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CVA and vulnerable options pricing by correlation expansions

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Abstract We consider the problem of computing the Credit Value Adjustment (CVA) of a European option in presence of the Wrong Way Risk (WWR) in a default intensity setting. Namely we model the asset price evolution as solution to a linear equation that might depend on different stochastic factors and we provide an approximate evaluation of the option's price, by exploiting a correlation expansion approach, introduced in [3]. We also extend our theoretical analysis to include some further value adjustments, for instance due to collateralization and funding costs. Finally, in the CVA case, we compare the numerical performance of our method with the one recently proposed by Brigo et al. ([11], [13]), in the case of a call option driven by a GBM correlated with a CIR default intensity. We additionally compare with the numerical evaluations obtained by other methods.

Keywords Credit Value Adjustment · Vulnerable Options · Conterparty Credit Risk · Wrong Way Risk · XVA · Affine Processes · Duhamel Principle · Girsanov Theorem

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1 Introduction

Vulnerable options are financial contracts that are subject to some default event concerning the solvency of the option's seller. The classical reference on this topic is the paper by Johnson and Stulz ([33]), the first to price European options with Counterparty Credit Risk (CCR). Their work was developed within the structural approach to credit risk and it considered the option as the sole liability of the counterparty. Later Klein, in [35], discussed more general liability structures and the presence of correlation between the option's underlying and the option's seller's assets, while in [36] interest rate risk was included and in [37] a (stochastic) default barrier depending on the value of the option was considered. In all these works default could happen only at maturity.

In the meantime reduced-form models to price bonds or options that might default at any time prior to maturity, started to be proposed. We refer the reader to Hull and White ([29]) and Jarrow and Turnbull ([32]) for the case of vulnerable options and to [20] and the references therein for a more general framework. Some more recent papers on vulnerable options include [17], [19], [22], [31] and [48].

Even before the last financial crisis (2007-2008), the focus on CCR started to increase remarkably (see [16]) and attention shifted to building a general framework for the evaluation of a premium to compensate a derivative's holder (in particular of Interest Rate Swaps) for taking (counterparty credit) risk. This risk premium was then clearly defined in a paper by Zhu and Pykhtin ([49]), under the name of Credit Value Adjustment (CVA). In the post-crisis era CVA became a key quantity to be taken into account when trading derivatives in the OTC markets and this spurred a lot of research in the field: see [7], [12] and [28] just to mention some. In practice, CVA is an adjustment of the default-free value of a portfolio, to reduce this price in order to include the default risk. Along the years, other value adjustments have been introduced leading to the acronym (X)VA, where X stands for D= debt, L= liquidity, F=funding, to include risks due to default of either party and/or to funding investment strategies, to lack of liquidity etc. An updated overview of the recent research directions under investigation is presented in [27], including the existing relationship between XVA and the theory of BSDE's, partially discussed also in the present paper.

In this work we treat in detail the plain vanilla (unilateral) CVA, but we also show that our methodology may be extended to include some forms of XVA. A correct evaluation of CVA is crucial when Wrong Way Risk (WWR) might occur, that is when a decrease in the credit quality of the counterparty produces a higher exposure of the derivative's holder. Under independence between the exposure and the credit quality of the counterparty, computation of CVA simplifies, while it becomes computationally much more delicate if dependence is assumed. To overcame this difficulty, several methods were proposed: Monte Carlo methods, from brute force to enhanced ones (see [30] and [47]), the copula method or *static* approach (see [18], [44]), a linear programming characterization leading to bounds for WWR (see [26]). Here, we propose a new method and we compare it with another recently investigated in [11].

More in detail, we employ the stochastic intensity approach for the time of default, when the investor might face either a total loss or a partial recovery of the investment's current value. Within this context, the computational difficulty in the evaluation of the CVA is twofold. First, the default time might be not completely measurable with respect to the information generated by the market prices, since it may reflect other exogenous factors, secondly even under full knowledge of the default time, the derivative's evaluation calls for the joint distributions of the random time and the price processes, usually very difficult to know.

Conditionally to the information generated by the market prices, under appropriate conditions the joint dynamics of the asset prices, of the default time intensity and of possibly other stochastic factors can be described as a Markovian system, whose components may exhibit correlation, that may be modeled by means of a set of parameters linking the processes driving the dynamics. In this framework, the usual theory of stochastic calculus allows to set up a PDE system, whose solution might be approximated. Several methods of approximations of PDE's are at disposal, most of them being based on some clever numerical discretization scheme, see e.g. [34]. Here we suggest an alternative method, introduced in the papers [3] and [4], which approximates the solution of the PDE system by a Taylor's polynomial with respect to the correlation parameters. Indeed, under quite general hypotheses, it is straightforward to verify that the solution to the PDE is regular with respect to the correlation parameters and therefore it can be developed in series around the zero value for all of them.The coefficients of the series are characterized, by using Duhamel's principle, as solutions to a chain of PDE problems and they are therefore identified by means of Feynman-Kac formulas and expressed as expectations. There are several advantages in using this method:

- the series coefficients are computed at the zero values of the correlation parameters and by consequence they are expectations of functionals of independent processes so they are easier to compute or to approximate;
- in many cases the zero-th term can be explicit computed, increasing the precision of the approximation;
- comparing with Finite Differences methods or Monte Carlo methods, often a comparable accuracy is reached by the first order expansion;
- consequently the computational times are very little;
- differently from other methods, ours extends quite easily to multi-factor models, as shown is Section 5.

In the next section we introduce the general problem and setting, in the third section we define our market model, while in the fourth we prove the convergence of the series in the case of a Future contract and we refer the reader to [6] for more general contracts. A stochastic interest rate is considered in the fifth section where it is showed that the method can cope also with this case. In the sixth section we provide, under some assumptions, the extension of our approach to include DVA, FVA and LVA. It follows a short section recalling the main features and results of the method [11], based on a change of measure technique, to which we compare in the numerical analysis run in the last section.

2 CVA Evaluation of Vulnerable Options in an Intensity Model

We consider a finite time interval $[0, T]$ and a complete probability space $(\Omega, \mathcal{F}, P)$, endowed with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, augmented with the $P-$null sets and made right continuous. We assume that all processes have a cádlág version.

The market is described by the interest rate process r_t determining the money market account denoted by $B(t, s) = \mathrm{e}^{\int_t^s r_u du}$ and by a process X_t representing an asset log-price (whose dynamics will be specified later), this process may depend also on additional stochastic factors. We assume

- that the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is rich enough to support all the aforementioned processes;
- to be in absence of arbitrage;
- that the given probability P is a risk neutral measure, already selected by some criterion.

In this market a defaultable European contingent claim paying $f(X_T)$ at maturity is traded, where f is a function whose regularity properties will be specified later. We denote by τ (not necessarily a stopping time w.r.t. the filtration $\mathcal{F}_t$) the default time of the contingent claim and by Z_t an $\mathcal{F}_t-$measurable bounded recovery process.

To properly evaluate this type of derivative we need to include the information generated by the default time. We denote by $\mathcal{G}_t$ the progressively enlarged filtration, that makes τ a $\mathcal{G}_t-$stopping time, that is $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau \le t\})$. From now on, we indicate by $H_t = \mathbf{1}_{\{\tau\le t\}}$, the process generating the filtration $\mathcal{H}_t$, so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$.

We make the fundamental assumption, known as the H-hypothesis (see e.g. [25] and [24] and the references therein), that

(H) Every $\mathcal{F}_t-$martingale remains a $\mathcal{G}_t-$martingale.

Under this assumption, we may affirm that $\mathrm{e}^{X_s}/B(t, s)$ for $s \ge t$ remains a $\mathcal{G}_s-$martingale under the unique extension of the risk neutral probability to the filtration $\mathcal{G}_s$. (To keep notation light, we do not indicate explicitly the probability we use for the expectations, assuming that we are always working with the one corresponding to the filtration in use).

In this setting, for any given time $t \in [0, T]$, the price of a defaultable claim, with positive final value $f(X_T)$, default time τ and recovery process Z_t, is given by

$$c^d(t, T) = \mathbf{E}[B^{-1}(t, T)f(X_T)1_{\{\tau>T\}} + B^{-1}(t, \tau)Z_\tau 1_{\{t<\tau\le T\}}|\mathcal{G}_t], \qquad (1)$$

while the corresponding default free value is

$$c(t,T) = \mathbf{E}[B^{-1}(t,T)f(X_T)|\mathcal{F}_t]. \tag{2}$$

Correspondingly the CVA, as a function of the running time and of the maturity, is given by

$$CVA(t,T) = 1_{\{\tau>t\}}[c(t,T) - c^d(t,T)]. \tag{3}$$

In many situations, investors do not know the default time and they may observe only whether it happened or not. The actual observable quantity is the asset price, therefore it is interesting to write the pricing formula (1) in terms of $\mathcal{F}_t$, rather than in terms of $\mathcal{G}_t$. For that we have the following Key Lemma, see [9] or [7].

Lemma 1 *For any integrable* $\mathcal{G}-$*measurable r.v.* Y*, the following equality holds*

$$\mathbf{E}\Big[\mathbf{1}_{\{\tau>t\}}Y|\mathcal{G}_t\Big] = P(\tau>t|\mathcal{G}_t)\frac{\mathbf{E}\Big[\mathbf{1}_{\{\tau>t\}}Y|\mathcal{F}_t\Big]}{P(\tau>t|\mathcal{F}_t)}. \tag{4}$$

Applying this lemma to the first and the second term of (1) and recalling that $1-H_t = \mathbf{1}_{\{\tau>t\}}$ is $\mathcal{G}_t-$measurable, we obtain

$$\mathbf{E}[B^{-1}(t,T)f(X_T)1_{\{\tau>T\}}|\mathcal{G}_t] = \mathbf{1}_{\{\tau>t\}}\frac{\mathbf{E}[B^{-1}(t,T)f(X_T)1_{\{\tau>T\}}|\mathcal{F}_t]}{P(\tau>t|\mathcal{F}_t)} \tag{5}$$

$$\mathbf{E}[B^{-1}(t,\tau)Z_\tau 1_{\{t<\tau\le T\}}|\mathcal{G}_t] = \mathbf{1}_{\{\tau>t\}}\frac{\mathbf{E}[B^{-1}(t,\tau)Z_\tau 1_{\{t<\tau\le T\}}|\mathcal{F}_t]}{P(\tau>t|\mathcal{F}_t)}, \tag{6}$$

which may be made more explicit by following the hazard process approach.

We denote the conditional distribution of the default time τ given $\mathcal{F}_t$ by

$$F_t = P(\tau \le t|\mathcal{F}_t), \qquad \forall\, t \ge 0, \tag{7}$$

whence, for $u \ge t$, $P(\tau \le u|\mathcal{F}_t) = \mathbf{E}(P(\tau \le u|\mathcal{F}_u)|\mathcal{F}_t) = \mathbf{E}(F_u|\mathcal{F}_t)$. If $F_t(\omega) < 1$ for all $t > 0$ (which automatically excludes that $\mathcal{G}_t \equiv \mathcal{F}_t$), we can well define the so called $\mathcal{F}$- hazard process of τ as

$$\varGamma_t := -\ln(1-F_t) \quad \Rightarrow \quad F_t = 1 - \mathrm{e}^{-\varGamma_t} \quad \forall\, t > 0, \qquad \varGamma_0 = 0, \tag{8}$$

moreover

$$S_t := 1 - F_t = \mathrm{e}^{-\varGamma_t} \quad \forall\, t > 0, \qquad S_0 = 1, \tag{9}$$

is the $\mathcal{F}$-survival process. We assume $\varGamma_t$ to be differentiable; its derivative, known as the intensity process and denoted by λ_t, is such that $\varGamma_t = \int_0^t \lambda_u du$.

Exploiting (5) and (6) to pass to the $\mathcal{F}_t$ filtration and assuming that $B(t,\cdot)^{-1}Z.$ is a bounded $\mathcal{F}-$ martingale (which is usually the case), by an extension of Proposition 5.1.1 of [8], as developed in [5], we may rewrite the pricing formula (1) as

$$\begin{aligned} c^d(t,T) &= 1_{\{\tau>t\}}\mathbf{E}[\mathrm{e}^{-\int_t^T (r_s+\lambda_s)ds} f(X_T)|\mathcal{F}_t] \\ &+ 1_{\{\tau>t\}}\mathbf{E}[\int_t^T Z_s\lambda_s \mathrm{e}^{-\int_t^s (r_u+\lambda_u)du} ds|\mathcal{F}_t], \end{aligned} \tag{10}$$

recovering formulas (3.1) and (3.3) in [40], that the author obtained by modeling directly the random time τ.

Remark 1 Indeed, we point out that if Z is an optional and uniformly integrable process, then it needs to be a martingale (Proposition 3.6 of [43]). Thus a bounded predictable process is a predictable martingale, which is the case considered in Proposition 5.1.1 of [8].

This formula can be specialized even further if we assume fractional recovery of the type $Z_t = Rc(t,T)$ for some $0 \leq R < 1$. Using the Optional Projection Theorem, see e.g. Theorem 4.16 in [43] , one gets to

$$\begin{aligned} c^d(t,T) = 1_{\{\tau>t\}}\Big[&R\mathbf{E}[\mathrm{e}^{-\int_t^T r_u du} f(X_T)|\mathcal{F}_t] \\ &+ (1-R)\mathbf{E}[\mathrm{e}^{-\int_t^T (r_u+\lambda_u)du} f(X_T)|\mathcal{F}_t]\Big], \end{aligned} \tag{11}$$

which can be interpreted as a convex combination of the default free price and the price with default.

An alternative choice for the recovery process could be made (i.e. fractional recovery of the market value as in Section 5.6 of [41] or in Section 4 of [15]), assuming that the market quotes, at default, an evaluation of the defaultable product based on the past information (i.e. $\mathcal{F}_t$). As we will mention later, this choice leads to setting a solvable Backward Stochastic Differential Equation (BSDE) that gives

$$c^d(t,T) = 1_{\{\tau>t\}}\mathbf{E}\Big[\mathrm{e}^{-\int_t^T (r_u+(1-R)\lambda_u)du} f(X_T)|\mathcal{F}_t\Big]. \tag{12}$$

It is evident that this formula is very similar to the second piece of (11), hence the method we are going to present is applicable to both choices.

As a consequence, from (3) we have an expression also for the unilateral CVA as

$$CVA(t,T) = 1_{\{\tau>t\}}(1-R)\mathbf{E}[\mathrm{e}^{-\int_t^T r_u du} f(X_T)(1-\mathrm{e}^{-\int_t^T \lambda_u du})|\mathcal{F}_t]. \tag{13}$$

Remark 2 Last formula, by means of the survival process, could be briefly rewritten as

$$CVA(t,T) = -1_{\{\tau>t\}}\frac{(1-R)}{S_t}\mathbf{E}[\int_t^T \frac{f(X_T)}{B(t,T)} dS_u|\mathcal{F}_t]. \tag{14}$$

If $G(t) = P(\tau > t) = \mathbf{E}[1_{\{\tau>t\}}]$ is the (deterministic) survival function, assuming it can be written as $G(t) = e^{-\int_0^t h_s ds}$, for some non-negative function h, then we have that $\mathbf{E}(S_t) = G(t)$ for all $t \geq 0$ (see (8) and (9)) and

$$dS_t = -\lambda_t S_t dt = \frac{\lambda_t S_t}{h_t G(t)} dG(t) = \zeta_t dG(t)$$

where we set $\zeta_t := \dfrac{\lambda_t S_t}{h_t G(t)}$. Consequently, using the optional projection theorem, the expectation in (14) may be rewritten as

$$\begin{aligned}
&\mathbf{E}[\int_t^T \frac{f(X_T)}{B(t,T)} dS_u|\mathcal{F}_t] = \mathbf{E}\Big[\int_t^T \frac{f(X_T)}{B(t,T)} \zeta_u dG(u)|\mathcal{F}_t\Big] \\
&= \mathbf{E}\Big[\int_t^T \mathbf{E}[\frac{f(X_T)}{B(t,T)}\zeta_u|\mathcal{F}_u] dG(u)|\mathcal{F}_t\Big] = \mathbf{E}\Big[\int_t^T \mathbf{E}[\frac{f(X_T)}{B(t,T)}|\mathcal{F}_u]\zeta_u] dG(u)|\mathcal{F}_t\Big] \\
&= \mathbf{E}\Big[\int_t^T \frac{c(u,T)\zeta_u}{B(t,u)} dG(u)|\mathcal{F}_t\Big] = \int_t^T \mathbf{E}[\frac{c(u,T)\zeta_u}{B(t,u)}|\mathcal{F}_t] dG(u)
\end{aligned}$$

and

$$CVA(t,T) = -1_{\{\tau>t\}} \frac{(1-R)}{S_t} \int_t^T \mathbf{E}[\frac{c(u,T)\zeta_u}{B(t,u)}|\mathcal{F}_t] dG(u). \tag{15}$$

For $t = 0$ and a generic portfolio price process V_t (the positive part V_t^+ coinciding in our case with $c(t,T)$, the default free price of the claim) this formula is the starting point of the analysis developed in [11].

Finally we remark that under independence between λ_t and (X_t, r_t), the second term in (11) simplifies further to

$$\mathbf{E}[e^{-\int_t^T (r_s+\lambda_s) ds} f(X_T)|\mathcal{F}_t] = \mathbf{E}[e^{-\int_t^T r_s ds} f(X_T)|\mathcal{F}_t]\mathbf{E}[e^{-\int_t^T \lambda_s ds}|\mathcal{F}_t]. \tag{16}$$

Correspondingly, we get a similar factorization for the CVA

$$\begin{aligned}
CVA(t,T) &= 1_{\{\tau>t\}}(1-R)\mathbf{E}[e^{-\int_t^T r_u du} f(X_T)|\mathcal{F}_t]\mathbf{E}[(1-e^{-\int_t^T \lambda_u du})|\mathcal{F}_t] \\
&= 1_{\{\tau>t\}}(1-R)c(t,T)\frac{P(t<\tau\leq T|\mathcal{F}_t)}{P(\tau\geq t|\mathcal{F}_t)},
\end{aligned} \tag{17}$$

where the last equality follows from the Key Lemma and the definition of hazard process (see e.g. [8], Sect. 8.2). In this case, the two factors are respectively the price of a European derivative and the price of a bond. Thus we may arrive at explicit formulas whenever the models for X and λ are appropriately chosen.

3 The model

We assume that in the given probability space, the following diffusion dynamics are satisfied

$$X_s = x + \int_t^s (r_u - \frac{\sigma^2}{2})du + \sigma(B_s - B_t), \qquad x \in \mathbb{R} \tag{18}$$

$$\lambda_s = \lambda + \int_t^s \gamma(\theta - \lambda_u)du + \eta \int_t^s \sqrt{\lambda_u} dY_u, \quad \lambda > 0 \tag{19}$$

$$r_s = r + \int_t^s k(\mu - r_u)du + \nu(W_s - W_t), \qquad r > 0, \tag{20}$$

where the parameters are such that $k, \theta, \eta, \sigma, \mu > 0$, $\gamma, \nu \geq 0$, $2\gamma\theta > \eta^2$ and B, Y, W are correlated Brownian motions with a given correlation matrix. To simplify calculations, in what follows we assume independence between the interest rate and default intensity, i.e. between Y and W; with this choice we may represent the triple B, Y, W as

$$B_t = \rho_1 B_t^1 + \rho_2 B_t^2 + \sqrt{1 - \rho_1^2 - \rho_2^2} B_t^3, \quad Y_t = B_t^1, \quad W_t = B_t^2;$$

where (B^1, B^2, B^3) is a 3-dimensional Brownian motion and $\rho_1^2 + \rho_2^2 \leq 1$.

We remark that under independence we have an explicit expression of the factor $E[\mathrm{e}^{-\int_t^T \lambda_s ds} | \mathcal{F}_t]$ appearing in (16), being the bond price with a CIR interest rate. The problem is then reduced to computing the other factor representing the price of the European derivative.

4 Correlation expansion

For the sake of simplicity, in this section we assume $R = 0$ and the short rate to be constant, $r_t \equiv r$ for all $t \in [0, T]$. To consider r a function in time is a straightforward generalization, while a stochastic interest rate will be considered specifically in the next section.

The model, which we write in flow notation, is hence reduced to

$$\begin{cases} X_s^{t,x,\lambda,\rho} = x + (r - \frac{\sigma^2}{2})(s - t) + \sigma \left[\rho(B_s^1 - B_t^1) + \sqrt{1 - \rho^2}(B_s^2 - B_t^2) \right] \\ \lambda_s^{t,\lambda} = \lambda + \int_t^s \gamma(\theta - \lambda_u^{t,\lambda})du + \int_t^s \eta \sqrt{\lambda_u^{t,\lambda}} dB_u^1. \end{cases} \tag{21}$$

The two-dimensional diffusion $\mathbf{U}_t^{t,x,\lambda,\rho} := (X_s^{t,x,\lambda,\rho}, \lambda_s^{t,\lambda})$ is a Markov process since the coefficients,

$$\mu(x, \lambda) := \begin{pmatrix} r - \frac{\sigma^2}{2} \\ \gamma(\theta - \lambda) \end{pmatrix} \quad \text{and} \quad \Sigma(x, \lambda) := \begin{pmatrix} \sigma\rho & \sigma\sqrt{1 - \rho^2} \\ \eta\sqrt{\lambda} & 0 \end{pmatrix}$$

are deterministic. This implies that the price $c^d(t,T)$ of any European defaultable derivative with integrable payoff $F(X_T^{t,x,\lambda,\rho})$ will be a deterministic function $u(\cdot)$ of all the initial data, that is

$$u(x,\lambda,t,T;\rho) = \mathrm{e}^{-r(T-t)}\mathbf{E}(\mathrm{e}^{-\int_t^T \lambda_s^{t,\lambda} ds} F(X_T^{t,x,\lambda,\rho})). \tag{22}$$

We remind that this computation is a crucial step towards the evaluation of the defaultable derivative (11) and of the corresponding CVA. We notice that, when $\rho = 0$, $\mu(x,\lambda)$ and

$$\Sigma(x,\lambda)\Sigma(x,\lambda)' = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \eta^2\lambda \end{pmatrix}$$

have components which are affine functions of (x,λ), therefore the vector process $\mathbf{U}_t^{t,x,\lambda,0}$ is an affine process (Thm. 2.2,[23]). On the contrary, when $\rho \neq 0$, the process $\mathbf{U}_t^{t,x,\lambda,\rho}$ is not so and the power of Fourier transform techniques, often exploited in the affine case, cannot be employed and one has to resort to an alternative method to evaluate (22). Here we adapt a technique introduced in [3] and [4], to approximate $u(x,\lambda,t,T;\rho)$ by means of a Taylor polynomial in ρ around 0. The following theorem shows the effectiveness of this approach.

Theorem 1 *Given the model* (21), *if* $F(x) = \mathrm{e}^x - K$ *(representing the payoff of a forward contract) then (22) can be approximated by the Taylor polynomial of order* n *around* $\rho = 0$,

$$u_n(x,\lambda,t,T;\rho) = \sum_{k=0}^{n} \frac{\partial^k u}{\partial \rho^k}\Big|_{\rho=0} \frac{\rho^k}{k!}, \tag{23}$$

with a remainder $R_n = u(x,\lambda,t,T;\rho) - u_n(x,\lambda,t,T;\rho)$ *bounded by*

$$|R_n| \leq M C^{n+1}, \tag{24}$$

where M, C *are positive constants (not depending on* n*) such that* $C < 1$ *if* $|\rho| < (2\sigma^2 T)^{-1/2}$.

Remark 3 This result can be extended to European call options, i.e. $F(x) = (\mathrm{e}^x - K)^+$, but the proof becomes more complicated, as it relies heavily on the careful use of the Faa di Bruno formula to get the estimates on the derivatives of u. A bound similar to (24) is achieved and a strictly positive convergence radius still exists. We refer to [6] for the complete and long proof, the present paper being focused on the applications of these results to CVA.

Proof . Without loss of generality, we set $t = 0$ and $r = 0$ and we omit the dependence on $t = 0$ in the formulas to come. Recalling (9) and (21), we have $S_T = \mathrm{e}^{-\int_0^T \lambda_s ds}$, $X_T(\rho) = x - \frac{\sigma^2}{2}T + \sigma\rho B_T^1 + \sigma(1-\rho^2)^{1/2} B_T^2$ and

$\mathcal{F}_T^1 = \sigma(\{B_s^1 : 0 \le s \le T\})$, so conditioning internally with respect to $\mathcal{F}_T^1$, equation (22) becomes

$$\begin{aligned} u(x,\lambda,T;\rho) &= \mathbf{E}(S_T(\mathrm{e}^{X_T(\rho)} - K)) = \mathbf{E}\Big(\mathbf{E}(S_T(\mathrm{e}^{X_T(\rho)} - K)|\mathcal{F}_T^1)\Big) \\ =&\mathbf{E}\Big(S_T \mathrm{e}^{x+\sigma B_T^1 \rho - \frac{\sigma^2 T}{2}\rho^2} E(\mathrm{e}^{\sigma\sqrt{1-\rho^2}B_T^2 - \frac{(1-\rho^2)\sigma^2 T}{2}}|\mathcal{F}_T^1)\Big) - K\mathbf{E}(S_T) \qquad (25) \\ =&\mathrm{e}^{-\frac{\sigma^2 T}{2}\rho^2}\mathbf{E}\Big(S_T \mathrm{e}^{x+\sigma B_T^1\rho}\Big) - K\mathbf{E}(S_T), \end{aligned}$$

since the inner factor is an exponential martingale independent of $\mathcal{F}_T^1$. When evaluating the remainder of order n only the first term counts, as the second does not depend on ρ. Denoting by $D = \frac{d}{d\rho}$, we have

$$\begin{aligned} D^n u(x,\lambda,T;\rho) &= \sum_{k=0}^n \binom{n}{k} D^k(\mathrm{e}^{-\frac{\sigma^2 T}{2}\rho^2}) D^{n-k}\Big(\mathbf{E}(S_T \mathrm{e}^{x+\sigma B_T^1\rho})\Big) \\ &= \sum_{k=0}^n \binom{n}{k} D^k(\mathrm{e}^{-\frac{\sigma^2 T}{2}\rho^2})\mathbf{E}\Big(S_T D^{n-k}(\mathrm{e}^{x+\sigma B_T^1\rho})\Big) \end{aligned}$$

and remarking that $|S_T| \le 1$,we may bound each term in the following way

$$\begin{aligned} |D^k(\mathrm{e}^{-\frac{\sigma^2 T}{2}\rho^2})| &= (\mathrm{e}^{-\frac{\sigma^2 T}{2}\rho^2})|He_k(\sigma\sqrt{T}\rho)| \quad \le 0.816\sqrt[4]{\pi}\sqrt{k!} \\ |D^{n-k}(\mathrm{e}^{x+\sigma B_T^1\rho})| &= |\sigma B_T^1|^{n-k}\mathrm{e}^{x+\sigma B_T^1\rho}, \end{aligned}$$

where we used the uniform bound on Hermite polynomials $He_k(z)$ that arise from the iterated derivatives of the function $\mathrm{e}^{-\frac{z^2}{2}}$ (see [1]). Plugging it back into the Lebnitz formula we get to

$$\begin{aligned} |D^n u(x,\lambda,T;\rho)| &\le 0.816\sqrt[4]{\pi}\mathrm{e}^x \sum_{k=0}^n \frac{n!}{k!(n-k)!}\sqrt{k!}\mathbf{E}(\mathrm{e}^{\sigma B_T^1\rho}|\sigma B_T^1|^{n-k}) \\ =&1.0864\mathrm{e}^x \sum_{k=0}^n \frac{n!}{\sqrt{k!}(n-k)!}\sigma^{n-k}\mathbf{E}(\mathrm{e}^{\sigma B_T^1\rho}|B_T^1|^{n-k}) \\ \le&1.0864\mathrm{e}^x \sum_{k=0}^n \frac{n!(\sqrt{2})^k\sigma^{n-k}}{\sqrt{k!}(n-k)!}\mathbf{E}(\mathrm{e}^{2\sigma B_T^1\rho})^{\frac{1}{2}}\mathbf{E}(|B_T^1|^{2(n-k)})^{\frac{1}{2}} \\ =&1.0864\mathrm{e}^{x+\sigma^2 T\rho^2} \sum_{k=0}^n \frac{n!\sqrt{(2(n-k)-1)!!}}{\sqrt{k!}(n-k)!}(\sigma\sqrt{T})^{n-k} \\ \le&1.0864\mathrm{e}^{x+\sigma^2 T\rho^2} \sum_{k=0}^n \frac{n!\sqrt{(2(n-k))!!}}{\sqrt{k!}(n-k)!}((\sigma\sqrt{T})^{n-k} \\ \le&1.0864\mathrm{e}^{x+\sigma^2 T\rho^2} \sum_{k=0}^n \frac{n!}{\sqrt{k!(n-k)!}}(\sqrt{2})^{n-k}(\sigma\sqrt{T})^{n-k} \\ \le&1.0864\mathrm{e}^{x+\sigma^2 T\rho^2} n!(\sqrt{\sigma^2 T})^n \sum_{k=0}^n (\sqrt{2})^k \le 1.0864\mathrm{e}^{x+\sigma^2 T\rho^2} n!(\sqrt{\sigma^2 T})^n(\sqrt{2})^{n+1}. \end{aligned}$$

Setting $M = 2 \cdot 1.0864 \mathrm{e}^{x+\sigma^2 T}$, we may conclude that the remainder R_{n-1} in Lagrange form is uniformly bounded by

$$\frac{1}{n!}|D^n u(x,\lambda,T;\rho)||\rho|^n \leq M(\sqrt{2\sigma^2 T})^n |\rho|^n \longrightarrow 0 \qquad \text{as } n \to +\infty$$

as long as $|\rho| < \dfrac{1}{\sqrt{2\sigma^2 T}}$. □

From now on we denote $g_k(x,\lambda,t,T) := \dfrac{\partial^k u}{\partial \rho^k}\Big|_{\rho=0}$. The series expansion gives a tool to approximate $u(x,\lambda,t,T;\rho)$, by stopping it at any chosen order. The coefficient $g_0(x,\lambda,t,T)$ equals $u(x,\lambda,t,T;0)$ and it can be computed in closed form. As we mentioned before, this corresponds to the independent case when the vector process $\mathbf{U}$ is affine. All the other coefficients $g_k(x,\lambda,t,T)$ can be iteratively computed by exploiting Duhamel's principle.

By the Feymann-Kac formulas, $u(x,\lambda,t,T;\rho)$ solves the parabolic PDE

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}^\rho u = 0 \\ u(x,\lambda,T,T;\rho) = (\mathrm{e}^x - K)^+, \end{cases} \tag{26}$$

where we denoted $\mathcal{L}^\rho = \mathcal{L}^{\mathbf{0}} + \rho \mathcal{A}$, with

$$\mathcal{L}^{\mathbf{0}} := \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \frac{\eta^2 \lambda}{2}\frac{\partial^2}{\partial \lambda^2} + (r - \frac{\sigma^2}{2})\frac{\partial}{\partial x} + \gamma(\theta - \lambda)\frac{\partial}{\partial \lambda} - r - \lambda \tag{27}$$

$$\mathcal{A} := \eta\sigma\sqrt{\lambda}\frac{\partial^2}{\partial x \partial \lambda}. \tag{28}$$

By differentiating and taking $\rho = 0$, it is readily seen that the coefficients $g_k(x,\lambda,t,T)$ must satisfy the following parabolic equations

$$\begin{cases} \frac{\partial g_0}{\partial t} + \mathcal{L}^{\mathbf{0}} g_0 = 0 \\ g_0(x,\lambda,T,T) = (\mathrm{e}^x - K)^+, \end{cases} \qquad \begin{cases} \frac{\partial g_k}{\partial t} + \mathcal{L}^{\mathbf{0}} g_k = -\mathcal{A} g_{k-1} \\ g_k(x,\lambda,T,T) = 0. \end{cases} \quad k \geq 1. \tag{29}$$

Once again, by the Markov property and Feymann-Kac formulas, $g_0(x,\lambda,t,T)$ admits the following representation

$$\begin{aligned} g_0(x,\lambda,t,T) &= \mathrm{e}^{-r(T-t)} \mathbf{E}(\mathrm{e}^{-\int_t^T \lambda_s^{t,\lambda} ds}(\mathrm{e}^{X_T^{t,x,\lambda}} - K)^+) \\ &= \mathbf{E}(\mathrm{e}^{-\int_t^T \lambda_s^{t,\lambda} ds})\mathrm{e}^{-r(T-t)}\mathbf{E}((\mathrm{e}^{X_T^{t,x}} - K)^+), \end{aligned} \tag{30}$$

where in the last passage we used the independence of the processes (X_t) and (λ_t) $(\rho = 0)$. The first factor is the bond price with a CIR process and presents an exponentially affine solution, while the second is the usual Black & Scholes price of a European call option, $c_{BS}(x,t,T)$, hence we have

$$\begin{aligned} &g_0(x,\lambda,t,T) = \mathrm{e}^{-B_1(T-t) - B_2(T-t)\lambda} c_{BS}(x,t,T) \\ =&\mathrm{e}^{-B_1(T-t) - B_2(T-t)\lambda}\left[\mathrm{e}^x N(d_1(x,T-t) - K\mathrm{e}^{-r(T-t)} N(d_2(x,T-t))\right], \end{aligned} \tag{31}$$

where $d_{1,2}(x,T-t) = \frac{x-\ln K+(r\pm\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$, $N(x)$ is the standard normal cumulative distribution function and

$$B_1(T-t) = \frac{2\gamma\theta}{\eta^2}\ln\left(\frac{2\beta e^{\frac{\gamma+\beta}{2}(T-t)}}{\beta-\gamma+(\gamma+\beta)e^{\beta(T-t)}}\right) \tag{32}$$

$$B_2(T-t) = \frac{2(e^{\beta(T-t)}-1)}{\beta-\gamma+(\gamma+\beta)e^{\beta(T-t)}}, \tag{33}$$

with $\beta = \sqrt{\gamma^2+\eta^2}$. The other equations of (29) can be solved by Duhamel's principle which states that

$$g_k(x,\lambda,t,T) = -\int_t^T g_k^\xi(x,\lambda,t)d\xi,$$

where $g_k^\xi(x,\lambda,t)$ is the solution to the PDE problem for any fixed $\alpha \in (t,T]$

$$\begin{cases} \dfrac{\partial g_k^\xi}{\partial t} + \mathcal{L}^0 g_k^\xi = 0, \qquad t<\xi \\ g_k^\xi(x,\lambda,\xi) = -\mathcal{A}g_{k-1}(x,\lambda,\xi,T). \end{cases} \tag{34}$$

This sets up an iterative procedure to compute theoretically the coefficients of any order, by means of a repeated application of Feymann-Kac formulas. Indeed for all $k \geq 1$ we have:

$$\begin{aligned} g_k(x,\lambda,t,T) &= -\int_t^T g_k^{\xi_k}(x,\lambda,t)d\xi_k \\ &= \int_t^T \mathbf{E}\Big(e^{-r(\xi_k-t)}e^{-\int_t^{\xi_k}\lambda_s^{t,\lambda}ds}\mathcal{A}g_{k-1}(X_{\xi_k}^{t,x},\lambda_{\xi_k}^{t,\lambda},\xi_k,T)\Big)d\xi_k \end{aligned}$$

and we can iterate the procedure arriving to a formula involving k integrals but depending only on g_0.

Inevitably, coefficients of higher order are harder to compute. In the hope to obtain good numerical results, we consider the first order approximation

$$\bar{u}(x,\lambda,t,T;\rho) := u(x,\lambda,t,T;0) + (\frac{\partial u}{\partial\rho}|_{\rho=0})\rho \equiv g_0(x,\lambda,t,T) + g_1(x,\lambda,t,T)\rho.$$

Using (31), we may explicitly compute

$$\begin{aligned} \mathcal{A}g_0(x,\lambda,t,T) &= \eta\sigma\sqrt{\lambda}\frac{\partial^2}{\partial x\partial\lambda}g_0(x,\lambda,t,T) \\ &= -\eta\sigma\sqrt{\lambda}B_2(T-t)e^{-B_1(T-t)-B_2(T-t)\lambda}\frac{\partial}{\partial x}c_{BS}(x,t,T) \\ &= -\eta\sigma\sqrt{\lambda}B_2(T-t)e^{-B_1(T-t)-B_2(T-t)\lambda}e^x N(d_1(x,T-t)), \end{aligned}$$

therefore

$$
\begin{aligned}
& g_1(x,\lambda,t,T) \\
&= -\int_t^T g_1^\xi(x,\lambda,t)d\xi = \int_t^T \mathbf{E}\Big(\mathrm{e}^{-r(\xi-t)}\mathrm{e}^{-\int_t^\xi \lambda_s^{t,\lambda}ds}\mathcal{A}g_0(X_\xi^{t,x},\lambda_\xi^{t,\lambda}\xi,T)\Big)d\xi \\
&= -\eta\sigma\int_t^T \mathbf{E}\Big[\sqrt{\lambda_\xi^{t,\lambda}}B_2(T-\xi)\mathrm{e}^{-B_1(T-\xi)-B_2(T-\xi)\lambda_\xi^{t,\lambda}}\mathrm{e}^{X_\xi^{t,x}}N(d_1(X_\xi^{t,x},T-\xi))\Big]d\xi \\
&= -\Gamma(t,\xi,T)\mathbf{E}\Big[\sqrt{\lambda_\xi^{t,\lambda}}\mathrm{e}^{-B_2(T-\zeta)\lambda_\xi^{t,\lambda}-\int_t^\xi \lambda_s^{t,\lambda}ds}\Big]\mathbf{E}\Big[\mathrm{e}^{X_\xi^{t,x}}N(d_1(X_\xi^{t,x},T-\xi))\Big],
\end{aligned}
\tag{35}
$$

where $\Gamma(t,\xi,T) \equiv \eta\sigma\mathrm{e}^{-r(\xi-t)}B_2(T-\xi)\mathrm{e}^{-B_1(T-\xi)} > 0$ and we used that the expectation in the integral is evaluated under independence of the processes X and λ. Hence we obtain the following approximation result for the CVA.

Proposition 1 *The price of the defaultable European call increases with ρ in a small interval around $\rho = 0$, moreover it holds*

$$
CVA(0,T) = c(0,T)P(\tau \le T) - g_1(x,\lambda,0,T)\rho + O(\rho^2) \tag{36}
$$

Proof The first statement follows from (35) which implies that $g_1(x,\lambda,t) < 0$. From (3) and (30) (and the remark following Theorem 1) for $t = 0$ we have

$$
\begin{aligned}
CVA(0,T) &= c(0,T) - c^d(0,T) \\
&= c(0,T) - g_0(x,\lambda,0,T) - g_1(x,\lambda,0,T)\rho + O(\rho^2) \\
&= c(0,T) - c(0,T)P(\tau > T) - g_1(x,\lambda,0,T)\rho + O(\rho^2) \\
&= c(0,T)P(\tau \le T) - g_1(x,\lambda,0,T)\rho + O(\rho^2). \qquad \square
\end{aligned}
$$

The first term on the right-hand side represents the CVA under independence between the default event and the exposure (see (17)). Hence $g_1(x,\lambda,0,T)$ measures the impact of the factor correlation on CVA.

We now focus on the first expectation in the last line of (35). We set $b_\xi := B_2(T-\xi)$ for shorthand and conditioning with respect to $\lambda_\xi^{t,\lambda}$, we obtain

$$
\begin{aligned}
\mathbf{E}\Big[\sqrt{\lambda_\xi^{t,\lambda}}\mathrm{e}^{-b_\xi\lambda_\xi^{t,\lambda}-\int_t^\xi \lambda_s^{t,\lambda}ds}\Big] &= \int_0^{+\infty}\mathbf{E}\Big[\sqrt{\lambda_\xi^{t,\lambda}}\mathrm{e}^{-b_\xi\lambda_\xi^{t,\lambda}-\int_t^\xi \lambda_s^{t,\lambda}ds}|\lambda_\xi^{t,\lambda} = \zeta\Big]f_{\lambda_\xi^{t,\lambda}}(\zeta)d\zeta \\
&= \int_0^{+\infty}\sqrt{\zeta}\mathrm{e}^{-b_\alpha\zeta}E\Big[\mathrm{e}^{-\int_t^\xi \lambda_s^{t,\lambda}ds}|\lambda_\xi^{t,\lambda} = \zeta\Big]f_{\lambda_\xi^{t,\lambda}}(\zeta)d\zeta.
\end{aligned}
$$

The density $f_{\lambda_\xi^{t,\lambda}}$ is explicitly known (see for instance [2]). Moreover in [42] or in [45] an explicit expression of the conditional moment generating function of $\int_t^\alpha \lambda_s^{t,\lambda}ds$ is provided

$$
E\Big[\mathrm{e}^{-\int_t^\xi \lambda_s^{t,\lambda}ds}|\lambda_\xi^{t,\lambda} = \zeta\Big] = \frac{M_{t,\xi}(\lambda,\zeta)}{f_{\lambda_\xi^{t,\lambda}}(\zeta)}I_\nu\Big(\frac{2\bar{\gamma}\sqrt{\zeta\lambda}}{\sigma^2\sinh\big(\frac{\bar{\gamma}(\xi-t)}{2}\big)}\Big),
$$

where $\nu = \frac{2\gamma\theta}{\sigma^2} - 1$, $\bar{\gamma} = \sqrt{\gamma^2 + 2\sigma^2}$,

$$I_\nu(z) \equiv (\frac{z}{2})^\nu \sum_{n=0}^{\infty} \frac{(\frac{z^2}{4})^n}{n!\Gamma(\nu + k + 1)}$$

is the modified Bessel function of the first kind and

$$M_{t,\xi}(\lambda,\zeta) = \frac{2\bar{\gamma}}{\sigma^2}(\frac{\zeta}{\lambda})^{\frac{\nu}{2}} \frac{\mathrm{e}^{-\frac{\bar{\gamma}(\xi-t)}{2} - \frac{1}{\sigma^2}[\bar{\gamma}(\lambda+\zeta)\frac{\mathrm{e}^{\bar{\gamma}(\xi-t)}+1}{\mathrm{e}^{\bar{\gamma}(\xi-t)}-1} - \gamma(\lambda-\zeta) - \theta\gamma^2(\xi-t)]}}{1 - \mathrm{e}^{-\bar{\gamma}(\xi-t)}}.$$

Setting $a_n(\nu) \equiv [2^{\nu+2n} n!\Gamma(\nu+n+1)]^{-1}$ and $z_{t,\xi}(\lambda,\zeta) = \frac{2\bar{\gamma}\sqrt{\zeta\lambda}}{\sigma^2 \sinh\left(\frac{\bar{\gamma}(\xi-t)}{2}\right)}$, we may write our expectation as a power series

$$\mathbf{E}\Big[\sqrt{\lambda_\xi^{t,\lambda}}\mathrm{e}^{-b_\alpha\lambda_\xi^{t,\lambda} - \int_t^\xi \lambda_s^{t,\lambda} ds}\Big] = \sum_{n=0}^{\infty} a_n(\nu) \int_0^{+\infty} \sqrt{\zeta}\mathrm{e}^{-b_\xi\zeta} M_{t,\xi}(\lambda,\zeta)[z_{t,\xi}(\lambda,\zeta)]^{\nu+2n} d\zeta$$

that can be truncated at any chosen order.

Since $X_\xi^{t,x} \sim N(x + (r - \frac{\sigma^2}{2})(\xi - t)), \sigma^2(\xi - t))$, the second expectation in (35) becomes

$$\mathbf{E}\Big[\mathrm{e}^{X_\xi^{t,x}} N(d_1(X_\xi^{t,x}, T-\xi))\Big] = \int_{\mathbb{R}} \mathrm{e}^y N(d_1(y, T-\xi)) \frac{\exp\left\{\frac{[y-x-(r-\frac{\sigma^2}{2})(\xi-t)]^2}{\sigma^2(\xi-t)}\right\}}{\sqrt{2\pi\sigma^2(\xi-t)}} dy.$$

5 A three-factor model

In this section we shortly present the correlation expansion for the more general market model (18), to show that the method can be easily extended to multi-factor models. Indeed the methodology remains the same and it is just a matter of handling slightly more complex calculations that lead nevertheless to computable formulas. As in the previous section we take $R = 0$.

Let $\bar{\rho} = (\rho_1, \rho_2)$ be the correlations vector, then by the Feymann-Kac theorem, the call price $u(x, \lambda, r, t, T; \bar{\rho})$ must solve the following parabolic PDE:

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}^{\bar{\rho}} u = 0 \\ u(x, \lambda, r, T, T; \bar{\rho}) = (\mathrm{e}^{X_T} - K)^+, \end{cases} \tag{37}$$

where

$$\mathcal{L}^{\bar{\rho}} \equiv \mathcal{L}^{\mathbf{0}} + \rho_1(\sigma\eta\sqrt{\lambda}\frac{\partial^2}{\partial x \partial\lambda}) + \rho_2(\sigma\nu\frac{\partial^2}{\partial x \partial r}) \equiv \mathcal{L}^{\mathbf{0}} + \bar{\rho}\cdot(\mathcal{A}_{\rho_1}, \mathcal{A}_{\rho_2})$$

and

$$\mathcal{L}^{\mathbf{0}} \equiv \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \frac{\eta^2\lambda}{2}\frac{\partial^2}{\partial\lambda^2} + \frac{\nu^2}{2}\frac{\partial^2}{\partial r^2} + (r - \frac{\sigma^2}{2})\frac{\partial}{\partial x} + \gamma(\theta-\lambda)\frac{\partial}{\partial\lambda} + k(\mu - r)\frac{\partial}{\partial r} - r - \lambda.$$

By definition, the first-order approximation of the call price is given by

$$\bar{u}(x,\lambda,r,t,T;\bar{\rho}) \equiv g_0(x,\lambda,r,t,T) + \bar{\rho}\cdot\bar{g}_1(x,\lambda,r,t,T) \tag{38}$$

where g_0 solves (37) with $\bar{\rho} = (0,0)$ and $\bar{g}_1 = (v,w)$ with functions $v = v(x,\lambda,r,t,T)$ and $w = w(x,\lambda,r,t,T)$ solving the following equations

$$\begin{cases} \frac{\partial v}{\partial t} + \mathcal{L}^{\mathbf{0}} v = -\mathcal{A}_{\rho_1} g_0, \\ v(x,\lambda,r,T,T) = 0 \end{cases} \qquad \begin{cases} \frac{\partial w}{\partial t} + \mathcal{L}^{\mathbf{0}} w = -\mathcal{A}_{\rho_2} g_0, \\ w(x,\lambda,r,T,T) = 0. \end{cases}$$

All the coefficients may be computed by the same method used in section 4 as we are showing below. Indeed, by the Feymann-Kac theorem and the independence of the processes at $\bar{\rho} = \mathbf{0}$, we first get explicitly g_0 as

$$\begin{aligned} g_0(x,\lambda,r,t,T) &= \mathbf{E}(\mathrm{e}^{-\int_t^T \lambda_s^{t,\lambda} ds})\mathbf{E}(\mathrm{e}^{-\int_t^T r_s^{t,r} ds}(\mathrm{e}^{X_T^{t,x}} - K)^+) \\ &= \mathrm{e}^{-B_1(T-t)-B_2(T-t)\lambda} c_{BS}^V(x,r,t,T), \end{aligned}$$

where $c_{BS}^V(x,r,t,T) = \mathrm{e}^x N(D_1) - KP^r(r,t,T)N(D_2)$. Here $P^r(r,t,T) = \mathrm{e}^{-A_1(T-t)-A_2(T-t)r}$ is the Vasicek ZCB price maturing at T and the functions $D_{1,2} = D_{1,2}(x,r,V(T-t))$ and $V(T-t)$ are known (see [46]). Then the derivatives $\frac{\partial}{\partial x}c_{BS}^V$ and $\frac{\partial}{\partial r}c_{BS}^V$ are also explicitly computable and so are the terms $\mathcal{A}_{\rho_1} g_0$ and $\mathcal{A}_{\rho_2} g_0$. By Duhamel's principle we get

$$v(x,\lambda,r,t,T) = -\int_t^T v^{(\xi)}(x,\lambda,r,t)d\xi, \quad w(x,\lambda,r,t,T) = -\int_t^T w^{(\xi)}(x,\lambda,r,t)d\xi,$$

where $v^{(\xi)}$ and $w^{(\xi)}$ solve the PDE's

$$\begin{cases} \frac{\partial v^{(\xi)}}{\partial t} + \mathcal{L}^0 v^{(\xi)} = 0, & t < \xi \\ v^{(\xi)}(x,\lambda,r,\xi) = -\mathcal{A}_{\rho_1} g_0(x,\lambda,r,\xi,T) \end{cases} \quad \begin{cases} \frac{\partial w^{(\xi)}}{\partial t} + \mathcal{L}^0 w^{(\xi)} = 0, & t < \xi \\ w^{(\xi)}(x,\lambda,r,\xi) = -\mathcal{A}_{\rho_2} g_0(x,\lambda,r,\xi,T), \end{cases}$$

explicitly given by

$$\begin{aligned} v^{(\xi)}(x,\lambda,r,t) =& \sigma\eta \mathrm{B}_2(T-\xi)\mathrm{e}^{-B_1(T-\xi)}\mathbf{E}\Big[\sqrt{\lambda_\xi^{t,\lambda}}\mathrm{e}^{-\int_t^\xi \lambda_s^{t,\lambda}ds - B_2(T-\xi)\lambda_\xi^{t,\lambda}}\Big] \\ & \times \mathbf{E}\Big[\mathrm{e}^{X_\xi^{t,x}} N(D_1(X_\xi^{t,x}, r_\xi^{t,r}, V(T-\xi)))\Big] \\ w^{(\xi)}(x,\lambda,r,t) =& -\sigma\nu\frac{A_2(T-\xi)}{\sqrt{V(T-\xi)}}\mathbf{E}\Big[\mathrm{e}^{-\int_t^\xi \lambda_s^{t,\lambda}ds - B_2(T-\xi)\lambda_\xi^{t,\lambda}}\Big] \\ & \times \mathbf{E}\Big[\mathrm{e}^{-\int_t^\xi r_s^{t,r}ds}\mathrm{e}^{X_\xi^{t,x}} N'(D_1(X_\xi^{t,x}, r_\xi^{t,r}, V(T-\xi)))\Big] \end{aligned}$$

We remark that all processes are evaluated for $\bar{\rho} = (0,0)$ and the expectations involving only the intensity process are similar to those of the previous section, while the other expectations are relative to Gaussian processes. Therefore (38) is numerically fully implementable.

6 Extension to XVA's

In this section we show that under appropriate conditions, the method can be extended to include several XVA's, such as bilateral CVA, DVA (Debt Value Adjustment), FVA (Funding Value Adjustment) and LVA (Liquidity Value Adjustment) due to collateralization. For the sake of completeness let us point out that an alternative path, which avoids the introduction of many credit adjustments, has been recently proposed in ([10]) for the pricing of defaultable European options.

Following [14], we remark that the adjusted value of a defaultable portfolio (with default risk of both parties), that takes into account the funding and collateralization costs verifies a, possibly nonlinear, BSDE that reduces either to formula (11) or to formula (12), when considering only a single CVA due to the defaultable counterparty.

To show this extension we refer to [15] and we consider the case of two parties exchanging some European claim with default free payoff $f(X_T)$, who can both default with respective times τ^C and τ^I. In this context we define the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^C \vee \mathcal{H}_t^I$, where $\mathcal{H}_t^C = \sigma(\mathbf{1}_{\{\tau^C \leq t\}})$, $\mathcal{H}_t^I = \sigma(\mathbf{1}_{\{\tau^I \leq t\}})$. As before, we assume that every $\mathcal{F}_t-$martingale remains a $\mathcal{G}_t-$martingale and that there exists a unique extension of the risk neutral probability, that we keep denoting by P.

As in the classical framework of [21], we postulate the default times to be conditionally independent with respect to $\mathcal{F}_t$, i.e. for any $t > 0$ and $t_1, t_2 \in [0, t]$, we assume that $P(\tau^C > t_1, \tau^I > t_2 | \mathcal{F}_t) = P(\tau^C > t_1 | \mathcal{F}_t) P(\tau^I > t_2 | \mathcal{F}_t)$. We may use an intensity approach for both $\mathcal{G}-$stopping times

$$P(\tau^C \leq t | \mathcal{F}_t) = 1 - \mathrm{e}^{-\int_0^t \lambda_u^C du}, \qquad P(\tau^I \leq t | \mathcal{F}_t) = 1 - \mathrm{e}^{-\int_0^t \lambda_u^I du}, \forall\, t \geq 0,$$

so that under conditional independence $\lambda_t = \lambda_t^C + \lambda_t^I$ is the intensity process of $\tau = \inf\{\tau_1, \tau_2\}$.

We therefore choose the following model for our state variables

$$X_s = x + \int_t^s (r_u - \frac{\sigma^2}{2}) du + \sigma(B_s - B_t) \qquad x \in \mathbb{R} \tag{39}$$

$$\lambda_s^C = \lambda_1 + \int_t^s \gamma_1(\theta_1 - \lambda_u^C) du + \eta_1 \int_t^s \sqrt{\lambda_u^C} dB_u^1, \quad \lambda_1 > 0 \tag{40}$$

$$\lambda_s^I = \lambda_2 + \int_t^s \gamma_2(\theta_2 - \lambda_u^I) du + \eta_2 \int_t^s \sqrt{\lambda_u^I} dB_u^2, \quad \lambda_2 > 0, \tag{41}$$

where the parameters verify $\gamma_i, \eta_i, \theta_i, \sigma > 0$, $2\gamma_i\theta_i > \eta_i^2$, $i = 1, 2$, (B^1, B^2, B^3) is a 3-dimensional Brownian motion and $B_s = \rho_1 B_s^1 + \rho_2 B_s^2 + \sqrt{1 - \rho_1^2 - \rho_2^2} B_s^3$ with $\rho_1^2 + \rho_2^2 \leq 1$. The risk free interest rate r, for the sake of simplicity, is taken to be bounded and deterministic and we therefore are in a situation similar to the three factor model considered in the previous section.

From [14], the $\mathcal{G}_t-$adapted adjusted value of a European claim $\bar{c}^a(t, T)$, should be written as

$$\bar{c}^a(t, T) = c(t, T) - CVA(t, T) + DVA(t, T) + LVA(t, T) + FVA(t, T), \tag{42}$$

with $CVA(t,T)$ and $DVA(t,T) \geq 0$ and $LVA(t,T), FVA(t,T) \in \mathbb{R}$.

As an example we assume we are analyzing a single contract between the two parties and that:

1. the parties are not investing in a repo market;
2. the claim pays no dividends;
3. the close out value at default is given by an $\mathcal{F}_t-$adapted process ϵ_t, evaluated at τ;
4. the collateral rate, r_c, and the funding rate, r_f, are deterministic;
5. the collateral account is proportional to the close-out value, $C_s = \alpha_s \epsilon_s$ for some deterministic function $\alpha : [0,T] \longrightarrow [0,1]$
6. $L_C, L_I \in (0,1)$ are the loss given defaults in case of default respectively of the counterparty and of the investor.

Proceeding along the same lines as in [15] (pages 42-47), after having applied the Key Lemma to (42), we may conclude that $\mathcal{F}_t-$adapted adjusted price of the European claim, $c^a(t,T)$, verifies the following BSDE on $\{\tau > t\}$

$$\begin{aligned} c^a(t,T) = \mathbf{E}\Big[&\mathrm{e}^{-\int_t^T (r_s+\lambda_s)ds} f(X_T) \\ + \int_t^T &\mathrm{e}^{-\int_t^s (r_u+\lambda_u)ds} [\lambda_s \epsilon_s - L_C \lambda_s^C (1-\alpha_s)\epsilon_s^+ - L_I \lambda_s^I (1-\alpha_s)\epsilon_s^-] ds \\ + \int_t^T &\mathrm{e}^{-\int_t^s (r_u+\lambda_u)ds} [(r_f(s) - r_c(s))\alpha_s \epsilon_s + (r_s - r_f(s)) c^a(s,T)] ds | \mathcal{F}_t\Big]. \end{aligned} \tag{43}$$

Remark 4 The above BSDE is linear or nonlinear depending on the choice of ϵ_s. In the literature there are fundamentally two possible choices: either $\epsilon_s = c(s,T)$ (the default free value of the claim) or $\epsilon_s = c^a(s,T)$.

The first choice will always give a solvable linear BSDE.

With the second choice, we might obtain a solvable linear BSDE if the adjusted value stays always non negative (or non positive), otherwise the nonlinearity due to the negative and positive parts will give a non linear BSDE, not explicitly solvable.

In absence of DVA, LVA and FVA, with constant α (as in the case of a call option), the first choice leads to the evaluation (11), the second to the evaluation (12). In both cases, the recovery constant will be $R = 1 - L(1-\alpha)$.

Since our method, aimed at pointing out the contribution of the correlations to the forming of the price, needs to have an explicit representation of zero-correlation term, in what follows we choose $\epsilon_s = c(s,T)$ (that corresponds to asking a collateralization proportional to the default free price rather than to the current price), to guarantee the solvability of the BSDE for all European claims.

Equation (43) hence becomes

$$c^a(t,T) = \mathbf{E}\Big[\mathrm{e}^{-\int_t^T (r_s+\lambda_s)ds} f(X_T)$$
$$+\int_t^T \mathrm{e}^{-\int_t^s (r_u+\lambda_u)ds}[\lambda_s c(s,T) - \lambda_s^C L_C(1-\alpha_s)c(s,T)^+ - L_I \lambda_s^I (1-\alpha_s)c(s,T)^-]ds$$
$$+\int_t^T \mathrm{e}^{-\int_t^s (r_u+\lambda_u)ds}[(r_f(s) - r_c(s))\alpha_s c(s,T) + (r_s - r_f(s))c^a(s,T)]ds|\mathcal{F}_t\Big]. \tag{44}$$

This equation can be solved with

$$c^a(t,T) = \mathbf{E}\Big[\mathrm{e}^{-\int_t^T (r_f(s)+\lambda_s)ds} f(X_T)$$
$$+\int_t^T \mathrm{e}^{-\int_t^s [r_f(u)+\lambda_u]ds}[\lambda_s c(s,T) - \lambda_s^C L_C(1-\alpha_s)c(s,T)^+ - L_I \lambda_s^I (1-\alpha_s)c(s,T)^-]ds$$
$$+\int_t^T \mathrm{e}^{-\int_t^s [r_f(u)+\lambda_u]ds}(r_f(s) - r_c(s))\alpha_s c(s,T)ds|\mathcal{F}_t\Big]. \tag{45}$$

The processes $X_t, \lambda_t^C, \lambda_t^I$ are Markovian, therefore $c(t,T)$ and $c^a(t,T)$ are deterministic functions respectively of the state variables X and X, λ^C, λ^I, that we are going to denote as $u^{df}(x,t)$ (the default free price) and $u(x,\lambda_1,\lambda_2,t,T;\bar{\rho})$, with $\bar{\rho} = (\rho_1,\rho_2)$ to point out the dependence on the correlation parameters.

From (45), by Feynman- Kaĉ formulas, u verifies the PDE

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}^{\bar{\rho}} u + \kappa = 0 \\ u(x,\lambda_1,\lambda_2,T,T;\bar{\rho}) = f(X_T), \end{cases} \tag{46}$$

where,

$$\mathcal{L}^{\bar{\rho}} \equiv \mathcal{L}^{\mathbf{0}} + \sigma\rho_1\eta_1\sqrt{\lambda_1}\frac{\partial^2}{\partial x\partial\lambda_1} + \sigma\rho_2\eta_2\sqrt{\lambda_2}\frac{\partial^2}{\partial x\partial\lambda_2} \equiv \mathcal{L}^{\mathbf{0}} + \bar{\rho}\cdot(\mathcal{A}_{\rho_1},\mathcal{A}_{\rho_2})$$

with $\mathbf{0} = (0,0)$ and

$$\begin{aligned} \mathcal{L}^{\mathbf{0}} \equiv& \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \frac{\eta_1^2\lambda_1}{2}\frac{\partial^2}{\partial\lambda_1^2} + \frac{\eta_2^2\lambda_2}{2}\frac{\partial^2}{\partial\lambda_2^2} \\ &+(r-\frac{\sigma^2}{2})\frac{\partial}{\partial x} + \gamma_1(\theta_1-\lambda_1)\frac{\partial}{\partial\lambda_1} + \gamma_2(\theta_2-\lambda_2)\frac{\partial}{\partial\lambda_2} - (r_f+\lambda_1+\lambda_2) \\ \kappa(x,\lambda_1,\lambda_2,t) =& [\lambda_1+\lambda_2+(r_f(t)-r_c(t))\alpha_s]u^{df}(x,t) \\ &- L_C\lambda_1(1-\alpha_s)u^{df}(x,t)^+ - L_I\lambda_s^I(1-\alpha_s)u^{df}(x,t)^-. \end{aligned}$$

To simplify the exposition, in what follows we take $r_f \equiv r$ and constant α and we denote $L_C^\alpha = L_C(1-\alpha)$ and $L_I^\alpha = L_I(1-\alpha)$. As before, we may construct the first-order approximation of the adjusted price

$$\bar{u}(x,\lambda_1,\lambda_2,t;\bar{\rho}) \equiv g_0(x,\lambda_1,\lambda_2,t,T) + \bar{\rho}\cdot\bar{g}_1(x,\lambda_1,\lambda_2,t,T) \tag{47}$$

with g_0 solving (46) with $\bar{\rho}=(0,0)$ and $\bar{g}_1=(v,w)$ solving

$$\begin{cases} \frac{\partial v}{\partial t}+\mathcal{L}^{\mathbf{0}}v+\kappa=-\mathcal{A}_{\rho_1}g_0, \\ v(x,\lambda_1,\lambda_2,T,T)=0 \end{cases} \qquad \begin{cases} \frac{\partial w}{\partial t}+\mathcal{L}^{\mathbf{0}}w+\kappa=-\mathcal{A}_{\rho_2}g_0, \\ w(x,\lambda_1,\lambda_2,T,T)=0. \end{cases}$$

Omitting the flow notation used in the previous sections, the zero-th coefficient by independence reduces to

$$\begin{aligned}
&g_0(x,\lambda_1,\lambda_2,r,t,T)=\mathbf{E}(\mathrm{e}^{-\int_t^T\lambda_s^C ds})\mathbf{E}(\mathrm{e}^{-\int_t^T\lambda_s^I ds})\mathbf{E}(\mathrm{e}^{-\int_t^T r_s ds}f(X_T))\\
&+\int_t^T\mathbf{E}\Big(\mathrm{e}^{-\int_t^s r_u du}u^{df}(X_s,s)\Big)\Big[(r_s-r_c(s))\alpha_s\mathbf{E}(\mathrm{e}^{-\int_t^s\lambda_u^C du})\mathbf{E}(\mathrm{e}^{-\int_t^s\lambda_u^I du})\\
&+\mathbf{E}(\lambda_s^C\mathrm{e}^{-\int_t^s\lambda_u^C du})\mathbf{E}(\mathrm{e}^{-\int_t^s\lambda_u^I du})+\mathbf{E}(\lambda_s^I\mathrm{e}^{-\int_t^s\lambda_u^I du}))\mathbf{E}(\mathrm{e}^{-\int_t^s\lambda_u^C du})\Big]ds\\
&-L_C^\alpha\int_t^T\mathbf{E}(\lambda_s^C\mathrm{e}^{-\int_t^s\lambda_u^C du})\mathbf{E}(\mathrm{e}^{-\int_t^s\lambda_u^I du})\mathbf{E}\Big(\mathrm{e}^{-\int_t^s r_u du}u^{df}(X_s,s)^+\Big)ds\\
&-L_I^\alpha\int_t^T\mathbf{E}(\lambda_s^I\mathrm{e}^{-\int_t^s\lambda_u^I du})\mathbf{E}(\mathrm{e}^{-\int_t^s\lambda_u^C du})\mathbf{E}\Big(\mathrm{e}^{-\int_t^s r_u du}u^{df}(X_s,s)^-\Big)ds\\
&=u^{df}(x,t)\Big\{\mathrm{e}^{-B_1^C(T-t)-B_2^C(T-t)\lambda_1}\mathrm{e}^{-B_1^I(T-t)-B_2^I(T-t)\lambda_2}\\
&+\alpha\int_t^T\Big[(r_s-r_c(s))\mathrm{e}^{-B_1^C(s-t)-B_2^C(s-t)\lambda_1}\mathrm{e}^{-B_1^I(s-t)-B_2^I(s-t)\lambda_2}\\
&+\mathrm{e}^{-B_1^C(s-t)-B_2^C(s-t)\lambda_1}\mathbf{E}(\lambda_s^I\mathrm{e}^{-\int_t^s\lambda_u^I du})+\mathrm{e}^{-B_1^I(s-t)-B_2^I(s-t)\lambda_1}\mathbf{E}(\lambda_s^C\mathrm{e}^{-\int_t^s\lambda_u^C du})\Big]ds\Big\}\\
&-L_C^\alpha\int_t^T\mathrm{e}^{-B_1^I(s-t)-B_2^I(s-t)\lambda_2})\mathbf{E}(\lambda_s^C\mathrm{e}^{-\int_t^s\lambda_u^C du})\mathbf{E}\Big(\mathrm{e}^{-\int_t^s r_u du}u^{df}(X_s,s)^+\Big)\Big]ds\\
&-L_I^\alpha\int_t^T\mathrm{e}^{-B_1^C(s-t)-B_2^C(s-t)\lambda_1}\mathbf{E}(\lambda_s^I\mathrm{e}^{-\int_t^s\lambda_u^I du})\mathbf{E}\Big(\mathrm{e}^{-\int_t^s r_u du}u^{df}(X_s,s)^-\Big)\Big]ds.
\end{aligned}$$

The expectations in the above terms are all explicitly computable. Indeed, once we choose a contract so that there exists a closed formula (possibly differentiable in time) for the default free price and for the discounted expectation of its positive part (such as a forward for instance), then we have that all the inner expectations to be evaluated are of the form

$$\int_t^T f_0(s)\mathrm{e}^{-f_1(s)-\lambda f_2(s)}\mathbf{E}(\lambda_s\mathrm{e}^{-\int_t^s\lambda_u du})ds,$$

where f_0,f_1,f_2 are deterministic and differentiable functions of time, while λ_s is a CIR process with $\lambda_t=\lambda$ (constant) and $\mathbf{E}(\mathrm{e}^{-\int_t^s\lambda_u du})=\mathrm{e}^{-f_1(s)-\lambda f_2(s)}$, with $f_1(t) = f_2(t) = 0$. So, we have

$$
\begin{aligned}
&\int_t^T f_0(s)\mathrm{e}^{-f_1(s)-\lambda f_2(s)}\mathbf{E}(\lambda_s \mathrm{e}^{-\int_t^s \lambda_u du})ds\\
&\mathbf{E}\Big(\int_t^T f_0(s)\mathrm{e}^{-f_1(s)-\lambda f_2(s)}\lambda_s \mathrm{e}^{-\int_t^s \lambda_u du}ds\Big)\\
=&\mathbf{E}\Big(f_0(t)\mathrm{e}^{-f_1(t)-\lambda f_2(t)} - \mathrm{e}^{-\int_t^T \lambda_u du} f_0(T)\mathrm{e}^{-f_1(T)-\lambda f_2(T)}\Big)\\
&+\mathbf{E}\Big(\int_t^T \mathrm{e}^{-\int_t^s \lambda_u du}[f_0'(s) - f_0(t)f_1'(s) - \lambda f_0(t) f_2'(s)]\mathrm{e}^{-f_1(s)-\lambda f_2(s)}ds\Big)
\end{aligned}
$$

having applied itegration by parts in the last passage. Using Fubini's theorem we may conclude

$$
\begin{aligned}
&\int_t^T f_0(s)\mathrm{e}^{-f_1(s)-\lambda f_2(s)}\mathbf{E}(\lambda_s \mathrm{e}^{-\int_t^s \lambda_u du})ds\\
=&f_0(t)\mathrm{e}^{-f_1(t)-\lambda f_2(t)} - f_0(T)\mathrm{e}^{-f_1(T)-\lambda f_2(T)}\mathbf{E}(\mathrm{e}^{-\int_t^T \lambda_u du})\\
&+\int_t^T [f_0'(s) - f_1'(s) - \lambda f_2'(t)]\mathrm{e}^{-f_1(s)-\lambda f_2(s)}\mathbf{E}(\mathrm{e}^{-\int_t^s \lambda_u du})ds\\
=&f_0(t)\mathrm{e}^{-f_1(t)-\lambda f_2(t)} - f_0(T)\mathrm{e}^{-2(f_1(T)+\lambda f_2(T))}\\
&+\int_t^T [f_0'(s) - f_0(t)f_1'(s) - \lambda f_0(t) f_2'(t)]\mathrm{e}^{-2[f_1(s)+\lambda f_2(s)]}ds.
\end{aligned}
$$

By Duhamel's principle we may compute the components of g_1

$$
\begin{aligned}
v(x,\lambda_1,\lambda_2,t,T) &= -\int_t^T v^{(\xi)}(x,\lambda_1,\lambda_2,t)d\xi,\\
w(x,\lambda_1,\lambda_2,t,T) &= -\int_t^T w^{(\xi)}(x,\lambda_1,\lambda_2,t)d\xi,
\end{aligned}
$$

where $v^{(\xi)}$ and $w^{(\xi)}$ solve the PDE's

$$
\begin{cases}
\frac{\partial v^{(\xi)}}{\partial t} + \mathcal{L}^0 v^{(\xi)} + \kappa = 0, & t < \xi\\
v^{(\xi)}(x,\lambda_1,\lambda_2,\xi) = -\mathcal{A}_{\rho_1} g_0(x,\lambda_1,\lambda_2,\xi,T)
\end{cases}
$$

$$
\begin{cases}
\frac{\partial w^{(\xi)}}{\partial t} + \mathcal{L}^0 w^{(\xi)} + \kappa = 0, & t < \xi\\
w^{(\xi)}(x,\lambda_1,\lambda_2,\xi) = -\mathcal{A}_{\rho_2} g_0(x,\lambda_1,\lambda_2,\xi,T),
\end{cases}
$$

explicitly given by

$$\begin{aligned}v^{(\xi)}(x,\lambda_1,\lambda_2,t) =&\sigma\eta_1\mathbf{E}\Big(\sqrt{\lambda_s^C}\mathrm{e}^{-\int_t^\xi(r_u+\lambda_u)du}\frac{\partial^2 g_0}{\partial x\partial\lambda_1}(X_\xi,\lambda_\xi^C,\lambda_\xi^I,\xi,T)\Big)\\&+\mathbf{E}\Big(\int_t^\xi \mathrm{e}^{-\int_t^s(r_u+\lambda_u)du}\kappa(X_s,\lambda_s^C,\lambda_s^I,s)ds\Big)\\w^{(\xi)}(x,\lambda_1,\lambda_2,t) =&\sigma\eta_2\mathbf{E}\Big(\sqrt{\lambda_s^I}\mathrm{e}^{-\int_t^\xi(r_u+\lambda_u)du}\frac{\partial^2 g_0}{\partial x\partial\lambda_2}(X_\xi,\lambda_\xi^C,\lambda_\xi^I,\xi,T)\Big)\\&+\mathbf{E}\Big(\int_t^\xi \mathrm{e}^{-\int_t^s(r_u+\lambda_u)du}\kappa(X_s,\lambda_s^C,\lambda_s^I,s)ds\Big)\end{aligned}$$

We remark that all processes are evaluated for $\bar\rho=(0,0)$: from the previous discussion it is quite evident that the partial derivatives of g_0 are explicitly computable. The final evaluation will require expectations involving the intensity processes similar to those of the previous section. Therefore (47) is numerically fully implementable. We postpone the explicit numerical computations for some specific European claim to future work.

7 CVA and the change of measure approach

Recently Brigo and Vrins [11], in a paper focusing only on CVA, proposed a method for its computation under WWR, based on a change of measures, e.g. Girsanov's theorem, in the stochastic-intensity default setup. Their starting point is the following formula for the time-zero CVA (compare with (15)) of the portfolio price process V_t

$$CVA(0,T) = -(1-R)\int_0^T \mathbf{E}[\frac{V_t^+}{B(0,t)}\zeta_t]dG(t), \tag{48}$$

where $\mathbf{E}[\cdot]$ is the expectation under the risk-neutral measure. The EPE (expected positive exposure) under WWR is the function

$$EPE(t) = \mathbf{E}[\frac{V_t^+}{B(0,t)}\zeta_t].$$

Girsanov's theorem is used to factorize the EPE. Indeed by defining an equivalent martingale measure $Q^{C^{\mathcal{F},t}}\sim Q$ as

$$Z_s^t := \frac{dQ^{C^{\mathcal{F},t}}}{dQ} = \frac{M_s^t}{M_0^t},\quad \text{where } M_s^t = \mathbf{E}[\frac{1}{B(0,t)}\lambda_t S_t|\mathcal{F}_s],\ s\in[0,t],$$

in [11] it is proved that

$$\mathbf{E}[\frac{V_t^+}{B(0,t)}\zeta_t] = \mathbf{E}^{C^{\mathcal{F},t}}[V_t^+]\mathbf{E}[\frac{\zeta_t}{B(0,t)}].$$

The measure $Q^{C^{\mathcal{F},t}}$ is called wrong-way measure and it is associated to the numéraire $C^{\mathcal{F},t}_{\cdot} = B(0,\cdot)M^t_{\cdot}$.

In order to apply such a methodology, it is therefore necessary to obtain the dynamics of V_t under the measure $Q^{C^{\mathcal{F},t}}$. By assuming a continuous dynamic for V_t under Q described by a SDE, the change of measure results in a drift adjustment, we refer to [11] for the full details.

In [13] Brigo et al. applied the results obtained in [11] to the calculation of CVA under WWR for a call option in the market model described by (21). The risk free rate being constant implies that $\mathbf{E}[B(0,t)^{-1}\zeta_t] = \mathrm{e}^{-rt}$. Moreover the explicit expression of the new drift is

$$\theta^s_t \equiv \theta^s_t(\lambda_t) = \rho\eta\sqrt{\lambda_t}\left(\frac{A^\lambda(s,t)B^\lambda_t(s,t)}{A^\lambda(s,t)B^\lambda_t(s,t)\lambda_t - A^\lambda_t(s,t)} - B^\lambda(s,t)\right), \tag{49}$$

the functions $\log A^\lambda = -B_1$ and $B^\lambda = B_2$ being as in (32). In order to be able to compute the expectations, it was necessary to replace the process λ_t with a deterministic proxy $\lambda(t)$ in (49). Once the chosen approximant is plugged into (49), the expression $EPE(t) = \mathrm{e}^{-rt}\mathbf{E}^{C^{\mathcal{F},t}}[c(t,T)]$ can be evaluated analytically leading to (see [13])

$$\begin{aligned} &E^{C^{\mathcal{F},t}}[\frac{c(t,T)}{B(0,t)}] \\ \approx &\mathrm{e}^{x_0+\sigma\Theta_t}N\left(\frac{\hat\alpha(t)+\beta(t)\sigma\sqrt{t}}{\sqrt{1+\beta^2(t)}}\right) - \mathrm{e}^{\kappa-rT}N\left(\frac{\hat\alpha(t)-\sigma\sqrt{T-t}}{\sqrt{1+\beta^2(t)}}\right), \end{aligned} \tag{50}$$

where

$$\Theta(t) = \int_0^t \theta(u,t)du, \quad \theta(u,t) = \theta^t_u(\lambda(u)), \quad \hat\alpha(t) = \alpha(t) + \frac{\Theta_t}{\sqrt{T-t}}$$
$$\alpha(t) = \frac{1}{\sigma\sqrt{T-t}}\left(x_0 - \kappa + \left(r + \frac{\sigma^2}{2}\right)T - \sigma^2 t\right), \quad \beta(t) = \sqrt{\frac{t}{T-t}}.$$

Two deterministic proxies $\lambda(t)$ were considered: $\mathbf{E}[\lambda_t]$ and $\mathbf{E}^{C^{\mathcal{F},t}}[\lambda_t]$. While the first is analytically known, the second requires a further approximation step (see [13]). Inserting (50) in (48) a numerical integration procedure gives the CVA under WWR.

Remark 5 It should be noticed that other methods based on the approximation of the process (λ_t) could be exploited in order to price a vulnerable call option in the market model (21), and hence its CVA. For instance, the volatility expansion method of Kim and Kunimoto, see [38], considers a Taylor expansion of the process (λ_t) in powers of η around $\eta = 0$. Taking the first order polynomial iand setting $\lambda(s) = \lambda\exp(-\gamma(s-t)) + \theta(1-\exp(-\gamma(s-t)))$, they have for all $s \geq t$ and $\lambda_t = \lambda$:

$$\lambda_s = \lambda(s) + \eta\int_t^s \mathrm{e}^{-\gamma(s-u)}\sqrt{\lambda(u)}(\rho dB^1_u + \sqrt{1-\rho^2}dB^2_u) + o(\eta). \tag{51}$$

Inserting the approximation (51) in the evaluation formula for the vulnerable call option, after some manipulations the following result is obtained

$$u(x,\lambda,t,T;\rho) \approx \mathrm{e}^{-\int_t^T \lambda(s)ds}\big[c_{BS}(x,t,T) - \rho\sigma\eta\mathrm{e}^{x-\frac{\sigma^2}{2}(T-t)}N(d_1)\Lambda(\lambda,t,T)\big] \quad (52)$$

with c_{BS} denoting the classical Black-Scholes price and

$$\Lambda(\lambda,t,T) = \int_t^T \int_u^T \mathrm{e}^{-\gamma(s-u)}\sqrt{\lambda(u)}duds.$$

In the next section we are going to provide a comparison of the numerical performances of the different methods which have been presented.

8 Numerical results

In this section we compare numerically our method to compute the CVA for a vulnerable option with the methods mentioned above, using the Monte Carlo approximations as a benchmark.

We considered model (21) with exogenously chosen parameters $\gamma = 0.2$, $\theta = 0.05$, $\lambda_0 = 0.04$ and $S_0 = 100$. Instead, we varied ρ, σ and η to check the performances of the methods. Positive correlation values relate to the WWR effect on the call option. The strike price is fixed to $K = 100$ and the maturity is $T = 1$: without loss of generality we also set the risk-free rate $r = 0$ and $t = 0$. All the pricing methods have been implemented in MatLab (R2017).

For the benchmark, Monte Carlo method was implemented with an Euler discretization with full-truncation of the CIR process, while the geometric Brownian motion was exactly simulated. In order to improve the Monte Carlo estimates, we implemented a control variate technique by using the default-free call price as a control. In these experiments we set $n = 1000$ time step points in $[0,T]$ and $M = 1\,000\,000$ samples.

For the approximation of the first order expansion in section 4, we computed g_0 analytically, while for g_1, we first computed the term g_1^α on a grid of equispaced points α_k in $[0,T]$ by using the adaptive Gauss-Kronrod (GK) quadrature algorithm and then the resulting vector was interpolated and finally integrated by using once again the GK algorithm to get g_1. On a Intel Core i7 (2.40 GHz), the whole procedure requires about 0.3 secs. Of course, the CVA approximation for different values of ρ is simply obtained by linearity, see eq. (36), without any further computational cost.

The drift adjustment method recalled in section 7 is based on the replacement of the process λ_t with a deterministic proxy in the drift (49). As it was pointed out, different choices can be made: we have chosen to implement $\lambda(t) = \mathbf{E}[\lambda_t]$. Inserting (50) in (48) a numerical integration procedure gives the CVA. This numerical approximation (taking about 0.6 secs in our implementation) must be repeated for every value of ρ.

The volatility expansion introduced in Remark 5 is easily implemented, all the terms being available in closed forms with the exception of $\Lambda(\lambda,0,T)$

ρ	Corr. exp.	Vol. exp.	Drift adj.	MC + control (C.I)
-0.9	0.11780 (0.00253)	0.11729 (0.00304)	0.12215 (-0.00181)	0.12034 (0.00009)
-0.7	0.12712 (0.00150)	0.12677 (0.00184)	0.12970 (-0.00108)	0.12861 (0.00010)
-0.5	0.13643 (0.00084)	0.13625 (0.00102)	0.13769 (-0.00042)	0.13727 (0.00012)
-0.3	0.14575 (0.00023)	0.14573 (0.00026)	0.14615 (-0.00017)	0.14598 (0.00013)
-0.1	0.15506 (0.00009)	0.15520 (-0.00004)	0.15508 (0.00008)	0.15516 (0.00014)
0.1	0.16438 (0.00004)	0.16468 (-0.00026)	0.16448 (-0.00006)	0.16443 (0.00015)
0.3	0.17369 (0.00014)	0.17416 (-0.00033)	0.17437 (-0.00053)	0.17383 (0.00015)
0.5	0.18301 (0.00062)	0.18364 (-0.00000)	0.18473 (-0.00110)	0.18364 (0.00015)
0.7	0.19233 (0.00156)	0.19312 (0.00077)	0.19558 (-0.00169)	0.19389 (0.00015)
0.9	0.20164 (0.00250)	0.20260 (0.00154)	0.20692 (-0.00277)	0.20414 (0.00014)

Table 1 Numerical results for varying ρ, $\sigma = 0.1$. In parenthesis the errors with respect to the MC values and, for the MC values, the 95% confidence interval length. The CIR volatility is $\eta = 0.1$.

ρ	Corr. exp.	Vol. exp.	Drift adj.	MC + control (C.I)
-0.9	0.04460 (0.02252)	0.03199 (0.03514)	0.07181 (-0.00468)	0.06713 (0.00015)
-0.7	0.06979 (0.01364)	0.06042 (0.02302)	0.08500 (-0.00156)	0.08344 (0.00020)
-0.5	0.09499 (0.00688)	0.08886 (0.01302)	0.10125 (0.00063)	0.10188 (0.00027)
-0.3	0.12018 (0.00247)	0.11729 (0.00536)	0.12100 (0.00166)	0.12265 (0.00034)
-0.1	0.14537 (0.00014)	0.14573 (-0.00050)	0.14462 (0.00060)	0.14522 (0.00041)
0.1	0.17057 (0.00025)	0.17416 (-0.00334)	0.17237 (-0.00155)	0.17082 (0.00047)
0.3	0.19576 (0.00251)	0.20260 (-0.00432)	0.20437 (-0.00610)	0.19827 (0.00052)
0.5	0.22095 (0.00689)	0.23103 (-0.00318)	0.24059 (-0.01275)	0.22784 (0.00056)
0.7	0.24614 (0.01360)	0.25946 (0.00029)	0.28087 (-0.02111)	0.25975 (0.00057)
0.9	0.27134 (0.02248)	0.28790 (0.00592)	0.32493 (-0.03111)	0.29382 (0.00056)

Table 2 Numerical results for varying ρ, $\sigma = 0.1$. In parenthesis the errors with respect to the MC values and, for the MC values, the 95% confidence interval length. The CIR volatility is $\eta = 0.3$.

ρ	Corr. exp.	Vol. exp.	Drift adj.	MC + control (C.I)
-0.9	0.00005 (0.04704)	-0.05330 (0.10029)	0.04566 (0.00132)	0.04698 (0.00016)
-0.7	0.03431 (0.02821)	-0.00591 (0.06844)	0.05762 (0.00489)	0.06252 (0.00024)
-0.5	0.06868 (0.01433)	0.04147 (0.04154)	0.07493 (0.00807)	0.08301 (0.00036)
-0.3	0.10305 (0.00530)	0.08886 (0.01950)	0.09960 (0.00875)	0.10836 (0.00049)
-0.1	0.13742 (0.00052)	0.13625 (0.00170)	0.13361 (0.00433)	0.13795 (0.00063)
0.1	0.17179 (0.00041)	0.18364 (-0.01143)	0.17841 (-0.00620)	0.17220 (0.00077)
0.3	0.20616 (0.00528)	0.23103 (-0.01959)	0.23451 (-0.02306)	0.21144 (0.00090)
0.5	0.24053 (0.01653)	0.27842 (-0.02135)	0.30140 (-0.04432)	0.25707 (0.00103)
0.7	0.27491 (0.02987)	0.32581 (-0.02102)	0.37783 (-0.07304)	0.30478 (0.00111)
0.9	0.30928 (0.05128)	0.37320 (-0.01263)	0.46226 (-0.10169)	0.36057 (0.00115)

Table 3 Numerical results for varying ρ, $\sigma = 0.1$. In parenthesis the errors with respect to the MC values and, for the MC values, the 95% confidence interval length. The CIR volatility is $\eta = 0.5$.

which was computed by a standard quadrature (GK) algorithm. The procedure is very fast (about 0.5×10^{-3} secs.) and since the approximation is linear in ρ, the estimated CVA is computed once for all values of ρ, as for the correlation expansion method.

The approximation methods are compared to MC (with control variates) estimates, the error being defined as $\widehat{CVA}_{MC} - \widehat{CVA}_{Method}$. A positive sign

ρ	Corr. exp.	Vol. exp.	Drift adj.	MC + control (C.I)
-0.9	0.34222 (0.00937)	0.34623 (0.00537)	0.35829 (-0.00667)	0.35160 (0.00030)
-0.7	0.37230 (0.00576)	0.37557 (0.00249)	0.38192 (-0.00386)	0.37806 (0.00036)
-0.5	0.40238 (0.00292)	0.40490 (0.00040)	0.40714 (-0.00184)	0.40530 (0.00040)
-0.3	0.43246 (0.00046)	0.43424 (-0.00132)	0.43403 (-0.00110)	0.43292 (0.00044)
-0.1	0.46254 (-0.00005)	0.46358 (-0.00109)	0.46262 (-0.00014)	0.46249 (0.00048)
0.1	0.49262 (0.00023)	0.49292 (-0.00006)	0.49299 (-0.00013)	0.49285 (0.00050)
0.3	0.52270 (0.00077)	0.52225 (0.00122)	0.52517 (-0.00169)	0.52348 (0.00051)
0.5	0.55278 (0.00288)	0.55159 (0.00407)	0.55921 (-0.00354)	0.55566 (0.00052)
0.7	0.58286 (0.00605)	0.58093 (0.00799)	0.59514 (-0.00622)	0.58892 (0.00057)
0.9	0.61294 (0.00922)	0.61027 (0.01190)	0.63300 (-0.01083)	0.62216 (0.00048)

Table 4 Numerical results for varying ρ, $\sigma = 0.3$. In parenthesis the errors with respect to the MC values and, for the MC values, the 95% confidence interval length. The CIR volatility is $\eta = 0.1$.

ρ	Corr. exp.	Vol. exp.	Drift adj.	MC + control (C.I)
-0.9	0.54936 (0.01904)	0.55828 (0.01012)	0.56840 (-0.01310)	0.56840 (0.00054)
-0.7	0.60299 (0.01159)	0.61017 (0.00441)	0.61459 (-0.00784)	0.61459 (0.00064)
-0.5	0.65663 (0.00519)	0.66207 (-0.00026)	0.66182 (-0.00453)	0.66182 (0.00073)
-0.3	0.71026 (0.00163)	0.71397 (-0.00208)	0.71189 (-0.00163)	0.71189 (0.00081)
-0.1	0.76390 (0.00089)	0.76587 (-0.00108)	0.76479 (0.00069)	0.76479 (0.00087)
0.1	0.81753 (-0.00007)	0.81777 (-0.00030)	0.81746 (-0.00078)	0.81746 (0.00092)
0.3	0.87117 (0.00269)	0.86966 (0.00419)	0.87386 (-0.00225)	0.87386 (0.00096)
0.5	0.92480 (0.00615)	0.92156 (0.00939)	0.93095 (-0.00689)	0.93095 (0.00096)
0.7	0.97844 (0.01078)	0.97346 (0.01576)	0.98922 (-0.01436)	0.98922 (0.00096)
0.9	1.03207 (0.01804)	1.02536 (0.02475)	1.05011 (-0.02337)	1.05011 (0.00091)

Table 5 Numerical results for varying ρ, $\sigma = 0.5$. In parenthesis the errors with respect to the MC values and, for the MC values, the 95% confidence interval length. The CIR volatility is $\eta = 0.1$.

η	0.1	0.2	0.3	0.4	0.5
$\|g_1\|$	0.0466	0.0898	0.1260	0.1532	0.1719

Table 6 The absolute value of g_1 for different volatilities η.

indicates an underestimation of the CVA with respect to MC. In our experiments (Tables (1) to (5)) we noticed that the three methods provide better approximation for small values of $|\rho|$: the correlation expansion, which is linear in ρ, provides a lower bound for CVA, while the drift adjustment gives a uniformly level of approximations which, however slightly worsens as the values of ρ become larger and positive (other choices of the $\lambda(t)$ tend to mitigate this effect, see [13]). In particular we experienced a systematic underestimation of the WWR effect for the correlation expansion method and an overestimation for the drift adjustment method, while the volatility expansion has not a definite behavior. This kind of pattern is still observed for the other parameter sets considered (see Figures (1), (2)).

As pointed out before, the contribution to the CVA due to the correlation ρ is quantified by g_1: its behavior is reported in Table (6) and it suggests an increasing impact of WWR as the volatility of the default intensity becomes larger.

	λ_0	γ	θ	η
Set 1	0.03	0.02	0.161	0.08
Set 3	0.01	0.8	0.02	0.2

Table 7 Parameter sets.

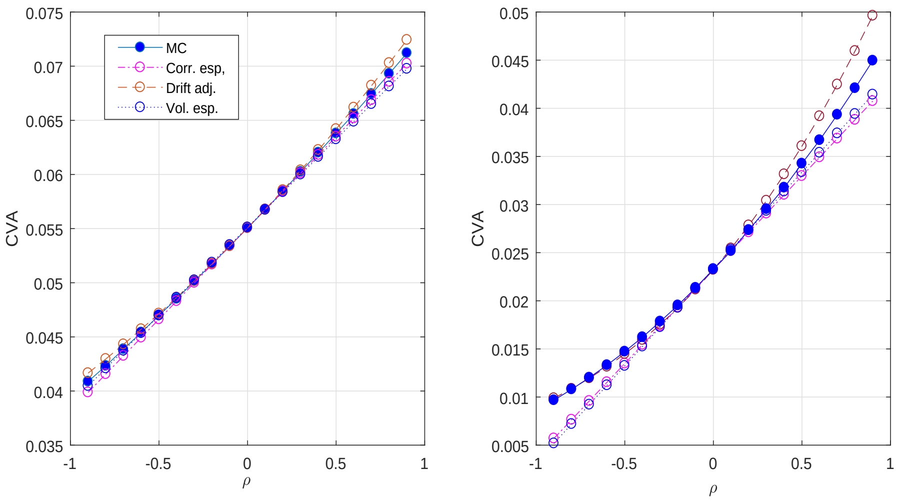

Fig. 1 Comparison of all methods for the set of parameters in Brigo et al. [13], maturity $T = 1$, parameter set 1 on the left and parameter set 3 on the right.

We further compared the approximation methods on the same two sets of parameters (set 1 and 3) used in [13] for the CIR dynamic, see Table (7). The results for $T = 1$ and $T = 5$ are reported graphically in Fig. (1) and Fig. (2), respectively confirming the behavior observed.

9 Conclusions

We considered the pricing problem for financial options subject to counterparty credit risk. The impact of a credit event is quantified by the Credit Value Adjustment, which we modeled in a stochastic intensity framework. This allows to represent the CVA as the expectation of the derivative's payoff discounted with a rate given by the sum of the risk-free and of the default intensity. Wrong Way Risk is accounted for by considering positive dependence between the exposure and the default event. The calculation of such a quantity may be tackled by classical Monte Carlo methods once the dynamics of the stochastic state variables (underlying, risk-free rate and default intensity) are chosen, but it is computationally very expensive. As an alternative to that, we proposed in this paper the correlation expansion method to evaluate CVA with WWR, when the underlying and the intensity dynamics are respectively given by a geometrical Brownian motion and a CIR process. We showed furthermore how

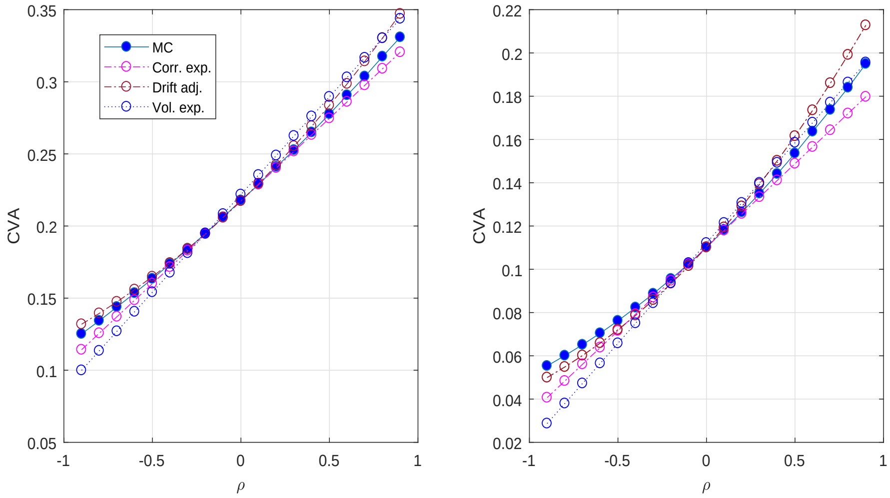

Fig. 2 Comparison of all methods for the set of parameters in Brigo et al. [13], maturity $T = 5$, parameter set 1 on the left and parameter set 3 on the right.

the correlation expansion method may be extended to cope with multi-factor models and to include several XVA's. Finally we compared the performance of our method with that of two other semi-analytical techniques: the drift adjustment introduced in [11] and the volatility expansion technique used in [38].

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