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HIGH-FREQUENCY ASYMPTOTICS FOR LIPSCHITZ–KILLING CURVATURES OF EXCURSION SETS ON THE SPHERE

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In this paper, we shall be concerned with geometric functionals and excursion probabilities for some nonlinear transforms evaluated on Fourier components of spherical random fields. In particular, we consider both random spherical harmonics and their smoothed averages, which can be viewed as random wavelet coefficients in the continuous case. For such fields, we consider smoothed polynomial transforms; we focus on the geometry of their excursion sets, and we study their asymptotic behaviour, in the high-frequency sense. We focus on the analysis of Euler–Poincaré characteristics, which can be exploited to derive extremely accurate estimates for excursion probabilities. The present analysis is motivated by the investigation of asymmetries and anisotropies in cosmological data. The statistics we focus on are also suitable to deal with spherical random fields which can only be partially observed, the canonical example being provided by the masking effect of the Milky Way on Cosmic Microwave Background (CMB) radiation data.

1. Introduction.

1.1. *Motivations and general framework.* In this paper, we shall be concerned with geometric functionals and excursion probabilities for some nonlinear transforms evaluated on Fourier components of spherical random fields. More precisely, let $\{T(x), x \in S^2\}$ denote a Gaussian, zero-mean isotropic spherical random field, that is, for some probability space $(\Omega, \Im, P)$ the application $T(x,\omega) \to \mathbb{R}$ is $\{\mathcal{B}(S^2) \times \Im\}$ measurable, where $\mathcal{B}(S^2)$ denotes the Borel σ-algebra on the sphere, and by isotropy we mean as usual that for all rotation $g \in \mathrm{SO}(3)$, the field $\{T(x)\}$ has the same law as $\{T^g(x) := T(gx)\}$. It is well known that the following representation holds in the mean square sense (see, e.g., [32–34]):

$$T(x) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(x) = \sum_{\ell} T_\ell(x), \qquad T_\ell(x) = \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x), \tag{1}$$

where $\{Y_{\ell m}(\cdot)\}$ denotes the family of spherical harmonics, and $\{a_{\ell m}\}$ the array of random spherical harmonic coefficients, which satisfy $\mathbb{E} a_{\ell m} \overline{a}_{\ell' m'} = C_\ell \delta_\ell^{\ell'} \delta_m^{m'}$;

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here, δ_a^b is the Kronecker delta function, and the sequence $\{C_\ell\}$ represents the angular power spectrum of the field. As pointed out in [35], under isotropy the sequence C_ℓ necessarily satisfies $\sum_\ell \frac{(2\ell+1)}{4\pi} C_\ell = \mathbb{E}T^2 < \infty$ and the random field $T(x)$ is mean square continuous. Under the slightly stronger assumption $\sum_{\ell \geq L}(2\ell+1)C_\ell \leq O(\log^{-2} L)$, the field can be shown to be a.s. continuous, an assumption that we shall exploit heavily below.

Our attention will be focused on the Fourier components $\{T_\ell(x)\}$, which represent random eigenfunctions of the spherical Laplacian:

$$\Delta_{S^2} T_\ell = -\ell(\ell+1)T_\ell, \qquad \ell = 1, 2, \ldots.$$

A lot of recent work has been focused on the characterization of geometric features for $\{T_\ell\}$, under Gaussianity assumptions; for instance, [58, 59] studied the asymptotic behaviour of the nodal domains, proving an earlier conjecture by Berry on the variance of (functionals of) the zero sets of T_ℓ. In an earlier contribution, [14] had focused on the *Defect* or signed area, that is, the difference between the positive and negative regions; a central limit theorem for these statistics and more general nonlinear transforms of Fourier components was recently established by [37]. These studies have been motivated, for instance, by the analysis of so-called Quantum Chaos (see again [14]), where the behaviour of random eigenfunctions is taken as an approximation for the asymptotics in deterministic case, under complex boundary conditions. More often, spherical eigenfunctions emerge naturally from the analysis of the Fourier components of spherical random fields, as in (1). In the latter circumstances, several functionals of T_ℓ assume a great practical importance: to mention a couple, the squared norm of T_ℓ provides an unbiased sample estimate for the angular power spectrum C_ℓ,

$$\mathbb{E}\left\{\int_{S^2} T_\ell^2(x)\,dx\right\} = (2\ell+1)C_\ell,$$

while higher-order power lead to estimates of the so-called polyspectra, which have a great importance in the analysis of non-Gaussianity (see, e.g., [34]).

The previous discussion shows that the analysis of nonlinear functionals of $\{T_\ell\}$ may have a great importance for statistical applications, especially in the framework of cosmological data analysis. In this area, a number of papers have searched for deviations of geometric functionals from the expected behaviour under Gaussianity. For instance, the so-called Minkowski functionals have been widely used as tools to probe non-Gaussianity of the field $T(x)$; see [38] and the references therein. On the sphere, Minkowski functionals correspond to the area, the boundary length and the Euler–Poincaré characteristic of excursion sets, and up to constants they correspond to the Lipschitz–Killing curvatures we shall consider in this paper; see [5], page 144. Many other works have also focused on local deviations from the Gaussianity assumption, mainly exploiting the properties of integrated higher order moments (polyspectra); see [46, 49].

In general, the works aimed at the analysis of local phenomena are often based upon wavelets-like constructions, rather than standard Fourier analysis. The astrophysical literature on these issues is vast; see, for instance, [40, 50] and the references therein. Indeed, the double localization properties of wavelets (in real and harmonic domain) turn out usually to be extremely useful when handling real data.

In this paper, we shall focus on sequence of spherical random fields which can be viewed as averaged forms of the spherical eigenfunctions, for example,

$$\beta_j(x) = \sum_\ell b\left(\frac{\ell}{B^j}\right) T_\ell(x), \qquad j = 1, 2, 3 \dots$$

for $b(\cdot)$ a weight function whose properties we shall discuss immediately. The fields $\{\beta_j(x)\}$ can indeed be viewed as a representation of the coefficients from a continuous wavelet transform from $T(x)$, at scale j. More precisely, consider the kernel

$$\begin{aligned}\Psi_j(\langle x, y\rangle) &:= \sum_\ell b\left(\frac{\ell}{B^j}\right)\frac{2\ell+1}{4\pi} P_\ell(\langle x, y\rangle) \\ &= \sum_\ell b\left(\frac{\ell}{B^j}\right) \sum_{m=-\ell}^{\ell} Y_{\ell m}(x)\overline{Y}_{\ell m}(y).\end{aligned}$$

Assuming that $b(\cdot)$ is smooth (e.g., C^∞), compactly supported in $[B^{-1}, B]$, and satisfying the partition of unity property $\sum_j b^2(\frac{\ell}{B^j}) = 1$, for all $\ell > B$, where B is a fixed "bandwidth" parameter s.t. $B > 1$. Then $\Psi_j(\langle x, y\rangle)$ can be viewed as a continuous version of the needlet transform, which was introduced by Narcowich et al. in [41], and considered from the point of view of statistics and cosmological data analysis by many subsequent authors, starting from [10, 36, 47]. In this framework, the following localization property is now well known (see, e.g., [41], Theorem 3.5., [26], Lemma 4.1 or [34], Proposition 10.5): for all $M \in \mathbb{N}$, there exists a constant C_M (independent of j) such that

$$|\Psi_j(\langle x, y\rangle)| \le \frac{C_M B^{2j}}{\{1 + B^j d(x, y)\}^M}, \tag{2}$$

where $d(x, y) = \arccos(\langle x, y\rangle)$ is the usual geodesic distance on the sphere. Hence, the needlet field

$$\begin{aligned}\beta_j(x) &= \int_{S^2} \Psi_j(\langle x, y\rangle) T(y)\,dy \\ &= \int_{S^2} \sum_{\ell m} b\left(\frac{\ell}{B^j}\right) Y_{\ell m}(x)\overline{Y}_{\ell m}(y) \sum_{\ell' m'} a_{\ell' m'} Y_{\ell' m'}(y)\,dy \\ &= \sum_{\ell m}\sum_{\ell' m'} b\left(\frac{\ell}{B^j}\right) a_{\ell' m'} Y_{\ell m}(x) \delta_\ell^{\ell'} \delta_m^{m'} = \sum_\ell b\left(\frac{\ell}{B^j}\right) T_\ell(x)\end{aligned} \tag{3}$$

is then only locally determined, that is, for B^j large enough its value depends only on the behaviour of $T(y)$ in a neighbourhood of x. This is a very important property, for instance, when dealing with spherical random fields which can only be partially observed, the canonical example being provided by the masking effect of the Milky Way on Cosmic Microwave Background (CMB) radiation.

It is hence very natural to produce out of $\{\beta_j(x)\}$ nonlinear statistics of great practical relevance. To provide a concrete example, a widely disputed theme in CMB data analysis concerns the existence of asymmetries in the angular power spectrum; it has been indeed often suggested that the angular power $\{C_\ell\}$ may exhibit different behaviour for different subsets of the sky, at least over some multipole range; see, for instance, [28, 46]. It is readily seen that

$$\mathbb{E}\{\beta_j^2(x)\} = \sum_\ell b\left(\frac{\ell}{B^j}\right)\frac{2\ell+1}{4\pi}C_\ell,$$

which hence suggests a natural "local" estimator for a binned form of the angular power spectrum (note that the right-hand side does not depend on x, as a consequence of isotropy). More precisely, it is natural to consider some form of averaging and introduce the process

$$\int_{S^2} K(\langle z, x\rangle)\beta_j^2(x)\,dx, \qquad z \in S^2, \tag{4}$$

where $K(\langle\cdot,\cdot\rangle)$ is some kernel function whose properties we will discuss below; for instance, should we consider the behaviour of the angular power spectrum on the northern and southern hemisphere, we might focus on $z = N, S$, where N, S denote, respectively, the North and South Poles (compare [12, 28, 46, 48] and the references therein). In the rest of this paper, we shall be concerned with centred and normalized versions of (4), that is, processes of the form

$$g_{j;q}(z) := \int_{S^2} K(\langle z, x\rangle)H_q\left(\frac{\beta_j(x)}{\sqrt{\mathbb{E}\beta_j^2(x)}}\right)dx, \tag{5}$$

where $H_q(\cdot)$ is the Hermite polynomial of qth order; for instance, for $q = 3$ these processes could be exploited to investigate local variation in Gaussian and non-Gaussian features (see [49] and below for more discussion and details).

1.2. *Main result.* The purpose of this paper is to study the asymptotic behaviour for the expected value of the Euler characteristic and other geometric functionals for the excursion regions of sequences of fields such as $\{g_{j;q}(\cdot)\}$, and to exploit these results to obtain excursion probabilities in non-Gaussian circumstances. Indeed, on one hand these geometric functionals are of interest by themselves, as they provide the basis for implementing goodness-of-fit tests (compare [38]); on the other hand, they provide the clue for approximations of the excursion probabilities for $\{g_{j;q}(\cdot)\}$, by means of some weak convergence results we shall establish, in combination with some now classical arguments described in detail in the monograph [5].

It is important to stress that our results are obtained under a setting which is quite different from usual. In particular, the asymptotic theory is investigated in the high frequency sense, for example, assuming that a single realization of a spherical random field is observed at higher and higher resolution as more and more refined experiments are implemented. This is the setting adopted in [34]; see also [7, 51] for the related framework of fixed-domain asymptotics.

Due to the nature of high-frequency asymptotics, we cannot expect the finite-dimensional distributions of the processes we focus on to converge. This will require a more general notion of weak convergence, as developed, for instance, by [21, 23]. By means of this, we shall indeed show how to evaluate asymptotically valid excursion probabilities, which provide a natural solution for hypothesis testing problems. Indeed, the main result of the paper, Theorem 20, provides a very explicit bound for the excursion probabilities of non-Gaussian fields such as (5), for example,

$$
\begin{aligned}
&\limsup_{j\to\infty}\Big|\Pr\Big\{\sup_{x\in S^2}\tilde{g}_{j;q}(x)>u\Big\}-\big\{2\big(1-\Phi(u)\big)+2u\phi(u)\lambda_{j;q}\big\}\Big| \\
&\quad\le \exp\Big(-\frac{\alpha u^2}{2}\Big),
\end{aligned}
\tag{6}
$$

where $\tilde{g}_{j;q}(x)$ has been normalized to have unit variance, $\phi(\cdot)$, $\Phi(\cdot)$ denote standard Gaussian density and distribution function, $\alpha > 1$ is some constant and the parameters $\lambda_{j;q}$ have analytic expressions in terms of generalized convolutions of angular power spectra; see (32), (27). See also [42] for some related results on the distribution of maxima of approximate Gaussian random fields; note, however, that our approach is quite different from theirs and the tools we use allow us to get much stronger results in terms of the uniform estimates.

1.3. *Plan of the paper.* The plan of the paper is as follows: In Section 2, we review some background results on random fields and geometry, mainly referring to the now classical monograph [5]. Section 3 specializes these results to spherical random fields, for which some background theory is also provided, and provides some simple evaluations for Lipschitz–Killing curvatures related to excursion sets for harmonic components of such fields. More interesting Gaussian subordinated fields are considered in Section 4, where some detailed computations for covariances in general Gaussian subordinated circumstances are also provided. Section 5 provides the main convergence results, that is, shows how the distribution of these random elements are asymptotically proximal (in the sense of [21]) to those of a Gaussian sequence with the same covariances. This result is then exploited in Section 6, to provide the proof of (6). A number of possible applications on real cosmological data sets are discussed throughout the paper.

2. Background: Random fields and geometry. This section is devoted to recall basic integral geometric concepts, to state the Gaussian kinematic fundamental formula, and to discuss its application in evaluating the excursion probabilities. This theory has been developed in a series of fundamental papers by R. J. Adler, J. E. Taylor and coauthors (see [1, 4, 16, 52–54]), and it is summarized in the monographs [5, 6] which are our main references in this Section (see also [8, 9] for a different approach, and [2, 3, 18, 55] for some further developments in this area; applications to the sphere have also been considered very recently by [17, 19]).

2.1. *Lipschitz–Killing curvatures and Gaussian Minkowski functionals*. There are a number of ways to define Lipschitz–Killing curvatures, but perhaps the easiest is via the so-called tube formula, which, in its original form is due to Hotelling [29] and Weyl [57]. To state the tube formula, let M be an m-dimensional smooth subset of $\mathbb{R}^n$ such that ∂M is a C^2 manifold endowed with the canonical Riemannian structure on $\mathbb{R}^n$. The tube of radius ρ around M is defined as

$$\mathrm{Tube}(M,\rho)=\{x\in\mathbb{R}^n : d(x,M)\le\rho\}, \tag{7}$$

where

$$d(x,M)=\inf_{y\in M}\|x-y\|. \tag{8}$$

Then according to Weyl's tube formula (see [5]), the Lebesgue volume of this constructed tube, for small enough ρ, is given by

$$\lambda_n\big(\mathrm{Tube}(M,\rho)\big)=\sum_{j=0}^{m}\rho^{n-j}\omega_{n-j}\mathcal{L}_j(M), \tag{9}$$

where ω_j is the volume of the j-dimensional unit ball and $\mathcal{L}_j(M)$ is the jth-Lipschitz–Killing curvature (LKC) of M. A little more analysis shows that $\mathcal{L}_m(M)=\mathcal{H}_m(M)$, the m-dimensional Hausdorff measure of M, and that $\mathcal{L}_0(M)$ is the Euler–Poincaré characteristic of M. Although the remaining LKCs have less transparent interpretations, it is easy to see that they satisfy simple scaling relationships, in that $\mathcal{L}_j(\alpha M)=\alpha^j\mathcal{L}_j(M)$ for all $1\le j\le m$, where $\alpha M=\{x\in\mathbb{R}^n : x=\alpha y \text{ for some } y\in M\}$. Furthermore, despite the fact that defining the $\mathcal{L}_j$ via (9) involves the embedding of M in $\mathbb{R}^n$, the $\mathcal{L}_j(M)$ are actually intrinsic, and so are independent of the ambient space.

Apart from their appearance in the tube formula (9), there are a number of other ways in which to define the LKCs. One such (nonintrinsic) way which signifies the dependence of the LKCs on the Riemannian metric is through the shape operator. Let M be an m-dimensional C^2 manifold embedded in $\mathbb{R}^n$; then

$$\begin{aligned}\mathcal{L}_k(M)&\\ &=K_{n,m,k}\int_M\int_{S(N_xM)}\mathrm{Tr}\big(S_\nu^{(m-k)}\big)1_{N_xM}(-\nu)\mathcal{H}_{n-m-1}(d\nu)\mathcal{H}_{m-1}(dx),\end{aligned} \tag{10}$$

where, $K_{n,m,k} = \frac{1}{(m-k)!} \frac{\Gamma((n-k)/2)}{(2\pi)^{(n-k)/2}}$, and $S(N_x M)$ denotes a sphere in the normal space $N_x M$ of M at the point $x \in M$.

Closely related to the LKCs are set functionals called the Gaussian Minkowski functionals (GMFs), which are defined via a Gaussian tube formula. Consider the Gaussian measure, $\gamma_n(dx) = (2\pi)^{-n/2} e^{-\|x\|^2/2}\, dx$, instead of the standard Lebesgue measure in (9); the Gaussian tube formula is then given by

$$\gamma_n\big((M, \rho)\big) = \sum_{k \geq 0} \frac{\rho^k}{k!} \mathcal{M}_k^{\gamma_n}(M), \tag{11}$$

where the coefficients $\mathcal{M}_k^{\gamma_n}(M)$'s are the GMFs (for technical details, we refer the reader to [5]). We note that these set functionals, like their counterparts in (9) can be expressed as integrals over the manifold and its normal space (cf. [5]).

2.2. *Excursion probabilities and the Gaussian kinematic fundamental formula*. A classical problem in stochastic processes is to compute the excursion probability or the suprema probability

$$P\Big(\sup_{x \in M} f(x) \geq u\Big),$$

where, as before, f is a random field defined on the parameter space M. In the case when f happens to be a centered Gaussian field with constant variance σ^2 defined on M, a piecewise smooth manifold, then by the arguments set forth in Chapter 14 of [5], we have that

$$\Big|P\Big\{\sup_{x \in M} f(x) \leq u\Big\} - \mathbb{E}\big\{\mathcal{L}_0\big(A_u(f; M)\big)\big\}\Big| < O\bigg(\exp\bigg(-\frac{\alpha u^2}{2\sigma^2}\bigg)\bigg), \tag{12}$$

where $\mathcal{L}_0(A_u(f; M))$ is, as defined earlier, the Euler–Poincaré characteristic of the excursion set $A_u(f; M) = \{x \in M : f(x) \geq u\}$, and $\alpha > 1$ is a constant, which depends on the field f and can be determined (see Theorem 14.3.3 of [5]).

At first sight, from (12) it may appear that we may have to deal with a hard task, for example, that of evaluating $\mathbb{E}\{\mathcal{L}_0(A_u(f; M))\}$. This task, however, is greatly simplified due to the *Gaussian kinematic fundamental formula* (Gaussian-KFF) (see Theorems 15.9.4–15.9.5 in [5]), which states that, for a smooth $M \subset \mathbb{R}^N$

$$\begin{aligned}
&\mathbb{E}\big(\mathcal{L}_i^f\big(A_u(f, M)\big)\big)\\
&\quad = \sum_{\ell=0}^{\dim(M)-i} \binom{i+\ell}{\ell} \frac{\Gamma(i/2+1)\Gamma(\ell/2+1)}{\Gamma((i+\ell)/2+1)} (2\pi)^{-\ell/2} \mathcal{L}_{i+\ell}^f(M) \mathcal{M}_\ell^\gamma\big([u, \infty)\big),
\end{aligned}$$

for example, in the special case of the Euler characteristic ($i = 0$)

$$\mathbb{E}\big\{\mathcal{L}_0^f\big(A_u(f; M)\big)\big\} = \sum_{j=0}^{\dim(M)} (2\pi)^{-j/2} \mathcal{L}_j^f(M) \mathcal{M}_j^\gamma\big([u, \infty)\big), \tag{13}$$

where $\mathcal{L}_j^f(M)$ is the jth LKC of M with respect to the induced metric g^f given by

$$g_x^f(Y_x, Z_x) = \mathbb{E}\{Yf(x) \cdot Zf(x)\},$$

for $X_x, Y_x \in T_x M$, the tangent space at $x \in M$. The Gaussian kinematic fundamental formula will play a crucial role in all the developments to follow in the subsequent sections.

3. Spherical Gaussian fields. In this section, we shall start from some simple results on the evaluation of the expected values of Lipschitz–Killing curvatures for sequences of spherical Gaussian processes. These results will be rather straightforward applications of the Gaussian kinematic fundamental formula (13), and are collected here for completeness and as a bridge toward the more complicated case of nonlocal transforms of Gaussian subordinated processes, to be considered later.

Note first that for a unit variance Gaussian field on the sphere $f : S^2 \to \mathbb{R}$, the expected value of the Euler–Poincaré characteristic of the excursion set $A_u(f; S^2) = \{x \in S^2 : f(x) \geq u\}$ is given by

$$\begin{aligned}
&\mathbb{E}\{\mathcal{L}_0(A_u(f, S^2))\} \\
&\quad = \mathcal{L}_0^f(S^2)\mathcal{M}_0^\gamma([u,\infty)) + (2\pi)^{-1/2}\mathcal{L}_1^f(S^2)\mathcal{M}_1^\gamma([u,\infty)) \\
&\qquad + (2\pi)^{-1}\mathcal{L}_2^f(S^2)\mathcal{M}_2^\gamma([u,\infty)),
\end{aligned}$$

for

$$\mathcal{M}_0^\gamma([u,\infty)) = \int_u^\infty \phi(x)\,dx, \qquad \mathcal{M}_j^\gamma([u,\infty)) = H_{j-1}(u)\phi(u),$$

where $\phi(\cdot)$ denotes the density of a real valued standard normal random variable, and $H_j(u)$ denotes the Hermite polynomials,

$$H_j(u) = (-1)^j (\phi(u))^{-1} \frac{d^j}{du^j}\phi(u) \quad \text{and} \quad H_{-1}(u) = 1 - \Phi(u),$$

while $\mathcal{L}_k^f(S^2)$ are the usual Lipschitz–Killing curvatures, under the induced Gaussian metric, that is,

$$\mathcal{L}_k^f(S^2) := \frac{(-2\pi)^{-(2-k)/2}}{2} \int_{S^2} \operatorname{Tr}(R^{(N-k)/2}) \operatorname{Vol}_{g^f};$$

here, R is the Riemannian curvature tensor and Vol_{g^f} is the volume form, under the induced Gaussian metric, given by

$$g^f(X, Y) := \mathbb{E}\{Xf \cdot Yf\} = XY\mathbb{E}(f^2).$$

We recall that $\mathcal{L}_0(M)$ is a topological invariant and does not depend on the metric; in particular, $\mathcal{L}_0(S^2) \equiv 2$. Moreover, because the sphere is an (even-) 2-dimensional manifold, $\mathcal{L}_1^f(S^2)$ is identically zero.

As mentioned before, we start from some very simple result on the Fourier components and wavelets transforms of Gaussian fields, for example, the expected value of the Euler–Poincaré characteristic for two forms of harmonic components, namely

$$T_\ell(x) = \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x) \quad \text{and} \quad \beta_j(x) = \sum_\ell b\left(\frac{\ell}{B^j}\right) T_\ell(x),$$

the first representing a Fourier component at the multipole ℓ, the second a field of continuous needlet/wavelet coefficients at scale j. We normalize these processes to unit variance by taking

$$\widetilde{T}_\ell(x) = \frac{T_\ell(x)}{\sqrt{((2\ell+1)/(4\pi))C_\ell}} \quad \text{and}$$

$$\widetilde{\beta}_j(x) = \frac{\beta_j(x)}{\sqrt{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell}}.$$

We start reporting some simple results on Lipschitz–Killing curvatures of excursion sets generated by spherical Gaussian fields (see [38] and the references therein for related expressions on $\mathbb{R}^2$ from an astrophysical point of view). These results are straightforward consequences of equation (13).

LEMMA 1. *We have*

$$\mathcal{L}_2^{\widetilde{\beta}_j}(S^2) = 4\pi \frac{\sum_\ell b^2(\ell/B^j)(2\ell+1)C_\ell(\ell(\ell+1)/2)}{\sum_\ell b^2(\ell/B^j)(2\ell+1)C_\ell}.$$

PROOF. Recall first that, in standard spherical coordinates,

$$P_\ell(\langle x, y\rangle) = P_\ell(\sin\vartheta_x \sin\vartheta_y \cos(\phi_x - \phi_y) + \cos\vartheta_x \cos\vartheta_y).$$

Some simple algebra then yields

$$\frac{\partial^2}{\partial\vartheta_x\,\partial\vartheta_y} P_\ell(\langle x, y\rangle)\Big|_{x=y} = \frac{\partial^2}{\sin\vartheta_x \sin\vartheta_y \partial\phi_x\,\partial\phi_y} P_\ell(\langle x, y\rangle)\Big|_{x=y} = P_\ell'(1)$$

and

$$\frac{\partial^2}{\sin\vartheta_x \partial\vartheta_y\,\partial\phi_x} P_\ell(\langle x, y\rangle)\Big|_{x=y} = 0.$$

The geometric meaning of the latter result is that the process is still isotropic under the new transformation, whence the derivatives along the two directions are still independent. As a consequence, writing $\mathbb{E}\{\widetilde{\beta}_j(x)\widetilde{\beta}_j(y)\} =: \Gamma_j(x, y)$ we have

$$\begin{aligned}\frac{\partial^2\Gamma_j(x,y)}{\partial\vartheta_x\,\partial\vartheta_y}\Big|_{x=y} &= \frac{\partial^2\Gamma_j(x,y)}{\sin\vartheta_x \sin\vartheta_y \partial\phi_x\,\partial\phi_y}\Big|_{x=y} \\ &= \frac{\sum_\ell b^2(\ell/B^j)C_\ell((2\ell+1)/(4\pi))P_\ell'(1)}{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell}\end{aligned}$$

and

$$\left.\frac{\partial^2 \Gamma_j(x,y)}{\sin\vartheta_x \partial\vartheta_y\,\partial\phi_x}\right|_{x=y} = 0.$$

We thus have that

$$\begin{aligned}
&\mathcal{L}_2^{\widetilde{\beta}_j}(S^2)\\
&= \int_{S^2}\left\{\det\begin{bmatrix} \left.\frac{\partial^2\Gamma_j(x,y)}{\partial\vartheta_x\,\partial\vartheta_y}\right|_{x=y} & \left.\frac{\partial^2\Gamma_j(x,y)}{\sin\vartheta_x\partial\phi_x\,\partial\vartheta_y}\right|_{x=y} \\ \left.\frac{\partial^2\Gamma_j(x,y)}{\sin\vartheta_y\partial\phi_y\,\partial\vartheta_x}\right|_{x=y} & \left.\frac{\partial^2\Gamma_j(x,y)}{\sin\vartheta_x\sin\vartheta_y\partial\phi_x\,\partial\phi_y}\right|_{x=y}\end{bmatrix}\right\}^{1/2}\sin\vartheta\, d\vartheta\, d\phi\\
&= 4\pi\frac{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell P'_\ell(1)}{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell}.
\end{aligned}$$

Now recall that $P'_\ell(1) = \frac{\ell(\ell+1)}{2}$, whence the claim is established. □

REMARK 2. Note that since the random field β_j is an isotropic Gaussian random field, the Lipschitz–Killing curvatures of S^2 under the metric induced by the field β_j are given by

$$\mathcal{L}_i^{\widetilde{\beta}_j}(S^2) = \lambda_j^{i/2}\mathcal{L}_i(S^2),$$

where $\mathcal{L}_i(S^2)$ is the ith LKC under the usual Euclidean metric, and λ_j is the second spectral moment of $\widetilde{\beta}_j$ (cf. [5]). This result is true for all isotropic and unit variance Gaussian random fields.

The second auxiliary result that we shall need follows immediately from Theorem 13.2.1 in [5], specialized to isotropic spherical random fields with unit variance. Analogous expressions have been given (among many other results) in the two recent papers [17, 19]. The computations are straightforward and we report them only for completeness.

LEMMA 3. *For the Gaussian isotropic field* $\widetilde{\beta}_j : S^2 \to \mathbb{R}$, *such that* $\mathbb{E}\widetilde{\beta}_j = 0$, $\mathbb{E}\widetilde{\beta}_j^2 = 1$, $\widetilde{\beta}_j \in C^2(S^2)$ *almost surely, we have that*

$$\begin{aligned}
&\mathbb{E}\{\mathcal{L}_0(A_u(\widetilde{\beta}_j(x), S^2))\}\\
&\quad = 2\{1-\Phi(u)\} + 4\pi\left\{\frac{\sum_\ell b^2(\ell/B^j)C_\ell((2\ell+1)/(4\pi))P'_\ell(1)}{\sum_\ell b^2(\ell/B^j)C_\ell((2\ell+1)/(4\pi))}\right\}\frac{ue^{-u^2/2}}{\sqrt{(2\pi)^3}},
\end{aligned}\tag{14}$$

$$\begin{aligned}
&\mathbb{E}\{\mathcal{L}_1(A_u(\widetilde{\beta}_j(x), S^2))\}\\
&\quad = \pi\left\{\frac{\sum_\ell b^2(\ell/B^j)C_\ell((2\ell+1)/(4\pi))P'_\ell(1)}{\sum_\ell b^2(\ell/B^j)C_\ell((2\ell+1)/(4\pi))}\right\}^{1/2} e^{-u^2/2},
\end{aligned}\tag{15}$$

and finally

$$\mathbb{E}\{\mathcal{L}_2(A_u(\widetilde{\beta}_j(x), S^2))\} = \{1 - \Phi(u)\}4\pi. \tag{16}$$

PROOF. We start by recalling that, from Theorem 13.2.1 in [5],

$$\mathbb{E}\{\mathcal{L}_i(A_u(\widetilde{\beta}_j(x), S^2))\} = \sum_{\ell=0}^{\dim(S^2)-i} \begin{bmatrix} i+\ell \\ \ell \end{bmatrix} \lambda^{\ell/2} \rho_\ell(u) \mathcal{L}_{i+\ell}(S^2),$$

where

$$\begin{bmatrix} i+\ell \\ \ell \end{bmatrix} := \binom{i+\ell}{\ell} \frac{\omega_{i+\ell}}{\omega_i \omega_\ell}, \qquad \omega_i = \frac{\pi^{i/2}}{\Gamma(i/2+1)},$$

$$\rho_\ell(u) = (2\pi)^{-\ell/2} \mathcal{M}_\ell^\gamma([u,\infty)) = (2\pi)^{-(\ell+1)/2} H_{\ell-1}(u) e^{-u^2/2},$$

so that

$$\rho_0(u) = (2\pi)^{-1/2}\sqrt{2\pi}(1-\Phi(u)) e^{u^2/2} e^{-u^2/2} = (1-\Phi(u)),$$

$$\rho_1(u) = \frac{1}{2\pi} e^{-u^2/2}, \qquad \rho_2(u) = \frac{1}{\sqrt{(2\pi)^3}} u e^{-u^2/2}.$$

Here,

$$\lambda = \mathbb{E}\beta_{j;\vartheta}^2 = \mathbb{E}\beta_{j;\phi}^2, \qquad \beta_{j;\vartheta} = \frac{\partial}{\partial\vartheta}\beta_j(\vartheta,\phi),$$

$$\beta_{j;\phi} = \frac{\partial}{\sin\vartheta\,\partial\phi}\beta_j(\vartheta,\phi),$$

$$\mathbb{E}\{\widetilde{\beta}_{j;\vartheta}^2\} = \frac{\partial^2}{\partial\vartheta^2}\mathbb{E}\{\widetilde{\beta}_j^2\} = \frac{\sum_\ell b^2(\ell/B^j) C_\ell((2\ell+1)/(4\pi)) P_\ell'(1)}{\sum_\ell b^2(\ell/B^j) C_\ell((2\ell+1)/(4\pi))},$$

whence

$$\mathbb{E}\{\mathcal{L}_0(A_u(\widetilde{\beta}_j(x), S^2))\}$$
$$= 2\{1-\Phi(u)\} + 4\pi \left\{ \frac{\sum_\ell b^2(\ell/B^j) C_\ell((2\ell+1)/(4\pi)) P_\ell'(1)}{\sum_\ell b^2(\ell/B^j) C_\ell((2\ell+1)/(4\pi))} \right\} \frac{u e^{-u^2/2}}{\sqrt{(2\pi)^3}}.$$

Also,

$$\mathbb{E}\{\mathcal{L}_1(A_u(\widetilde{\beta}_j(x), S^2))\} = \pi \left\{ \frac{\sum_\ell b^2(\ell/B^j) C_\ell((2\ell+1)/(4\pi)) P_\ell'(1)}{\sum_\ell b^2(\ell/B^j) C_\ell((2\ell+1)/(4\pi))} \right\}^{1/2} e^{-u^2/2}.$$

Finally,

$$\mathbb{E}\{\mathcal{L}_2(A_u(\widetilde{\beta}_j(x), S^2))\} = \rho_0(u)\mathcal{L}_2(S^2) = \{1-\Phi(u)\}4\pi,$$

which completes the proof. □

In the case of spherical eigenfunctions, the previous lemma takes the following simpler form; the proof is entirely analogous, and hence omitted.

COROLLARY 4. *For the field* $\{T_\ell(\cdot)\}$, *we have that*

$$(17)\qquad \begin{aligned}\mathbb{E}\{\mathcal{L}_0(A_u(\widetilde{T}_\ell(\cdot), S^2))\} &= 2\{1-\Phi(u)\} + \frac{\ell(\ell+1)}{2}\frac{ue^{-u^2/2}}{\sqrt{(2\pi)^3}}4\pi,\\ \mathbb{E}\{\mathcal{L}_1(A_u(\widetilde{T}_\ell(\cdot), S^2))\} &= \pi\left\{\frac{\ell(\ell+1)}{2}\right\}^{1/2} e^{-u^2/2}\end{aligned}$$

and

$$\mathbb{E}\{\mathcal{L}_2(A_u(\widetilde{T}_\ell(\cdot), S^2))\} = 4\pi \times \{1-\Phi(u)\}.$$

REMARK 5. Using the differential geometric definition of the Lipschitz–Killing curvatures, it is easy to observe that

$$2\mathbb{E}\{\mathcal{L}_1(A_u(\widetilde{T}_\ell(\cdot), S^2))\} = \mathbb{E}\{\mathrm{len}(\partial A_u(\widetilde{T}_\ell(\cdot), S^2))\},$$

where $\mathrm{len}(\partial A_u(\widetilde{T}_\ell(\cdot), S^2))$ is the usual length of the boundary region of the excursion set, in the usual Hausdorff sense, which can also be expressed as $\mathcal{L}_1(\partial A_u(T_\ell(\cdot), S^2))$. Hence,

$$\mathbb{E}\{\mathrm{len}(\partial A_u(\widetilde{T}_\ell(\cdot), S^2))\} = 2\pi\left\{\frac{\ell(\ell+1)}{2}\right\}^{1/2} e^{-u^2/2},$$

which for $u = 0$ fits with well-known results on the expected value of nodal lines for random spherical eigenfunctions (see [59] and the references therein). Likewise

$$(18)\qquad \begin{aligned}&\mathbb{E}\{\mathrm{len}(\partial A_u(\widetilde{\beta}_j(\cdot), S^2))\}\\ &\quad= 2\pi\left\{\frac{\sum_\ell b^2(\ell/B^j)C_\ell((2\ell+1)/(4\pi))P_\ell'(1)}{\sum_\ell b^2(\ell/B^j)C_\ell((2\ell+1)/(4\pi))}\right\}^{1/2} e^{-u^2/2}.\end{aligned}$$

These formulae can be made more explicit by setting a specific form for the behaviour of the angular power spectrum $\{C_\ell\}$ and the weighting kernel $b(\cdot)$, see [25] for numerical results under conditions of astrophysical interest.

4. Gaussian subordinated fields.

4.1. *Local transforms of* $\beta_j(\cdot)$. For statistical applications, it is often more interesting to consider nonlinear transforms of random fields. For instance, in a CMB related environment a lot of efforts have been spent to investigate local fluctuations of angular power spectra; to this aim, moving averages of squared wavelet/needlet coefficients are usually computed; see, for instance, [46] and the references therein. Our purpose here is to derive some rigorous results on the behaviour of these statistics.

To this aim, let us consider first the simple squared field

$$H_{2j}(x) := H_2(\widetilde{\beta}_j(x)) = \frac{\beta_j^2(x)}{\sigma_{\beta_j}^2} - 1,$$

$$\sigma_{\beta_j}^2 := \sum_\ell b^2\left(\frac{\ell}{B^j}\right) C_\ell \frac{2\ell+1}{4\pi} = \mathbb{E}\beta_j^2(x).$$

The expected value of Lipschitz–Killing curvatures for the excursion regions of such fields is easily derived, indeed by the general Gaussian kinematic formula we have, for $u \geq -1$

$$\begin{aligned}
&\mathbb{E}\{\mathcal{L}_0^{\widetilde{\beta}_j}(A_u(H_2; S^2))\} \\
&\quad = \sum_{k=0}^{2} (2\pi)^{-k/2} \mathcal{L}_k^{\widetilde{\beta}_j}(S^2) \mathcal{M}_k^{\mathcal{N}}((-\infty, -\sqrt{u+1}) \cup (\sqrt{u+1}, \infty)) \\
&\quad = \sum_{k=0}^{2} (2\pi)^{-k/2} \mathcal{L}_k^{\widetilde{\beta}_j}(S^2) 2\mathcal{M}_k^{\mathcal{N}}((\sqrt{u+1}, \infty)) \\
&\quad = 4(1 - \Phi(\sqrt{u+1})) \\
&\qquad + \frac{1}{2\pi} \frac{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi)) C_\ell P_\ell'(1)}{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi)) C_\ell} \mathcal{L}_2(S^2) \frac{e^{-(u+1)/2}}{\sqrt{2\pi}} 2\sqrt{u+1}.
\end{aligned}$$

Likewise

$$\begin{aligned}
&\mathbb{E}\{\mathcal{L}_1^{\widetilde{\beta}_j}(A_u(H_2; S^2))\} \\
&\quad = \sum_{k=0}^{1} (2\pi)^{-k/2} \begin{bmatrix} k+1 \\ k \end{bmatrix} \mathcal{L}_{k+1}^{\widetilde{\beta}_j}(S^2) \mathcal{M}_k^{\mathcal{N}}((-\infty, -\sqrt{u+1}) \cup (\sqrt{u+1}, \infty)) \\
&\quad = \mathcal{L}_1^{\widetilde{\beta}_j}(S^2) \mathcal{M}_0^{\mathcal{N}}((-\infty, -\sqrt{u+1}) \cup (\sqrt{u+1}, \infty)) \\
&\qquad + (2\pi)^{-1/2} \frac{\pi}{2} \mathcal{L}_2^{\widetilde{\beta}_j}(S^2) \mathcal{M}_1^{\mathcal{N}}((-\infty, -\sqrt{u+1}) \cup (\sqrt{u+1}, \infty)) \\
&\quad = (2\pi)^{-1/2} \frac{\pi}{2} \left(4\pi \times \frac{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi)) C_\ell P_\ell'(1)}{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi)) C_\ell}\right) 2 \frac{e^{-(u+1)/2}}{\sqrt{2\pi}} \\
&\quad = 2\pi \left(\frac{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi)) C_\ell P_\ell'(1)}{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi)) C_\ell}\right) e^{-(u+1)/2},
\end{aligned}$$

which implies for the Euclidean LKC

$$\mathbb{E}\{\mathcal{L}_1(A_u(H_2; S^2))\} = 2\pi \left\{\frac{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi)) C_\ell P_\ell'(1)}{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi)) C_\ell}\right\}^{1/2} e^{-(u+1)/2}$$

and, therefore,

$$\mathbb{E}\{\mathcal{L}_1(\partial A_u(H_2; S^2))\} = 4\pi \left\{ \frac{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell P'_\ell(1)}{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell} \right\}^{1/2} e^{-(u+1)/2}.$$

Finally,

$$\begin{aligned}
&\mathbb{E}\{\mathcal{L}_2^{\widetilde{\beta}_j}(A_u(H_2; S^2))\} \\
&\quad = \mathcal{L}_2^{\widetilde{\beta}_j}(S^2)\mathcal{M}_0^{\mathcal{N}}((-\infty, -\sqrt{u+1}) \cup (\sqrt{u+1}, \infty)) \\
&\quad = 4\pi \left\{ \frac{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell P'_\ell(1)}{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell} \right\} 2(1 - \Phi(\sqrt{u+1}))
\end{aligned}$$

entailing a Euclidean LKC

$$\mathbb{E}\{\mathcal{L}_2(A_u(H_2; S^2))\} = 4\pi \times 2(1 - \Phi(\sqrt{u+1})).$$

It should be noted that the tail decay for the Euler characteristic and the boundary length is much slower than in the Gaussian case. This is consistent with the elementary fact that polynomial transforms shift angular power spectra at higher frequencies, hence yielding a rougher path behaviour. Likewise, for cubic transforms we have

$$\begin{aligned}
&\mathbb{E}\{\mathcal{L}_0^{\widetilde{\beta}_j}(A_u(\widetilde{\beta}_j^3(x); S^2))\} \\
&\quad = 2(1 - \Phi(\sqrt[3]{u})) \\
&\qquad + \frac{1}{2\pi} \frac{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell P'_\ell(1)}{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell} \mathcal{L}_2(S^2) \frac{e^{-(\sqrt[3]{u})^2/2}}{\sqrt{2\pi}} \sqrt[3]{u}, \\
&\mathbb{E}\{\mathcal{L}_1^{\widetilde{\beta}_j}(A_u(\widetilde{\beta}_j^3(x); S^2))\} \\
&\quad = \pi \left(\frac{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell P'_\ell(1)}{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell} \right) e^{-(\sqrt[3]{u})^2/2}, \\
&\mathbb{E}\{\mathcal{L}_1(\partial A_u(\widetilde{\beta}_j^3(x); S^2))\} \\
&\quad = 2\pi \left\{ \frac{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell P'_\ell(1)}{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell} \right\}^{1/2} e^{-(\sqrt[3]{u})^2/2},
\end{aligned}$$

and finally

$$\begin{aligned}
&\mathbb{E}\{\mathcal{L}_2^{\widetilde{\beta}_j}(A_u(\widetilde{\beta}_j^3(x); S^2))\} \\
&\quad = 4\pi \left\{ \frac{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell P'_\ell(1)}{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell} \right\} 2(1 - \Phi(\sqrt[3]{u}))
\end{aligned}$$

entailing an expected value for the excursion area given by

$$\mathbb{E}\{\mathcal{L}_2(A_u(\widetilde{\beta}_j^3(x); S^2))\} = 4\pi(1 - \Phi(\sqrt[3]{u})).$$

Similar results could be easily derived for higher order polynomial transforms; numerical evidence and astrophysical applications can be found in [25]. However, as motivated above we believe it is much more important to focus on transforms that entail some form of local averaging, as these are likely to be more relevant for practitioners. To this issue, we devote the rest of this section and a large part of the paper.

4.2. *Nonlocal transforms of* $\beta_j(\cdot)$. We now consider the case of smoothed nonlinear functionals. We are interested, for instance, in studying the LKCs for local estimates of the angular power spectrum, which as mentioned before have already found many important applications in a CMB related framework. To this aim, we introduce, for every $x \in S^2$,

$$g_{j;q}(x) := \int_{S^2} K(\langle x, y\rangle) H_q(\widetilde{\beta}_j(y))\, dy; \tag{19}$$

throughout the sequel, we shall assume that the following finite-order expansion holds:

$$K(u) = \sum_{\ell=1}^{L_K} \frac{2\ell+1}{4\pi} \kappa(\ell) P_\ell(u) \qquad \text{some fixed } L_K \in \mathbb{N}, u \in [-1, 1]. \tag{20}$$

Here, as before we write $H_q(\cdot)$ for the Hermite polynomials. For $q = 1$, we just get the smoothed Gaussian process

$$g_j(x) := g_{j;1}(x) = \int_{S^2} K(\langle x, y\rangle) \widetilde{\beta}_j(y)\, dy. \tag{21}$$

The practical importance of the analysis of fields such as $g_{j;q}(\cdot)$ can be motivated as follows. A crucial topic when dealing with cosmological data is the analysis of isotropy properties. For instance, in a CMB related framework a large amount of work has focused on the possible existence of asymmetries in the behaviour of angular power spectra or bispectra across different hemispheres (see, e.g., [46, 49]). In these papers, powers of wavelet coefficients at some frequencies j are averaged over different hemispheres to investigate the existence of asymmetries/anisotropies in the CMB distribution; some evidence has been reported, for instance, for power asymmetries with respect to the Milky Way plane for frequencies corresponding to angular scales of a few degrees (such effects are related in the cosmological literature to widely debated anomalies known as *the Cold Spot* and *the Axis of Evil*; see [12, 48] and the references therein). To investigate these anomalies, statistics which can be viewed as discretized versions of $\sup_{x\in S^2} g_{j;q}(x)$ have been evaluated; their significance is typically tested against Monte Carlo simulations, under the null of isotropy. Our results below will provide the first rigorous derivation of asymptotic properties in this settings.

Our first lemma is an immediate application of spherical Fourier analysis techniques.

LEMMA 6. *The field $g_j(x)$ is zero-mean, finite variance and isotropic, with covariance function*

$$\mathbb{E}\{g_j(x_1)g_j(x_2)\} = \frac{1}{\sigma^2_{\beta_j}} \sum_\ell b^2\left(\frac{\ell}{B^j}\right)\kappa^2(\ell)\frac{2\ell+1}{4\pi}C_\ell P_\ell(\langle x_1, x_2\rangle).$$

PROOF. Note first that

$$\begin{aligned}
&\mathbb{E}\{g_j(x_1)g_j(x_2)\} \\
&\quad = \frac{1}{\sigma^2_{\beta_j}}\left\{\int_{S^2\times S^2} K(\langle x_1, y_1\rangle)K(\langle x_2, y_2\rangle)\mathbb{E}\{\beta_j(y_1)\beta_j(y_2)\}\,dy_1\,dy_2\right\} \\
&\quad = \frac{1}{\sigma^2_{\beta_j}}\int_{S^2\times S^2} K(\langle x_1, y_1\rangle)K(\langle x_2, y_2\rangle)\sum_\ell b^2\left(\frac{\ell}{B^j}\right)\frac{2\ell+1}{4\pi}C_\ell P_\ell(\langle y_1, y_2\rangle).
\end{aligned}$$

Recall the reproducing kernel formula (see, e.g., [34], pages 248–249)

$$\begin{aligned}
\int_{S^2} P_\ell(\langle x_1, y_1\rangle)P_\ell(\langle y_1, y_2\rangle)\,dy_1 &= \frac{4\pi}{2\ell+1}P_\ell(\langle x_1, y_2\rangle), \\
\int_{S^2} P_{\ell_1}(\langle x_1, y_1\rangle)P_{\ell_2}(\langle y_1, y_2\rangle)\,dy_1 &= 0, \qquad \ell_1 \neq \ell_2,
\end{aligned}$$

whence

$$\begin{aligned}
&\int_{S^2\times S^2} K(\langle x_1, y_1\rangle)K(\langle x_2, y_2\rangle)\sum_\ell b^2\left(\frac{\ell}{B^j}\right)\frac{2\ell+1}{4\pi}C_\ell P_\ell(\langle y_1, y_2\rangle) \\
&\quad = \int_{S^2\times S^2}\sum_{\ell_1}\frac{2\ell_1+1}{4\pi}\kappa(\ell_1)P_{\ell_1}(\langle x_1, y_1\rangle)\sum_{\ell_2}\frac{2\ell_2+1}{4\pi}\kappa(\ell_2)P_{\ell_2}(\langle x_2, y_2\rangle) \\
&\qquad\qquad \times \sum_\ell b^2\left(\frac{\ell}{B^j}\right)\frac{2\ell+1}{4\pi}C_\ell P_\ell(\langle y_1, y_2\rangle)\,dy_1\,dy_2 \\
&\quad = \int_{S^2}\sum_\ell b^2\left(\frac{\ell}{B^j}\right)\frac{2\ell+1}{4\pi}C_\ell\sum_{\ell_1}\kappa(\ell_1)\sum_{\ell_2}\frac{2\ell_2+1}{4\pi}\kappa(\ell_2)P_{\ell_2}(\langle x_2, y_2\rangle) \\
&\qquad\qquad \times \int_{S^2}\frac{2\ell_1+1}{4\pi}P_{\ell_1}(\langle x_1, y_1\rangle)P_\ell(\langle y_1, y_2\rangle)\,dy_1\,dy_2 \\
&\quad = \sum_\ell b^2\left(\frac{\ell}{B^j}\right)\kappa(\ell)\frac{2\ell+1}{4\pi}C_\ell\sum_{\ell_2}\frac{2\ell_2+1}{4\pi}\kappa(\ell_2)
\end{aligned}$$

$$\times \int_{S^2} P_{\ell_2}(\langle x_2, y_2\rangle) P_\ell(\langle x_1, y_2\rangle)\, dy_2$$
$$= \sum_\ell b^2\left(\frac{\ell}{B^j}\right)\kappa^2(\ell)\frac{2\ell+1}{4\pi} C_\ell P_\ell(\langle x_1, x_2\rangle),$$

as claimed. □

The derivation of analogous results in the case of $q \geq 2$ requires more work and extra notation. In particular, we shall need the Wigner's $3j$ coefficients, which are defined by [for $m_1 + m_2 + m_3 = 0$, see [56], expression (8.2.1.5)]

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$
$$:= (-1)^{\ell_1+m_1}\sqrt{2\ell_3+1}\left[\frac{(\ell_1+\ell_2-\ell_3)!(\ell_1-\ell_2+\ell_3)!(\ell_1-\ell_2+\ell_3)!}{(\ell_1+\ell_2+\ell_3+1)!}\right]^{1/2}$$
$$\times \left[\frac{(\ell_3+m_3)!(\ell_3-m_3)!}{(\ell_1+m_1)!(\ell_1-m_1)!(\ell_2+m_2)!(\ell_2-m_2)!}\right]^{1/2}$$
$$\times \sum_z \frac{(-1)^z(\ell_2+\ell_3+m_1-z)!(\ell_1-m_1+z)!}{z!(\ell_2+\ell_3-\ell_1-z)!(\ell_3+m_3-z)!(\ell_1-\ell_2-m_3+z)!},$$

where the summation runs over all z's such that the factorials are nonnegative. This expression becomes somewhat neater for $m_1 = m_2 = m_3 = 0$, where we have

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \tag{22}$$
$$= \begin{cases} 0, & \text{for } \ell_1+\ell_2+\ell_3 \text{ odd}, \\ (-1)^{(\ell_1+\ell_2-\ell_3)/2} \\ \quad \times \dfrac{[(\ell_1+\ell_2+\ell_3)/2]!}{[(\ell_1+\ell_2-\ell_3)/2]![(\ell_1-\ell_2+\ell_3)/2]![(-\ell_1+\ell_2+\ell_3)/2]!} \\ \quad \times \left\{\dfrac{(\ell_1+\ell_2-\ell_3)!(\ell_1-\ell_2+\ell_3)!(-\ell_1+\ell_2+\ell_3)!}{(\ell_1+\ell_2+\ell_3+1)!}\right\}^{1/2}, \\ \quad \text{for } \ell_1+\ell_2+\ell_3 \text{ even}. \end{cases}$$

It is occasionally more convenient to focus on Clebsch–Gordan coefficients, which are related to the Wigner's by a simple change of normalization, for example,

$$C_{\ell_1 m_1 \ell_2 m_2}^{\ell_3 m_3} := \frac{(-1)^{\ell_3-m_3}}{\sqrt{2\ell_3+1}}\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}. \tag{23}$$

Wigner's $3j$ coefficients are elements of unitary matrices which intertwine alternative reducible representations of the group of rotations SO(3), and because of this emerge naturally in the evaluation of multiple integrals of spherical harmonics (see Section 3.5.2 of [34]) . As a consequence, they also appear in the covariances of nonlinear transforms; for $q = 2$, we have indeed

LEMMA 7. *The field $g_{j;2}(x)$ is zero-mean, finite variance and isotropic, with covariance function*

$$\begin{aligned}\mathbb{E}\{g_{j;2}(x_1)g_{j;2}(x_2)\} &= \frac{2}{\sigma^4_{\beta_j}}\sum_\ell \kappa^2(\ell)\frac{2\ell+1}{4\pi}\sum_{\ell_1\ell_2} b^2\left(\frac{\ell_1}{B^j}\right)b^2\left(\frac{\ell_2}{B^j}\right)\\ &\quad\times\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi}\\ &\quad\times C_{\ell_1}C_{\ell_2}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ 0 & 0 & 0\end{pmatrix}^2 P_\ell(\langle x_1,x_2\rangle).\end{aligned}$$

PROOF. Note first that

$$\begin{aligned}\mathbb{E}\{g_{j;2}(x_1)g_{j;2}(x_2)\} &= \mathbb{E}\left\{\int_{S^2} K(\langle x_1,y_1\rangle)H_2(\widetilde{\beta}_j(y_1))\,dy_1\int_{S^2}K(\langle x_2,y_2\rangle)H_2(\widetilde{\beta}_j(y_2))\,dy_2\right\}\\ &= \int_{S^2\times S^2} K(\langle x_1,y_1\rangle)K(\langle x_2,y_2\rangle)\mathbb{E}\{H_2(\widetilde{\beta}_j(y_1))H_2(\widetilde{\beta}_j(y_2))\}\,dy_1\,dy_2\\ &= \frac{2}{\sigma^4_{\beta_j}}\int_{S^2\times S^2} K(\langle x_1,y_1\rangle)K(\langle x_2,y_2\rangle)\\ &\qquad\times\left\{\sum_\ell b^2\left(\frac{\ell}{B^j}\right)\frac{2\ell+1}{4\pi}C_\ell P_\ell(\langle y_1,y_2\rangle)\right\}^2 dy_1\,dy_2\\ &= \frac{2}{\sigma^4_{\beta_j}}\int_{S^2\times S^2}\sum_{\ell_1}\frac{2\ell_1+1}{4\pi}\kappa(\ell_1)P_{\ell_1}(\langle x_1,y_1\rangle)\sum_{\ell_2}\frac{2\ell_2+1}{4\pi}\kappa(\ell_2)P_{\ell_2}(\langle x_2,y_2\rangle)\\ &\qquad\times\sum_{\ell_3\ell_4}b^2\left(\frac{\ell_3}{B^j}\right)b^2\left(\frac{\ell_4}{B^j}\right)\frac{2\ell_3+1}{4\pi}\frac{2\ell_4+1}{4\pi}\\ &\qquad\times C_{\ell_3}C_{\ell_4}P_{\ell_3}(\langle y_1,y_2\rangle)P_{\ell_4}(\langle y_1,y_2\rangle)\,dy_1\,dy_2,\end{aligned}$$

where in the third step we have used the covariance formula for Hermite polynomials in zero-mean, unit variance Gaussian variables (see, e.g., [34], Remark 4.10)

$$\mathbb{E}\{H_q(X)H_{q'}(Y)\} = \delta_q^{q'} q!\{\mathbb{E}XY\}^q, \tag{24}$$

which in this case yields

$$\mathbb{E}\{H_2(\widetilde{\beta}_j(y_1))H_2(\widetilde{\beta}_j(y_2))\} = \frac{2}{\sigma^4_{\beta_j}}\left\{\sum_\ell b^2\left(\frac{\ell}{B^j}\right)\frac{2\ell+1}{4\pi}C_\ell P_\ell(\langle y_1,y_2\rangle)\right\}^2.$$

Now recall that

$$\begin{aligned}
&\int_{S^2} P_{\ell_1}(\langle x_1, y_1\rangle) P_{\ell_3}(\langle y_1, y_2\rangle) P_{\ell_4}(\langle y_1, y_2\rangle)\, dy_1 \\
&\quad = \frac{(4\pi)^3}{(2\ell_1+1)(2\ell_3+1)(2\ell_4+1)} \\
&\qquad \times \int_{S^2} \sum_{m_1 m_3 m_4} Y_{\ell_1 m_1}(y_1)\overline{Y}_{\ell_1 m_1}(x_1) Y_{\ell_3 m_3}(y_1)\overline{Y}_{\ell_3 m_3}(y_2) \\
&\qquad\qquad \times Y_{\ell_4 m_4}(y_1)\overline{Y}_{\ell_4 m_4}(y_2)\, dy_1 \\
&\quad = \biggl(\frac{(4\pi)^5}{(2\ell_1+1)(2\ell_3+1)(2\ell_4+1)}\biggr)^{1/2} \\
&\qquad \times \sum_{m_1 m_3 m_4} \begin{pmatrix} \ell_1 & \ell_3 & \ell_4 \\ m_1 & m_3 & m_4 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_3 & \ell_4 \\ 0 & 0 & 0 \end{pmatrix} \\
&\qquad \times \overline{Y}_{\ell_1 m_1}(x_1)\overline{Y}_{\ell_3 m_3}(y_2)\overline{Y}_{\ell_4 m_4}(y_2).
\end{aligned}$$

Likewise

$$\begin{aligned}
&\int_{S^2} P_{\ell_2}(\langle x_2, y_2\rangle)\overline{Y}_{\ell_3 m_3}(y_2)\overline{Y}_{\ell_4 m_4}(y_2)\, dy_2 \\
&\quad = \frac{4\pi}{2\ell_2+1} \int_{S^2} \sum_{m_2} \overline{Y}_{\ell_2 m_2}(y_2) Y_{\ell_2 m_2}(x_2)\overline{Y}_{\ell_3 m_3}(y_2)\overline{Y}_{\ell_4 m_4}(y_2)\, dy_2 \\
&\quad = \sqrt{\frac{(4\pi)(2\ell_3+1)(2\ell_4+1)}{2\ell_2+1}} \\
&\qquad \times \sum_{m_2} \begin{pmatrix} \ell_2 & \ell_3 & \ell_4 \\ m_2 & m_3 & m_4 \end{pmatrix} \begin{pmatrix} \ell_2 & \ell_3 & \ell_4 \\ 0 & 0 & 0 \end{pmatrix} Y_{\ell_2 m_2}(x_2).
\end{aligned}$$

Using the orthonormality properties of Wigner's $3j$ coefficients (see again [34], Chapter 3.5), we have

$$\sum_{m_3 m_4} \begin{pmatrix} \ell_1 & \ell_3 & \ell_4 \\ m_1 & m_3 & m_4 \end{pmatrix} \begin{pmatrix} \ell_2 & \ell_3 & \ell_4 \\ m_2 & m_3 & m_4 \end{pmatrix} = \frac{\delta_{m_1}^{m_2}\delta_{\ell_1}^{\ell_2}}{(2\ell_1+1)},$$

whence we get

$$\begin{aligned}
&\mathbb{E}\{g_{j;2}(x_1) g_{j;2}(x_2)\} \\
&\quad = \frac{2}{\sigma_{\beta_j}^4} \sum_{\ell} \kappa^2(\ell) \frac{2\ell+1}{4\pi} \\
&\qquad \times \sum_{\ell_1 \ell_2} b^2\biggl(\frac{\ell_1}{B^j}\biggr) b^2\biggl(\frac{\ell_2}{B^j}\biggr) \\
&\qquad\qquad \times \frac{(2\ell_1+1)(2\ell_2+1)}{4\pi} C_{\ell_1} C_{\ell_2} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix}^2 P_{\ell}(\langle x_1, x_2\rangle),
\end{aligned}$$

as claimed. As a special case, the variance is provided by

$$\mathbb{E}g_{j;2}^2(x) = \frac{2}{\sigma_{\beta_j}^4} \sum_{\ell} \kappa^2(\ell) \frac{2\ell+1}{4\pi} \times \sum_{\ell_1 \ell_2} b^2\left(\frac{\ell_1}{B^j}\right) b^2\left(\frac{\ell_2}{B^j}\right) \times \frac{(2\ell_1+1)(2\ell_2+1)}{4\pi} C_{\ell_1} C_{\ell_2} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix}^2. \qquad \square$$

REMARK 8. Since the field $\{g_{j;2}(\cdot)\}$ has finite-variance and it is isotropic, it admits itself a spectral representation. Indeed, it is a simple computation to show that the corresponding angular power spectrum is provided by

$$C_{\ell;j,2} := \frac{2}{\sigma_{\beta_j}^4} \kappa^2(\ell) \sum_{\ell_1 \ell_2} b^2\left(\frac{\ell_1}{B^j}\right) b^2\left(\frac{\ell_2}{B^j}\right) \frac{(2\ell_1+1)(2\ell_2+1)}{4\pi} \times C_{\ell_1} C_{\ell_2} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix}^2, \tag{25}$$

for $\ell = 1, 2, \ldots$. This result will have a great relevance for the practical implementation of the findings in the next sections.

4.2.1. *Higher-order transforms.* The general case of nonlinear transforms with $q \geq 3$ can be dealt with analogous lines; the main difference being the appearance of multiple integrals of spherical harmonics of order greater than 3, and hence so-called higher order Gaunt integrals and convolutions of Clebsch–Gordan coefficients. For brevity's sake, we provide only the basic details; we refer to [34] for a more detailed discussion on nonlinear transforms of Gaussian spherical harmonics. Here, we simply recall the definition of the multiple Gaunt integral (see [34], Remark 6.30 and Theorem 6.31), which is given by

$$\mathcal{G}(\ell_1, m_1; \ldots \ell_q, m_q; \ell, m) := \int_{S^2} Y_{\ell_1 m_1}(x) \cdots Y_{\ell_q m_q}(x) Y_{\ell m}(x)\, d\sigma(x),$$

where the coefficients $\mathcal{G}(\ell_1, m_1; \ldots \ell_q, m_q; \ell, m)$ can be expressed as multiple convolution of Wigner/Clebsch–Gordan terms (see 23),

$$\mathcal{G}(\ell_1, m_1; \ldots \ell_q, m_q; \ell, m) = (-1)^m \sqrt{\frac{(2\ell_1+1)\cdots(2\ell_q+1)}{(4\pi)^{q-1}(2\ell+1)}} \times \sum_{\lambda_1 \cdots \lambda_{q-2}} C_{\ell_1 0 \ell_2 0}^{\lambda_1 0} C_{\lambda_1 0 \ell_3 0}^{\lambda_2 0} \cdots C_{\lambda_{q-2} 0 \ell_q 0}^{\ell 0} \times \sum_{\mu_1 \cdots \mu_{q-2}} C_{\ell_1 m_1 \ell_2 m_2}^{\lambda_1 \mu_1} C_{\lambda_1 \mu_1 \ell_3 m_3}^{\lambda_2 \mu_2} \cdots C_{\lambda_{q-2} \mu_{q-2} \ell_q m_q}^{\ell m}.$$

Following also [34], equation (6.40), let us introduce the shorthand notation

$$C^{\lambda_1\cdots\lambda_{q-2}\ell 0}_{\ell_1 0\ell_2 0\cdots\ell_q 0} := C^{\lambda_1 0}_{\ell_1 0\ell_2 0} C^{\lambda_2 0}_{\lambda_1 0\ell_3 0}\cdots C^{\ell 0}_{\lambda_{q-2}0\ell_q 0},$$
$$\mathcal{C}(\ell_1,\dots,\ell_q,\ell) := \sum_{\lambda_1\cdots\lambda_{q-2}} \{C^{\lambda_1\cdots\lambda_{q-2}\ell 0}_{\ell_1 0\ell_2 0\cdots\ell_q 0}\}^2. \tag{26}$$

It should be noted that, from the unitary properties of Clebsch–Gordan coefficients

$$\sum_\ell \mathcal{C}(\ell_1,\dots,\ell_q,\ell) = \sum_{\lambda_1\cdots\lambda_{q-2}} \{C^{\lambda_1 0}_{\ell_1 0\ell_2 0}\}^2 \cdots \sum_\ell \{C^{\ell 0}_{\lambda_{q-2}0\ell_q 0}\}^2 = \cdots = 1.$$

LEMMA 9. *For general $q \geq 3$, the field $g_{j;q}(x)$ is zero-mean, finite variance and isotropic, with covariance function*

$$\mathbb{E}\{g_{j;q}(x_1)g_{j;q}(x_2)\} = \frac{q!}{\sigma^{2q}_{\beta_j}} \sum_\ell \kappa^2(\ell) \sum_{\ell_1\cdots\ell_q} \mathcal{C}(\ell_1,\dots,\ell_q,\ell) \times \left[\prod_{k=1}^q b^2\left(\frac{\ell_k}{B^j}\right)\frac{2\ell_k+1}{4\pi} C_{\ell_k}\right] P_\ell(\langle x_1,x_2\rangle).$$

PROOF. We have

$$\begin{aligned}\mathbb{E}g^2_{j;q}(x) &= \mathbb{E}\left\{\int_{S^2}\int_{S^2} K(\langle x,y_1\rangle)K(\langle x,y_2\rangle)H_q(\widetilde{\beta}_j(y_1))H_q(\widetilde{\beta}_j(y_2))\,dy_1\,dy_2\right\} \\ &= \frac{q!}{\sigma^{2q}_{\beta_j}}\int_{S^2}\int_{S^2} K(\langle x,y_1\rangle)K(\langle x,y_2\rangle) \\ &\qquad \times \left\{\sum_\ell b^2\left(\frac{\ell}{B^j}\right)\frac{2\ell+1}{4\pi}P_\ell(\langle y_1,y_2\rangle)\right\}^q dy_1\,dy_2,\end{aligned}$$

where we have used the covariance formula for Hermite polynomials (24). It is convenient here to view $T_\ell(x)$, $\beta_j(x)$ as isonormal processes of the form

$$T_\ell(x) = \int_{S^2}\sqrt{\frac{2\ell+1}{4\pi}C_\ell}\,P_\ell(\langle x,y\rangle)\,dW(y),$$

$$\beta_j(x) = \frac{1}{\sigma_{\beta_j}}\int_{S^2}\sum_\ell b\left(\frac{\ell}{B^j}\right)\sqrt{\frac{2\ell+1}{4\pi}C_\ell}\,P_\ell(\langle x,y\rangle)\,dW(y),$$

where $dW(y)$ denotes a Gaussian white noise measure on the sphere, whence

$$\begin{aligned} &H_q(\beta_j(x)) \\ &\quad = \frac{1}{\sigma_{\beta_j}^q} \sum_{\ell_1 \cdots \ell_q} b\left(\frac{\ell_1}{B^j}\right) \cdots b\left(\frac{\ell_q}{B^j}\right) \sqrt{\prod_{i=1}^q \left\{\frac{2\ell_i+1}{4\pi} C_{\ell_i}\right\}} \\ &\qquad \times \int_{\{S^2 \times \cdots \times S^2\}'} P_{\ell_1}(\langle x, y_1\rangle) \cdots P_{\ell_q}(\langle x, y_q\rangle)\, dW(y_1) \cdots dW(y_q). \end{aligned}$$

Here, the domain of integration excludes the "diagonals," that is,

$$\{S^2 \times \cdots \times S^2\}' := \{(x_1, \dots, x_q) \in S^2 \times \cdots \times S^2 : x_i \neq x_j \text{ for all } i \neq j\},$$

and we are using the characterization of Hermite polynomials as multiple Wiener–Itô integrals; see, for instance, Theorem 2.7.7 in [44]. We are thus led to

$$\begin{aligned} g_{j;q}(z) = \frac{1}{\sigma_{\beta_j}^q} \int_{S^2} \sum_{\ell} \kappa(\ell) \frac{2\ell+1}{4\pi} P_\ell(\langle z, x\rangle) \\ \times \sum_{\ell_1 \cdots \ell_q} b\left(\frac{\ell_1}{B^j}\right) \cdots b\left(\frac{\ell_q}{B^j}\right) \sqrt{\prod_{i=1}^q \left\{\frac{2\ell_i+1}{4\pi} C_{\ell_i}\right\}} \\ \times \int_{S^2 \times \cdots \times S^2} P_{\ell_1}(\langle x, y_1\rangle) \cdots \\ \times P_{\ell_q}(\langle x, y_q\rangle)\, dW(y_1) \cdots dW(y_q)\, dx. \end{aligned}$$

Using the isometry property of stochastic integrals, it follows easily that

$$\begin{aligned} &\mathbb{E}\{g_{j;q}(z_1) g_{j;q}(z_2)\} \\ &\quad = \frac{q!}{\sigma_{\beta_j}^{2q}} \int_{S^2 \times S^2} \sum_{\ell_1 \ell_2} \frac{2\ell_1+1}{4\pi} \kappa(\ell_1) \frac{2\ell_2+1}{4\pi} \kappa(\ell_2) P_{\ell_1}(\langle z_1, x_1\rangle) P_{\ell_2}(\langle z_2, x_2\rangle) \\ &\qquad \times \sum_{\ell_1 \cdots \ell_q} b^2\left(\frac{\ell_1}{B^j}\right) \cdots b^2\left(\frac{\ell_q}{B^j}\right) \sqrt{\prod_{i=1}^q \left\{\frac{2\ell_i+1}{4\pi} C_{\ell_i}\right\}} \\ &\qquad \times P_{\ell_1}(\langle x_1, x_2\rangle) \cdots P_{\ell_q}(\langle x_1, x_2\rangle)\, dx_1\, dx_2. \end{aligned}$$

Now write

$$\begin{aligned} &\frac{(2\ell_1+1)\cdots(2\ell_q+1)}{(4\pi)^q} P_{\ell_1}(\langle x_1, x_2\rangle) \cdots P_{\ell_q}(\langle x_1, x_2\rangle) \\ &\quad = \sum_{m_1 \cdots m_q} Y_{\ell_1 m_1}(x_1) \cdots Y_{\ell_q m_q}(x_1) \overline{Y}_{\ell_1 m_1}(x_2) \cdots \overline{Y}_{\ell_q m_q}(x_2) \end{aligned}$$

so that

$$\begin{aligned}\frac{(2\ell_1+1)\cdots(2\ell_q+1)}{(4\pi)^q}&\int_{S^2\times S^2} P_{\ell_1}(\langle z_1,x_1\rangle)P_{\ell_2}(\langle z_2,x_2\rangle)\\&\qquad\times P_{\ell_1}(\langle x_1,x_2\rangle)\cdots P_{\ell_q}(\langle x_1,x_2\rangle)\,dx_1\,dx_2\\&=\sum_{\mu_1\mu_2}\sum_{m_1\cdots m_q}\mathcal{G}(\ell_1,m_1;\dots\ell_q,m_q;\ell_1,\mu_1)\\&\qquad\times\mathcal{G}(\ell_1,m_1;\dots\ell_q,m_q;\ell_2,\mu_2)\left\{\frac{4\pi}{2\ell+1}Y_{\ell_1\mu_1}(z_1)\overline{Y}_{\ell_2\mu_2}(z_2)\right\}\\&=\frac{4\pi}{2\ell+1}\sum_{\mu_1\mu_2}Y_{\ell_1\mu_1}(z_1)\overline{Y}_{\ell_2\mu_2}(z_2)\delta_{\ell_1}^{\ell_2}\delta_{\mu_1}^{\mu_2}=P_{\ell_1}(\langle z_1,z_2\rangle).\end{aligned}$$

The general case $q\geq 3$ hence yields (see also [34], Theorem 7.5 for a related computation)

$$\begin{aligned}\mathbb{E}g_{j;q}^2(x)&\\&=\frac{q!}{\sigma_{\beta_j}^{2q}}\sum_\ell\kappa^2(\ell)\sum_{\ell_1\cdots\ell_q}\mathcal{C}(\ell_1,\dots,\ell_q,\ell)b^2\left(\frac{\ell_1}{B^j}\right)\cdots b^2\left(\frac{\ell_q}{B^j}\right)\frac{2\ell_1+1}{4\pi}\cdots\\&\qquad\times\frac{2\ell_q+1}{4\pi}C_{\ell_1}\cdots C_{\ell_q}\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\{g_{j;q}(x)g_{j;q}(y)\}&\\&=\frac{q!}{\sigma_{\beta_j}^{2q}}\sum_\ell\kappa^2(\ell)\sum_{\ell_1\cdots\ell_q}\mathcal{C}(\ell_1,\dots,\ell_q,\ell)b^2\left(\frac{\ell_1}{B^j}\right)\cdots b^2\left(\frac{\ell_q}{B^j}\right)\\&\qquad\times\frac{2\ell_1+1}{4\pi}\cdots\frac{2\ell_q+1}{4\pi}C_{\ell_1}\cdots C_{\ell_q}P_\ell(\langle x_1,x_2\rangle),\end{aligned}$$

as claimed. □

REMARK 10. It is immediately checked that the angular power spectrum of $g_{j;q}(y)$ is given by [see (26)]

$$\begin{aligned}C_{\ell;j,q}&:=\frac{q!}{\sigma_{\beta_j}^{2q}}\frac{4\pi}{2\ell+1}\kappa^2(\ell)\\&\qquad\times\sum_{\ell_1\cdots\ell_q}\mathcal{C}(\ell_1,\dots,\ell_q,\ell)\prod_{k=1}^q\left[b^2\left(\frac{\ell_k}{B^j}\right)\frac{2\ell_k+1}{4\pi}C_{\ell_k}\right].\end{aligned}\tag{27}$$

As a special case, for $q = 2$ we recover the previous result (25)

$$
\begin{aligned}
C_{\ell;j,2} &= \frac{2!}{\sigma_{\beta_j}^4}\kappa^2(\ell)\sum_{\ell_1\ell_2} b^2\Big(\frac{\ell_1}{B^j}\Big) b^2\Big(\frac{\ell_2}{B^j}\Big)\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi} \\
&\qquad \times C_{\ell_1}C_{\ell_2}\begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix}^2 \\
&= \frac{2!}{\sigma_{\beta_j}^4}\kappa^2(\ell)\frac{4\pi}{2\ell+1}\sum_{\ell_1\ell_2} \mathcal{C}(\ell_1,\ell_2,\ell) b^2\Big(\frac{\ell_1}{B^j}\Big) b^2\Big(\frac{\ell_2}{B^j}\Big) \\
&\qquad \times \frac{(2\ell_1+1)}{4\pi}\frac{(2\ell_2+1)}{4\pi} C_{\ell_1}C_{\ell_2},
\end{aligned} \tag{28}
$$

because

$$
\mathcal{C}(\ell_1,\ell_2,\ell) = \{C_{\ell_1 0 \ell_2 0}^{\ell 0}\}^2 = (2\ell+1)\begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix}^2.
$$

5. Weak convergence. In this section, we provide our main convergence results. It must be stressed that the convergence we study here is in some sense different from the standard theory as presented, for instance, by [13], but refers instead to the broader notion developed by [20, 21]; see also [23], Chapter 11.

We start first from the following conditions (see, e.g., [10, 34, 39]):

CONDITION 11. *The angular power spectrum has the form*

$$
C_\ell = G(\ell)\ell^{-\alpha}, \qquad \ell = 1, 2, \ldots,
$$

where $\alpha > 2$ and $G : \mathbb{R}^+ \to \mathbb{R}^+$ is such that, for all $u > 0$,

$$
0 < c_0 \le G(\cdot) \le d_0,
$$

$$
\left| \frac{d^r}{du^r} G(u) \right| \le c_r u^{-r}, \qquad r = 1, 2, \ldots, M \in \mathbb{N}.
$$

CONDITION 12. *The Kernel $K(\cdot)$ and the field $\{\beta_j(\cdot)\}$ are such that, for all $j = 1, 2, 3, \ldots$*

$$
\operatorname{Var}\left\{ \int_{S^2} K(\langle x, y\rangle) H_q(\widetilde{\beta}_j(y))\, dy \right\} = \sigma_j^2 B^{-2j} \qquad \textit{for all } j = 1, 2, \ldots
$$

and there exist positive constants c_1, c_2 such that $c_1 \le \sigma_j^2 \le c_2$ (note that the right-hand side does not depend on x by isotropy).

These assumptions are mild and it is easy to find many physical examples such that they are fulfilled. In particular, Condition 11 is fulfilled when $G(\ell) = P(\ell)/Q(\ell)$ and $P(\ell), Q(\ell) > 0$ are two positive polynomials of the same order.

In the now dominant Bardeen's potential model for the angular power spectrum of the cosmic microwave background radiation (which is theoretically justified by the so-called inflationary paradigm for the Big Bang Dynamics; see, e.g., [22, 24]) one has $C_\ell \sim (\ell(\ell+1))^{-1}$ for the observationally relevant range $\ell \leq 5 \times 10^3$ (the decay becomes faster at higher multipoles, in view of the so-called Silk damping effect, but these multipoles are far beyond observational capacity). This is clearly in good agreement with Condition 11. On the other hand, assuming that Condition 11 holds and taking, for instance, $K(\langle x, y\rangle) \equiv 1$ [e.g., focusing on the integral of the field $\{H_q(\widetilde{\beta}_j(y))\}$], Condition 12 has been shown to be satisfied by [15]. Indeed, it is readily checked that $\{H_q(\widetilde{\beta}_j(y))\}$ is a polynomial of finite order (the integer part of $B^{q(j+1)}$), and we can hence consider the following heuristic argument: we have

$$\int_{S^2} K(\langle x, y\rangle) H_q(\widetilde{\beta}_j(y))\, dy = \int_{S^2} H_q(\widetilde{\beta}_j(y))\, dy$$
$$= \sum_{k \in \mathcal{X}_j} H_q(\widetilde{\beta}_j(\xi_{jk}))\lambda_{jk},$$

where $\{\xi_{jk}, \lambda_{jk}\}$ are a set of cubature points and weights (see [11, 41]); indeed, because the $\beta_j(\cdot)$ are band-limited (polynomial) functions, this Riemann sum approximations can be constructed to be exact (by the so-called cubature formula established in [41]; see also [11] for some discussion), with weights λ_{jk} of order $\simeq B^{-2j}$. It is now known that under Condition 11, it is possible to establish a fundamental decorrelation inequality which will play a crucial role in our proof below (see also [10, 31, 39]). Indeed, exploiting (24) and (2) we have that for any $M \in \mathbb{N}$, there exists a constant C_M such that

$$\operatorname{Cov}\{H_q(\widetilde{\beta}_j(\xi_{jk_1})), H_q(\widetilde{\beta}_j(\xi_{jk_2}))\} \leq \frac{C_M q!}{\{1 + B^j d(\xi_{jk_1}, \xi_{jk_2})\}^{qM}},$$

entailing that the terms $H_q(\beta_j(\xi_{jk}))$ can be treated as asymptotically uncorrelated, for large j. Hence, heuristically

$$\operatorname{Var}\biggl\{\sum_{k \in \mathcal{X}_j} H_q(\widetilde{\beta}_j(\xi_{jk}))\lambda_{jk}\biggr\} \simeq \sum_{k \in \mathcal{X}_j} \operatorname{Var}\{H_q(\widetilde{\beta}_j(\xi_{jk}))\}\lambda_{jk}^2$$
$$\simeq C_q \sum_{k \in \mathcal{X}_j} \lambda_{jk}^2 \simeq C_q B^{-2j},$$

because $\sum_{k \in \mathcal{X}_j} \lambda_{jk} = 4\pi$.

EXAMPLE 13. For $q = 2$, we obtain

$$\operatorname{Var}\biggl\{\int_{S^2} (\widetilde{\beta}_j^2(y) - 1)\, dy\biggr\}$$
$$= \operatorname{Var}\biggl\{\int_{S^2} \widetilde{\beta}_j^2(y)\, dy\biggr\}$$

$$= \frac{1}{\{\sum_\ell b^2(\ell/B^j)(2\ell+1)C_\ell\}^2} \operatorname{Var}\left\{\int_{S^2}\left[\sum_{\ell m} b\left(\frac{\ell}{B^j}\right) a_{\ell m} Y_{\ell m}(y)\right]^2 dy\right\}$$

$$= \frac{1}{\{\sum_\ell b^2(\ell/B^j)(2\ell+1)C_\ell\}^2} \operatorname{Var}\left\{\sum_\ell b^2\left(\frac{\ell}{B^j}\right) \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2\right\},$$

where we have used (1), (3) and the ortho-normality properties of spherical harmonics, that is,

$$\int_{S^2} Y_{\ell m}(y)\overline{Y}_{\ell' m'}(y)\, dy = \delta_\ell^{\ell'} \delta_m^{m'}.$$

Now write

$$\widehat{C}_\ell := \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2,$$

the so-called sample angular power spectrum; it is readily verified that $\widehat{C}_\ell / C_\ell$ obeys a chi-square law with $(2\ell+1)$ degrees of freedom, whence we obtain

$$\operatorname{Var}\left\{\int_{S^2} \widetilde{\beta}_j^2(y)\, dy\right\}$$

$$= \frac{\operatorname{Var}\{\sum_\ell b^2(\ell/B^j)(2\ell+1)\widehat{C}_\ell\}}{\{\sum_\ell b^2(\ell/B^j)(2\ell+1)C_\ell\}^2} = \frac{\sum_\ell b^4(\ell/B^j)(2\ell+1)^2 \operatorname{Var}(\widehat{C}_\ell)}{\{\sum_\ell b^2(\ell/B^j)(2\ell+1)C_\ell\}^2}$$

$$= \frac{2\sum_{\ell=B^{j-1}}^{B^{j+1}} b^4(\ell/B^j)(2\ell+1)C_\ell^2}{\{\sum_\ell b^2(\ell/B^j)(2\ell+1)C_\ell\}^2} \simeq \frac{B^{j(2-2\alpha)}}{\{B^{j(2-\alpha)}\}^2} \simeq B^{-2j},$$

as claimed.

5.1. *Finite-dimensional distributions*. The general technique we shall exploit to establish the central limit theorem is based upon sharp bounds on normalized fourth-order cumulants. Note that, in view of results from [43], this will actually entail a stronger form of convergence, more precisely in total variation norm (see [43]).

We start by recalling that the field $\{\widetilde{\beta}_j(\cdot)\}$ can be expressed in terms of the isonormal Gaussian process, for example, as a stochastic integral

$$\widetilde{\beta}_j(y) := \frac{1}{\sigma_{\beta_j}} \sum_\ell b\left(\frac{\ell}{B^j}\right) T_\ell(y)$$

$$= \frac{1}{\sigma_{\beta_j}} \sum_\ell b\left(\frac{\ell}{B^j}\right) \sqrt{\frac{(2\ell+1)C_\ell}{4\pi}} \int_{S^2} P_\ell(\langle y, z\rangle) W(dz),$$

where $W(A)$ is a white noise Gaussian measure on the sphere, which satisfies

$$\mathbb{E}W(A) = 0, \qquad \mathbb{E}\{W(A)W(B)\} = \int_{A\cap B} dz \qquad \text{for all } A, B \in \mathcal{B}(S^2).$$

It thus follows immediately that the transformed process $\{H_q(\widetilde{\beta}_j(\cdot))\}$ belongs to the qth order Wiener chaos; see [43, 44] for more discussion and detailed definitions. Let us now recall the definition of the *total variation* distance between the laws of two random variables X and Z, which is given by

$$d_{\mathrm{TV}}(X,Z)=\sup_{A\in\mathcal{B}(\mathbb{R})}\bigl|\Pr(W\in A)-\Pr(X\in A)\bigr|.$$

When Z is a standard Gaussian and X is a zero-mean, unit variance random variable which belongs to the qth order Wiener chaos of a Gaussian measure, the following remarkable inequality holds for the total variation distance

$$d_{\mathrm{TV}}(X,Z)\le\sqrt{\frac{q-1}{3q}\mathrm{cum}_4(X)};$$

see again [43, 44] for more discussion and a full proof.

From now on, we shall normalize the fields $\{g_{j;q}\}$ to make them unit variance, that is, we shall define

$$\widetilde{g}_{j;q}(x):=\frac{g_{j;q}(x)}{\sqrt{\mathbb{E}g_{j;q}^2(x)}};$$

also, we introduce an isotropic zero-mean Gaussian process $f_{j;q}$, with the same covariance function as that of $\widetilde{g}_{j;q}$. Our next result will establish the asymptotic convergence of the finite-dimensional distributions for $\widetilde{g}_{j;q}$ and $f_{j;q}$. In particular, we have the following.

LEMMA 14. *For any fixed vector* $(x_1,\dots,x_p)$ *in* S^2, *we have that*

$$d_{\mathrm{TV}}\bigl((\widetilde{g}_{j;q}(x_1),\dots,\widetilde{g}_{j;q}(x_p)),(f_{j;q}(x_1),\dots,f_{j;q}(x_p))\bigr)=o(1),$$

as $j\to\infty$.

PROOF. For notational simplicity, we shall focus on the univariate case. In this case, the Nourdin–Peccati inequality [43, 44] can be restated as

$$d_{\mathrm{TV}}\biggl(\frac{g_{j;q}(x)}{\sqrt{\mathbb{E}g_{j;q}^2(x)}},N(0,1)\biggr)\le\sqrt{\frac{q-1}{3q}\mathrm{cum}_4\biggl(\frac{g_{j;q}(x)}{\sqrt{\mathbb{E}g_{j;q}^2(x)}}\biggr)}.\tag{29}$$

In view of (29), for the central limit theorem to hold we shall only need to study the limiting behaviour of the normalized fourth-order cumulant of $g_{j;q}$. Let us then consider

$$\begin{aligned}\mathrm{cum}_4&\{g_{j;q}(x)\}\\&=\int_{\{S^2\}^{\otimes 4}}K(\langle x,y_1\rangle)\cdots K(\langle x,y_4\rangle)\\&\qquad\times\mathrm{cum}_4\{H_q(\widetilde{\beta}_j(y_1)),\dots,H_q(\widetilde{\beta}_j(y_4))\}\,dy_1\cdots dy_4.\end{aligned}$$

We now need to provide a bound on the cumulant inside the integral; to this aim, we need to recall the *diagram formula* (see, e.g., [45], Chapter 7 or [34], Proposition 4.15 for further details). In particular, fix a set of integers $\alpha_1, \dots, \alpha_p$; a *diagram* is a graph with α_1 vertices labeled by 1, α_2 vertices labeled by $2, \dots, \alpha_p$ vertices labeled by p, such that each vertex has degree 1. We can view the vertices as belonging to p different rows; the edges may connect only vertices with different labels, that is, there are no ("flat") edges connecting two vertices on the same row. The set of such diagrams that are connected (i.e., such that it is not possible to partition the rows into two subsets A and B such that no edge connect a vertex in A with a vertex in B) is denoted by $\Gamma_c(\alpha_1, \dots, \alpha_p)$. Given a diagram $\gamma \in \Gamma_c$, $\eta_{ik}(\gamma)$ is the number of edges between the vertices labeled by i and the vertices labeled by k in γ. The *diagram formula* for Hermite polynomials states the following; let $(Z_1, \dots, Z_p)$ be a centered Gaussian vector whose components have unit variance, and let $H_{l_1}, \dots, H_{l_p}$ be Hermite polynomials of degrees $l_1, \dots, l_p$ (≥ 1), respectively. Then

$$\operatorname{cum}\bigl(H_{l_1}(Z_1), \dots, H_{l_p}(Z_p)\bigr) = \sum_{\gamma \in \Gamma_c(l_1, \dots, l_p)} \prod_{1 \leq i \leq j \leq p} \{\mathbb{E}[Z_i Z_j]\}^{\eta_{ij}(\gamma)}.$$

For a proof, see [45], Section 7.3. A simple application in our case then yields

$$\begin{aligned} &\operatorname{cum}_4\{H_q(\widetilde{\beta}_j(y_1)), \dots, H_q(\widetilde{\beta}_j(y_4))\} \\ &\quad = \sum_{\gamma \in \Gamma_c(q,q,q,q)} \prod_{1 \leq s \leq t \leq 4} \{\mathbb{E}[\widetilde{\beta}_j(y_s)\widetilde{\beta}_j(y_t)]\}^{\eta_{st}(\gamma)} \\ &\quad \leq \sum_{\gamma \in \Gamma_c(q,q,q,q)} |\rho_j(y_1, y_2)|^{\eta_{12}(\gamma)} |\rho_j(y_2, y_3)|^{\eta_{23}(\gamma)} |\rho_j(y_3, y_4)|^{\eta_{34}(\gamma)} \\ &\qquad \times |\rho_j(y_4, y_1)|^{\eta_{41}(\gamma)} |\rho_j(y_1, y_3)|^{\eta_{13}(\gamma)} |\rho_j(y_2, y_4)|^{\eta_{24}(\gamma)}, \end{aligned} \tag{30}$$

where

$$\rho_j(y_1, y_2) = \frac{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell P_\ell(y_1, y_2)}{\sum_\ell b^2(\ell/B^j)((2\ell+1)/(4\pi))C_\ell} \leq \frac{C_M}{\{1 + B^j d(y_1, y_2)\}^M},$$

in view of (11) and the decorrelation inequality provided by [10]; see also [31, 39]. Note that in our circumstances, the total number of "edges" satisfies

$$\sum_{t=1}^{4} \eta_{st}(\gamma) = q \qquad \text{for all } s = 1, \dots, 4 \quad \text{and} \quad \sum_{1 < s < t \leq 4} \eta_{st}(\gamma) = 2q.$$

It is simple to see that for any $\gamma \in \Gamma_c(q, q, q, q)$ and any given s, there must exist two distinct indexes t, t' such that $\eta_{st}(\gamma), \eta_{st'}(\gamma) > 0$. Indeed, assume by contradiction that this is not the case for some s; then there must exist $t \neq s$ such that such that $\eta_{st}(\gamma) = q$, and hence $\eta_{s't}(\gamma) = 0$ for all $s \neq s'$. It follows that γ

cannot be connected, yielding the desired contradiction. Hence, up to a relabeling of the indices there must necessarily exist a "spanning cycle," that is, a sequence

$$\eta_{12}(\gamma), \eta_{23}(\gamma), \eta_{34}(\gamma), \eta_{41}(\gamma) > 0,$$

where the inequality is strict. Since the correlations are bounded by unity, it follows that

$$\begin{aligned}
&|\rho_j(y_1, y_2)|^{\eta_{12}(\gamma)} |\rho_j(y_2, y_3)|^{\eta_{23}(\gamma)} |\rho_j(y_3, y_4)|^{\eta_{34}(\gamma)} \\
&\quad \times |\rho_j(y_4, y_1)|^{\eta_{41}(\gamma)} |\rho_j(y_1, y_3)|^{\eta_{13}(\gamma)} |\rho_j(y_2, y_4)|^{\eta_{24}(\gamma)} \\
&\quad \le |\rho_j(y_1, y_2)| |\rho_j(y_2, y_3)| |\rho_j(y_3, y_4)| |\rho_j(y_4, y_1)|.
\end{aligned}$$

Therefore, writing $C(q)$ as the cardinality of set $\Gamma_c(q, q, q, q)$, which is the set of all connected graphs of a given order, we get

$$\begin{aligned}
&\mathrm{cum}_4\{H_q(\widetilde{\beta}_j(y_1)), \ldots, H_q(\widetilde{\beta}_j(y_4))\} \\
&\qquad \le \sharp\{\Gamma_c(q, q, q, q)\} \times |\rho_j(y_1, y_2)| |\rho_j(y_2, y_3)| |\rho_j(y_3, y_4)| |\rho_j(y_4, y_1)| \\
&\qquad = C(q) \times |\rho_j(y_1, y_2)| |\rho_j(y_2, y_3)| |\rho_j(y_3, y_4)| |\rho_j(y_4, y_1)|,
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mathrm{cum}_4\{g_{j;q}(x)\} \le C(q) \int_{\{S^2\}^{\otimes 4}} &|K(\langle x, y_1\rangle) \cdots K(\langle x, y_4\rangle)| |\rho_j(y_1, y_2)| \\
&\times |\rho_j(y_2, y_3)| |\rho_j(y_3, y_4)| |\rho_j(y_4, y_1)| \, dy_1 \cdots dy_4.
\end{aligned}$$

Now standard computations yield

$$\begin{aligned}
\int_{S^2} |\rho(y_1, y_2)| \, dy_2 &\le \int_{S^2} \frac{C_M}{\{1 + B^j d(y_1, y_2)\}^M} \, dy_2 \\
&\le \int_{y_2 : d(y_1, y_2) \le B^{-j}} \frac{C_M}{\{1 + B^j d(y_1, y_2)\}^M} \, dy_2 \\
&\quad + \int_{y_2 : d(y_1, y_2) \ge B^{-j}} \frac{C_M}{\{1 + B^j d(y_1, y_2)\}^M} \, dy_2 \\
&\le C B^{-2j}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{\{S^2\}^{\otimes 4}} |\rho(y_1, y_2)| |\rho(y_2, y_3)| |\rho(y_3, y_4)| |\rho(y_4, y_1)| \, dy_1 \cdots dy_4 \\
&\qquad \le \int_{\{S^2\}^{\otimes 4}} |\rho(y_1, y_2)| |\rho(y_2, y_3)| |\rho(y_3, y_4)| \, dy_1 \cdots dy_4 \le C B^{-6j}
\end{aligned}$$

and

$$\mathrm{cum}_4\{\widetilde{g}_{j;q}(x)\} = O(B^{-2j}),$$

entailing that for every fixed $x \in S^2$,

$$d_{\mathrm{TV}}\big(\widetilde{g}_{j;q}(x), N(0,1)\big) = O\big(B^{-2j}\big),$$

and hence the univariate central limit theorem, as claimed. The proof in the multivariate case is analogous, and hence omitted for the sake of brevity. □

5.2. *Tightness*. We now focus on asymptotic tightness for both sequences $\{g_{j;q}\}$ and $\{f_{j;q}\}$. We shall exploit the following criterion from [30].

PROPOSITION 15 ([30]). *Let $g_j : M \to D$ be a sequence of stochastic processes, where M is compact and D is complete and separable. Assume that the finite-dimensional distributions of g_j converge to the those of g, and that (tightness)*

$$\lim_{h\to 0} \lim_{j\to\infty} \sup \mathbb{E}\Big(\sup_{d(x,y)\le h} \big|g_j(x) - g_j(y)\big| \wedge 1\Big) = 0.$$

Then $g_j \Rightarrow g$.

We are hence able to establish the following.

LEMMA 16. *For every $q \in \mathbb{N}$, the sequences $\{\widetilde{g}_{j;q}\}$ and $\{f_{j;q}\}$ are tight.*

PROOF. Write $\{a_{\ell m}(f_{j;q})\}$ for the spherical harmonic coefficients of the fields $\{f_{j;q}\}$. For any $x_1, x_2 \in S^2$, we have

$$\begin{aligned}
&\mathbb{E}\Big\{\sup_{d(x_1,x_2)\le\delta} \big|f_{j;q}(x_1) - f_{j;q}(x_2)\big|\Big\} \\
&\quad = \mathbb{E}\Big\{\sup_{d(x_1,x_2)\le\delta} \Big|\sum_{\ell m} a_{\ell m}(f_{j;q})\{Y_{\ell m}(x_1) - Y_{\ell m}(x_2)\}\Big|\Big\} \\
&\quad \le \sum_{\ell m} \{\mathbb{E}|a_{\ell m}(f_{j;q})|\}\Big\{\sup_{d(x_1,x_2)\le\delta} \big|\{Y_{\ell m}(x_1) - Y_{\ell m}(x_2)\}\big|\Big\}.
\end{aligned}$$

Now

$$\sup_{d(x_1,x_2)\le\delta} \big|\{Y_{\ell m}(x_1) - Y_{\ell m}(x_2)\}\big| \le c\ell^2\delta$$

and

$$\sum_{\ell m} \{\mathbb{E}|a_{\ell m}(f_{j;q})|\} \le \sum_{\ell m} \sqrt{\{\mathbb{E}|a_{\ell m}(f_{j;q})|^2\}} = \sum_{\ell} (2\ell+1)\sqrt{C_\ell(f_{j;q})}$$

and because $K(\cdot)$ is compactly supported in harmonic space (and hence, again, a finite-order polynomial)

$$\le \Big\{\sum_{\ell}^{L_K} (2\ell+1)\Big\}^{1/2} \sqrt{\sum_{\ell}^{L_K} (2\ell+1) C_\ell(f_{j;q})} \le O(L_K),$$

whence

$$\mathbb{E}\Big\{\sup_{d(x_1,x_2)\le\delta}\big|f_{j;q}(x_1)-f_{j;q}(x_2)\big|\Big\}\le CL_K^3\delta,$$

for some $C>0$, uniformly over j, and thus the result follows [once again, recall that L_K is fixed by assumption (20)]. The proof for $\{\widetilde{g}_{j;q}\}$ is analogous. □

5.3. *Asymptotic proximity of distributions*. Our discussion above shows that the finite-dimensional distributions of the non-Gaussian sequence of random fields $\{\widetilde{g}_{j;q}\}$ converge to those of the Gaussian sequence $\{f_{j;q}\}$ as j tends to infinity; moreover, both sequences are tight. However, the finite-dimensional distributions of neither processes converge to a well-defined limit. In view of this situation, we need a broader notion of convergence than the one envisaged in standard treatment such as [13]; this extended form of convergence is provided by the notion of *Asymptotic Proximity*, or *Merging*, of distributions, as discussed, for instance, by [20, 21, 23] and others.

DEFINITION 17 (Asymptotic proximity of distribution [20, 21, 23]). Let g_n, f_n be two sequences of random elements in some metric space (X,ρ), possibly defined on two different probability spaces. We say that the laws of g_n, f_n are *asymptotically merging*, or *asymptotically proximal*, (denoted as $g_n\Rightarrow f_n$) if and only if as $n\to\infty$

$$\big|\mathbb{E}h(g_n)-\mathbb{E}h(f_n)\big|\to 0,$$

for all continuous and bounded functionals $h\in\mathcal{C}_b(X,\mathbb{R})$.

In view of the results provided in the previous subsection, it is immediate to establish that the sequences $\{\widetilde{g}_{j;q}\},\{f_{j;q}\}$ are proximal. Indeed,

THEOREM 18. *As $j\to\infty$*

$$\widetilde{g}_{j;q}\Longrightarrow f_{j;q},$$

that is, for all $h=h:\mathcal{C}(S^2,\mathbb{R})\to\mathbb{R}$, h continuous and bounded, we have

$$\big|\mathbb{E}h(\widetilde{g}_{j;q})-\mathbb{E}h(f_{j;q})\big|\to 0.$$

PROOF. Applying to our circumstances the characterization of asymptotic proximity provided by [21], we find that the sequences $\{\widetilde{g}_{j;q}\},\{f_{j;q}\}$ are asymptotically proximal if and only if they are both tight and their finite-dimensional distribution converge, that is, for all $n\ge 1$, $x_1,\dots,x_n\in K$, we have that

$$\big|\Pr\{(\widetilde{g}_{j;q}(x_1),\dots,\widetilde{g}_{j;q}(x_n))\in A\}-\Pr\{(f_{j;q}(x_1),\dots,f_{j;q}(x_n))\in A\}\big|\to 0$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$. Now convergence of the finite-dimensional distributions was established in Section 5.1, while tightness was established in Section 5.2; thus the result follows immediately. □

As a simple application of the asymptotic proximity result, we have

$$\mathbb{E}\left\{\frac{\sup \widetilde{g}_{j;q}}{1+\sup \widetilde{g}_{j;q}}\right\} \to \mathbb{E}\left\{\frac{\sup f_{j;q}}{1+\sup f_{j;q}}\right\}.$$

It should be noted that asymptotically proximal sequences do not enjoy all the same properties as in the standard weak convergence case. For instance, it is known that the Portmanteau lemma does not hold in general, that is, it is not true that, for every Borel set such that $\Pr\{g_n \in \partial A\} = \Pr\{f_n \in \partial A\} = 0$, we have

$$\left|\Pr\{g_n \in A\} - \Pr\{f_n \in A\}\right| \to 0;$$

as a counterexample, it is enough to consider the sequences $f_n = -n^{-1}$ and $g_n = n^{-1}$. However, it is indeed possible to obtain more stringent characterizations when the subsequences are asymptotically Gaussian. We have the following.

PROPOSITION 19. *For every $A \in \mathcal{B}(\mathbb{R})$, we have that*

$$\left|\Pr\left\{\sup_{x\in S^2} \widetilde{g}_{j;q}(x) \in A\right\} - \Pr\left\{\sup_{x\in S^2} f_{j;q} \in A\right\}\right| \to 0.$$

PROOF. We shall argue again by contradiction. Assume that there exists a subsequence j_n' such that for some $\varepsilon > 0$

$$\left|\Pr\left\{\sup_{x\in S^2} \widetilde{g}_{j_n';q}(x) \in A\right\} - \Pr\left\{\sup_{x\in S^2} f_{j_n';q} \in A\right\}\right| > \varepsilon. \tag{31}$$

By relative compactness, there exists a subsequence j_n'' and a limiting process $g_{\infty;q}$ such that

$$\left|\Pr\left\{\sup_{x\in S^2} \widetilde{g}_{j_n'';q}(x) \in A\right\} - \Pr\left\{\sup_{x\in S^2} \widetilde{g}_{\infty;q} \in A\right\}\right| \to 0.$$

Likewise, consider $\{j_n'''\} \subset \{j_n''\}$; again by relative compactness there exist $f_{\infty;q}$ such that $f_{j_n''';q} \Rightarrow f_{\infty;q}$, and hence

$$\left|\Pr\left\{\sup_{x\in S^2} f_{j_n''';q}(x) \in A\right\} - \Pr\left\{\sup_{x\in S^2} f_{\infty;q} \in A\right\}\right| \to 0.$$

Note that $f_{\infty;q}$, $\widetilde{g}_{\infty;q}$ are isotropic and continuous Gaussian random fields; indeed for $\widetilde{g}_{\infty;q}$ it suffices to recall that the finite-dimensional distributions of $\{\widetilde{g}_{j;q}\}$ are asymptotically Gaussian (Section 5.1), so if a weak limit exists it must be Gaussian as well. Hence, the supremum is necessarily a continuous random variable, and no problems with nonzero boundary probabilities can arise. Also, the finite-dimensional distributions are a determining class, whence the two Gaussian processes $f_{\infty;q}, \widetilde{g}_{\infty;q}$ must necessarily have the same distribution. Hence,

$$\left|\Pr\left\{\sup_{x\in S^2} f_{j_n''';q}(x) \in A\right\} - \Pr\left\{\sup_{x\in S^2} \widetilde{g}_{j_n''';q}(x) \in A\right\}\right| \to 0,$$

yielding a contradiction with (31). □

This result immediately suggests two alternative ways to achieve the ultimate goal of this paper, for example, the evaluation of excursion probabilities on the non-Gaussian sequence of random fields $\{g_{j;q}\}$. On one hand, it follows immediately that these probabilities may be evaluated by simulations, by simply sampling realizations of a Gaussian field with known angular power spectrum; for $q = 2$, for example, $f_{j;q}$ is simply a Gaussian process with angular power spectrum given by (28). There exist now very efficient techniques, based on packages such as HealPix [27], for the numerical simulation of Gaussian fields with a given power spectra; here the only burdensome step can be the numerical evaluation of expressions like (28), but this is in any case much faster and simpler than the Monte Carlo evaluation of smoothed non-Gaussian fields. Therefore, our result has an immediate applied relevance.

One can try, however, to be more ambitious than this, and verify whether these excursion probabilities can indeed be evaluated analytically, rather than by Gaussian simulations. This is in fact the purpose of the next, and final, section.

6. Asymptotics for the excursion probabilities. The purpose of this final section is to show how the previous weak convergence results allow for very neat characterizations of excursion probabilities, even in non-Gaussian circumstances. In particular, our main result is the following.

THEOREM 20. *There exists constants $\alpha > 1$ and $\mu^+ > 0$ such that, for $u > \mu^+$*

$$\limsup_{j\to\infty}\left|\Pr\left\{\sup_{x\in S^2} \tilde{g}_{j;q}(x) > u\right\} - \left\{2\bigl(1-\Phi(u)\bigr) + 2u\phi(u)\lambda_{j;q}\right\}\right| \le \exp\left(-\frac{\alpha u^2}{2}\right),$$

where [*see* (27)]

$$\lambda_{j;q} = \frac{\sum_{\ell=1}^{L}((2\ell+1)/(4\pi))C_{\ell;j,q}P_\ell'(1)}{\sum_{\ell=1}^{L}((2\ell+1)/(4\pi))C_{\ell;j,q}}. \tag{32}$$

PROOF. Note that

$$\left|\Pr\left\{\sup_{x\in S^2}\tilde{g}_{j;q}(x)>u\right\}-\left\{2\left(1-\Phi(u)\right)+2u\phi(u)\lambda_{j;q}\right\}\right|$$
$$\leq\left|\Pr\left\{\sup_{x\in S^2}\tilde{g}_{j;q}(x)>u\right\}-\Pr\left\{\sup_{x\in S^2}\tilde{f}_{j;q}(x)>u\right\}\right| \tag{33}$$
$$+\left|\Pr\left\{\sup_{x\in S^2}\tilde{f}_{j;q}(x)>u\right\}-\left\{2\left(1-\Phi(u)\right)+2u\phi(u)\lambda_{j;q}\right\}\right|,$$

where $\tilde{f}_{j;q}$ is as defined in the previous section. Observe that by Proposition 19 the first part of the right-hand side of the above inequality converges to 0, therefore, we need only prove the required estimate for the second part of the right-hand side.

We shall mainly exploit Theorem 14.3.3 of [5], with some modifications to adapt it to our needs. For each $x_0\in S^2$, let us define the corresponding pivoted random field as

$$\begin{aligned}\widehat{f}^{x_0}_{j;q}(x)=\frac{1}{1-\rho(x,x_0)}\biggl\{&f_{j;q}(x)-\rho(x,x_0)f_{j;q}(x_0)\\&-\operatorname{Cov}\left(f_{j;q}(x),\frac{\partial}{\partial\vartheta}f_{j;q}(x_0)\right)\\&\times\operatorname{Var}\left(\frac{\partial}{\partial\vartheta}f_{j;q}(x)\right)\frac{\partial}{\partial\vartheta}f_{j;q}(x)\\&-\operatorname{Cov}\left(f_{j;q}(x),\frac{\partial}{\sin\vartheta\,\partial\phi}f_{j;q}(x_0)\right)\\&\times\operatorname{Var}\left(\frac{\partial}{\sin\vartheta\,\partial\phi}f_{j;q}(x)\right)\frac{\partial}{\sin\vartheta\,\partial\phi}f_{j;q}(x)\biggr\},\end{aligned} \tag{34}$$

where $\rho(x,x_0)=\mathbb{E}(f_{j;q}(x)f_{j;q}(x_0))$. Next define

$$\mu_j^+=\sup_{x_0}\mathbb{E}\left(\sup_{x\neq x_0}\widehat{f}^{x_0}_{j;q}(x)\right)$$

and

$$\sigma_j^2=\sup_{x_0}\sup_{x\neq x_0}\operatorname{Var}(\widehat{f}^{x_0}_{j;q}(x)).$$

Then from page 371 of [5], we know that for $u\geq\mu_j^+$

$$\left|\Pr\left\{\sup_{x\in S^2}f_{j;q}(x)>u\right\}-\mathbb{E}\mathcal{L}_0\left(A_u(f_{j;q},S^2)\right)\right|$$
$$\leq Kue^{-(u-\mu_j^+)^2/2(1+1/(2\sigma_j^2))}\sum_{i=0}^{2}\left\{\mathbb{E}\left|\det_i\left(-\nabla^2 f_{j;q}-f_{j;q}I_2\right)\right|^2\right\}^{1/2}, \tag{35}$$

where I_2 is the 2×2 identity matrix, $\det_i$ of a matrix is the sum over all the i-minors of the matrix under consideration, and K is a constant not depending on j. Note that the expression on page 371 of [5] also involves an integral over the parameter space with the metric induced by the second-order spectral moment. However, under (20) this integral is easily seen to be uniformly bounded with respect to j, so that we can get rid of it by invoking the isotropy of the field $f_{j;q}$, and absorbing the arising constant into K upfront.

Our goal is to get a uniform bound for the right-hand side of (35). Clearly, $\sum_{i=0}^{2} \mathbb{E}|\det_i(-\nabla^2 f_{j;q} - f_{j;q} I_2)|^2$ is bounded above by a universal constant, largely because of the finite expansion for the kernel $K(\cdot,\cdot)$ used to define the field $g_{j;q}$. Next, to get a uniform bound for μ_j^+, we shall resort to a Slepian inequality type of argument, and use the standard techniques of estimating the expected value of supremum of a Gaussian random field using *metric entropy*.

In particular, we shall prove Proposition 21 in the Appendix that the assumed regularity conditions on the kernel K ensure the following:

$$\mathbb{E}\big(\widehat{f}_{j;q}^{x_0}(x_2) - \widehat{f}_{j;q}^{x_0}(x_1)\big)^2 \le c(L_K, q)|x_2 - x_1|. \tag{36}$$

Then using this uniform bound and a Slepian type of comparison argument, we get a uniform (over j) bound on the metric entropy corresponding to various $\widehat{f}_{j;q}^{x_0}$, which in turn ensures that there exist finite constants $\alpha > 1$ and $\mu^+ = \sup_j \mu_j^+ < \infty$, such that, for $u > \mu^+$,

$$\Big|\Pr\Big\{\sup_{x\in S^2} f_{j;q}(x) > u\Big\} - \mathbb{E}\mathcal{L}_0\big(A_u(f_{j;q}, S^2)\big)\Big| \le \exp\Big(-\frac{\alpha u^2}{2}\Big), \tag{37}$$

uniformly over j, where

$$\mathbb{E}\mathcal{L}_0\big(A_u(f_{j;q}, S^2)\big) = 2\big(1 - \Phi(u)\big) + 2u\phi(u)\lambda_{j;q},$$

which proves the result. $\square$

APPENDIX

All of this section is devoted to the proof of the following proposition.

PROPOSITION 21. *Under the assumption that the kernel K appearing in the definition of $\tilde{g}_{j;q}$ is of the form* (20), *the field $\widehat{f}_{j;q}^{x_0}$ satisfies the following*:

$$\mathbb{E}\big(\widehat{f}_{j;q}^{x_0}(x_2) - \widehat{f}_{j;q}^{x_0}(x_1)\big)^2 \le c(L_K, q)|x_2 - x_1|, \tag{38}$$

where the constant $c(L_K, q)$ depends on q and ℓ, but does not depend on j.

As a by-product of the proof, we shall also obtain a uniform upper bound on σ_j^2. For notational simplicity and without loss of generality, we take the coefficients $\{k_i \frac{2i+1}{4\pi}\}$ in (20) to be identically equal to one.

Writing $\rho(x, y) = \mathrm{cov}(f_{j;q}(x), f_{j;q}(y))$, and $\partial_{\phi_x}, \partial_{\theta_x}$ as directional derivatives at x in the normalized spherical coordinate directions, we have

$$
\begin{aligned}
&\mathrm{cov}\big(\widehat{f}_{j;q}^{x_0}(x_1), \widehat{f}_{j;q}^{x_0}(x_2)\big) \\
&\quad = \frac{1}{(1-\rho(x_0,x_1))(1-\rho(x_0,x_2))} \\
&\qquad \times \big(\rho(x_1,x_2) - \rho(x_0,x_1)\rho(x_0,x_2) \\
&\qquad\quad - \mathrm{cov}\big(f_{j;q}(x_1), \partial_{\theta_{x_0}} f_{j;q}(x_0)\big)\,\mathrm{cov}\big(f_{j;q}(x_2), \partial_{\theta_{x_1}} f_{j;q}(x_1)\big) \\
&\qquad\qquad \times \mathrm{cov}\big(\partial_{\theta_{x_1}} f_{j;q}(x_1), \partial_{\theta_{x_1}} f_{j;q}(x_1)\big) \\
&\qquad\quad - \mathrm{cov}\big(f_{j;q}(x_1), \partial_{\phi_{x_0}} f_{j;q}(x_0)\big)\,\mathrm{cov}\big(f_{j;q}(x_2), \partial_{\phi_{x_1}} f_{j;q}(x_1)\big) \\
&\qquad\qquad \times \mathrm{cov}\big(\partial_{\phi_{x_1}} f_{j;q}(x_1), \partial_{\phi_{x_1}} f_{j;q}(x_1)\big) \\
&\qquad\quad - \rho(x_0,x_1)\rho(x_0,x_2) + \rho(x_0,x_1)\rho(x_0,x_2)\rho(x_0,x_0) \\
&\qquad\quad + \rho(x_0,x_2)\,\mathrm{cov}\big(f_{j;q}(x_1), \partial_{\theta_{x_0}} f_{j;q}(x_0)\big)\,\mathrm{cov}\big(f_{j;q}(x_0), \partial_{\theta_{x_1}} f_{j;q}(x_1)\big) \\
&\qquad\qquad \times \mathrm{cov}\big(\partial_{\theta_{x_1}} f_{j;q}(x_1), \partial_{\theta_{x_1}} f_{j;q}(x_1)\big) \\
&\qquad\quad + \rho(x_0,x_2)\,\mathrm{cov}\big(f_{j;q}(x_1), \partial_{\phi_{x_0}} f_{j;q}(x_0)\big)\,\mathrm{cov}\big(f_{j;q}(x_0), \partial_{\phi_{x_1}} f_{j;q}(x_1)\big) \\
&\qquad\qquad \times \mathrm{cov}\big(\partial_{\phi_{x_1}} f_{j;q}(x_1), \partial_{\phi_{x_1}} f_{j;q}(x_1)\big) \\
&\qquad\quad - \mathrm{cov}\big(f_{j;q}(x_2), \partial_{\theta_{x_0}} f_{j;q}(x_0)\big)\,\mathrm{cov}\big(\partial_{\theta_{x_2}} f_{j;q}(x_2), f_{j;q}(x_1)\big) \\
&\qquad\qquad \times \mathrm{cov}\big(\partial_{\theta_{x_2}} f_{j;q}(x_2), \partial_{\theta_{x_2}} f_{j;q}(x_2)\big) \\
&\qquad\quad + \rho(x_0,x_1)\,\mathrm{cov}\big(f_{j;q}(x_2), \partial_{\theta_{x_0}} f_{j;q}(x_0)\big)\,\mathrm{cov}\big(\partial_{\theta_{x_2}} f_{j;q}(x_2), f_{j;q}(x_0)\big) \\
&\qquad\qquad \times \mathrm{cov}\big(\partial_{\theta_{x_2}} f_{j;q}(x_2), \partial_{\theta_{x_2}} f_{j;q}(x_2)\big) \\
&\qquad\quad + \big(\mathrm{var}\big(\partial_{\theta_{x_1}} f_{j;q}(x_1)\big)\big)^2\,\mathrm{cov}\big(f_{j;q}(x_1), \partial_{\theta_{x_0}} f_{j;q}(x_0)\big) \\
&\qquad\qquad \times \mathrm{cov}\big(f_{j;q}(x_2), \partial_{\theta_{x_0}} f_{j;q}(x_0)\big) \\
&\qquad\qquad \times \mathrm{cov}\big(\partial_{\theta_{x_1}} f_{j;q}(x_1), \partial_{\theta_{x_2}} f_{j;q}(x_2)\big) \\
&\qquad\quad + \mathrm{var}\big(\partial_{\theta_{x_1}} f_{j;q}(x_1)\big)\,\mathrm{var}\big(\partial_{\phi_{x_2}} f_{j;q}(x_2)\big)\,\mathrm{cov}\big(f_{j;q}(x_1), \partial_{\phi_{x_0}} f_{j;q}(x_0)\big) \\
&\qquad\qquad \times \mathrm{cov}\big(f_{j;q}(x_2), \partial_{\theta_{x_0}} f_{j;q}(x_0)\big)\,\mathrm{cov}\big(\partial_{\theta_{x_1}} f_{j;q}(x_1), \partial_{\phi_{x_2}} f_{j;q}(x_2)\big) \\
&\qquad\quad - \mathrm{cov}\big(f_{j;q}(x_2), \partial_{\phi_{x_0}} f_{j;q}(x_0)\big)\,\mathrm{var}\big(\partial_{\phi_{x_2}} f_{j;q}(x_2)\big) \\
&\qquad\qquad \times \mathrm{cov}\big(\partial_{\phi_{x_2}} f_{j;q}(x_2), f_{j;q}(x_1)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\rho(x_0,x_1)\operatorname{cov}\big(f_{j;q}(x_2),\partial_{\phi_{x_0}}f_{j;q}(x_0)\big)\operatorname{var}\big(\partial_{\phi_{x_2}}f_{j;q}(x_2)\big)\\
&\quad\times\operatorname{cov}\big(\partial_{\phi_{x_2}}f_{j;q}(x_2)f_{j;q}(x_0)\big)\\
&+\operatorname{var}\big(\partial_{\theta_{x_1}}f_{j;q}(x_1)\big)\operatorname{var}\big(\partial_{\phi_{x_2}}f_{j;q}(x_2)\big)\\
&\quad\times\operatorname{cov}\big(f_{j;q}(x_1),\partial_{\theta_{x_0}}f_{j;q}(x_0)\big)\\
&\quad\times\operatorname{cov}\big(f_{j;q}(x_2),\partial_{\phi_{x_0}}f_{j;q}(x_0)\big)\operatorname{cov}\big(\partial_{\theta_{x_1}}f_{j;q}(x_1),\partial_{\phi_{x_2}}f_{j;q}(x_2)\big)\\
&+\big(\operatorname{var}\big(\partial_{\phi_{x_1}}f_{j;q}(x_1)\big)\big)^2\operatorname{cov}\big(f_{j;q}(x_1),\partial_{\phi_{x_0}}f_{j;q}(x_0)\big)\\
&\quad\times\operatorname{cov}\big(f_{j;q}(x_2),\partial_{\phi_{x_0}}f_{j;q}(x_0)\big)\operatorname{cov}\big(\partial_{\phi_{x_1}}f_{j;q}(x_1),\partial_{\phi_{x_2}}f_{j;q}(x_2)\big)\big).
\end{aligned}
$$

Note that $\rho(x_1,x_2)$ can be assumed to have $P_l(\langle x_1,x_2\rangle)$ as the leading polynomial (uniform over all j). Then, taking $x_1=x_2$ in the above computation, and going through some more (but simple) calculations, one can show that there exists a constant $M>0$ such that $\operatorname{Var}(\widehat{f}^{x_0}_{j;q}(x))\le M$ uniformly over all j, which in turn, together with the assumption of isotropy, proves that $\sigma_j^2\le M'$, for some $M'<\infty$.

Next, to prove Proposition 21 we begin with

$$
\begin{aligned}
&\mathbb{E}\big(\widehat{f}^{x_0}_{j;q}(x_2)-\widehat{f}^{x_0}_{j;q}(x_1)\big)^2\\
&\quad=\operatorname{var}\big(\widehat{f}^{x_0}_{j;q}(x_1)\big)+\operatorname{var}\big(\widehat{f}^{x_0}_{j;q}(x_2)\big)-2\operatorname{cov}\big(\widehat{f}^{x_0}_{j;q}(x_1),\widehat{f}^{x_0}_{j;q}(x_2)\big).
\end{aligned}
$$

We shall analyze each pair of the terms in the above expression separately. Let us, for instance, consider (together) one of the, seemingly, more involved term of the expression which is the last term of the covariance and the corresponding term in $\operatorname{var}(\widehat{f}^{x_0}_{j;q}(x_1))$. At the expense of introducing more notation, let us write $C_{\ell;\phi\phi}=\operatorname{var}(\partial_{\phi_x}f_{j;q}(x))$ (note that due to isotropy, the variance does not depend on the spatial point x), then the difference between the last term of $\operatorname{Var}(\widehat{f}^{x_0}_{j;q}(x_1))$ and the last term of $\operatorname{Cov}(\widehat{f}^{x_0}_{j;q}(x_1),\widehat{f}^{x_0}_{j;q}(x_2))$, can be written as, for all $x_1,x_2\in(B(x_0,\varepsilon))^c$ that is, outside a ball of size ε around the point x_0, we shall have

$$
\begin{aligned}
&\frac{1}{(1-\rho(x_0,x_1))^2(1-\rho(x_0,x_2))}\\
&\qquad\times\big(C^3_{\ell;\phi\phi}\big(\operatorname{cov}\big(f_{j;q}(x_1),\partial_{\phi_{x_0}}f_{j;q}(x_0)\big)\big)^2\big(1-\rho(x_0,x_2)\big)\\
&\qquad\quad-C^2_{\ell;\phi\phi}\operatorname{cov}\big(f_{j;q}(x_1),\partial_{\phi_{x_0}}f_{j;q}(x_0)\big)\\
&\qquad\qquad\times\operatorname{cov}\big(f_{j;q}(x_2),\partial_{\phi_{x_0}}f_{j;q}(x_0)\big)\\
&\qquad\qquad\quad\times\operatorname{cov}\big(\partial_{\phi_{x_1}}f(x_1),\partial_{\phi_{x_2}}f_{j;q}(x_2)\big)\big(1-\rho(x_0,x_1)\big)\big)\\
&\qquad=\frac{C^2_{\ell;\phi\phi}\partial_{\phi_{x_0}}\rho(\langle x_1,x_0\rangle)}{(1-\rho(x_0,x_1))^2(1-\rho(x_0,x_2))}
\end{aligned}
$$

$$\times \big(C_{\ell;\phi\phi}\big(1-\rho(x_0,x_2)\big)\partial_{\phi_{x_0}}\rho(x_1,x_0) \\ -\big(1-\rho(x_0,x_1)\big)\partial_{\phi_{x_0}}\rho(x_2,x_0)\partial_{\phi_{x_1}}\partial_{\phi_{x_2}}\rho(x_1,x_2)\big) \\ = \frac{C^2_{\ell;\phi\phi}\partial_{\phi_{x_0}}\rho(\langle x_1,x_0\rangle)}{(1-\rho(x_0,x_1))^2(1-\rho(x_0,x_2))} \\ \times \big(\big(\partial_{\phi_{x_0}}\rho(x_1,x_0)-\partial_{\phi_{x_0}}\rho(x_2,x_0)\big)C_{\ell;\phi\phi}\big(1-\rho(x_0,x_2)\big) \\ +\partial_{\phi_{x_0}}\rho(x_2,x_0)\big(C_{\ell;\phi\phi}\big(1-\rho(x_0,x_2)\big) \\ -\big(1-\rho(x_0,x_1)\big)\partial_{\phi_{x_1}}\partial_{\phi_{x_2}}\rho(x_1,x_2)\big)\big).$$

Recall that the covariance function ρ does depend on j, but since we are assuming the kernel $K(x,y)$ to have finite expansion, thus the corresponding Legendre polynomial expansion of $\rho(x_1,x_2)$ can be assumed to have a $P_\ell(\langle x_1,x_2\rangle)$ (uniform over j) which is the leading polynomial. Then, taking the modulus of the above expression, and considering all $x_1,x_2\in(B(x_0,\varepsilon))^c$ that is, outside a ball of size ε around the point x_0, we shall have

$$\left|\frac{C^2_{\ell;\phi\phi}\partial_{\phi_{x_0}}P_\ell(\langle x_1,x_0\rangle)}{[1-P_\ell(\langle x_0,x_1\rangle)]^2[1-P_\ell(\langle x_0,x_2\rangle)]}\right| \\ \times \big|\big(\big\{\partial_{\phi_{x_0}}P_\ell\big(\langle x_1,x_0\rangle\big)-\partial_{\phi_{x_0}}P_\ell\big(\langle x_2,x_0\rangle\big)\big\}C_{\ell;\phi\phi}\big[1-P_\ell\big(\langle x_0,x_2\rangle\big)\big] \\ +\partial_{\phi_{x_0}}P_\ell\big(\langle x_2,x_0\rangle\big)\big\{C_{\ell;\phi\phi}\big[1-P_\ell\big(\langle x_0,x_2\rangle\big)\big] \\ -\big[1-P_\ell\big(\langle x_0,x_1\rangle\big)\big]\partial_{\phi_{x_1}}\partial_{\phi_{x_2}}P_\ell\big(\langle x_1,x_2\rangle\big)\big\}\big)\big| \\ \leq \left|\frac{C^2_{\ell;\phi\phi}P'_\ell(\langle x_1,x_0\rangle)}{(1-P_\ell(\langle x_0,x_1\rangle))^2(1-P_\ell(\langle x_0,x_2\rangle))}\right| \\ \times \big(\big|\big(P'_\ell\big(\langle x_1,x_0\rangle\big)\big(-\sin\theta_{x_1}\sin(\phi_{x_1}-\phi_{x_0})\big) \\ -P'_\ell\big(\langle x_2,x_0\rangle\big)\big(-\sin\theta_{x_2}\sin(\phi_{x_2}-\phi_{x_0})\big)\big)\big|\cdot\varepsilon C_{\ell;\phi\phi} \\ +\big|P'_\ell\big(\langle x_2,x_0\rangle\big)\big(-\sin\theta_{x_2}\sin(\phi_{x_2}-\phi_{x_0})\big)\big| \\ \times\big|\big(C_{\ell;\phi\phi}\big[1-P_\ell\big(\langle x_0,x_2\rangle\big)\big]-\big[1-P_\ell\big(\langle x_0,x_1\rangle\big)\big] \\ \times\big\{P''_\ell\big(\langle x_1,x_2\rangle\big)\sin\theta_{x_1}\sin\theta_{x_2}\sin^2(\phi_{x_1}-\phi_{x_2}) \\ +P'_\ell\big(\langle x_1,x_2\rangle\big)\cos(\phi_{x_1}-\phi_{x_2})\big\}\big)\big|\big) \\ \leq C^2_{\ell;\phi\phi}M(\varepsilon,\ell) \\ \times\big(\big\{\big|P'_\ell\big(\langle x_1,x_0\rangle\big)\big| \\ \times\big|\big(\sin\theta_{x_2}\sin(\phi_{x_2}-\phi_{x_0})-\sin\theta_{x_1}\sin(\phi_{x_1}-\phi_{x_0})\big)\big| \\ +\big|\big(P'_\ell\big(\langle x_2,x_0\rangle\big)-P'_\ell\big(\langle x_1,x_0\rangle\big)\big)\big|\cdot\big|\sin\theta_{x_2}\sin(\phi_{x_2}-\phi_{x_0})\big|\big\}\times\varepsilon C_{\ell;\phi\phi}$$

$$
\begin{aligned}
&\quad + M_1(\varepsilon,\ell) C_{\ell;\phi\phi} \big| P_\ell(\langle x_0, x_1\rangle) - P_\ell(\langle x_0, x_2\rangle)\big| \\
&\quad + M_1(\varepsilon,\ell)\big|1 - P_\ell(\langle x_0, x_1\rangle)\big| \\
&\quad \times \big|C_{\ell,\phi\phi} - P_\ell''(\langle x_1, x_2\rangle)\sin\theta_{x_1}\sin\theta_{x_2}\sin^2(\phi_{x_1}-\phi_{x_2}) \\
&\qquad\qquad - P_\ell'(\langle x_1, x_2\rangle)\cos(\phi_{x_1}-\phi_{x_2})\big|\big) \\
&\le C^2_{\ell,\phi\phi} M(\varepsilon,\ell) \\
&\quad \times \big(\varepsilon C_{\ell,\phi\phi} M_2(\ell,\varepsilon) \\
&\qquad \times \big(|\sin\theta_{x_2}|\cdot\big|\sin(\phi_{x_2}-\phi_{x_0}) - \sin(\phi_{x_1}-\phi_{x_0})\big| \\
&\qquad\quad + \big|\sin(\phi_{x_1}-\phi_{x_0})\big|\cdot|\sin\theta_{x_2} - \sin\theta_{x_1}| + M_3(\ell,\varepsilon)|x_2 - x_1|\big) \\
&\qquad + M_1'(\varepsilon,\ell)|x_2 - x_1| + M_1''(\varepsilon,\ell)\cdot|\sin\theta_{x_1}\sin\theta_{x_2}|\cdot\sin^2(\phi_{x_1}-\phi_{x_2}) \\
&\qquad\qquad + M_1''(\varepsilon,\ell)\times\big|C_{\ell,\phi\phi} - P_\ell'(\langle x_1, x_2\rangle)\cos(\phi_{x_1}-\phi_{x_2})\big|\big) \\
&\le C^2_{l\phi\phi} M(\varepsilon,\ell) \\
&\quad \times \big(\varepsilon C_{\ell,\phi\phi} M_2(\varepsilon,\ell) M_4(\varepsilon,\ell)\cdot\big|\sin(\phi_{x_2}-\phi_{x_1}) - \sin(\phi_{x_1}-\phi_{x_1})\big| \\
&\qquad + M_4(\varepsilon,\ell)\cdot|\sin\theta_{x_2} - \sin\theta_{x_1}| + M_3(\varepsilon,\ell)|x_2 - x_1| \\
&\qquad + M_1'(\varepsilon,\ell)|x_2 - x_1| \\
&\qquad + M_1'''(\varepsilon,\ell)\sin^2(\theta_{x_2}-\theta_{x_1}) + M_1''(\varepsilon,\ell)\cdot\big|C_{\ell,\phi\phi} - P_\ell'(\langle x_1, x_2\rangle)\big| \\
&\qquad\qquad + M_1^{(iv)}(\varepsilon,\ell)\big|1 - \cos(\phi_{x_2}-\phi_{x_1})\big|\big).
\end{aligned}
$$

Now note that $C_{\ell,\phi\phi}$ is precisely equal to $P_\ell'(1)$, which can be rewritten as $P_\ell'(\langle x_1, x_1\rangle)$. Replacing this in the last part of the above expression, we get the following:

$$
\begin{aligned}
&\left|\frac{C^2_{\ell;\phi\phi}\partial_{\phi_{x_0}} P_\ell(\langle x_1, x_0\rangle)}{[1 - P_\ell(\langle x_0, x_1\rangle)]^2[1 - P_\ell(\langle x_0, x_2\rangle)]}\right| \\
&\quad \times \big|\big(\{\partial_{\phi_{x_0}} P_\ell(\langle x_1, x_0\rangle) - \partial_{\phi_{x_0}} P_\ell(\langle x_2, x_0\rangle)\} C_{\ell;\phi\phi}[1 - P_\ell(\langle x_0, x_2\rangle)] \\
&\qquad + \partial_{\phi_{x_0}} P_\ell(\langle x_2, x_0\rangle)\{C_{\ell;\phi\phi}[1 - P_\ell(\langle x_0, x_2\rangle)] \\
&\qquad\qquad - [1 - P_\ell(\langle x_0, x_1\rangle)]\partial_{\phi_{x_1}}\partial_{\phi_{x_2}} P_\ell(\langle x_1, x_2\rangle)\}\big)\big| \\
&\le C^2_{l\phi\phi} M(\varepsilon,\ell) \\
&\quad \times \big(\varepsilon C_{\ell,\phi\phi} M_2(\varepsilon,\ell) M_4(\varepsilon,\ell)\cdot\big|\sin(\phi_{x_2}-\phi_{x_1}) - \sin(\phi_{x_1}-\phi_{x_1})\big| \\
&\qquad + M_4(\varepsilon,\ell)\cdot|\sin\theta_{x_2} - \sin\theta_{x_1}| + M_3(\varepsilon,\ell)|x_2 - x_1| \\
&\qquad + M_1'(\varepsilon,\ell)|x_2 - x_1| \\
&\qquad + M_1'''(\varepsilon,\ell)\sin^2(\theta_{x_2}-\theta_{x_1}) + M_1''(\varepsilon,\ell)\cdot\big|P_\ell'(\langle x_1, x_1\rangle) - P_\ell'(\langle x_1, x_2\rangle)\big|
\end{aligned}
$$

$$+ M_1^{(iv)}(\varepsilon, \ell)\big|1 - \cos(\phi_{x_2} - \phi_{x_1})\big|\big)$$

$$\leq c(\varepsilon, L_K)|x_1 - x_2|.$$

By replicating these set of calculations for each pair of terms in $\mathbb{E}(\widehat{f}_{j;q}^{x_0}(x_2) - \widehat{f}_{j;q}^{x_0}(x_1))^2$, we conclude that for every $x_1, x_2 \in B(x_0, \varepsilon)$,

$$\mathbb{E}\big(\widehat{f}_{j;q}^{x_0}(x_2) - \widehat{f}_{j;q}^{x_0}(x_1)\big)^2 \leq c(\varepsilon, L_K)|x_2 - x_1|.$$

Next, we wish to extend this to points inside the set $B(x_0, \varepsilon) \setminus \{x_0\}$, but the Lipschitz coefficient $c(\varepsilon, L_K)$ needs to be controlled. Observing that $c(\varepsilon, L_K)$ depends on ε through the distance of points x_1, x_2 to x_0, note that $\operatorname{cov}(\widehat{f}_{j;q}^{x_0}(x_1), \widehat{f}_{j;q}^{x_0}(x_2))$ grows rapidly as either of x_1 or x_2 approach x_0, whereas when x_1 and x_2 simultaneously approach x_0, then the expression assumes the form of an indeterminate form, for which one can use the standard l'Hôpital's rule to get a precise form of the expression. Thus, let us first examine the following:

$$\begin{aligned}
&\lim_{x\to x_0} \operatorname{var}\big(\widehat{f}_{j;q}^{x_0}(x)\big) \\
&\quad = \lim_{x\to x_0} \frac{1}{(1-\rho(x_0,x))^2} \\
&\qquad \times \big(1 - \rho^2(x_0,x) + 2\rho(x_0,x)\partial_{\theta_{x_0}}\rho(x_0,x)\partial_{\theta_x}\rho(x_0,x)\partial^2_{\theta_x}\rho(x,x) \\
&\qquad\quad + 2\rho(x_0,x)\partial_{\phi_{x_0}}\rho(x_0,x)\partial_{\phi_x}\rho(x_0,x)\partial^2_{\phi_x}\rho(x,x) \\
&\qquad\quad + \big\{\partial_{\theta_{x_0}}\rho(x_0,x)\big\}^2\big\{\partial^2_{\theta_x}\rho(x,x)\big\}^3 \\
&\qquad\quad + \big\{\partial_{\phi_{x_0}}\rho(x_0,x)\big\}^2\big\{\partial^2_{\phi_x}\rho(x,x)\big\}^3\big).
\end{aligned}$$

Let us do the limit computations for just the first term of the variance expression:

$$\begin{aligned}
&\lim_{x\to x_0} \frac{1-\rho^2(x_0,x)}{(1-\rho(x_0,x))^2} \\
&\quad = \lim_{x\to x_0} \frac{-2\rho(x_0,x)\partial_{\theta_x}\rho(x_0,x)}{(-2)(1-\rho(x_0,x))\partial_{\theta_x}\rho(x_0,x)} \\
&\quad = \lim_{x\to x_0} \frac{\rho(x_0,x)\partial^2_{\theta_x}\rho(x_0,x) + (\partial_{\theta_x}\rho(x_0,x))^2}{(1-\rho(x_0,x))\partial^2_{\theta_x}\rho(x_0,x) - (\partial_{\theta_x}\rho(x_0,x))^2} \\
&\quad = \lim_{x\to x_0} \frac{\rho(x_0,x)\partial^3_{\theta_x}\rho(x_0,x) + 3\partial_{\theta_x}\rho(x_0,x)\partial^2_{\theta_x}\rho(x_0,x)}{(1-\rho(x_0,x))\partial^3_{\theta_x}\rho(x_0,x) - 3\partial_{\theta_x}\rho(x_0,x)\partial^2_{\theta_x}\rho(x_0,x)} \\
&\quad = \lim_{x\to x_0} \big(\rho(x_0,x)\partial^4_{\theta_x}\rho(x_0,x) + \partial_{\theta_x}\rho(x_0,x)\partial^3_{\theta_x}\rho(x_0,x) \\
&\qquad\quad + 3\partial^2_{\theta_x}\rho(x_0,x)\partial^2_{\theta_x}\rho(x_0,x) + 3\partial_{\theta_x}\rho(x_0,x)\partial^3_{\theta_x}\rho(x_0,x)\big)
\end{aligned}$$

$$/\big((1-\rho(x_0,x))\partial^4_{\theta_x}\rho(x_0,x) - 4\partial_{\theta_x}\rho(x_0,x)\partial^3_{\theta_x}\rho(x_0,x) \\ - 3\big(\partial^2_{\theta_x}\rho(x_0,x)\big)^2\big),$$

where we have applied l'Hôpital's rule at each step (four times), and we note that the final expression is indeed a nontrivial, determinate limit.

We note that we have assumed $\rho(x_0,x) = P_\ell(\langle x_0,x\rangle)$, and hence the derivatives above have the following form:

$$\partial_{\theta_x} P_\ell(\langle x_0,x\rangle) = P'_\ell(\cdot)\big(\cos\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) - \sin\theta_x\cos\theta_{x_0}\big),$$

$$\partial^2_{\theta_x} P_\ell(\langle x_0,x\rangle) = P''_\ell(\cdot)\big(\cos\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) - \sin\theta_x\cos\theta_{x_0}\big)^2 \\ + P'(\cdot)\big(-\sin\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) - \cos\theta_x\cos\theta_{x_0}\big),$$

$$\partial^3_{\theta_x} P_\ell(\langle x_0,x\rangle) \\ = P'''_\ell(\cdot)\big(\cos\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) - \sin\theta_x\cos\theta_{x_0}\big)^3 \\ + 2P''_\ell(\cdot)\big(\cos\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) - \sin\theta_x\cos\theta_{x_0}\big) \\ \times \big(-\sin\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) - \cos\theta_x\cos\theta_{x_0}\big) \\ + P'(\cdot)\big(-\cos\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) + \sin\theta_x\cos\theta_{x_0}\big),$$

$$\partial^4_{\theta_x} P_\ell(\langle x_0,x\rangle) = P^{(iv)}_\ell(\cdot)\big(\cos\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) - \sin\theta_x\cos\theta_{x_0}\big)^4 \\ + 3P'''_\ell(\cdot)\big(\cos\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) - \sin\theta_x\cos\theta_{x_0}\big) \\ \times \big(-\sin\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) - \cos\theta_x\cos\theta_{x_0}\big) \\ + 2P'''_\ell(\cdot)\big(\cos\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) - \sin\theta_x\cos\theta_{x_0}\big)^2 \\ \times \big(-\sin\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) - \cos\theta_x\cos\theta_{x_0}\big) \\ + 2P''_\ell(\cdot)\big(-\sin\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) - \cos\theta_x\cos\theta_{x_0}\big)^2 \\ - 2P''_\ell(\cdot)\big(\cos\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) - \sin\theta_x\cos\theta_{x_0}\big)^2 \\ + P''(\cdot)\big(\cos\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) - \sin\theta_x\cos\theta_{x_0}\big) \\ \times \big(-\cos\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) + \sin\theta_x\cos\theta_{x_0}\big) \\ + P'(\cdot)\big(\sin\theta_x \sin\theta_{x_0}\cos(\phi_x-\phi_{x_0}) + \cos\theta_x\cos\theta_{x_0}\big).$$

Thus, we conclude that

$$P_\ell(\langle x_0,x\rangle)|_{x=x_0} = 1,$$

$$\partial_{\theta_x} P_\ell(\langle x_0,x\rangle)|_{x=x_0} = 0,$$

$$\partial^2_{\theta_x} P_\ell(\langle x_0,x\rangle)|_{x=x_0} = -P'(1),$$

$$\partial^3_{\theta_x} P_\ell(\langle x_0,x\rangle)|_{x=x_0} = 0,$$

$$\partial_{\theta_x}^4 P_\ell(\langle x_0, x\rangle)|_{x=x_0}, = 2P_\ell''(1) + P_\ell'(1).$$

Subsequently, we shall argue that by continuity, and the fact the field $\widehat{f}_{j;q}^{x_0}$ appears to be singular at x_0, we conclude that for $x_1, x_2 \in B(x_0, \varepsilon)$ and a small enough ε,

$$\sup_{x_1,x_2\in B(x_0,\varepsilon)} \mathbb{E}\big(\widehat{f}_{j;q}^{x_0}(x_2) - \widehat{f}_{j;q}^{x_0}(x_1)\big)^2 = \lim_{(x_1,x_2)\to(x_0,x_0)} \mathbb{E}\big(\widehat{f}_{j;q}^{x_0}(x_2) - \widehat{f}_{j;q}^{x_0}(x_1)\big)^2.$$

The limit on the right-hand side can again be evaluated by applying l'Hôpital's rule, and thus, the (uniform) Lipschitz behaviour is justified. Thereafter, we note that by the isotropy of the underlying field $f_{j;q}$, the $\mathbb{E}(\sup_{x\in S^2\setminus\{x_0\}} \widehat{f}_{j;q}^{x_0}(x))$ does not depend on x_0, and thus we get a uniform (over j and x_0) Lipschitz bound, as claimed.

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