cf. [1, Cap. II, p. 52]):

$$\mathcal{A} := \partial_{V,V'}\hat{\alpha}_V : V \to 2^{V'} \tag{3.13}$$

and (cf. [9, Ex. 2.1.4, p. 21])

$$\partial_H\hat{\alpha}_H : H \to 2^H \tag{3.14}$$

are maximal monotone operators. Then, for $u, \vartheta \in H$, we have (see, e.g., [9, Ex. 2.1.3, p. 52]) that

$$\vartheta \in \partial_H \hat{\alpha}_H(u) \quad \text{if and only if} \quad \vartheta \in \alpha(u) = \partial\hat{\alpha}(u) \quad \text{a.e. in } \Omega. \tag{3.15}$$

Hence, due to (3.12) and the definitions of $\mathcal{A}$ and $\partial_H \hat{\alpha}_H$, there holds (cf. [2, Section 2] for the proofs of these results),

$$\partial_H \hat{\alpha}_H(u) \subseteq H \cap \mathcal{A}(u) \quad \forall u \in V. \tag{3.16}$$

As a consequence of (3.15) and (3.16), for $\vartheta \in H$ and $u \in V$ there holds:

$$\vartheta \in \alpha(u) \text{ a.e. in } \Omega \quad \Longrightarrow \quad \vartheta \in \mathcal{A}(u) \text{ in } V'. \tag{3.17}$$

Finally, recalling (3.15), with an abuse of notation, from now on, we will use the same symbols β, α both for the subdifferentials $\partial\hat{\beta}, \partial\hat{\alpha} : \mathbb{R} \to 2^{\mathbb{R}}$ and for the subdifferentials $\partial_H \hat{\beta}_H, \partial_H \widehat{\alpha}_H : H \to 2^H$ (cf. (3.14)).

Now, let us introduce our problem.

Problem 3.3. Find $(u, \chi, \vartheta, \xi)$ with the regularity properties:

$$u \in H^1(0,T;V') \cap L^2(0,T;V), \tag{3.18}$$
$$\chi \in H^1(0,T;H) \cap L^\infty(0,T;V), \tag{3.19}$$
$$\vartheta \in L^2(0,T;V'), \quad k * \vartheta \in L^2(0,T;V) \cap C^0\big([0,T];H\big), \tag{3.20}$$
$$\xi \in L^\infty\big(0,T;L^{6/(5-\eta)}(\Omega)\big), \tag{3.21}$$

and satisfying,

$$\partial_t(u+\chi) + Au + A(k*\vartheta) = F \quad \text{in } V', \text{ a.e. in } (0,T), \tag{3.22}$$
$$\partial_t \chi + \nu A(\chi) + \xi + \sigma'(\chi) - \vartheta = 0 \quad \text{in } V', \text{ a.e. in } (0,T), \tag{3.23}$$
$$\vartheta \in \mathcal{A}(u) \text{ in } V' \text{ a.e. in } (0,T), \quad \xi \in \beta(\chi) \text{ a.e. in } Q, \tag{3.24}$$
$$u(0) = u_0, \quad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{3.25}$$

In order to prove the existence of solutions to Problem 3.3 we must regularize it with a suitable approximating Problem 3.4, depending on a small parameter $\varepsilon > 0$. Then, we pass to the limit as $\varepsilon \searrow 0$, by means of suitable a-priori estimates and compactness/monotonicity arguments recovering the desired solution to Problem 3.3. In order to introduce this regularized problem, let us consider the Yosida approximations α_ε and β_ε of the maximal monotone operators α and β (cf. [9]), then for $\varepsilon > 0$ and $T > 0$, the approximating problem is stated as follows:

Problem 3.4. Find $(u_\varepsilon, \chi_\varepsilon, \vartheta_\varepsilon, \xi_\varepsilon)$ with the regularity properties:

$$u_\varepsilon, \vartheta_\varepsilon \in H^1(0,T;V') \cap L^2(0,T;V) \cap C^0\big([0,T];H\big), \tag{3.26}$$
$$\chi_\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \tag{3.27}$$
$$k * \vartheta_\varepsilon \in L^2(0,T;V) \cap C^0\big([0,T];H\big), \tag{3.28}$$
$$\xi_\varepsilon \in L^2(Q), \tag{3.29}$$

and satisfying,

$$\partial_t(u_\varepsilon + \chi_\varepsilon) + Au_\varepsilon + A(k*\vartheta_\varepsilon) = F \quad \text{in } V', \text{ a.e. in } (0,T), \tag{3.30}$$
$$\partial_t \chi_\varepsilon + \nu A(\chi_\varepsilon) + \xi_\varepsilon + \sigma'(\chi_\varepsilon) - \vartheta_\varepsilon = 0 \quad \text{a.e. in } Q, \tag{3.31}$$
$$\vartheta_\varepsilon = \alpha_\varepsilon(u_\varepsilon) \text{ a.e. in } Q, \quad \xi_\varepsilon = \beta_\varepsilon(\chi_\varepsilon) \text{ a.e. in } Q, \tag{3.32}$$
$$u_\varepsilon(0) = u_0, \quad \chi_\varepsilon(0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{3.33}$$

The existence and uniqueness of a solution for Problem 3.4 is stated by the following proposition, whose proof can be found e.g. in [8] (cf. also [6]).

Proposition 3.5. *Let the assumptions* (3.3)–(3.9) *hold and* α_ε, β_ε *be the Yosida approximation of* α *and* β (*so that they are in particular Lipschitz-continuous functions*). *Then, there exists a unique solution to Problem* 3.4.

Now, we present our existence result for Problem 3.3, whose proof will be performed in Section 4. Note that we actually find $(u, \chi, \vartheta, \xi)$ solving Problem 3.3 as limit solution of the approximating Problem 3.4 when $\varepsilon \searrow 0$.

Theorem 3.6. *Let T be a positive final time and let assumptions* (3.3)–(3.9) *be satisfied, then Problem* 3.3 *has at least a solution* $(u, \chi, \vartheta, \xi)$ *on the time interval* $(0, T)$, *which is obtained as the limit of some subsequence* (*still denoted by the index* ε *for the sake of simplicity*) *of the quadruple* $(u_\varepsilon, \chi_\varepsilon, \vartheta_\varepsilon, \xi_\varepsilon)$ *solving Problem* 3.4, *as* $\varepsilon \searrow 0$. *The limit is intended in the following sense*:

$$u_\varepsilon \to u \quad \text{weakly in } H^1(0,T;V') \cap L^2(0,T;V) \\ \text{and strongly in } L^2\big(0,T;L^{6-\eta}(\Omega)\big), \tag{3.34}$$

$$\chi_\varepsilon \to \chi \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \\ \text{and strongly in } C^0\big(0,T;L^{6-\eta}(\Omega)\big) \cap L^2(0,T;V), \tag{3.35}$$

$$\alpha_\varepsilon(u_\varepsilon) = \partial_t(1 * \alpha_\varepsilon(u_\varepsilon)) \to \vartheta \quad \text{weakly star in } L^2(0,T;V'), \tag{3.36}$$

$$1 * \alpha_\varepsilon(u_\varepsilon) \to 1 * \vartheta \quad \text{weakly in } L^\infty(0,T;V) \\ \text{and strongly in } C^0\big(0,T;L^{6-\eta}(\Omega)\big), \tag{3.37}$$

$$\xi_\varepsilon \to \xi \quad \text{weakly star in } L^\infty\big(0,T;L^{6/(5-\eta)}(\Omega)\big). \tag{3.38}$$

Notice that under the general assumption (3.3) on $\widehat{\alpha}$, the relation $\vartheta \in \mathcal{A}(u)$ can be intended only in V' due to the low regularity of ϑ and to the generality of $\hat{\alpha}$. Under more restrictive assumptions on $\hat{\alpha}$ we have the following result, whose proof is given in [2, Corollary 3.2, p.1247].

Proposition 3.7. *Suppose, beside* (3.3)–(3.9), *that the domain* $D(\hat{\alpha})$ *of* $\hat{\alpha}$ *fulfils*:

$$D(\hat{\alpha}) \equiv \mathbb{R}. \tag{3.39}$$

Then, $\vartheta(t) \in L^1(\Omega)$ *for all* $t \in [0, T]$ *and the relation* $\vartheta \in \mathcal{A}(u)$ *in* (3.24) *can be rewritten as*

$$\vartheta \in \alpha(u) \quad \text{a.e. in } Q.$$

Remark 3.8. Assumption (3.39) (which is satisfied, e.g., in case $\widehat{\alpha}(u) = \exp(u)$) yields the statements made in Section 2 rigorous, indeed, according to Proposition 3.7, in this case we are allowed to state the relation $\vartheta \in \alpha(u)$ a.e. in Q.

Finally, in general, we cannot prove that the solution given by Theorem 3.6 is unique. This is mainly due to the doubly nonlinear character of the system and the lack of regularity for the ϑ-component of the solution (see also the subsequent Remark 3.9).

Remark 3.9. The main difficulty in the uniqueness proof relies in the doubly nonlinear character of the system (and in particular in the nonlinear coupling between the phase and the temperature equations). Indeed, only some spatial regularity properties on the u-variable can be deduced by Eq. (3.22) (cf. also (3.18)), while on $\vartheta (\in \alpha(u))$ we cannot state any further regularity (see (3.23)). Indeed, the inclusion $\vartheta \in \mathcal{A}(u)$ can be intended here only in the duality $V' - V$ as we have underlined above. It is this lack of regularity in the ϑ-component of the solution which prevents us from proving the uniqueness of solutions for a general nonlinear function α, while such a result can be achieved in case α is a Lipschitz continuous function (cf. the following Proposition 3.10). Concerning the regularity of solutions, we also point out that the spatial regularity of χ (3.19) is due to the strict positivity of the coefficient ν in (3.23) (cf. assumption (3.7)).

However, assuming further regularity on α we can improve the regularity of ϑ and then prove uniqueness of solutions to Problem 3.3.

Proposition 3.10. *Let us prescribe, in addition to* (3.3)–(3.9)*, the following hypothesis*:

$$\alpha \text{ is a Lipschitz continuous function on } \mathbb{R}$$

(*so that in particular* (3.39) *is satisfied*). *Then, Problem* 3.3 (*where relation* $\vartheta \in \mathcal{A}(u)$ *in* V' *can be rewritten as* $\vartheta \in \alpha(u)$ *a.e. in* Q*, due to Proposition* 3.7) *admits a unique solution with the further regularity of* ϑ *prescribed by* (3.26).

The proof of the above proposition is detailed in [8] (cf. also [6]), we refer to for details.

Remark 3.11. Observe that Proposition 3.10 still holds true under the following weaker assumptions on the data:

$$k \in W^{1,1}(0,T),\, k(0) \geqslant 0,\, k \equiv 0 \text{ if } k(0) = 0,$$
$$\beta \text{ satisfying only assumption (3.4)},$$
$$F \in L^2(0,T;V'),\, \nu \geqslant 0,$$

and actually yields a more regular solution than in Theorem 3.6.

Remark 3.12. Let us note that in case we choose $\alpha(u) = \exp(u)$ we just recover the model studied first in [6] for some particular choices of $\hat{\beta}$ and σ, and then in [8] for more general graphs $\hat{\beta}$ and σ also from the long-time behaviour viewpoint. In this case more regularity of the solution is obtained and the graph α can be defined as the standard subdifferential of the convex analysis almost everywhere. However, also in this case uniqueness remains an open problem (cf. also the Introduction above for further comments on this topic). Another choice we can make for the function α is $\alpha : (0,+\infty) \to (-\infty, 0)$ such that $\alpha(w) = -1/w$. In this case, regarding w as the absolute temperature of the system, we recover the well-known Penrose–Fife model ([27]) with memory, which, in case $k \equiv 0$ (Penrose–Fife model with Fourier heat flux law) and if $\hat{\beta} + \sigma$ is the standard double-well potential, has been studied in [25]. Also in that case more regularity on the solution is achieved by means of suitable estimates of $\vartheta = \alpha(u)$ in some Orlicz space. Note that, also for this particular choice of α (which is not Lipschitz continuous!) uniqueness is still an open problem.

Finally, we state here our main result regarding the long-time behaviour of solutions. Let us point out that the above Theorem 3.6 allows us to infer that there exists at least a solution to Problem 3.3, obtained as limit of solutions to Problem 3.4, and defined on the whole time interval $(0,+\infty)$ (cf. the following Remark 3.13) in the following sense: there exists $(u, \chi, \vartheta, \xi)$ solving Problem 3.3 on $(0,T)$ for any fixed $T > 0$. However, we cannot deduce that every solutions to (3.18)–(3.25) in some interval $(0,T)$ can be extended to the whole $(0,+\infty)$, because of the lack of regularity of the solutions. Thus, we restrict our asymptotic analysis (for $T \nearrow \infty$) to those solutions holding on the whole $(0,+\infty)$ which are the limit of solutions to Problem 3.4 (cf. Theorem 3.6). More precisely, we perform a priori estimates—uniform in time and with respect to ε—on the approximating system and then pass to the limit as $\varepsilon \searrow 0$ and $T \nearrow \infty$.

Remark 3.13. Theorem 3.6 ensures that there exists a solution to Problem 3.3 on the whole $(0,+\infty)$ in the sense specified above. Let us briefly detail how one can get a solution on the whole $(0,+\infty)$. Fix a sequence of times $T_k \nearrow +\infty$. Let $T_0 > 0$ be the first term of our sequence. We can consider a solution $(u, \chi, \vartheta, \xi)$ of Problem 3.3 on $(0, T_0)$ which is the limit of a sequence $(u_{\varepsilon_k}, \chi_{\varepsilon_k}, \vartheta_{\varepsilon_k}, \xi_{\varepsilon_k})$ of approximated solutions to Problem 3.4 (whose existence is ensured by Theorem 3.6). It is straightforward to deduce from Proposition 3.5 that, for $\varepsilon > 0$ fixed, the unique solution of the Problem 3.4 actually extends to the whole $(0,+\infty)$ (this is mainly due to the regularity of the solution (3.26)–(3.27)). Thus, we can consider $(u_{\varepsilon_k}, \chi_{\varepsilon_k}, \vartheta_{\varepsilon_k}, \xi_{\varepsilon_k})$ solving Problem 3.4 on $(0, T_1)$ and find a subsequence of $(u_{\varepsilon_k}, \chi_{\varepsilon_k}, \vartheta_{\varepsilon_k}, \xi_{\varepsilon_k})$ converging (in the sense specified by (3.34)–(3.38)) to some $(\widetilde{u}, \widetilde{\chi}, \widetilde{\vartheta}, \widetilde{\xi})$ on $(0, T_1)$. Note that by construction $(\widetilde{u}, \widetilde{\chi}, \widetilde{\vartheta}, \widetilde{\xi})$ actually extends $(u, \chi, \vartheta, \xi)$. Hence, we can iterate the procedure for any time T_k, finding a solution to Problem 3.3 which is obtained from $(u_{\varepsilon_k}, \chi_{\varepsilon_k}, \vartheta_{\varepsilon_k}, \xi_{\varepsilon_k})$ through a diagonalization argument. Eventually, we get a solution to Problem 3.3 on the whole time interval $(0,+\infty)$, from which we start with our asymptotic analysis (cf. Theorem 3.17 below).

Now, we need prescribe further assumptions on the data. In particular, we let k be a strongly positive kernel (cf. Remark 3.2 and Lemma 3.14 below) and assume that, besides (3.3)–(3.9), the following hypotheses hold true:

$$k \in W^{1,1}(0,\infty) \text{ is of strongly positive type, i.e. there exists } \delta > 0 \text{ s.t.} \tag{3.40}$$

$$\widetilde{k}(t) := k(t) - \delta \exp(-t) \text{ is of positive type,}$$

$$F \in L^\infty\bigl(0,+\infty; L^1(\Omega)\bigr) \cap L^2(0,+\infty; H) \cap L^1\bigl(0,+\infty; L^\infty(\Omega)\bigr) \cap L^1(0,+\infty; V), \tag{3.41}$$

$$\lim_{|r|\to+\infty} |r|^{-2}\hat{\beta}(r) = +\infty. \tag{3.42}$$

Using condition (3.40) on k, it turns out that the operator $v \mapsto k * v$ enjoys the following additional property that we will use in the sequel (cf. [14]).

Lemma 3.14. *Assume that k fulfils* (3.40). *If $v \in L^2(0,T;H)$ for each $T \in (0,+\infty)$, then there exists a positive constant $\kappa > 0$ such that*

$$\int_0^T \int_\Omega |k * v|^2 \leqslant \kappa \int_0^T \int_\Omega (k * v)v, \quad \forall T \in (0,+\infty),$$

where κ depends only on $|k|_{W^{1,1}(0,+\infty)}$ and δ.

Now, let us take a solution (u,χ,ξ,ϑ) to (3.18)–(3.25) in $(0,+\infty)$, the existence of which is ensured by Theorem 3.6. Our next result (whose proof is contained in Section 5.1) contains some uniform bounds with respect to time.

Theorem 3.15. *Assume that, besides* (3.3)–(3.9), *hypotheses* (3.40)–(3.42) *hold. Then, there exists a positive constant $C_{\mathcal{L}}$, depending only on the data of the problem, but not on t, such that, for all $t > 0$ and any (u,χ,ξ,ϑ) solving* (3.18)–(3.25) *in* $(0,+\infty)$ *in the sense of Theorem* 3.6, *it follows*:

$$\mathcal{L}(t) := \int_0^t \Bigl(\bigl|\chi_t(s)\bigr|_H^2 + \bigl|u_t(s)\bigr|_{V'}^2 + \bigl|\vartheta(s)\bigr|_{V'}^2 + \bigl|k * \nabla\vartheta(s)\bigr|_H^2\Bigr)\,\mathrm{d}s + \bigl|\chi(t)\bigr|_V^2 + \bigl|u(t)\bigr|_H^2 \leqslant C_{\mathcal{L}}. \tag{3.43}$$

Moreover, fixing $T > 0$, there holds:

$$|k * \vartheta|_{L^2(t,t+T;V)} + |u|_{L^2(t,t+T;V)} + |\xi|_{L^\infty(t,t+T;L^{6/(5-\eta)}(\Omega))} \leqslant C(T), \tag{3.44}$$

for any $t \geqslant 0$ and for some positive constant $C(T)$, depending on the data of the problem and possibly on T.

Now, using Theorem 3.15, we investigate the *long time behaviour* of solutions to (3.18)–(3.25) on $(0,+\infty)$. To this aim, we introduce the ω-limit set $\omega(u,\chi) \subset V' \times H$ of a single trajectory (u,χ) defined in $(0,+\infty)$ and approximated (in the sense of Theorem 3.6) by a sequence of solutions $(u_\varepsilon,\chi_\varepsilon)$ to Problem 3.4 (cf. also Remark 3.13):

$$\omega(u,\chi) := \bigl\{(u_\infty,\chi_\infty) \in H \times V\text{: there exists } t_n \nearrow \infty,\ \bigl(u(t_n),\chi(t_n)\bigr) \to (u_\infty,\chi_\infty) \text{ in } V' \times H\bigr\}. \tag{3.45}$$

Remark 3.16. Note that in the case when (3.18)–(3.25) has a unique solution (cf., e.g., Proposition 3.10), the trajectory $(u,\chi): (0,+\infty) \to H \times V$ is uniquely determined by the initial datum (u_0,χ_0) so that in this case $\omega(u,\chi)$ is replaced by $\omega(u_0,\chi_0)$. In the general situation we cannot prove uniqueness of the solution and thus the ω-limit set is intended to depend on u and χ as well as on the initial data (u_0,χ_0), even if it is not specified in the notation (cf. [8, Section 6] for related results).

Our main result (which we prove in Section 5.2) can be now stated as follows:

Theorem 3.17. *Under the assumptions of Theorem* 3.15, *let $(u,\chi): (0,+\infty) \to H \times V$ be a solution to* (3.22)–(3.25) *on* $(0,+\infty)$ *in the sense of Theorem* 3.6. *Then, the ω-limit set $\omega(u,\chi)$ is a nonempty, compact, and connected* *subset of* $V' \times H$. *Moreover, for any* $(u_\infty, \chi_\infty) \in \omega(u, \chi)$ *there exist* $\vartheta_\infty \in V'$ *and* $\xi_\infty \in L^{6/(5-\eta)}(\Omega)$ *such that* $(u_\infty, \chi_\infty, \xi_\infty, \vartheta_\infty)$ *solves the corresponding stationary problem*:

$$u_\infty = \frac{1}{|\Omega|}\left(-\int_\Omega \chi_\infty + c_0 + m\right) \quad \text{a.e. in } \Omega, \tag{3.46}$$

$$\nu A\chi_\infty + \xi_\infty + \sigma'(\chi_\infty) - \vartheta_\infty = 0 \quad \text{in } V', \tag{3.47}$$

$$\xi_\infty \in \beta(\chi_\infty) \quad \text{a.e. in } \Omega, \qquad \vartheta_\infty \in \mathcal{A}\left(\frac{1}{|\Omega|}\left(-\int_\Omega \chi_\infty + c_0 + m\right)\right) \quad \text{in } V', \tag{3.48}$$

where

$$c_0 := \int_\Omega u_0 + \int_\Omega \chi_0 \quad \text{and} \quad m := \int_0^\infty \left(\int_\Omega R(s) + \int_\Gamma g(s)\right) \mathrm{d}s. \tag{3.49}$$

Remark 3.18. Note that only the u_∞-component of the solution to problem (3.46)–(3.48) turns out to be a constant and that uniqueness of the solution to the limit equations is not deduced. Consequently, we cannot conclude that, in general, the whole trajectory $\{(u(t), \chi(t))t \geqslant 0\}$ tends to (u_∞, χ_∞) weakly in $H \times V$ and strongly in $V' \times H$ as $t \nearrow \infty$. This is mainly due to the presence of the anti-monotone term $\sigma'(\chi_\infty)$ in (3.23) and to the generality of our maximal monotone graph α. On the contrary, in the case when, e.g.,

$$\sigma'(\chi) \equiv \vartheta_c, \tag{3.50}$$

and under additional assumptions on α and β, something more can be deduced, as stated by the following Proposition 3.19 (cf. also [8]).

Proposition 3.19. *Under the assumptions of Theorem* 3.15, *letting* (3.50) *hold true, and supposing that*

$$\alpha \text{ is not multivalued,} \tag{3.51}$$

we can conclude in addition to Theorem 3.15 *that* χ_∞ *is constant a.e. in* Ω.

Moreover, if we assume in addition that

$$\beta + \widetilde{\alpha} \text{ is injective, where } \widetilde{\alpha}(\cdot) := -\alpha\left(-(\cdot) + \frac{1}{|\Omega|}(c_0 + m)\right), \tag{3.52}$$

then the couple $(u_\infty, \chi_\infty) \in \omega(u, \chi)$ *is uniquely determined as the solution of the following system*:

$$\begin{aligned} &u_\infty = -\chi_\infty + \frac{1}{|\Omega|}(c_0 + m), \\ &\beta(\chi_\infty) - \alpha\left(-\chi_\infty + \frac{1}{|\Omega|}(c_0 + m)\right) \ni -\vartheta_c \quad \text{a.e. in } \Omega, \end{aligned} \tag{3.53}$$

being c_0 *and* m *defined by* (3.49). *In particular, the whole trajectory* $(u(t), \chi(t))$ *tends to* (u_∞, χ_∞) *weakly in* $H \times V$ *and strongly in* $V' \times H$ *as* $t \nearrow \infty$.

The proof of this last result is performed in Section 5.3.

Remark 3.20. First, we shall observe here that, due to the assumptions (3.3) and (3.51) on α, $\widetilde{\alpha}:\mathbb{R} \to \mathbb{R}$ is still a monotone function. Then, we have to note that, under assumption (3.51), Proposition 3.7 guarantees that relation (3.53) can be intended to hold true a.e. in Ω and in this framework the operator $\mathcal{A}$ can be substituted by α.

4. Proof of Theorem 3.6 (existence of solutions)

The proof of Theorem 3.6 is based on the following scheme: first we make a priori estimates on the solution to Problem 3.4, independent of $\varepsilon > 0$, and pass to the limit as $\varepsilon \searrow 0$ proving convergences (3.34)–(3.38), which, by means of monotonicity arguments will allow us to obtain at the limit a solution to Problem 3.3. Before proceeding we recall that the Yosida regularization of α is the subdifferential $\alpha_\varepsilon = \partial\hat{\alpha}_\varepsilon$, where $\hat{\alpha}_\varepsilon(x) := \min_y\{\hat{\alpha}(y) + \frac{1}{2\varepsilon}|y-x|^2\}$.

4.1. A priori estimates

To simplify notation, we will not use the index ε in the following estimates except in case it is necessary. Moreover, we use the symbol c for some positive constants (may be also different from line to line), depending on the data of the problem, but not on ε.

4.1.1. First estimate

Test (3.30) by ϑ and (3.31) by $\partial_t\chi$, sum up the resulting equations and integrate over $(0,t)$, being $t \in (0,T)$. Let us point out that two terms cancel. Thus, after integrating by parts in time, applying the Young inequality, and using the fact that (cf. (3.7))

$$k * \vartheta = k(0)(1 * \vartheta) + k' * 1 * \vartheta, \tag{4.1}$$

we get:

$$\begin{aligned}
&\int_\Omega \hat{\alpha}_\varepsilon\big(u(t)\big) + \int_0^t\int_\Omega \alpha'_\varepsilon(u)|\nabla u|^2 + \frac{k(0)}{2}\big|\nabla(1*\vartheta)(t)\big|_H^2 + \int_0^t\int_\Omega |\partial_t\chi|^2 + \frac{\nu}{2}\int_\Omega \big|\nabla\chi(t)\big|^2 + \int_\Omega \widehat{\beta}\big(\chi(t)\big)\\
&\quad\leqslant \int_\Omega \hat{\alpha}_\varepsilon(u_0) + \frac{\nu}{2}\int_\Omega |\nabla\chi_0|^2 + \int_\Omega \hat{\beta}(\chi_0) + \int_0^t \langle F,\vartheta\rangle - \int_0^t\int_\Omega \sigma'(\chi)\partial_t\chi - \int_\Omega \nabla(1*\vartheta)(t)\cdot\nabla(k'*1*\vartheta)(t)\\
&\qquad + \int_0^t\int_\Omega \nabla(1*\vartheta)\cdot\nabla(k'*\vartheta)\\
&\quad\leqslant c + \int_0^t\langle F,\vartheta\rangle + \frac{1}{2}\int_0^t\int_\Omega |\partial_t\chi|^2 + c\int_0^t\int_\Omega |\chi|^2 + \frac{k(0)}{4}\big|\nabla(1*\vartheta)(t)\big|_H^2 + \int_0^t\int_\Omega \big|k'(0)\big|\big|\nabla(1*\vartheta)\big|^2\\
&\qquad + \int_0^t\int_\Omega \big(k''*\nabla(1*\vartheta)\big)\cdot\nabla(1*\vartheta) + \frac{|k'|^2_{L^2(0,T)}}{k(0)}\big|\nabla(1*\vartheta)\big|^2_{L^2(0,t;H)}\\
&\quad\leqslant \int_0^t\langle F,\vartheta\rangle + \frac{1}{2}\int_0^t\int_\Omega |\partial_t\chi|^2 + c\int_0^t |\partial_t\chi|^2_{L^2(0,\tau;H)}\,\mathrm{d}\tau + \frac{k(0)}{4}\big|\nabla(1*\vartheta)(t)\big|_H^2\\
&\qquad + \left(\frac{|k'|^2_{L^2(0,T)}}{k(0)} + \big|k'(0)\big| + |k''|_{L^1(0,T)}\right)\big|\nabla(1*\vartheta)\big|^2_{L^2(0,t;H)} + c,
\end{aligned}\tag{4.2}$$

where $\vartheta = \alpha_\varepsilon(u)$. Note that, in (4.2) we have exploited (3.6), the standard Young inequality and the Young inequality for convolution,

$$|a*b|_{L^r(0,\tau;B)} \leqslant |a|_{L^p(0,\tau)}|b|_{L^q(0,\tau;B)},$$

holding for $\tau > 0$, B Banach space, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, $r, p, q \in [1,+\infty]$, and the fact that

$$\big|\chi(t)\big|_H^2 \leqslant c\left(1 + \int_0^t |\partial_t\chi|_H^2\right) \quad \forall t \in (0,T).$$

Using here a technique already exploited in [13], we split the term $\int_0^t \langle F, \vartheta \rangle$ as follows:

$$\int_0^t \langle F, \vartheta \rangle = \int_0^t \langle F, \vartheta - \vartheta_\Omega \rangle + \int_0^t \langle F, \vartheta_\Omega \rangle, \tag{4.3}$$

where ϑ_Ω stands for the mean value of ϑ. We recall that for a function ζ its mean value is defined by:

$$\zeta_\Omega := \frac{1}{|\Omega|} \int_\Omega \zeta(x)\,\mathrm{d}x. \tag{4.4}$$

Hence, to handle (4.3), we begin estimating the first integral, after reminding one form of the standard Poincaré–Wirtinger inequality (cf., e.g., [10] and (4.4)), that is

$$|v - v_\Omega|_H \leqslant c|\nabla v|_H \quad \forall v \in V. \tag{4.5}$$

Note that (4.5) holds for some positive constant c depending only on Ω. Now, integrating by parts in time and using (4.5) yield:

$$\begin{aligned} \int_0^t \langle F, \vartheta - \vartheta_\Omega \rangle &= -\int_0^t \bigl\langle \partial_t F, 1 * (\vartheta - \vartheta_\Omega)\bigr\rangle + \bigl\langle F(t), 1 * (\vartheta - \vartheta_\Omega)(t)\bigr\rangle \\ &\leqslant c \int_0^t |\partial_t F|_{V'} \bigl|\nabla(1 * \vartheta)\bigr|_H + \delta\bigl|1 * \nabla\vartheta(t)\bigr|_H^2 + c_\delta |F|^2_{L^\infty(0,T;V')}, \end{aligned} \tag{4.6}$$

for all positive δ and for some positive constant c_δ. Now, in order to handle with the term containing ϑ_Ω, we use Eq. (3.31), testing it with the constant function 1 and obtaining (cf. (3.4)–(3.6)):

$$|\Omega|\bigl|\vartheta_\Omega(t)\bigr| \leqslant c\left(1 + \int_\Omega \bigl|\partial_t \chi(t)\bigr| + \int_\Omega \hat\beta\bigl(\chi(t)\bigr) + \int_\Omega \bigl|\chi(t)\bigr|\right). \tag{4.7}$$

Hence, using (4.7), we can proceed estimating the second term on the right-hand side of (4.3) as follows:

$$\begin{aligned} \left|\int_0^t \langle F, \vartheta_\Omega \rangle\right| &\leqslant \int_0^t |F|_{V'} |\vartheta_\Omega| \\ &\leqslant c \int_0^t |F|_{V'} \left(1 + \int_\Omega \bigl|\partial_t \chi(t)\bigr| + \int_\Omega \hat\beta\bigl(\chi(t)\bigr) + \int_\Omega \bigl|\chi(t)\bigr|\right) \\ &\leqslant c|F|^2_{L^2(0,T;V')} + \frac{1}{4}|\partial_t \chi|^2_{L^2(0,t;H)} + c\int_0^t |F|_{V'} \int_\Omega \hat\beta\bigl(\chi(t)\bigr) + c\int_0^t |\chi_t|^2_{L^2(0,s;H)}\,\mathrm{d}s + c. \end{aligned} \tag{4.8}$$

Using (4.2), (4.6)–(4.8) with δ sufficiently small, we can apply a standard version of Gronwall's lemma (cf. [9, Lemme A.3, A.5]) to get:

$$\begin{aligned} &\bigl|\hat\alpha_\varepsilon(u)\bigr|_{L^\infty(0,T;L^1(\Omega))} + \bigl|\nabla(1 * \vartheta)\bigr|_{L^\infty(0,T;H)} + |\chi_t|^2_{L^2(0,T;H)} \\ &\quad + |\chi|^2_{L^\infty(0,T;V)} + \bigl|\hat\beta\bigl(\chi(t)\bigr)\bigr|_{L^\infty(0,T;L^1(\Omega))} \leqslant c. \end{aligned} \tag{4.9}$$

Note also that from (4.7) we shall deduce a bound in $L^2(0,T;L^1(\Omega))$ for $\alpha_\varepsilon(u_\varepsilon)$.

4.1.2. Second estimate

Now, using (3.5) and the continuous embedding of V in $L^6(\Omega)$ holding true in case the Ω-dimension $d = 1, 2, 3$, from (4.9) we deduce (cf. (3.5)),

$$\left|\beta(\chi)\right|_{L^\infty(0,T;L^{6/(5-\eta)}(\Omega))} \leqslant c\left(1 + |\chi|^{5-\eta}_{L^\infty(0,T;L^6(\Omega))}\right) \leqslant c\left(1 + |\chi|^{5-\eta}_{L^\infty(0,T;V)}\right) \leqslant c.$$

Note that as $L^{6/5}(\Omega) = (L^6(\Omega))'$ is continuously embedded in V', we eventually get:

$$\left|\beta(\chi)\right|_{L^\infty(0,T;L^{6/(5-\eta)}(\Omega))} + \left|\beta(\chi)\right|_{L^\infty(0,T;V')} \leqslant c. \tag{4.10}$$

Hence, a comparison in (3.23) leads to

$$|\vartheta|_{L^2(0,T;V')} \leqslant c. \tag{4.11}$$

Finally, we observe that the following inequalities hold true (cf. (4.9) and (4.11)),

$$\left|1 * \vartheta(t)\right|^2_V \leqslant c\left(|1 * \nabla\vartheta|^2_{L^\infty(0,T;H)} + \int_0^t \langle\vartheta, 1 * \vartheta\rangle\right) \leqslant c\left(1 + \int_0^t |\vartheta|_{V'}|1 * \vartheta|_V\right). \tag{4.12}$$

Thus, the standard Gronwall's lemma (cf. [9, Lemme A.5]) ensures that

$$|1 * \vartheta|_{L^\infty(0,T;V)} \leqslant c. \tag{4.13}$$

4.1.3. Third estimate

Test now (3.30) by u and integrate over $(0, t)$. After an integration by parts in time and applying the Young inequality, due to (4.9) and the regularity of k (cf. (3.7) and (4.1)), we find (cf. also (3.8)) the following estimate,

$$\left|u(t)\right|^2_H + |\nabla u|^2_{L^2(0,t;H)} \leqslant c\left(|k * \nabla\vartheta|^2_{L^2(0,t;H)} + |\chi_t|^2_{L^2(0,t;H)} + |F|^2_{L^2(0,T;V')}\right) \leqslant c,$$

so that we can easily deduce:

$$|u|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leqslant c. \tag{4.14}$$

4.1.4. Fourth estimate

By comparison in (3.30) and using again (4.9), (4.14) with (4.1), we get:

$$|\partial_t u|_{L^2(0,T;V')} \leqslant c. \tag{4.15}$$

4.2. Passage to the limit as $\varepsilon \searrow 0$

In this section, we aim to pass to the limit in (3.30)–(3.33) as $\varepsilon \searrow 0$. Hence, we consider a solution $(u_\varepsilon, \chi_\varepsilon, \vartheta_\varepsilon = \alpha_\varepsilon(u_\varepsilon), \xi_\varepsilon \in \beta(\chi_\varepsilon))$ to Problem 3.4 and make use of the above uniform estimates (4.9), (4.11)–(4.15). We combine these estimates with the compactness lemma [28, Cor. 5, p. 86] leading to the existence of four functions u, χ, ϑ, ξ such that the following convergences hold, at least for a subsequence of $\varepsilon \searrow 0$ and for $\eta > 0$ given in (3.5),

$$u_\varepsilon \to u \quad \text{weakly in } H^1(0,T;V') \cap L^2(0,T;V) \text{ and strongly in } L^2\left(0,T;L^{6-\eta}(\Omega)\right), \tag{4.16}$$

$$\chi_\varepsilon \to \chi \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \text{ and strongly in } C^0\left(0,T;L^{6-\eta}(\Omega)\right), \tag{4.17}$$

$$\alpha_\varepsilon(u_\varepsilon) = \partial_t\left(1 * \alpha_\varepsilon(u_\varepsilon)\right) \to \vartheta \quad \text{weakly star in } L^2(0,T;V'), \tag{4.18}$$

$$1 * \alpha_\varepsilon(u_\varepsilon) \to 1 * \vartheta \quad \text{weakly in } L^\infty(0,T;V)$$
$$\text{and strongly in } C^0\big(0,T;L^{6-\eta}(\Omega)\big), \tag{4.19}$$
$$\xi_\varepsilon \to \xi \quad \text{weakly star in } L^\infty\big(0,T;L^{6/(5-\eta)}(\Omega)\big). \tag{4.20}$$

Note that all these convergences really hold true only for a subsequence of $\varepsilon \searrow 0$, due to the fact that, in general, we do not get uniqueness of solutions for the limit problem (cf. Remark 3.9). Now, we can pass to the limit in our system (3.30)–(3.31). In particular, let us observe that we can identify the weak limit of $\sigma'(\chi_\varepsilon)$, as $\sigma'(\chi_\varepsilon) \to \sigma'(\chi)$ a.e. due to the strong convergence of χ_ε and the regularity of σ (cf. (3.6) and (4.17)). We get in V' and a.e. in $(0,T)$:

$$\partial_t(u+\chi) + Au + A(1*\vartheta) = F, \tag{4.21}$$
$$\partial_t\chi + A\chi + \xi + \sigma'(\chi) = \vartheta. \tag{4.22}$$

Moreover, due to the strong convergences in (4.16) and (4.17), we can pass to the limit in (3.33). Hence, to pass to the limit in (3.32), we first observe that we can identify $\xi \in \beta(\chi)$ using (3.4) along with convergences (4.17) (e.g., with $\eta = 2$), (4.20), and the result by [1, Lemma 1.3, p. 42].

Remark 4.1. Let us point out that to identify $\xi \in \beta(\chi)$ we could not use a semicontinuity argument testing (3.31) (and using (4.22)) by χ_ε, as we have that $\alpha_\varepsilon(u_\varepsilon)$ converges only in V' while for χ_ε-component of the solution we have only (4.17). Thus, we need to prove that ξ_ε is bounded (and weakly converges) in the dual of a space in which χ_ε strongly converges. For χ-component of the solution we can prove a bound in $L^\infty(0,T;V)$ (and this does not depend on the dimension of Ω). Thus, in a 3D domain, we can use the Sobolev embedding of V into $L^6(\Omega)$ (and the compact embedding of V in L^p for $p < 6$), and so (4.20)—which directly follows from assumption (3.5) on β—is required (cf. (4.10)). On the contrary, in the case of a 2D domain we have the continuous and compact embedding of V into $L^p(\Omega)$ for every $p \in [1,\infty)$. Thus, it would be sufficient to ensure the following bound for ξ_ε (in place of (4.10))—holding true for $p \in [1,+\infty)$—

$$|\xi_\varepsilon|_{L^\infty(0,T;L^{p/p-1}(\Omega))} + |\xi_\varepsilon|_{L^\infty(0,T;V')} \leqslant c. \tag{4.23}$$

Note that to get (4.23), in place of (3.5) we would need only the weaker condition,

$$|\xi| \leqslant c_\beta + c'_\beta \min\big\{|r|^{p-1}, \big|\hat\beta(r)\big|\big\} \quad \forall r \in \mathbb{R}, \xi \in \partial\hat\beta(r), \text{ for some } p \in [1,+\infty). \tag{4.24}$$

Finally, let us consider the case of a 1D domain Ω. In this case, we can substitute the assumption (3.5) by:

$$|\xi| \leqslant c_\beta + c'_\beta\big|\hat\beta(r)\big| \quad \forall r \in \mathbb{R}, \xi \in \partial\hat\beta(r). \tag{4.25}$$

Indeed, in this case one may use the continuous and compact embedding of V in $C^0(\bar\Omega)$ and only the following $L^1(\Omega)$-estimate for ξ_ε is required,

$$|\xi_\varepsilon|_{L^\infty(0,T;L^1(\Omega))} + |\xi_\varepsilon|_{L^\infty(0,T;V')} \leqslant c. \tag{4.26}$$

This estimate follows immediately from (4.25) and (4.9). Finally, note that (4.25) turns out to be satisfied by a function β with at most an exponential growth at $+\infty$, and so in the 1D case we have obtained a result holding true for a quite general class of functions β.

Secondly, we would like to identify also $\vartheta \in \mathcal{A}(u)$ in V' (see (3.24)). We first recall that by definition of the Yosida approximation we have $\vartheta_\varepsilon \in \alpha(J_\varepsilon u_\varepsilon)$ a.e. in Q, where $J_\varepsilon u_\varepsilon$ is the resolvent of α defined as the unique solution of (cf., e.g., [1, (1.23), p. 41]),

$$J_\varepsilon u_\varepsilon - u_\varepsilon + \varepsilon\alpha(J_\varepsilon u_\varepsilon) \ni 0.$$

Hence, as u_ε is bounded in $L^2(0,T;V)$, then $J_\varepsilon u_\varepsilon$ is bounded in $L^2(0,T;V)$, too (see (4.16)). Thus, recalling that $\vartheta_\varepsilon = \alpha_\varepsilon(u_\varepsilon)$ belongs to $L^2(0,T;H)$, we can deduce (cf. (3.17)) $\vartheta_\varepsilon \in \mathcal{A}(J_\varepsilon u_\varepsilon)$. Moreover, we can infer that $J_\varepsilon u_\varepsilon \rightharpoonup u$ in $L^2(0,T;V)$. Indeed, using the fact that $\alpha(J_\varepsilon u_\varepsilon)$ is bounded in $L^2(0,T;V')$ we have that $J_\varepsilon u_\varepsilon - u_\varepsilon \to 0$ in $L^2(0,T;V')$ and we can identify the weak limit of $J_\varepsilon u_\varepsilon$ in $L^2(0,T;V)$. Hence, to identify $\vartheta \in \mathcal{A}(u)$ we can apply [1, Lemma. 1.3, p. 42] if we prove the following inequality:

$$\limsup_{\varepsilon\searrow 0} \int_0^t \langle\vartheta_\varepsilon, J_\varepsilon u_\varepsilon\rangle \leqslant \int_0^t \langle\vartheta, u\rangle. \tag{4.27}$$

To this aim, we use (3.31) leading to,

$$\begin{aligned}\limsup_{\varepsilon\searrow 0}\int_0^t \langle \alpha_\varepsilon(u_\varepsilon), J_\varepsilon u_\varepsilon\rangle &= \limsup_{\varepsilon\searrow 0}\int_0^t \langle \alpha_\varepsilon(u_\varepsilon), u_\varepsilon - u_\varepsilon + J_\varepsilon u_\varepsilon\rangle \\ &= \limsup_{\varepsilon\searrow 0}\Bigg[-\frac{1}{\varepsilon}\int_0^t |J_\varepsilon u_\varepsilon - u_\varepsilon|_V^2 + \int_0^t\int_\Omega \partial_t\chi_\varepsilon u_\varepsilon \\ &\quad + \nu\int_0^t\int_\Omega \nabla\chi_\varepsilon\nabla u_\varepsilon + \int_0^t\int_\Omega \xi_\varepsilon u_\varepsilon + \int_0^t\int_\Omega \sigma'(\chi_\varepsilon)u_\varepsilon\Bigg].\end{aligned} \tag{4.28}$$

We first point out that, exploiting (4.16) and (4.17) (and the properties of σ'), we directly have:

$$\lim_{\varepsilon\searrow 0}\left(\int_0^t\int_\Omega \partial_t\chi_\varepsilon u_\varepsilon + \int_0^t\int_\Omega \sigma'(\chi_\varepsilon)u_\varepsilon\right) = \int_0^t\int_\Omega \partial_t\chi u + \int_0^t\int_\Omega \sigma'(\chi)u, \tag{4.29}$$

and, due to the strong convergence in (4.16) and to (4.20), we have:

$$\lim_{\varepsilon\searrow 0}\int_0^t\int_\Omega \xi_\varepsilon u_\varepsilon = \int_0^t\int_\Omega \xi u. \tag{4.30}$$

It remains to treat the term $\nu\int_0^t\int_\Omega \nabla\chi_\varepsilon\nabla u_\varepsilon$ for which we need a strong convergence for $\nabla\chi_\varepsilon$ in $L^2(0,T;H)$. Thus, we are going to prove that χ_ε strongly converges to χ in $L^2(0,T;V)$ (cf. (4.16)).

To this aim, we use a Cauchy argument. Let us consider the difference between (3.31), written for two approximating indices ε' and ε'', and test it by $\chi_{\varepsilon'} - \chi_{\varepsilon''}$, obtaining:

$$\begin{aligned}&\frac{1}{2}\big|(\chi_{\varepsilon'} - \chi_{\varepsilon''})(t)\big|_H^2 + \nu\big|\nabla(\chi_{\varepsilon'} - \chi_{\varepsilon''})\big|_{L^2(0,t;H)}^2 + \int_0^t\int_\Omega \big(\beta(\chi_{\varepsilon'}) - \beta(\chi_{\varepsilon''})\big)(\chi_{\varepsilon'} - \chi_{\varepsilon''}) \\ &\quad\leqslant -\int_0^t\int_\Omega \big(\sigma'(\chi_{\varepsilon'}) - \sigma'(\chi_{\varepsilon''})\big)(\chi_{\varepsilon'} - \chi_{\varepsilon''}) + \int_0^t\int_\Omega \big(\alpha_{\varepsilon'}(u_{\varepsilon'}) - \alpha_{\varepsilon''}(u_{\varepsilon''})\big)(\chi_{\varepsilon'} - \chi_{\varepsilon''}) \\ &\quad\leqslant c\int_0^t |\chi_{\varepsilon'} - \chi_{\varepsilon''}|_H^2 + \int_0^t\int_\Omega \big(\alpha_{\varepsilon'}(u_{\varepsilon'}) - \alpha_{\varepsilon''}(u_{\varepsilon''})\big)(\chi_{\varepsilon'} - \chi_{\varepsilon''}).\end{aligned} \tag{4.31}$$

Now, the term involving β on the left-hand side is non-negative (due to the monotonicity of β), while the first term on the right-hand side can be treated using a standard version of Gronwall's lemma. It remains to consider the last integral on the right-hand side. We can integrate it by parts in time obtaining:

$$\begin{aligned}\int_0^t\int_\Omega \big(\alpha_{\varepsilon'}(u_{\varepsilon'}) - \alpha_{\varepsilon''}(u_{\varepsilon''})\big)(\chi_{\varepsilon'} - \chi_{\varepsilon''}) &= \int_\Omega \Big(1 * \big(\alpha_{\varepsilon'}(u_{\varepsilon'}) - \alpha_{\varepsilon''}(u_{\varepsilon''})\big)\Big)(t)(\chi_{\varepsilon'} - \chi_{\varepsilon''})(t) \\ &\quad - \int_0^t\int_\Omega \Big(1 * \big(\alpha_{\varepsilon'}(u_{\varepsilon'}) - \alpha_{\varepsilon''}(u_{\varepsilon''})\big)\Big)\partial_t(\chi_{\varepsilon'} - \chi_{\varepsilon''}).\end{aligned} \tag{4.32}$$

The first scalar product on the right-hand side tends to zero as $\varepsilon', \varepsilon'' \searrow 0$ due to (4.17) and (4.19). Analogously, we treat the last time integral on the right-hand side, which also converges to zero due to the strong against weak convergences in $L^2(0,T;H)$ of $1 * \alpha_\varepsilon(u_\varepsilon)$ and $\partial_t\chi_\varepsilon$. These considerations leads to the strong convergence,

$$\chi_\varepsilon \to \chi \quad \text{strongly in } L^2(0,T;V). \tag{4.33}$$

Now we are in the position of proving (4.27) using (4.29)–(4.33) in (4.28), which concludes the proof of (3.24) and of Theorem 3.6.

5. Long-time behaviour of solutions

In this section, we investigate the long-time behaviour of those solutions $(u, \chi, \vartheta, \xi)$ of Problem 3.3 which are defined in $(0, +\infty)$, obtained as limit of solutions to Problem 3.4, and whose existence is ensured by Theorem 3.6. Moreover, we characterize the associated ω-limit. To this aim we first prove the uniform (in time) estimates stated in Theorem 3.15, then we conclude identifying the elements of the ω-limit proving Theorems 3.17 and 3.19.

5.1. Proof of Theorem 3.15

In this subsection we prove the uniform—in time—estimates stated in Theorem 3.15. In the asymptotic analysis we are dealing with the whole time interval $(0, +\infty)$, thus, from now on, the same symbol c will stand for—possibly different—positive constants depending on the data of the problem, but not on T, while we denote by $c(T)$ positive constants (possibly) depending (besides the data of the problem) increasingly on T. We perform the estimates on Problem 3.4, but we prefer, for simplicity of notation, not to use the index ε. The estimates will be in any case independent of ε.

5.1.1. First estimate

We test (3.30) by ϑ, (3.31) by χ_t, and (3.31) by $-F$ and integrate over $(0, t)$, $t \in (0, T)$. After adding the resulting equations and proceeding as in the First estimate of Section 4 (cf. (4.2)), applying the Young inequality, and using assumption (3.5), some terms cancel out and we get:

$$\begin{aligned}
&\int_\Omega \hat{\alpha}\big(u(t)\big) + \int_0^t \int_\Omega \alpha'(u)|\nabla u|^2 + c|1 * \nabla\vartheta|^2_{L^2(0,t;H)} + \int_0^t \int_\Omega |\chi_t|^2 + \frac{\nu}{2} \int_\Omega \big|\nabla\chi(t)\big|^2 + \int_\Omega \big(\hat{\beta}\big(\chi(t)\big) + \sigma\big(\chi(t)\big)\big) \\
&\quad \leqslant c + \int_0^t |\chi_t|_H |F|_H, + \int_0^t |A\chi|_{V'} |F|_V + \int_0^t \big|\beta(\chi)\big|_{L^1(\Omega)} |F|_{L^\infty(\Omega)} + \int_0^t \int_\Omega \big|\sigma'(\chi)\big||F| \\
&\quad \leqslant c + \frac{1}{2} \int_0^t \int_\Omega |\chi_t|^2 + |F|^2_{L^2(0,+\infty;H)} + c\int_0^t \big(1 + |\chi|_V\big)|F|_V + c\int_0^t \big|1 + \hat{\beta}(\chi)\big|_{L^1(\Omega)} |F|_{L^\infty(\Omega)}.
\end{aligned} \tag{5.1}$$

Note that (3.42) implies that for any $\delta > 0$ there exists c_δ—independent of T—such that

$$r^2 \leqslant \delta\hat{\beta}(r) + c_\delta \quad \forall r \in \mathbb{R}. \tag{5.2}$$

Moreover, from (3.6), it follows that there exists a positive constant M_σ—depending only on σ—such that

$$|\sigma(r)| \leqslant M_\sigma\big(1 + r^2\big) \quad \forall r \in \mathbb{R}. \tag{5.3}$$

Consequently, we get:

$$\int_\Omega \sigma\big(\chi(t)\big) \geqslant -\delta \int_\Omega \hat{\beta}\big(\chi(t)\big) - c_\delta. \tag{5.4}$$

Thus we deduce that there exist c_σ and c'_σ—depending only on σ—such that

$$\int_\Omega \big(\hat{\beta}\big(\chi(t)\big) + \sigma\big(\chi(t)\big)\big) \geqslant (1 - \delta) \int_\Omega \hat{\beta}\big(\chi(t)\big) - c_\sigma \geqslant c'_\sigma \int_\Omega \big(\hat{\beta}\big(\chi(t)\big) + \chi^2(t)\big) - c'_\sigma. \tag{5.5}$$

Hence, in (5.1) we have used the fact that k is a strongly positive kernel to infer that

$$\int_0^t \int_\Omega (k * \nabla\vartheta)\nabla\vartheta \geqslant c|1 * \nabla\vartheta|^2_{L^2(0,t;H)}.$$

Now, exploiting (3.41), (5.2), and (5.5), we can apply the Gronwall's lemma (cf. [9, Lemme A.5]) to (5.1), yielding:

$$\left|\hat{\alpha}(u)\right|_{L^\infty(0,+\infty;L^1(\Omega))} + \left|\nabla(1*\vartheta)\right|_{L^2(0,+\infty;H)} \leqslant c, \tag{5.6}$$

$$|\chi_t|_{L^2(0,+\infty;H)} + |\chi|_{L^\infty(0,+\infty;V)} \leqslant c, \tag{5.7}$$

$$\left|\hat{\beta}(\chi)\right|_{L^\infty(0,+\infty;L^1(\Omega))} \leqslant c. \tag{5.8}$$

Notice that, since all the constants above are independent of T and, due to assumption (3.41), we can use the Gronwall's lemma [9, Lemme A.5] for all $T \in (0,+\infty)$ and this is the reason why we have obtained the uniform in time estimates (5.6)–(5.8). Note, moreover, that, due to the regularity of k, we also have:

$$|k * \nabla\vartheta|_{L^2(0,+\infty;H)} \leqslant c. \tag{5.9}$$

5.1.2. Second estimate

Now, we proceed as in [8], to which we mainly refer for some details. We first estimate $|\int_\Omega u(t)|$ uniformly with respect to time. We first test (3.30) by 1 and integrate over $(0,t)$, $t \in (0,+\infty)$, yielding

$$\int_\Omega u(t) \leqslant c + \left|\int_\Omega \chi(t) + \int_\Omega R(t)\right| \leqslant c, \tag{5.10}$$

and, analogously, testing by -1 one eventually gets:

$$\left|\int_\Omega u(t)\right| \leqslant c. \tag{5.11}$$

Hence, testing (3.30) by u and using (5.11) lead to,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|u(t)\right|_H^2 + \int_\Omega |\nabla u|^2 + \left|\int_\Omega u(t)\right|^2 \leqslant c + \left(\left|\partial_t\chi(t)\right|_H + \left|F(t)\right|_H\right)\left|u(t)\right|_H + \left|k*\nabla\vartheta(t)\right|_H\left|\nabla u(t)\right|_H$$
$$\leqslant \varepsilon\left|u(t)\right|_V^2 + c\left(\left|\partial_t\chi(t)\right|_H^2 + \left|F(t)\right|_H^2 + \left|k*\nabla\vartheta(t)\right|_H^2\right).$$

Note that $|\partial_t\chi(t)|_H^2 + |F(t)|_H^2 + |k*\nabla\vartheta(t)|_H^2$ is bounded in $L^1(0,+\infty)$ due to (5.7), (5.9), and (3.41). Using the Poincaré inequality and a uniform version of the Gronwall's lemma (see, e.g., [21, Lemma 2.5]), we eventually get:

$$|u|_{L^\infty(0,+\infty;H)} \leqslant c \quad \text{and} \quad |u|_{L^2(t,t+T;V)} \leqslant c(T). \tag{5.12}$$

By comparison in (3.30), we deduce:

$$|\partial_t u|_{L^2(0,+\infty;V')} \leqslant c.$$

Finally, proceeding analogously to the way we proved (4.10), we can deduce from (5.8) and using assumption (3.5) that

$$|\xi|_{L^\infty(t,t+T;L^{6/(5-\eta)}(\Omega))} \leqslant c(T),$$

and this concludes the proof of Theorem 3.15 because the same estimates are preserved at the limit for $\varepsilon \searrow 0$. Let us only note that in order to obtain the estimates (3.43)–(3.44) on $\xi(\in\beta(\chi))$ and $\vartheta(\in\mathcal{A}(u))$, we should first identify the limits of ξ_ε and ϑ_ε respectively with $\beta(\chi)$ (a.e. in Q) and $\mathcal{A}(u)$ (in V') (being χ and u the limits of χ_ε and u_ε, respectively). However, we prefer not to detail here the proof of this identification because it can be done exactly like in Section 4 (cf., in particular, formulas (4.27)–(4.33)).

5.2. Proof of Theorem 3.17

Now, we are in the position of considering the ω-limit $\omega(u,\chi)$, defined in (3.45), as a non-empty and compact subset of $V' \times H$. Indeed, (5.7) and (5.12) imply that the set $\{(u(t),\chi(t)), t \geqslant 0\}$ of solutions to Problem 3.3 is bounded in $H \times V$ and thus relatively compact in $V' \times H$. We also point out that, since $(u, \chi) \in C^0([0, +\infty); V' \times H)$, we can infer that ω is connected in $V' \times H$. Now, we are allowed to take:

$$(u_\infty, \chi_\infty) \in \omega(u, \chi), \text{ such that } \big(u(t_n), \chi(t_n)\big) \to (u_\infty, \chi_\infty) \text{ in } V' \times H, \text{ as } t_n \nearrow +\infty.$$

Then, for $n \geqslant 1$ and $t \geqslant 0$, we define:

$$\begin{aligned} &u_n(t) := u(t_n + t), \quad \vartheta_n(t) := \vartheta(t_n + t), \\ &\chi_n(t) := \chi(t_n + t), \quad \xi_n(t) := \xi(t_n + t), \\ &\zeta_n(t) := (k * \vartheta)(t_n + t). \end{aligned}$$

In particular, the reader can observe that $\zeta_n \neq k * \vartheta_n$. Then, fixing a final time $T > 0$ and applying Theorem 3.6, it follows that (3.22)–(3.24) are solved in $(0, T)$ by $(u_n, \chi_n, \vartheta_n, \xi_n)$, where in place of F we are taking $F_n := F(t_n + t)$. In this section, we refer to these equations depending on the index n. Thus, from (3.43)–(3.44) we obtain:

$$|u_n|_{H^1(0,T;V') \cap L^2(0,T;V)} \leqslant c, \tag{5.13}$$

$$|\chi_n|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \leqslant c, \tag{5.14}$$

$$|\vartheta_n|_{L^2(0,T;V')} + |\nabla \zeta_n|_{L^2(0,T;H)} \leqslant c, \tag{5.15}$$

$$|\xi_n|_{L^\infty(0,T;L^{6/(5-\eta)}(\Omega))} \leqslant c, \tag{5.16}$$

where the constants c may depend increasingly on T but not on n. Thus, using compactness tools (cf. [28]), we deduce that the following convergences hold true for some suitable subsequences of $n \nearrow +\infty$,

$$\begin{aligned} u_n \to u \quad &\text{weakly in } H^1(0, T; V') \cap L^2(0, T; V) \\ &\text{and strongly in } C^0(0, T; V') \cap L^2\big(0, T; H^{1-\varepsilon}(\Omega)\big), \end{aligned} \tag{5.17}$$

$$\begin{aligned} \chi_n \to \chi \quad &\text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \\ &\text{and strongly in } C^0\big(0, T; H^{1-\varepsilon}(\Omega)\big), \end{aligned} \tag{5.18}$$

$$\vartheta_n \to \vartheta \quad \text{weakly star in } L^2(0, T; V'), \tag{5.19}$$

$$\nabla \zeta_n \to \nabla \zeta \quad \text{weakly in } L^2(0, T; H), \tag{5.20}$$

$$\xi_n \to \xi \quad \text{weakly star in } L^\infty\big(0, T; L^{6/(5-\eta)}(\Omega)\big), \tag{5.21}$$

for any $\varepsilon > 0$ and η introduced in (3.5). Now, let us show that $u = u_\infty$ and $\chi = \chi_\infty$. Indeed, using the fact that, in case X is a Banach space and $p < \infty$, from $|v|_{L^p(0,+\infty;X)} \leqslant c$ it follows $v_n(t) := v(t + t_n) \to 0$ in $L^p(0, +\infty; X)$, we can immediately deduce from (5.17) and (5.19) that

$$\partial_t u_n \to \partial_t u = 0 \quad \text{in } L^2(0, T; V'), \qquad \partial_t \chi_n \to \partial_t \chi = 0 \quad \text{in } L^2(0, T; H). \tag{5.22}$$

From (5.22) it follows immediately that both u and χ do not depend on time. Thus, for any t, we may infer that (recall the strong convergence of u_n in (5.17)),

$$u(t) = u(0) = \lim_{n \nearrow \infty} u_n(0) = \lim_{n \nearrow \infty} u(t_n) = u_\infty,$$

where the last equality holds true by the definition of the ω-limit. At this point, we can pass to the limit in (3.22)–(3.23) (written for the index n). Proceeding in a similar way for χ_n, using (5.18), we get:

$$\chi(t) = \chi(0) = \lim_{n \nearrow \infty} \chi_n(0) = \lim_{n \nearrow \infty} \chi(t_n) = \chi_\infty.$$

Moreover, due to (5.18) and (5.21), it is now a standard matter to infer that $\xi \in \beta(\chi)$. Finally, we may identify $\vartheta \in \mathcal{A}(u)$ in V'. In order to apply [1, Lemma 1.3, p. 42], we aim to prove that (cf. (5.17) and (5.19) and recall that $\vartheta_n \in \mathcal{A}(u_n)$ in V' for all n),

$$\limsup_{n \nearrow \infty} \int_0^t \langle \vartheta_n, u_n \rangle \leqslant \int_0^t \langle \vartheta, u \rangle. \tag{5.23}$$

We test (3.23) with index n by u_n and integrate over $(0,t)$ getting:

$$\limsup_{n\nearrow\infty}\int_0^t\langle\vartheta_n,u_n\rangle=\limsup_{n\nearrow\infty}\left[\int_0^t\int_\Omega\partial_t\chi_n u_n+\nu\int_0^t\int_\Omega\nabla\chi_n\nabla u_n+\int_0^t\int_\Omega\xi_n u_n+\int_0^t\int_\Omega\sigma'(\chi_n)u_n\right].\tag{5.24}$$

We first point out that, exploiting (5.17) and (5.18) (and the properties of σ'), we directly obtain:

$$\lim_{n\searrow\infty}\left(\int_0^t\int_\Omega\partial_t\chi_n u_n+\int_0^t\int_\Omega\sigma'(\chi_n)u_n\right)=\int_0^t\int_\Omega\partial_t\chi u+\int_0^t\int_\Omega\sigma'(\chi)u,\tag{5.25}$$

and due to the strong convergences in (5.17) and (5.21), we get:

$$\lim_{n\nearrow\infty}\int_0^t\int_\Omega\xi_n u_n=\int_0^t\int_\Omega\xi u.\tag{5.26}$$

It remains to treat the term $\nu\int_0^t\int_\Omega\nabla\chi_n\nabla u_n$ for which we need strong convergence for $\nabla\chi_n$, namely to prove that χ_n strongly converges to χ in V (cf. (5.17)). To this aim, we exploit an analogous Cauchy argument as in (4.31)–(4.32), leading to

$$\chi_n\to\chi\quad\text{strongly in } L^2(0,T;V).\tag{5.27}$$

Now we are in the position of proving (5.23). Indeed, using (5.25)–(5.27) in (5.24), we immediately deduce (5.23).

Finally, due to (5.15), we get:

$$\nabla\zeta_n\to 0\quad\text{in } L^2(0,T;H),$$

thus, passing to the limit in (3.22) (written for the index n), we obtain $Au_\infty=0$, and hence u_∞ turns out to be constant also in space. Now, let us test (3.22) with index n by 1 and integrate over $(0,t_n)$. We have:

$$\int_\Omega u(t_n)+\int_\Omega\chi(t_n)=\int_0^{t_n}\left(\int_\Omega R+\int_\Gamma g\right)+c_0,\tag{5.28}$$

where $c_0=\int_\Omega(u_0+\chi_0)$ (cf. 3.49). We let $n\nearrow\infty$ in (5.28) and get—due to our assumptions on the data and to (5.17)–(5.18)—

$$u_\infty=\frac{1}{|\Omega|}\left(-\int_\Omega\chi_\infty+c_0+m\right).\tag{5.29}$$

Hence, we can write down the limit equation for χ_∞ (holding true in V'),

$$A\chi_\infty+\beta(\chi_\infty)+\sigma'(\chi_\infty)-\mathcal{A}(u_\infty)\ni 0.\tag{5.30}$$

This concludes the proof of Theorem 3.17.

5.3. *Proof of Proposition 3.19*

In the general case, we cannot deduce that χ_∞ is constant in space, i.e. $A\chi_\infty=0$ a.e. in Ω. Hence, to prove uniqueness for the stationary limit problem requires some further conditions on the operators α and β. Let us now assume that (3.50)–(3.51) hold. Then, we test (5.30) by $A\chi_\infty$. Due to the fact that α is not multivalued, then $\alpha(u_\infty)$ is a constant both in time and space (cf. (3.46)) and thus we get:

$$\int_\Omega\alpha(u_\infty)A\chi_\infty=0.$$

Then, exploiting the monotonicity of β, we obtain:

$$\int_\Omega \beta(\chi_\infty) A\chi_\infty \left(= \int_\Omega \nabla\beta(\chi_\infty)\nabla\chi_\infty\right) \geqslant 0,$$

where we have put the first equality between parentheses because in order to write it down rigorously we should first approximate β by, e.g., its Lipschitz continuous Yosida regularization and then pass to the limit. Since the sign of the derivative of the approximation is again non-negative, we skip this passage, because the right inequality is preserved at the limit. Thus, it follows,

$$\int_\Omega |A\chi_\infty|^2 \leqslant 0,$$

i.e. $A\chi_\infty = 0$ a.e. in Ω, so that we have:

$$\chi_\infty = \text{const} \quad \text{a.e.} \tag{5.31}$$

Then, (3.47) (recall that $A\chi_\infty = 0$ a.e. in Ω), leads to

$$\beta(\chi_\infty) - \alpha\left(-\chi_\infty + \frac{1}{|\Omega|}(c_0 + m)\right) \ni -\vartheta_c. \tag{5.32}$$

Note that $\widetilde{\alpha}(x) := -\alpha(-x + c)$ is monotone. Thus—due to [1, Thm. 17]—$\beta + \widetilde{\alpha}$ is a maximal monotone graph and, consequently, there exists at least a solution to (5.32). Moreover, assumption (3.52) ensures that such a solution to (5.32) is also uniquely determined, which concludes the proof.

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