UNIQUENESS OF BUBBLING SOLUTIONS OF MEAN FIELD EQUATIONS

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ABSTRACT. We prove uniqueness of blow up solutions of the mean field equation as $\rho_n \to 8\pi m$, $m \in \mathbb{N}$. If $u_{n,1}$ and $u_{n,2}$ are two sequences of bubbling solutions with the same $\rho_n$ and the same (non degenerate) blow up set, then $u_{n,1} = u_{n,2}$ for sufficiently large $n$. The proof of the uniqueness requires a careful use of some sharp estimates for bubbling solutions of mean field equations [24] and a rather involved analysis of suitably defined Pohozaev-type identities as recently developed in [47] in the context of the Chern-Simons-Higgs equations. Moreover, motivated by the Onsager statistical description of two dimensional turbulence, we are bound to obtain a refined version of an estimate about $\rho_n - 8\pi m$ in case the first order evaluated in [24] vanishes.

1. INTRODUCTION

Let $(M, ds)$ be a compact Riemann surface with volume $|M| = 1$ and $\rho_n > 0$ be a sequence satisfying $\lim_{n \to +\infty} \rho_n = 8\pi m$ for some positive integer $m \geq 1$. We denote by $d\mu$ the volume form, by $\Delta_M$ the Laplace-Beltrami operator on $(M, ds)$, and consider the following mean field type problem:

\[
\begin{cases}
\Delta_M u_n + \rho_n \left( \frac{h(x)u_n^p(x)}{\int_M h u_n^p d\mu} - 1 \right) = 0 \text{ in } M, \\
\int_M u_n d\mu = 0, \quad u_n \in C^\infty(M),
\end{cases}
\]

where $h(x) = h_*(x)e^{-4\pi \sum_{j=1}^N a_j G(x,y)} \geq 0$, $p$ are distinct points, $a_j \in \mathbb{N}$, $h_* > 0$, $h_* \in C^2(M)$, and $G$ is the Green function, which satisfies,

\[-\Delta_M G(x, p) = \delta_p - 1 \text{ in } M, \quad \text{and } \int_M G(x, p) d\mu(x) = 0.\]

The mean field equation (1.1) and the corresponding Dirichlet problem (see (5.1) below) have attracted a lot of attention in recent years because of their applications to several issues of interest in Mathematics and Physics, such as Electroweak and Chern-Simons self-dual vortices [53], [55], [60], conformal metrics on surfaces with [58] or without conical singularities [38], statistical mechanics of two-dimensional turbulence [17] and of self-gravitating systems [59] and cosmic strings [51], and more recently the theory of hyperelliptic curves [19] and of the Painlevé equations [22].

In spite of the many results at hand, and with few exceptions (see [21], [54], and more recently [4], [8]), we still don’t know much about the qualitative behaviour of the global bifurcation diagram of solutions of (1.1) and (5.1).

What one can infer from the above mentioned results in the sub-critical/critical regime is that solutions of (5.1) are unique if $\rho_n < 8\pi$ and are unique whenever they exist if $\rho_n = 8\pi$, see [54] and [8], [21], [36]. The same question can be asked about (1.1) whose answer is still not well understood, see [41], [44] and [37]. However, solutions of either (1.1) or (5.1) are expected to be generically non unique for $\rho_n > 8\pi$, see [12], [13], [29].

Our aim here is to contribute in this direction by showing that blow up solutions of (1.1) and (5.1) are unique for $n$ large enough.

**Definition 1.1.** Let $u_n$ be a sequence of solutions of (1.1). We say that $u_n$ blows up at the points $q_j \notin \{p_1, \cdots, p_N\}$, $j = 1, \cdots, m$, if $\frac{h(x)u_n^p(x)}{\int_M h u_n^p d\mu} \to 8\pi \sum_{j=1}^m \delta_{q_j}$ weakly in the sense of measure in $M$.
Let $K(x)$ be the Gaussian curvature at $x \in M$ and $R(x, y)$ denote the regular part (see section 2 below), of the Green function $G(x, y)$. For $q = (q_1, \ldots, q_m) \in M \times \cdots \times M$, we denote by,

$$G^*_j(x) = 8\pi R(x, q_j) + 8\pi \sum_{i \neq j}^1 G(x, q_i),$$

(1.2)

$$\ell(q) = \sum_{j=1}^m |\Delta_m \log h(q_j) + 8m \pi - 2K(q_j)| h(q_j)e^{G^*_j(q_j)},$$

and

$$f_{q,j}(x) = 8\pi \left[ R(x, q_j) - R(q_j, q_j) + \sum_{i \neq j}^1 (G(x, q_i) - G(q_j, q_j)) \right] + \log \frac{h(x)}{h(q_j)}.$$  

(1.3)

We will denote by $B^m_s(q)$ the geodesic ball of radius $r$ centred at $q \in M$, while $U^m_s(q)$ will denote the pre image of the Euclidean ball of radius $r$, $B_s(q) \subset \mathbb{R}^2$, in a suitably defined isothermal coordinate system (see section 2 below for further details). If $m \geq 2$ we fix a constant $r_0 \in (0, \frac{1}{2})$ and a family of open sets $M_j$ satisfying, $M_j \cap M_j = \emptyset$ if $j \neq k$, $\bigcup_{j=1}^m M_j = M$, $U^m_s(q_j) \subset M_j$, $j = 1, \ldots, m$. Then, let us define,

$$D(q) = \lim_{r \to 0} \sum_{j=1}^m h(q_j)e^{G^*_j(q_j)} \left( \int_{M_j \setminus U^m_s(q_j)} e^{\Phi_j(x, q_j)} d\mu(x) - \frac{\pi}{r_j} \right),$$

(1.5)

where $M_1 = M$ if $m = 1$, $r_j = r/\sqrt{8h(q_j)e^{G^*_j(q_j)}}$ and,

$$\Phi_j(x, q) = \sum_{i=1}^m 8\pi G(x, q_i) - G^*_j(q_j) + \log h(x) - \log h(q_j).$$

(1.6)

The quantity $D(q)$ was first introduced in [21, 28]. For $(x_1, \ldots, x_m) \in M \times \cdots \times M$, we also define,

$$f_m(x_1, x_2, \cdots, x_m) = \sum_{j=1}^m \left[ \log(h(x_j)) + 4\pi R(x_j, x_j) \right] + 4\pi \sum_{i \neq j}^1 G(x_i, x_j),$$

(1.7)

and let $D^2 m$ be its Hessian tensor field on $M$. Then we have,

**Theorem 1.1.** Let $u_n^{(1)}$ and $u_n^{(2)}$ be two sequence of solutions of (1.1), blowing up at the points $q_j \notin \{p_1, \cdots, p_N\}$, $j = 1, \cdots, m$, where $q = (q_1, \cdots, q_m)$ is a critical point of $f_m$ and $\det(D^2 m f_m(q)) \neq 0$. Assume that $\rho_n^{(1)} = \rho_n = \rho_n^{(2)}$ and that, either,

1. $\ell(q) \neq 0$, or,
2. $\ell(q) = 0$ and $D(q) \neq 0$.

Then there exists $n_0 \leq 1$ such that $u_n^{(1)} = u_n^{(2)}$ for all $n \geq n_0$.

The proof of Theorem 1.1 is worked out by an adaptation of an argument recently proposed in [47]. In that paper Lin and Yang prove uniqueness for blow up solutions of the Chern-Simons-Higgs equation. In particular, it is claimed in [47] that the method adopted there does the job also in the case of the mean field equation (1.1) and in fact our aim is to prove that claim. However it seems that the adaptation of that argument to our problem is not straightforward.

First of all, the cornerstone of the proof is the description of the blow up behavior of solutions established in [24]. In that case the leading order of the expansion of $\rho_n - 8\pi m$ as well as of the reminder term of blow up solutions is proportional to $\ell(q)$, see section 2 below. By means of these estimates, if $\ell(q) \neq 0$, we can prove that the difference of the blow up rates (which we denote by $\lambda^{(1)}_{n,j} - \lambda^{(2)}_{n,j}$) is small for large $n$, see Lemma 3.1. This is why the case $\ell(q) = 0$ is more subtle and this is why we are bound to derive an improved version of the estimate concerning $\rho_n - 8\pi m$. A full generalization of the estimates in [24] to the case $\ell(q) = 0$, that is, including the reminder term of blow up solutions, at least to our knowledge has been derived only in case $m = 1$ and only for the Dirichlet problem, see [21].

**Remark 1.2.** Far from being just a mathematical problem, the case $\ell(q) = 0$ often arise in the study of geometric and physical problems, as for example in the Onsager statistical mechanical description of two dimensional turbulence, see [17] and more recently [4]. Motivated by this problem, in the final part of this paper we will discuss the uniqueness result relative to the Dirichlet problem (5.1), see Theorem 5.2 below. Indeed, inspired by a recent result [4], we believe that, in the non degenerate setting of Theorem 5.2 and for large enough $n$, 1-point blow up solutions could be parametrized by their Dirichlet energy.

In particular, on domains of second kind [17], [21], we believe that this fact would imply the existence of a full interval of
strict convexity of the entropy, see [4]. We will discuss this problem in a forthcoming paper [6]. However it is crucial to the understanding of this application to establish uniqueness in case \( \ell(q) = 0 \).

Therefore we derive the following improvement of Theorem 1.1 in [24].

**Theorem 1.3.** Let \( u_n \) be a sequence of solutions of (1.1) which blows up at the points \( q_j \notin \{ p_1, \ldots, p_N \} \), \( j = 1, \ldots, m \), \( \delta > 0 \) be a fixed constant and \( \lambda_{n,j} = \max_{R_B(q_j)} \left( u_n - \log(\int h e^{u_n}) \right) \) for \( j = 1, \ldots, m \).

Then, for any \( n \) large enough, the following estimate holds,

\[
\rho_n - 8\pi m = \frac{2\ell(q) e^{-\lambda_{n,1}}}{m h^2(q_1) e^{G_1(q_1)}} \left( \lambda_{n,1} + \log \rho_n h^2(q_1) e^{G_1(q_1)} \delta^2 - 2 \right) \frac{8e^{-\lambda_{n,1}}}{h^2(q_1) e^{G_1(q_1)} \pi m} \left( D(q) + O(\delta^e) \right) + O(\lambda_{n,1}^2 e^{-\frac{1}{2} \lambda_{n,1}}) + O(e^{-\left(1-\frac{1}{2}\right) \lambda_{n,1}}),
\]

where \( \sigma \) is fixed by the assumption \( h_\sigma \in C^{2,\sigma}(M) \).

The proof of Theorem 1.3 relies on a careful improvement of an argument first proposed in [24]. By using Theorem 1.3, we succeed in showing that \( \lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)} \) is asymptotically small if \( \ell(q) = 0 \) and \( D(q) \neq 0 \) as well, see Lemma 3.1.

Then, as in [47], we analyze the asymptotic behavior of \( \zeta_n = \frac{u_n^{(1)} - u_n^{(2)}}{\|u_n^{(1)} - u_n^{(2)}\|_{L^p(M)}} \). Near each blow up point \( q_j \), and after a suitable scaling, \( \zeta_n \) converges to an entire solution of the linearized problem associated to the Liouville equation:

\[
\Delta v + e^v = 0 \quad \text{in} \quad \mathbb{R}^2.
\]

Solutions of (1.9) are completely classified [23] and take the form,

\[
v(z) = v_{\mu,a}(z) = \log \left( \frac{8e^\mu}{(1 + |z|^2)^2} \right), \quad \mu \in \mathbb{R}, \quad a = (a_1, a_2) \in \mathbb{R}^2.
\]

The freedom in the choice of \( \mu \) and \( a \) is due to the well known invariance of equation (1.9) under dilations and translations. The linearized operator \( L \) relative to \( v_{\mu,a} \) is defined by,

\[
L \phi := \Delta \phi + \frac{8}{(1 + |z|^2)^2} \phi \quad \text{in} \quad \mathbb{R}^2.
\]

It is well known, see [2, Proposition 1], that the kernel of \( L \) has real dimension 3 with eigenfunctions \( Y_0, Y_1, Y_2 \), where,

\[
Y_0(z) = \frac{1 - |z|^2}{1 + |z|^2} = \frac{\partial v_{\mu,a}}{\partial \mu} \bigg|_{(\mu,a)=(0,0)}, \quad Y_1(z) = \frac{z_1}{1 + |z|^2} = -\frac{1}{4} \frac{\partial^2 v_{\mu,a}}{\partial \mu \partial a} \bigg|_{(\mu,a)=(0,0)}, \quad Y_2(z) = \frac{z_2}{1 + |z|^2} = -\frac{1}{4} \frac{\partial^2 v_{\mu,a}}{\partial a \partial a} \bigg|_{(\mu,a)=(0,0)}.
\]

The second crucial point of the proof of Theorem 1.1 is to show that, after scaling and for large \( n \), \( \zeta_n \) is orthogonal to \( Y_0, Y_1 \) and \( Y_2 \). As in [47] this is done by a rather delicate analysis of various suitably defined Pohozaev-type identities. However, compared with [47], we face a truly new difficulty, since the difference of the blow up rates (that is \( \lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)} \)) in our case can be of the same order of \( \lambda_{n,j}^{(2)} \), a situation which cannot occur in the Chern-Simons-Higgs problem discussed in [47]. In order to overcome this difficulty we have to carry out an higher order expansion of \( u_n^{(1)} - u_n^{(2)} \) by using Green’s representation formula. The leading order of that expansion has to be determined explicitly by using the explicit form of entire solutions of (1.9) (see Lemma 3.4). Besides, the main estimates relies on a series of subtle cancellations, see Lemma 4.2 and Lemma 4.3.

**Remark 1.4.** The assumption \( a_j \in \mathbb{N} \) is used to guarantee that \( u \in C^\infty(M) \), which in turn allows a simplified discussion of the already very technical proof. However, since by assumption \( q_j \notin \{ p_1, \ldots, p_m \} \), then we may relax that assumption and let \( a_j \in (-1, +\infty) \). Indeed, the sharp local estimates in [24] still hold in this more general setting, just with minor changes relative to the regularity class of \( u_n \). In other words, Theorem 1.1 still holds if we allow \( a_j \in (-1, +\infty) \), \( j = 1, \ldots, m \).

This paper is organized as follows. In section 2 we review some known sharp estimates for blow up solutions of (1.1). In section 3 we analyse the limit behavior of \( \zeta_n \) on each region \( U_M^{(j)}(q_j) \) and \( M \setminus \cup_{j=1}^m U_M^{(j)}(q_j) \). In section 4 we prove Theorem 1.1 by the analysis of some suitably derived Pohozaev-type identities. In section 5 we discuss the uniqueness of solutions of the Dirichlet problem. In section 6 we prove Theorem 1.3.
2. PRELIMINARY

In this section we recall some sharp estimates for blow up solutions of (1.1). Suppose that \( u_n \) is a sequence of blow-up solutions of (1.1) which blows up at \( \tilde{q}_j \neq \{ q_1, \ldots, q_N \} \), \( j = 1, \ldots, m \). Let

\[
\tilde{u}_n(x) = u_n(x) - \log \left( \int_M h \mu \right).
\]

Then it is easy to see that,

\[
\Delta_m \tilde{u}_n + \rho_n(h(x)) \tilde{u}_n(x) - 1 = 0 \quad \text{in} \ M, \quad \text{and}
\int_M h \tilde{u}_n \, d\mu = 1. \tag{2.1}
\]

We denote by,

\[
\lambda_n = \max_M \tilde{u}_n, \quad \text{and}
\lambda_{n,j} = \max_{B_{r}(\tilde{q}_j)} \tilde{u}_n(x_{n,j}) \quad \text{for} \ j = 1, \ldots, m, \quad \text{where} \ \delta > 0 \ \text{is a fixed constant.}
\]

Next, let us introduce some notations for local computations. We introduce a local isothermal coordinate system \( \tilde{x} = T_j(x) \in \mathbb{R}^2 \), such that \( \tilde{y}_j = T_j(q_j) \), \( \tilde{x}_n = T_j(x_n) \) and \( ds^2 = e^{2\varphi_j(\tilde{x})} |d\tilde{x}|^2 \) with \( \varphi_j(\tilde{x}_{n,j}) = 0 \) and \( \nabla \varphi_j(\tilde{x}_{n,j}) = 0 \). It will be also useful to denote by \( U_n^M(x_0) = T_j^{-1}(B_{\delta}(\tilde{x}_m)) \), the pre-image of \( B_{\delta}(\tilde{x}_m) \), where \( \tilde{x}_m = T_j(0) \) and \( B_{\delta}(\tilde{x}_m) \subset \mathbb{R}^2 \) denotes the Euclidean ball of radius \( r \) centred at \( \tilde{x}_m \in \mathbb{R}^2 \). Therefore, when evaluated in \( U_n^M(x_{n,j}) \), in local coordinates (2.1) takes the form,

\[
\Delta \tilde{u}_n + \rho_n e^{2\varphi_j(h(x))} e^{\delta_n(x)} - 1 = 0 \quad \text{in} \ x \in B_{\delta}(\tilde{x}_{n,j}),
\]

where \( \Delta = \sum_{i=1}^{2} \partial_{x_i}^2 \) denotes the standard Laplacian in \( \mathbb{R}^2 \).

For later use we recall that \( r_0 > 0 \) is defined as right after (1.4) to guarantee that,

\[
U_{2r_0}^M(q_j) \subset M_j, \quad j = 1, \ldots, m. \tag{2.3}
\]

**Remark 2.1.** To simplify the exposition we will use the expressions \( \tilde{u}, h, R, G, K, \ldots \) to denote those function in both global and local coordinates. It will be clear time to time which one of the functions involved is being used.

**Remark 2.2.** We will often need to take back local estimates into globally defined quantities. Therefore we fix an atlas whose local maps are denoted by \( \{ T_n \} \), and whenever for some \( k \geq 1 \) we have \( g = g(\tilde{x}_1, \ldots, \tilde{x}_m) \) with \( \tilde{x}_m = T_n(x_i), \ i = 1, \ldots, k \), then we will denote by \( T_n^{-1}(g(\tilde{x}_1, \ldots, \tilde{x}_m)) = g(T_n(x_1), \ldots, T_n(x_k)) \).

It is well known that the conformal factor \( \varphi_a \) is a solution of,

\[
-\Delta \varphi_a = e^{2\varphi_a} K, \quad \tilde{x} \in B_{\delta}(\tilde{x}_m). \tag{2.4}
\]

The regular part of the Green function \( R(x, y) \) is defined in a local isothermal coordinate system \( \tilde{x} = T_n(x) \) as follows. For \( y = T_n(y) \) fixed, we can choose the conformal factor \( e^{2\varphi_a(\tilde{x})} \) so that \( \varphi_a(y) = 0 \). Then \( R(x, y) \) is defined to be the solution of

\[
\Delta R(\tilde{x}, \tilde{y}) = e^{2\varphi_a(\tilde{x})}, \quad \tilde{y} \in B_{\delta}(\tilde{x}_m). \tag{2.5}
\]

and therefore it is not difficult to check that it also satisfies,

\[
R(\tilde{x}, \tilde{y}) = \frac{1}{2\pi} \log |\tilde{x} - \tilde{y}| + G(\tilde{x}, \tilde{y}).
\]

Next, let us define,

\[
U_{n,j}(\tilde{x}) = \log \frac{e^{\lambda_{n,j}}}{(1 + \rho_n(h(\tilde{x}_{n,j})) \lambda_{n,j} |\tilde{x} - \tilde{x}_{n,j}|^2)^2}, \quad \tilde{x} \in \mathbb{R}^2, \tag{2.6}
\]

where the point \( \tilde{x}_{n,j} \) is chosen to satisfy,

\[
\nabla U_{n,j}(\tilde{x}_{n,j}) = \nabla (\log h(\tilde{x}_{n,j})).
\]

Then, it is not difficult to check that,

\[
|\tilde{x}_{n,j} - \tilde{x}_{n,j}| = O(e^{-\lambda_{n,j}}). \tag{2.7}
\]
Let us also define,
\[
\eta_{n,j}(x) = \partial_\mu(x) - U_{n,j}(x) - (G_\mu^j(x) - G_\mu^j(x_{n,j})), \quad x \in B_\delta(x_{n,j}).
\]  
(2.8)

It has been proved in [24, Theorem 1.4] that, for \( x \in B_\delta(x_{n,j}) \), it holds,
\[
\eta_{n,j}(x) = \frac{-8}{\rho_n h(x_{n,j})} [\Delta \log h(x_{n,j}) + 8\pi m - 2K(x_{n,j})] e^{-\lambda_n} [\log(R_{n,j}x - x_{n,j}) + 2]^2
\]
\[
+ O(\log(R_{n,j}|x - x_{n,j}| + 2)e^{-\lambda_n}) + O(\lambda_{n,j} e^{-\lambda_n}) = O(\lambda_{n,j} e^{-\lambda_n}),
\]  
(2.9)

where \( R_{n,j} = \sqrt{\rho_n h(x_{n,j}) / 8} \). It has also been proved in [24, Corollary 2.4] that one can find constants \( c > 0 \) and \( c_\delta > 0 \) such that,
\[
|\lambda_n - \lambda_{n,j}| \leq c \quad \text{for} \quad j = 1, \cdots, m, \quad |\tilde{u}_n(x) + \lambda_n| \leq c_\delta \quad \text{for} \quad x \in M \setminus \bigcup_{j=1}^m B_\delta(q_j).
\]  
(2.10)

Moreover, see [24, section 3], we have,
\[
e^{\lambda_n h(x_n,j)} e^{G_j^\lambda(x_n,j)} = e^{\lambda_n} e^{2\lambda_n^2} (1 + O(e^{\lambda_n})) = \lambda_{n,j} e^{-\lambda_n^j}.
\]  
(2.11)

and in particular, see [24, Theorem 1.4], the following estimate holds,
\[
\lambda_{n,j} + \int_M \tilde{u}_n d\mu + 2 \log \frac{\rho_n h(x_{n,j})}{8} + G_j^\lambda(x_{n,j})
\]
\[
= -\frac{2}{\rho_n h(x_{n,j})} (\Delta h(x_{n,j}) + 8\pi m - 2K(x_{n,j})) \lambda_{n,j} e^{-\lambda_n^j} + O(\lambda_{n,j} e^{-\lambda_n}).
\]  
(2.12)

Let us also recall, see [24, Lemma 5.4], that,
\[
\nabla_M |\log h(x) + G_j^\lambda(x)| \big|_{x=x_{n,j}} = O(\lambda_{n,j} e^{-\lambda_n}).
\]  
(2.13)

where \( \nabla_M \) is a suitable gradient vector field on \( M \), which, together with the assumption \( \det(D^2f_m(q)) \neq 0 \), shows that,
\[
|x_{n,j} - q_j| = O(\lambda_{n,j} e^{-\lambda_n}).
\]  
(2.14)

**Remark 2.3.** We remark that, since in any local isothermal coordinate system it holds \( \Delta_M = e^{-2\psi} \Delta \), then, in view of (2.14) and \( \varphi_j(x_{n,j}) = 0, \nabla \varphi_j(x_{n,j}) = 0 \), we find that,
\[
\Delta_M \log h(x_{n,j}) = e^{-2\psi} \Delta \log h(x_{n,j}) = e^{-2\psi} \nabla \log h(q_j) = O(\lambda_{n,j} e^{-\lambda_n}) = \Delta_M \log h(q_j) + O(\lambda_{n,j} e^{-\lambda_n}).
\]  

This fact will be often used in the many estimates involved.

The local masses corresponding to the blow up of \( \tilde{u}_n \) at \( q_j, 1 \leq j \leq m \), are defined as follows,\[\]
\[
\rho_{n,j} = \rho_n \int_{\partial B_{\delta}(q_j)} \rho e^{\tilde{u}_n} d\mu,
\]  
(2.15)

and we will use the following estimate proved in [24, section 3],
\[
\rho_{n,j} - 8\pi = \frac{16\pi}{\rho_n h(x_{n,j})} (\Delta \log h(x_{n,j}) + \rho_n - 2K(x_{n,j}) \lambda_{n,j} e^{-\lambda_n^j} + O(e^{-\lambda_n})),
\]  
(2.16)

In particular, see [24, Theorem 1.1], we have:
\[
\rho_n - 8\pi m = \frac{2}{m} \sum_{j=1}^m h^{-1}(x_{n,j}) |\Delta \log h(x_{n,j}) + 8\pi m - 2K(x_{n,j}) |\lambda_{n,j} e^{-\lambda_n^j} + O(e^{-\lambda_n})
\]
\[
= \frac{2}{m} \sum_{j=1}^m \lambda_{n,j} e^{-\lambda_n^j} \sum_{j=1}^m [\Delta \log h(x_{n,j}) + 8\pi m - 2K(x_{n,j})] h(x_{n,j}) e^{G_j^\lambda(x_{n,j})} + O(e^{-\lambda_n})
\]  
(2.17)

\[
\frac{2}{m} \sum_{j=1}^m \lambda_{n,j} e^{-\lambda_n^j} \ell(q) + O(e^{-\lambda_n}).
\]
The asymptotic behavior of $\tilde{u}_n$ on $M \setminus \cup_{j=1}^{m} U^M_{\delta}(q_j)$, is well described in terms of the auxiliary function,
\begin{equation}
w_n(x) = \tilde{u}_n(x) - \sum_{j=1}^{m} \rho_{n,j} G(x, x_{n,j}) - \int_M \tilde{u}_n \, d\mu.
\end{equation}
which satisfies, see [24, Lemma 5.3],
\begin{equation}
w_n = o(e^{-\frac{\lambda_{n,j}}{n}}) \text{ on } C^1(M \setminus \cup_{j=1}^{m} U^M_{\delta}(q_j)).
\end{equation}

3. Uniqueness of the Blow up Solutions with Mass Concentration

To prove Theorem 1.1 we argue by contradiction and assume that (1.1) has two different solutions $u_n^{(1)}$ and $u_n^{(2)}$, with $\rho_n^{(1)} = \rho_n = \rho_n^{(2)}$, which blow up at $q_j$, $j = 1, \ldots, m$. We will use $x_{n,j}$, $\lambda_{n,j}$, $\tilde{u}_n^{(i)}$, $R_{n,j}$, $U_n^{(i)}$, $x_{n,j,s}$, $w_n^{(i)}$, $\rho_{n,j}$ to denote $x_{n,j}$, $\lambda_{n,j}$, $\tilde{u}_n$, $R_{n,j}$, $U_n$, $x_{n,j,s}$, as defined in section 2, corresponding to $u_n^{(i)}$, $i = 1, 2$, respectively.

Our first result is an estimate about $|\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}|$ and $\|\tilde{u}_n^{(1)} - \tilde{u}_n^{(2)}\|_{L^\infty(M)}$.

**Lemma 3.1.** (i) $|\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}| = O \left( \sum_{i=1}^{2} \frac{1}{\lambda_{n,j}^{(i)}} \right)$ for all $1 \leq j \leq m$.

(ii) There exists a constant $c > 1$ such that:
\begin{equation}
\frac{1}{c} |\lambda_{n,1}^{(1)} - \lambda_{n,1}^{(2)}| + O \left( \sum_{i=1}^{2} \lambda_{n,j}^{(i)} e^{-\frac{\lambda_{n,j}^{(i)}}{n}} \right) \leq \|u_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)} \leq c |\lambda_{n,1}^{(1)} - \lambda_{n,1}^{(2)}| + O \left( \sum_{i=1}^{2} \lambda_{n,j}^{(i)} e^{-\frac{\lambda_{n,j}^{(i)}}{n}} \right).
\end{equation}

**Proof.** (i) In view of (2.8) and (2.9), we see that, for $x \in B_i(q_j)$, it holds,
\begin{equation}
\begin{aligned}
\tilde{u}_n^{(1)}(x) - \tilde{u}_n^{(2)}(x) &= U_n^{(1)}(x) - U_n^{(2)}(x) + G_j^*(x_{n,j}) - G_j^*(x_{n,j}) + \eta_{n,j}^{(1)}(x) - \eta_{n,j}^{(2)}(x) \\
&= U_n^{(1)}(x) - U_n^{(2)}(x) + G_j^*(x_{n,j}) - G_j^*(x_{n,j}) + O \left( \sum_{i=1}^{2} (\lambda_{n,j}^{(i)})^2 e^{-\lambda_{n,j}^{(i)}} \right).
\end{aligned}
\end{equation}

By the definition of $U_n^{(i)}$, we find,
\begin{equation}
U_n^{(1)}(x) - U_n^{(2)}(x) = 2 \log \left( 1 + \frac{\rho_{n,j} (x_{n,j}^{(1)} - x_{n,j}^{(2)})^2}{8} \lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)} \right)
\end{equation}
while, by (2.7) and (2.14), we also have,
\begin{equation}
|\xi_{n,j}^{(1)} - \xi_{n,j}^{(2)}| = O \left( \sum_{i=1}^{2} \lambda_{n,j}^{(i)} e^{-\lambda_{n,j}^{(i)}} \right), \quad \text{and} \quad |\xi_{n,j,s}^{(1)} - \xi_{n,j,s}^{(2)}| = O \left( \sum_{i=1}^{2} \lambda_{n,j}^{(i)} e^{-\lambda_{n,j}^{(i)}} \right) \text{ for any } 1 \leq j \leq m.
\end{equation}

At this point we conclude the proof of Lemma 3.1 by considering two distinct cases:

Case 1. $\ell(q) \neq 0$. From (2.17) we have,
\begin{equation}
\frac{2}{m} \lambda_{n,1}^{(1)} e^{-\lambda_{n,1}^{(1)}} \ell(q) + O(e^{-\lambda_{n,1}^{(1)}}) = \frac{2}{m} \lambda_{n,2}^{(2)} e^{-\lambda_{n,2}^{(2)}} \ell(q) + O(e^{-\lambda_{n,2}^{(2)}}),
\end{equation}
that is, dividing by $\frac{2}{m} \lambda_{n,1}^{(1)} e^{-\lambda_{n,1}^{(1)}} \ell(q)$, and in view of (3.3),
\begin{equation}
\frac{\lambda_{n,1}^{(1)} e^{-\lambda_{n,1}^{(1)}}}{\lambda_{n,2}^{(2)} e^{-\lambda_{n,2}^{(2)}}} = 1 + O \left( \sum_{i=1}^{2} \frac{1}{\lambda_{n,j}^{(i)}} \right),
\end{equation}
which in turn implies that,
\begin{equation}
-(\lambda_{n,1}^{(1)} - \lambda_{n,1}^{(2)}) + \log \frac{\lambda_{n,1}^{(1)}}{\lambda_{n,1}^{(2)}} = \log \left( 1 + O \left( \sum_{i=1}^{2} \frac{1}{\lambda_{n,j}^{(i)}} \right) \right) = O \left( \sum_{i=1}^{2} \frac{1}{\lambda_{n,j}^{(i)}} \right).
\end{equation}
As a consequence, by using also (2.11), we conclude that
\[ |\lambda_{n,1}^{(1)} - \lambda_{n,1}^{(2)}| = O \left( \sum_{i=1}^{2} \frac{1}{\lambda_{n,1}^{(i)}} \right). \tag{3.6} \]

As a consequence, by using also (2.11), we conclude that
\[ |\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}| = O \left( \sum_{i=1}^{2} \frac{1}{\lambda_{n,1}^{(i)}} \right) \quad \text{for all } 1 \leq j \leq m, \tag{3.7} \]

whenever \( \ell(q) \neq 0 \), as claimed.

Case 2. If \( \ell(q) = 0 \) and \( D(q) \neq 0 \): In view of (1.8), we have,
\[ \frac{8}{h^2(q_1)e^\frac{n}{q_1}(q_1)} D(q) + O(\delta^\nu) = O\left( \sum_{i=1}^{2} (\lambda_{n,1}^{(i)})^2 e^{-\frac{2}{n} \lambda_{n,1}^{(i)}} \right) + O\left( \sum_{i=1}^{2} e^{-\frac{1}{n} \lambda_{n,1}^{(i)}} \right), \tag{3.8} \]

and then the same argument used in Case 1 above shows that if \( \ell(q) = 0 \) and \( D(q) \neq 0 \), then (3.7) holds as well.

(ii) Next, in view of (3.7) and (3.8), we see that,
\[ h(x_{n,j}^{(1)}) e^{\lambda_{n,j}^{(1)}} |x - x_{n,j}^{(1)}|^2 - h(x_{n,j}^{(2)}) e^{\lambda_{n,j}^{(2)}} |x - x_{n,j}^{(2)}|^2 \]
\[ = O(e^{\lambda_{n,j}^{(1)}}) \left( \sum_{j=1}^{m} |\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}|^2 \right), \]

which, together with (3.2), (3.3), allows us to conclude that,
\[ \|u_{n,j}^{(1)} - u_{n,j}^{(2)}\|_{L^{\infty}(B_d(q_j))} = O(1) \left( \sum_{i=1}^{2} \lambda_{n,j}^{(i)} e^{-\frac{\lambda_{n,j}^{(i)}}{2}} \right). \tag{3.9} \]

From (3.1), (3.3), and (3.9), we finally obtain that,
\[ \|\tilde{\mu}_n^{(1)} - \tilde{\mu}_n^{(2)}\|_{L^{\infty}(B_d(q_j))} = O(1) \left( \sum_{i=1}^{2} \lambda_{n,j}^{(i)} e^{-\frac{\lambda_{n,j}^{(i)}}{2}} \right) \quad \text{for all } 1 \leq j \leq m. \tag{3.10} \]

Next we estimate \( \tilde{\mu}_n^{(1)} - \tilde{\mu}_n^{(2)} \) in \( M \setminus \cup_{j=1}^{m} U_{\delta}^{(i)}(q_j) \). By the Green’s representation formula, we see that, for \( x \in M \setminus \cup_{j=1}^{m} U_{\delta}^{(i)}(q_j) \), it holds,
\[ \tilde{\mu}_n^{(1)}(x) - \tilde{\mu}_n^{(2)}(x) = \int_M (\tilde{\mu}_n^{(1)} - \tilde{\mu}_n^{(2)}) d\mu = \rho_n \int_M G(y, x) h(y) (\tilde{\mu}_n^{(1)}(y) - \tilde{\mu}_n^{(2)}(y)) d\mu(y) \]
\[ = \rho_n \int_M G(y, x) h(y) (\tilde{\mu}_n^{(1)}(y) - \tilde{\mu}_n^{(2)}(y)) d\mu(y) \]
\[ + \sum_{j=1}^{m} G(x_{n,j}^{(1)}, x) \int_{U_{\delta}^{(i)}(q_j)} \rho_n h(y) (\tilde{\mu}_n^{(1)}(y) - \tilde{\mu}_n^{(2)}(y)) d\mu(y) \]
\[ + \rho_n \int_M G(y, x) h(y) (\tilde{\mu}_n^{(1)}(y) - \tilde{\mu}_n^{(2)}(y)) d\mu(y). \]

In view of (2.15) and (2.10), we find that, for \( x \in M \setminus \cup_{j=1}^{m} U_{\delta}^{(i)}(q_j) \),
\[ \tilde{\mu}_n^{(1)}(x) - \tilde{\mu}_n^{(2)}(x) = \int_M (\tilde{\mu}_n^{(1)} - \tilde{\mu}_n^{(2)}) d\mu = \rho_n \sum_{j=1}^{m} G(x_{n,j}^{(1)}, x) \rho_n h(y) (\tilde{\mu}_n^{(1)}(y) - \tilde{\mu}_n^{(2)}(y)) d\mu(y) \]
\[ + \sum_{j=1}^{m} G(x_{n,j}^{(1)}, x) (\rho_n^{(1)}(x) - \rho_n^{(2)}(x)) + O\left( \sum_{i=1}^{2} e^{-\frac{\lambda_{n,j}^{(i)}}{2}} \right). \tag{3.11} \]
We also have, from (2.16), (3.3), and (3.7),
\[ \frac{1}{\rho_n} \left( \rho_n h(x) \right) - \frac{1}{\rho_n} \left( \rho_n h(x) \right) = 16\pi \{ \Delta \log h(x) + \rho_n - 2K(x) \} \lambda_n^{(1)} e^{-\lambda_n^{(1)}} \]
\[ - 16\pi \{ \Delta \log h(x) + \rho_n - 2K(x) \} \lambda_n^{(2)} e^{-\lambda_n^{(2)}} + O \left( \sum_{i=1}^{\infty} e^{-\lambda_n^{(i)}} \right) = O \left( \sum_{i=1}^{\infty} e^{-\lambda_n^{(i)}} \right). \] (3.12)

By using (3.11), (3.12), and (2.9), we have, for \( x \in M \setminus \bigcup_{j=1}^{\infty} U_{\delta}^{(i)}(q_j) \),
\[ \int_{M} \left( \hat{\alpha}_n^{(1)}(x) - \hat{\alpha}_n^{(2)}(x) \right) d\mu = \frac{\rho_n}{\rho_n h(x)} \int_{M} \left( G(y, x) - G(x^{(1)}_{n,j}, x) \right) h(y) \left( e^{\hat{\alpha}_n^{(1)}}(x) - e^{\hat{\alpha}_n^{(2)}}(x) \right) d\mu(y) + O \left( \sum_{i=1}^{\infty} e^{-\lambda_n^{(i)}} \right) \]
\[ = \sum_{j=1}^{\infty} \int_{B_{\delta}^{(i)}(x^{(1)}_{n,j})} \left( \sum_{i=1}^{\infty} \frac{1}{(1 + e^{\hat{\alpha}_n^{(1)}}(x)) |y - x^{(1)}_{n,j}|^2} \right) dy + O \left( \sum_{i=1}^{\infty} e^{-\lambda_n^{(i)}} \right) = O \left( \sum_{i=1}^{\infty} e^{-\lambda_n^{(i)}} \right). \] (3.13)

Therefore, we see from (2.12), (3.3), and (3.6) that,
\[ \int_{M} \left( \hat{\alpha}_n^{(1)}(x) - \hat{\alpha}_n^{(2)}(x) \right) d\mu = - \left( \lambda_n^{(1)} - \lambda_n^{(2)} \right) + O \left( \sum_{i=1}^{\infty} \lambda_n^{(i)} e^{-\lambda_n^{(i)}} \right), \] (3.14)
which, together with (3.13), shows that,
\[ \hat{\alpha}_n^{(1)}(x) - \hat{\alpha}_n^{(2)}(x) = - \left( \lambda_n^{(1)} - \lambda_n^{(2)} \right) + O \left( \sum_{i=1}^{\infty} e^{-\lambda_n^{(i)}} \right), \] (3.15)
for \( x \in M \setminus \bigcup_{j=1}^{\infty} U_{\delta}^{(i)}(q_j) \). Clearly (3.15) and (3.10) prove (ii), and so the proof of Lemma 3.1 is completed.

Let us define,
\[ \xi_n(x) = \frac{\hat{\alpha}_n^{(1)}(x) - \hat{\alpha}_n^{(2)}(x)}{\| \hat{\alpha}_n^{(1)} - \hat{\alpha}_n^{(2)} \|_{L^\infty(M)}} \quad \text{for} \quad x \in M, \]
\[ f_n(x) = \frac{\rho_n h(x)}{\| \hat{\alpha}_n^{(1)} - \hat{\alpha}_n^{(2)} \|_{L^\infty(M)}} \left( e^{\hat{\alpha}_n^{(1)}}(x) - e^{\hat{\alpha}_n^{(2)}}(x) \right), \quad x \in M, \]
\[ c_n(x) = \frac{e^{\hat{\alpha}_n^{(1)}}(x)}{\| \hat{\alpha}_n^{(1)} - \hat{\alpha}_n^{(2)} \|_{L^\infty(M)}} \left( e^{\hat{\alpha}_n^{(1)}}(x) - e^{\hat{\alpha}_n^{(2)}}(x) \right) = e^{\hat{\alpha}_n^{(1)}}(x) \left( 1 + O(\| \hat{\alpha}_n^{(1)} - \hat{\alpha}_n^{(2)} \|_{L^\infty(M)}) \right), \quad x \in M. \] (3.16)

Clearly \( \xi_n \) satisfies
\[ \Delta \xi_n + f_n(x) = \Delta \xi_n + \rho_n h(x) c_n(x) \xi_n(x) = 0, \quad x \in M. \] (3.17)

Finally, let us define \( \xi_n \) to be the local coordinate expression of \( \xi_n \) for \( x \in B_{\delta}(x^{(1)}_{n,j}) \) and,
\[ \xi_n_{n,j}(z) = \xi_n \left( e^{-\lambda_n^{(1)}} z + x^{(1)}_{n,j} \right), \quad |z| < \delta e^{-\lambda_n^{(1)}}, \quad 1 \leq j \leq m. \]

Our next result is about the limit of \( \xi_n_{n,j} \).

**Lemma 3.2.** There exists constants \( b_{j,0}, b_{j,1}, \text{ and } b_{j,2} \) such that,
\[ \xi_n_{n,j}(z) \rightarrow b_{j,0} \psi_{j,0}(z) + b_{j,1} \psi_{j,1}(z) + b_{j,2} \psi_{j,2}(z) \quad \text{in } C_0^\infty(\mathbb{R}^2), \]
where
\[ \psi_{j,0}(z) = \frac{1 - \pi m h(q_j) |z|^2}{1 + \pi m h(q_j) |z|^2}, \quad \psi_{j,1}(z) = \sqrt{\frac{\pi m h(q_j) |z|^2}{1 + \pi m h(q_j) |z|^2}}, \quad \psi_{j,2}(z) = \frac{\sqrt{\pi m h(q_j) |z|^2}}{1 + \pi m h(q_j) |z|^2}. \]
Proof. By Lemma 3.1 and (2.8), we have, for \( x \in B_\delta(\mathbf{x}_{n,j}^{(1)}) \),
\[
\zeta_n(x) = e^{u_n^{(1)}(x)} \left( 1 + O \left( \sum_{i=1}^{2} \frac{1}{\lambda_{n,j}^{(1)}} \right) \right).
\]
(3.18)

Then, in view of (2.9) and (2.7), we see that (3.18) implies that,
\[
e^{-\lambda_{n,j}^{(1)} \zeta_n(x)} e^{\lambda_{n,j}^{(1)} \lambda_{n,j}^{(1)} \zeta_n(x)} \left( 1 + O \left( \sum_{i=1}^{2} \frac{1}{\lambda_{n,j}^{(1)}} \right) \right).
\]
(3.19)

Since \( |\zeta_{n,j}| \leq 1 \), and in view of (3.17) and (3.19), we also find, \( \zeta_{n,j}(z) \to \zeta_j(z) \) in \( C^0_{loc}(\mathbb{R}^2) \), where,
\[
\Delta \zeta_j + \frac{8 \pi \rho h(q_j)}{(1 + \pi \rho h(q_j) |z|^2) |z|^2} \zeta_j(z) = 0 \text{ in } \mathbb{R}^2, \quad |z| \leq 1.
\]

By [2, Proposition 1], \( \zeta_j = b_{j,0} \varphi_{j,0} + b_{j,1} \varphi_{j,1} + b_{j,2} \varphi_{j,2} \) which proves Lemma 3.2.

Next, let us set,
\[
h_j(x) = h(x) e^{2 \varphi_j(x)}, \quad x \in B_\delta(\mathbf{x}_{n,j}^{(1)}).
\]

For any subset \( \Lambda \subseteq M \), we denote by,
\[
1_A(x) = \begin{cases} 
1 & \text{if } x \in A, \\
0 & \text{if } x \notin A,
\end{cases}
\]
while, for any \( r > 0 \), we also denote by,
\[
\Lambda_{n,j,r}^- = r e^{-\lambda_{n,j}^{(1)}/2},
\]
\[
\Lambda_{n,j,r}^+ = r e^{\lambda_{n,j}^{(1)}/2}.
\]
(3.20)

Next we prove an estimate which will be needed in section 4.

**Lemma 3.3.**
\[
\zeta_n(x) - \int_M \zeta_n \, d\mu = \sum_{j=1}^{m} A_{n,j} G(x_{n,j}^{(1)}, x) + \sum_{j=1}^{m} \frac{\sqrt{2\pi h(q_j)}}{\rho h(q_j)} \int_{\mathbb{R}^2} \frac{|z|^2}{1 + |z|^2} \, dz
\]
+ \( o(e^{-\lambda_{n,j}^{(1)}}) \) in \( C^1(\mathbb{R} \setminus \bigcup_{j=1}^{m} U_{\delta}(x_{n,j}^{(1)}) \)),

where \( \theta > 0 \) is a suitable small constant, \( \partial_{\Lambda} G(y,x) = \frac{\partial G(y,x)}{\partial n} \), \( y = (y_1, y_2) \), and,
\[
A_{n,j} = \int_{M_j} f_n^*(y) \, d\mu(y).
\]
Moreover, there is a constant \( C > 0 \), which do not depend by \( R > 0 \), which satisfies,
\[
\left| \zeta_n(x) - \int_M \zeta_n \, d\mu - \sum_{j=1}^{m} A_{n,j} G(x_{n,j}^{(1)}, x) \right| \leq C \sum_{j=1}^{m} e^{-\lambda_{n,j}^{(1)}/4} \left( \frac{1}{(T_j(x) - T_j(x_{n,j}^{(1)}))} + \frac{1}{M_j \setminus U_{\delta}(x_{n,j}^{(1)})} \right),
\]
(3.22)
for \( x \in M \setminus \bigcup_{j=1}^{m} U_{\Lambda_{n,j,R}^{(1)}}(x_{n,j}^{(1)}) \), where \( r_0 \) is fixed as in (2.3).

Proof. By the Green representation formula we find that,
\[
\zeta_n(x) - \int_M \zeta_n \, d\mu = \int_M G(y,x) f_n^*(y) \, d\mu(y)
\]
= \( \sum_{j=1}^{m} A_{n,j} G(x_{n,j}^{(1)}, x) + \sum_{j=1}^{m} \int_{M_j} (G(y,x) - G(x_{n,j}^{(1)}, x)) f_n^*(y) \, d\mu(y) \).
(3.23)
For $x \in M \setminus \bigcup_{j=1}^{m} U_{\theta}^{H}(x^{(1)}_{n,j})$, let $z = T(x)$ denote any suitable local isothermal coordinate system. Then we see from (2.9), (2.10), and (3.3) that,

$$
\int_{M} (G(y,x) - G(x^{(1)}_{n,j},x)) f^{n}_{\theta}(y) d\mu(y) = \int_{B_{\delta_{n}}(z^{(1)}_{n,j})} \partial_{x_{n,j}} G(y,x) \left|_{y = z^{(1)}_{n,j}} \right. y - z^{(1)}_{n,j} > f^{n}_{\theta}(y) e^{2\theta_{1}} d\gamma(y) + O(1) \left( \int_{B_{\delta_{n}}(z^{(1)}_{n,j})} \frac{|y - z^{(1)}_{n,j}|^{2} e^{\lambda_{n,j}}}{(1 + e^{\lambda_{n,j}} |y - z^{(1)}_{n,j}|^{2})^{2}} d\gamma(y) \right) + O(e^{-\lambda_{n,j}}) \right)
$$

(3.24)

for a suitable $r > 0$. Next, by Lemma 3.1 we find that,

$$
\frac{e^{\theta_{1}} - e^{\theta_{2}}}{\|e^{\theta_{1}} - e^{\theta_{2}}\|_{L_{\infty}(M)}} = e^{\theta_{1}} - \frac{1}{\lambda_{n,j}} \left( 1 + O \left( \frac{1}{\lambda_{n,j}} \right) \right).
$$

(3.25)

At this point, setting $\delta_{n} = e^{-\theta_{1}}$, and using (2.9), (3.3), then after scaling we see that, for $x \in M \setminus \bigcup_{j=1}^{m} U_{\theta}^{H}(x^{(1)}_{n,j})$, it holds,

$$
\int_{B_{\delta_{n}}(z^{(1)}_{n,j})} \partial_{x_{n,j}} G(y,x) \left|_{y = z^{(1)}_{n,j}} \right. y - z^{(1)}_{n,j} > f^{n}_{\theta}(y) e^{2\theta_{1}} d\gamma(y) = \delta_{n} \int_{B_{\delta_{n}}(z^{(1)}_{n,j})} \partial_{x_{n,j}} G(y,x) \left|_{y = z^{(1)}_{n,j}} \right. y - z^{(1)}_{n,j} > \rho_{n} h_{j}(\delta_{n} z + z^{(1)}_{n,j}) e^{\theta_{1}} d\gamma(y) + O(\lambda_{n,j}) dz + o(\delta_{n}).
$$

(3.26)

In view of Lemma 3.2, we see that, for $x \in M \setminus \bigcup_{j=1}^{m} U_{\theta}^{H}(x^{(1)}_{n,j})$, $z = T(x)$, it holds,

$$
\int_{B_{\delta_{n}}(z^{(1)}_{n,j})} \partial_{x_{n,j}} G(y,x) \left|_{y = z^{(1)}_{n,j}} \right. y - z^{(1)}_{n,j} > f^{n}_{\theta}(y) e^{2\theta_{1}} d\gamma(y) = e^{-\theta_{1}} \sum_{h=1}^{2} \partial_{x_{n,j}} G(y,x) \left|_{y = z^{(1)}_{n,j}} \right. y - z^{(1)}_{n,j} > \frac{4\sqrt{8}}{\rho_{n} h_{j}(z^{(1)}_{n,j})} \int_{\mathbb{R}^{2}} \frac{\left| z \right|^{2}}{(1 + \left| z \right|^{2})^{2}} d\gamma(y) + o(e^{-\lambda_{n,j}}).
$$

From (3.23)-(3.26), we see that the estimate (3.21) holds in $C^{1}(M \setminus \bigcup_{j=1}^{m} U_{\theta}^{H}(x^{(1)}_{n,j}))$. The proof of the fact that (3.21) holds in $C^{1}(M \setminus \bigcup_{j=1}^{m} U_{\theta}^{H}(x^{(1)}_{n,j}))$ is similar and we skip it here to avoid repetitions.

From (3.25), (2.10), and (2.9), and a suitable scaling, we see that there exist $C > 0$, independent of $R > 0$, such that, for $z \in B_{2\delta_{0}}(z^{(1)}_{n,j}) \setminus B_{\delta_{n}}(z^{(1)}_{n,j})$, $x = T_{j}^{-1}(z)$, it holds,

$$
\left| \xi_{n}(x) - \int_{M} \xi_{n} d\mu - \sum_{j=1}^{m} A_{n,j} G(x^{(1)}_{n,j},x) \right| 
\leq \sum_{j=1}^{m} \int_{U_{\theta}^{H}(z^{(1)}_{n,j})} (G(y,x) - G(x^{(1)}_{n,j},x)) f^{n}_{\theta}(y) d\mu(y) \right) + O(e^{-\lambda_{n,j}}) 
\leq \sum_{j=1}^{m} \frac{1}{2\pi} \int_{B_{2\delta_{0}}(z^{(1)}_{n,j})} \log \frac{|x - z^{(1)}_{n,j}|}{|x - y|} f^{n}_{\theta}(y) e^{2\theta_{1}} d\gamma(y) + O \left( \int_{B_{2\delta_{0}}(z^{(1)}_{n,j})} \frac{|y - z^{(1)}_{n,j}|^{2} e^{\lambda_{n,j}}}{(1 + e^{\lambda_{n,j}} |y - z^{(1)}_{n,j}|^{2})^{2}} d\gamma(y) \right) + O(e^{-\lambda_{n,j}})
$$
\[ \leq \sum_{j=1}^{m} O(1) \left( \int_{B_{h,\epsilon \eta_{m,\epsilon 0}}^1(0)} \left| \frac{\log |x - \eta_{m,\epsilon 0}| - \log |x - e^{\frac{i h}{2} z} - \eta_{m,\epsilon 0}|}{(1 + |z|^2)^2} \right| \, dz \right) + O(e^{-\frac{i h}{2} z}) \]
\[ \leq O(1) \left( \int_{B_{h,\epsilon \eta_{m,\epsilon 0}}^1(0)} \left| \frac{\log |x - \eta_{m,\epsilon 0}| - \log |x - e^{\frac{i h}{2} z} - \eta_{m,\epsilon 0}|}{(1 + |z|^2)^2} \right| \, dz \right) + O(e^{-\frac{i h}{2} z}) \]
\[ + \sum_{j=1}^{m} O(1) \left( \int_{B_{h,\epsilon \eta_{m,\epsilon 0}}^1(0)} \left| \frac{e^{-\frac{i h}{2} z}}{|x - \eta_{m,\epsilon 0}|(1 + |z|^2)^2} \right| \, dz \right) + O(e^{-\frac{i h}{2} z}) \]
\[ \leq O(1) \left( \frac{e^{-\frac{i h}{2} z}}{|x - \eta_{m,\epsilon 0}|} \right) + O(1) \left( \frac{|z||z|^2}{|x - \eta_{m,\epsilon 0}|} \right) + O(e^{-\frac{i h}{2} z}) \leq C \left( e^{-\frac{i h}{2} z} \right). \]  
(3.27)

By (3.23), (3.25), (2.9), and (2.10), we also see that, for \( x \in M \setminus \bigcup_{j=1}^{m} U_{\epsilon \eta_{m}}^1(q_{j}) \), it holds,
\[ \left| \zeta_n(x) - \int_M \zeta_n \mu - \sum_{j=1}^{m} A_{n,j} G(x,\eta_{m,j}, x) \right| = O \left( \sum_{j=1}^{m} \int_{B_{\epsilon \eta_{m,j}}^1(0)} \frac{|y - \eta_{m,j}| e^{\frac{i h}{2} y}}{(1 + e^{\frac{i h}{2} y}) |y - \eta_{m,j}|^2} \, dy \right) + O(e^{-\frac{i h}{2} y}) = O(e^{-\frac{i h}{2} y}). \]  
(3.28)

By (3.27) and (3.28) we obtain (3.22), which concludes the proof of Lemma 3.3.

From now on, to simplify the notations, we will set
\[ f(z) = f(e^{\frac{i h}{2} z} + \eta_{m,j}), \quad |z| < \delta e^{\frac{i h}{2} z} \quad \text{for any function} \quad f : B_{\delta}^1(\eta_{m,j}) \to \mathbb{R}. \]

Our next aim is to obtain a detailed description of the asymptotic behavior of \( \zeta_n \) on \( U_{\epsilon \eta_{m,j}}^1(q_{j}) \) and on \( M \setminus \bigcup_{j=1}^{m} U_{\epsilon \eta_{m,j}}^1(q_{j}) \) for a suitable small \( c > 0 \). This task has been already worked out in [47] for the Chern-Simons-Higgs equation and we will follow that approach here. However, as mentioned in the introduction, our case is in some respect more involved, since if \( |\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}| \) is not asymptotically small enough, then the argument in [47] does not work. To overcome this difficulty, we have to use the Green representation formula and carry out a rather delicate set of estimates.

**Lemma 3.4.** There is a constant \( b_0 \), such that \( b_{j,0} = b_0 \) for \( j = 1, \ldots, m \). Moreover, for any \( c > 0 \) small enough, we have,
\[ \zeta_n(x) = -b_0 + o(1) \quad \text{for any} \quad x \in M \setminus \bigcup_{j=1}^{m} U_{\epsilon \eta_{m}}^1(q_{j}). \]

**Proof.** Let us recall that,
\[ \Delta_M \zeta_n + \rho_n h c_n \zeta_n = \Delta_M \zeta_n + \frac{\rho_n h(x)}{\|\tilde{h}_n^{(1)} - \tilde{h}_n^{(2)}\|_{L^\infty(M)}} \left( e^{\tilde{h}_n^{(1)}(x)} - e^{\tilde{h}_n^{(2)}(x)} \right) = 0 \quad \text{in} \quad M. \]

By (2.10) and Lemma 3.1, we have \( c_n \to 0 \) in \( C_{\text{loc}}(M \setminus \{q_1, \ldots, q_m\}) \).

Since \( \|\zeta_n\|_{L^\infty(M)} \leq 1 \), we see that \( \zeta_n \to \xi_0 \) in \( C_{\text{loc}}(M \setminus \{q_1, \ldots, q_m\}) \), where,
\[ \Delta_M \xi_0 = 0 \quad \text{in} \quad M \setminus \{q_1, \ldots, q_m\}. \]  
(3.29)

Moreover, since \( \|\zeta_n\|_{L^\infty(M)} \leq 1 \), then we have \( \|\xi_0\|_{L^\infty(M)} \leq 1 \). Therefore \( \xi_0 \) is smooth near \( q_i, i = 1, \ldots, m \), and we can extend (3.29) to \( M \). Then \( \xi_0 \equiv -b_0 \) in \( M \), where \( b_0 \) is a constant and in particular we find,
\[ \zeta_n \to -b_0 \quad \text{in} \quad C_{\text{loc}}(M \setminus \{q_1, \ldots, q_m\}). \]  
(3.30)

At this point, we consider the following two cases separately:

**Case 1.** \( |\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}| \leq o \left( \frac{1}{\lambda_{n,j}^{(1)}} \right) \).

In this situation, we can follow the argument adopted in [47]. We sketch the proof here for readers convenience.
Let \( \mathcal{X} = T_j(x) \in B_{x_j}(x, \eta), \psi_{n,j}(\mathcal{X}) = \frac{1 - \Phi(h_{n,j}(\mathcal{X}))}{1 + \Phi(h_{n,j}(\mathcal{X}))} |e_j^{n,j}(\mathcal{X})| \) and let us fix \( d \in (0, \delta) \). Then, in view of (2.7), we find,

\[
\int_{\partial B_{\delta}(x_{n,j})} \left( \psi_{n,j} \partial_{n,j} - \zeta_n \partial_{n,j} \right) d\sigma = \int_{\partial B_{\delta}(x_{n,j})} \left( \psi_{n,j} \Delta \zeta_n - \zeta_n \Delta \psi_{n,j} \right) d\mathcal{X}
\]

\[
= \int_{B_{\delta}(x_{n,j})} \left\{ - \rho_n \zeta_n \psi_{n,j} \left( e_{\delta}^{n,j}(1) - e_{\delta}^{n,j}(2) \right) + \rho_n \zeta_n \psi_{n,j} \left( x_{n,j}^{(1)} \right) e_{n,j}^{(1)} \left( 1 + \frac{\Phi(h_{n,j}(\mathcal{X}))}{1 + \Phi(h_{n,j}(\mathcal{X}))} \right) \right\} d\mathcal{X}
\]

\[
= \int_{B_{\delta}(x_{n,j})} \rho_n \zeta_n \psi_{n,j} \left\{ - h_{e}^{n,j} \left( 1 + O(|\mathcal{X}|_{2}) \right) + h_{n,j}^{(1)} e_{n,j}^{(1)} \left( 1 + O(e^{-\gamma_{n,j}^{(1)}}) \right) \right\} d\mathcal{X}
\]

Therefore, by a suitable scaling and by using (2.9), we see that,

\[
\int_{\partial B_{\delta}(x_{n,j})} \left( \psi_{n,j} \partial_{n,j} - \zeta_n \partial_{n,j} \right) d\sigma = \int_{\partial B_{\Lambda_{n,j}^{-1}}(0)} \rho_n \zeta_n(z) \psi_{n,j}(z) \left( \frac{1}{\lambda_{n,j}^{-1}} \right) \left( 1 + \frac{\rho_n h_{n,j}(\mathcal{X})}{8} \right) \left( \frac{1}{\lambda_{n,j}^{-1}} \right) e_{n,j}^{(1)} \left( 1 + O(e^{-\gamma_{n,j}^{(1)}}) \right) | \mathcal{X} | d\mathcal{X}
\]

In view of Lemma 3.1(ii) and since we are concerned with the case \( |\lambda_{n,j}^{-1} - \lambda_{n,j}^{-2}| \leq o \left( \frac{1}{\lambda_{n,j}^{-1}} \right) \), then we obtain,

\[
\int_{\partial B_{\delta}(x_{n,j})} \left( \psi_{n,j} \partial_{n,j} - \zeta_n \partial_{n,j} \right) d\sigma = o \left( \frac{1}{\lambda_{n,j}^{-1}} \right)
\]

(3.31)

Let \( \zeta_{n,j}^{*}(r) = \int_{0}^{2\pi} \zeta_n(r, \theta) d\theta \), where \( r = |x - x_{n,j}^{(1)}| \). Then (3.31) yields,

\[
(\zeta_{n,j}^{*})'(r) \psi_{n,j}(r) - \dot{\zeta}_{n,j}^{*}(r) \psi_{n,j}(r) = o \left( \frac{1}{\lambda_{n,j}^{-1}} \right) \frac{r}{\lambda_{n,j}^{-1}} \quad \forall r \in (\Lambda_{n,j}^{-1}, R, \delta]
\]

For any \( R > 0 \) large enough and for any \( r \in (\Lambda_{n,j}^{-1}, R, \delta] \), we also obtain that,

\[
\psi_{n,j}(r) = -1 + O \left( \frac{e^{-\gamma_{n,j}^{(1)}}}{r^2} \right) \,, \quad \psi_{n,j}'(r) = O \left( \frac{e^{-\gamma_{n,j}^{(1)}}}{r^3} \right)
\]

and so we conclude that,

\[
(\zeta_{n,j}^{*})'(r) = o \left( \frac{1}{\lambda_{n,j}^{-1}} \right) \frac{r}{\lambda_{n,j}^{-1}} + O \left( \frac{e^{-\gamma_{n,j}^{(1)}}}{r^3} \right) \quad \text{for all } r \in (\Lambda_{n,j}^{-1}, R, \delta].
\]

(3.32)

Integrating (3.32), we obtain,

\[
\zeta_{n,j}^{*}(r) = \zeta_{n,j}^{*}(\Lambda_{n,j}^{-1}) + o(1) + o \left( \frac{1}{\lambda_{n,j}^{-1}} \right) R + O(R^{-2}) \quad \text{for all } r \in (\Lambda_{n,j}^{-1}, R, \delta].
\]

(3.33)

By using Lemma 3.2, we find,

\[
\zeta_{n,j}^{*}(\Lambda_{n,j}^{-1}) = -2\pi b_{j,0} + o_{R}(1) + o_{n}(1),
\]

where \( \lim_{R \to \infty} o_{R}(1) = 0 \) and \( \lim_{n \to \infty} o_{n}(1) = 0 \) and then (3.33) shows that,

\[
\zeta_{n,j}^{*}(r) = -2\pi b_{j,0} + o_{R}(1) + o_{n}(1)(1 + O(R)), \quad \text{for all } r \in (\Lambda_{n,j}^{-1}, R, \delta],
\]

(3.34)
where \( \lim_{n \to +\infty} a_n(1) = 0 \). In view of (3.30), we see that,

\[
\zeta_{n,j} = -2\pi b_0 + a_n(1) \text{ in } C_{loc}(M \setminus \{q_1 \cdots q_m\}),
\]

which implies that \( b_{j,0} = b_0 \) for \( j = 1, \cdots, m \), whenever \( |\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}| = o\left(\frac{1}{\lambda_{n,j}}\right) \), as claimed.

Case 2. \( \frac{1}{C_{\lambda_{n,j}^{(1)}}} \leq |\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}| \leq \frac{C}{\lambda_{n,j}^{(2)}} \) for some constant \( C > 1 \). In this case, the argument in [47] as outlined above does not yield the desired result. Indeed, since \( |\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}| \) is “not small enough”, then \( \zeta_{n,j}^{*}(r) - \zeta_{n,j}^{*}(\Lambda_{n,j,R}) \) is not as small as we would need, see (3.30). So we adopt a different approach based on the Green representation formula.

Fix \( d \in (0, \delta) \), and let \( \Lambda_{n,j,R} \leq |x_1 - x_{n,j}^{(1)}| \leq |x_2 - x_{n,j}^{(1)}| \leq d \), then,

\[
\zeta_n(x_1) - \zeta_n(x_2) = \rho_n \int_M (G(x_1, y) - G(x_2, y)) h(y) \frac{\rho^2 h_j(y) e^{\alpha h_j(y)}}{||\bar{u}^{(1)} - \bar{u}^{(2)}||_{L^\infty(M)}} \, dy
\]

\[
= \frac{1}{2\pi} \sum_{j=1}^n \int_{B_{2\rho}(x_{n,j}^{(1)})} \log \left| \frac{x_2 - y}{x_1 - y} \right| \rho_n h_j(y) e^{\alpha h_j(y)} \frac{1 - e^{\bar{u}^{(2)} - \bar{u}^{(1)}}}{||\bar{u}^{(1)} - \bar{u}^{(2)}||_{L^\infty(M)}} \, dy + O(|x_1 - x_2|).
\]

By the usual scaling, \( y = \delta_n z + x_{n,j}^{(1)} \), where \( \delta_n = e^{-\frac{\lambda_{n,j}^{(1)}}{\lambda_{n,j}^{(2)}}} \), we see that,

\[
\int_{B_{2\rho}(x_{n,j}^{(1)})} \log \left| \frac{x_2 - y}{x_1 - y} \right| \rho_n h_j(y) e^{\alpha h_j(y)} \frac{1 - e^{\bar{u}^{(2)} - \bar{u}^{(1)}}}{||\bar{u}^{(1)} - \bar{u}^{(2)}||_{L^\infty(M)}} \, dy
\]

\[
= \int_{B_{2\lambda_{n,j}^{(1)}}(0)} \log \left| \frac{x_2 - x_{n,j}^{(1)} - \delta_n z}{x_1 - x_{n,j}^{(1)} - \delta_n z} \right| \rho_n h_j(z) e^{\alpha h_j(z)} e^{-\lambda_{n,j}^{(1)}} \frac{1 - e^{\bar{u}^{(2)} - \bar{u}^{(1)}}}{||\bar{u}^{(1)} - \bar{u}^{(2)}||_{L^\infty(M)}} \, dz
\]

\[
= \int_{B_{2\lambda_{n,j}^{(1)}}(0)} \log \left| \frac{\delta_n^{r-1}(x_2 - x_{n,j}^{(1)}) - z}{\delta_n^{r-1}(x_1 - x_{n,j}^{(1)}) - z} \right| \rho_n h_j(z) e^{\alpha h_j(z)} e^{-\lambda_{n,j}^{(1)}} \frac{1 - e^{\bar{u}^{(2)} - \bar{u}^{(1)}}}{||\bar{u}^{(1)} - \bar{u}^{(2)}||_{L^\infty(M)}} \, dz.
\]

Fix \( \alpha \in (0, \frac{1}{2}) \). We will use the following inequality (see [20, Theorem 4.1]): let \( g : \mathbb{R}^2 \to \mathbb{R} \) satisfies \( \int_{\mathbb{R}^2} g^2(1 + |z|)^{2+\alpha} \, dz < +\infty \). Then there exists a constant \( c > 0 \), independent of \( \chi \in \mathbb{R}^2 \setminus B_2(0) \) and \( g \), such that,

\[
\left| \int_{\mathbb{R}^2} (\log |x| - \log |z|) g(z) \, dz \right| \leq c |x|^{-\frac{\alpha}{2}} (\log |x| + 1) g(z) (1 + |z|)^{1+\frac{\alpha}{2}} ||g||_{L^2(\mathbb{R}^2)}.
\]

In view of (3.36) and (3.37), we find that,

\[
\int_{B_{2\rho}(x_{n,j}^{(1)})} \log \left| \frac{x_2 - y}{x_1 - y} \right| \rho_n h_j(y) e^{\alpha h_j(y)} \frac{1 - e^{\bar{u}^{(2)} - \bar{u}^{(1)}}}{||\bar{u}^{(1)} - \bar{u}^{(2)}||_{L^\infty(M)}} \, dy
\]

\[
= \int_{B_{2\lambda_{n,j}^{(1)}}(0)} \left( \log \frac{\delta_n^{r-1}(x_2 - x_{n,j}^{(1)})}{\delta_n^{r-1}(x_1 - x_{n,j}^{(1)})} + \sum_{i=1}^2 (-1)^i \log \frac{|\delta_n^{r-1}(x_2 - x_{n,j}^{(1)}) - z|}{|\delta_n^{r-1}(x_1 - x_{n,j}^{(1)}) - z|} \right) \rho_n h_j(z) e^{\alpha h_j(z)} e^{-\lambda_{n,j}^{(1)}} \frac{1 - e^{\bar{u}^{(2)} - \bar{u}^{(1)}}}{||\bar{u}^{(1)} - \bar{u}^{(2)}||_{L^\infty(M)}} \, dz
\]

\[
= \log \frac{x_2 - x_{n,j}^{(1)}}{x_1 - x_{n,j}^{(1)}} \int_{B_{2\lambda_{n,j}^{(1)}}(0)} \rho_n h_j(z) e^{\alpha h_j(z)} e^{-\lambda_{n,j}^{(1)}} \frac{1 - e^{\bar{u}^{(2)} - \bar{u}^{(1)}}}{||\bar{u}^{(1)} - \bar{u}^{(2)}||_{L^\infty(M)}} \, dz
\]

\[
+ O\left( \sum_{i=1}^2 |e^{\frac{\alpha}{2}} (x_2 - x_{n,j}^{(1)})|^{-\frac{\alpha}{2}} \log |x_2 - x_{n,j}^{(1)}| \right).
\]
From (2.7)-(2.9), we also see that,

\[
\int_{B_{2n_{\delta, d}}^+}(0) \rho_\delta h(z) e^{\overline{a}_{\delta n}^{(1)}(z)} e^{-\lambda_{n,j}^{(1)}} \left(1 - e^{\lambda_{n,j}^{(2)} - \overline{a}_{\delta n}^{(1)}} \right) \frac{1}{\|\overline{\frac{A_{\delta n}}{2}}\|_{L^\infty(M)}} \, dz \\
= \int_{B_{2n_{\delta, d}}^+}(0) \rho_\delta h(\delta_n z + \overline{\lambda}_{n,j}^{(1)}) e^{\overline{a}_{\delta n}^{(1)} + \overline{G}^{(1)}(\delta_n z + \overline{\lambda}_{n,j}^{(1)}) - \overline{G}^{(1)}(\overline{\lambda}_{n,j}^{(1)})} \left(1 - e^{\overline{a}_{\delta n}^{(2)} - \overline{a}_{\delta n}^{(1)}} \right) \frac{1}{\|\overline{\frac{A_{\delta n}}{2}}\|_{L^\infty(M)}} \, dz \\
= \int_{B_{2n_{\delta, d}}^+}(0) \rho_\delta h(\overline{\lambda}_{n,j}^{(1)}) \left(1 + O(\delta_n) + O(\delta_n |z|)\right) \left(1 - e^{\overline{a}_{\delta n}^{(2)} - \overline{a}_{\delta n}^{(1)}} \right) \frac{1}{\|\overline{\frac{A_{\delta n}}{2}}\|_{L^\infty(M)}} \, dz + O(\delta_n).
\]

(3.39)

On the other side, from (2.8) and (2.9), we find that,

\[
\overline{\lambda}_{n,j}^{(1)}(x) - \overline{\lambda}_{n,j}^{(2)}(x) = U^{(1)}_{n,j}(x) - U^{(2)}_{n,j}(x) + G^{(1)}(\overline{\lambda}_{n,j}^{(1)}(x) - \overline{\lambda}_{n,j}^{(2)}(x)) - G^{(1)}(\overline{\lambda}_{n,j}^{(1)}(x)) + \eta_{n,j}^{(1)}(x) - \eta_{n,j}^{(2)}(x) \\
= \lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)} + 2 \log \left(1 + \frac{\rho_\delta(h(z))^{(1)}}{\lambda_{n,j}^{(2)}} e^{-\lambda_{n,j}^{(2)}} \left|\delta_n z + \overline{\lambda}_{n,j}^{(1)} - \overline{\lambda}_{n,j}^{(2)}\right|^2 \right) + O(\left(\lambda_{n,j}^{(1)} \lambda_{n,j}^{(2)}\right)^2 e^{-\lambda_{n,j}^{(2)}}),
\]

which in turn implies that,

\[
\overline{\lambda}_{n,j}^{(1)}(z) - \overline{\lambda}_{n,j}^{(2)}(z) = \lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)} + 2 \log \left(1 + \frac{\rho_\delta(h(z))^{(1)}}{\lambda_{n,j}^{(2)}} e^{-\lambda_{n,j}^{(2)}} \left|\delta_n z + \overline{\lambda}_{n,j}^{(1)} - \overline{\lambda}_{n,j}^{(2)}\right|^2 \right) + O(\left(\lambda_{n,j}^{(1)} \lambda_{n,j}^{(2)}\right)^2 e^{-\lambda_{n,j}^{(2)}}).
\]

(3.40)

By (2.7) and (3.3), we also see that,

\[
2 \log \left(1 + \frac{\rho_\delta(h(z))^{(1)}}{\lambda_{n,j}^{(2)}} e^{-\lambda_{n,j}^{(2)}} \left|\delta_n z + \overline{\lambda}_{n,j}^{(1)} - \overline{\lambda}_{n,j}^{(2)}\right|^2 \right) = 2 \log \left(1 + \frac{\rho_\delta(h(z))^{(1)}}{\lambda_{n,j}^{(2)}} e^{-\lambda_{n,j}^{(2)}} \left|\delta_n z + \overline{\lambda}_{n,j}^{(1)} - \overline{\lambda}_{n,j}^{(2)}\right|^2 \right) + O(\lambda_{n,j}^{(1)} e^{-\lambda_{n,j}^{(2)}}).
\]

(3.40)

which, together with (3.40), allows us to conclude that,

\[
\overline{\lambda}_{n,j}^{(1)}(z) - \overline{\lambda}_{n,j}^{(2)}(z) \\
= \left(1 - \frac{\rho_\delta(h(z))^{(1)}}{\lambda_{n,j}^{(2)}} \right) \left(\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}\right) + \left(\frac{\rho_\delta(h(z))^{(1)}}{\lambda_{n,j}^{(2)}} \right) \left(\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}\right)^2 + O(\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}),
\]

(3.40)
and thus,
\[
\overline{\mu}_n^{(1)}(z) - \overline{\mu}_n^{(2)}(z) = \frac{(\alpha_n^{(1)} - \alpha_n^{(2)})}{2} + \left( 1 - \frac{\mu h(z_1)\|z\|^2}{8} \right) (\lambda_n^{(1)} - \lambda_n^{(2)}) + O\left( |\lambda_n^{(1)} - \lambda_n^{(2)}|^3 \right) + O\left( e^{-\frac{|z|^2}{4}} \right).
\]

From (3.39)-(3.41), we deduce that,
\[
\int_{B_{2\Lambda_{n,j}^+(0)}} \rho_n \overline{\nu}_j(z) e^{i\mu_1^{(1)}(z)} e^{-\lambda_n^{(1)}} \frac{1 - e^{i\mu_1^{(2)} - \mu_1^{(1)}}}{\|\overline{\mu}_n^{(1)} - \overline{\mu}_n^{(2)}\|_{L^\infty(M)}} dz
\]
\[
= \int_{B_{2\Lambda_{n,j}^+(0)}} \rho_n h(z_1) \frac{1 - \mu h(z_1)\|z\|^2}{8} \left( \frac{(\alpha_n^{(1)} - \alpha_n^{(2)})}{2} + \left( 1 - \frac{\mu h(z_1)\|z\|^2}{8} \right) (\lambda_n^{(1)} - \lambda_n^{(2)}) + O\left( |\lambda_n^{(1)} - \lambda_n^{(2)}|^3 \right) + O\left( e^{-\frac{|z|^2}{4}} \right) \right) dz
\]
\[
+ O\left( \frac{|\lambda_n^{(1)} - \lambda_n^{(2)}|^3}{\|\overline{\mu}_n^{(1)} - \overline{\mu}_n^{(2)}\|_{L^\infty(M)}} \right) + O\left( e^{-\frac{|z|^2}{4}} \right) + O\left( \|\overline{\mu}_n^{(1)} - \overline{\mu}_n^{(2)}\|_{L^\infty(M)} \right).
\]

At this point we note that, for any fixed \( t > 0 \), it holds,
\[
\int_{B_t(0)} \frac{8}{(1 + |z|^2)^2} \frac{1 - |z|^2}{1 + |z|^2} dz = \frac{8\pi t^2}{(t^2 + 1)^2}, \quad \text{and}
\]
\[
\int_{B_t(0)} \frac{8}{(1 + |z|^2)^2} \frac{1 - |z|^2}{2(1 + |z|^2)^2} dz = \frac{4\pi t^2(t^2 - 1)}{(t^2 + 1)^3}.
\]

Since \( \Lambda_{n,j}^+ = d e^{i\varphi} \), then (3.42)-(3.44) imply that,
\[
\int_{B_{2\Lambda_{n,j}^+(0)}} \rho_n \overline{\nu}_j(z) e^{i\mu_1^{(1)}(z)} e^{-\lambda_n^{(1)}} \frac{1 - e^{i\mu_1^{(2)} - \mu_1^{(1)}}}{\|\overline{\mu}_n^{(1)} - \overline{\mu}_n^{(2)}\|_{L^\infty(M)}} dz
\]
\[
= O\left( \frac{|\lambda_n^{(1)} - \lambda_n^{(2)}|}{\|\overline{\mu}_n^{(1)} - \overline{\mu}_n^{(2)}\|_{L^\infty(M)}} \right) + O\left( \frac{|\lambda_n^{(1)} - \lambda_n^{(2)}|^3}{\|\overline{\mu}_n^{(1)} - \overline{\mu}_n^{(2)}\|_{L^\infty(M)}} \right) + O\left( e^{-\frac{|z|^2}{4}} \right) + O\left( \|\overline{\mu}_n^{(1)} - \overline{\mu}_n^{(2)}\|_{L^\infty(M)} \right).
\]

At this point, from Lemma 3.1 and our assumption \( \frac{1}{C \lambda_{n,j}^{(1)}} \leq |\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}| \leq \frac{C}{\lambda_{n,j}^{(1)}} \), we can find a constant \( c_0 > 1 \) such that,
\[
\frac{1}{C_0 C \lambda_{n,j}^{(1)}} \leq \frac{|\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}|}{C_0 \lambda_{n,j}^{(1)}} \leq \|\overline{\mu}_n^{(1)} - \overline{\mu}_n^{(2)}\|_{L^\infty(M)} \leq c_0 |\lambda_{n,j}^{(1)} - \lambda_{n,j}^{(2)}| \leq \frac{C_0 C}{\lambda_{n,j}^{(1)}}.
\]

By (3.45), we obtain,
\[
\int_{B_{2\Lambda_{n,j}^+(0)}} \rho_n \overline{\nu}_j(z) e^{i\mu_1^{(1)}(z)} e^{-\lambda_n^{(1)}} \frac{1 - e^{i\mu_1^{(2)} - \mu_1^{(1)}}}{\|\overline{\mu}_n^{(1)} - \overline{\mu}_n^{(2)}\|_{L^\infty(M)}} dz = o\left( \frac{1}{\lambda_{n,j}^{(1)}} \right).
\]
Lemma 4.1 showing that Recall the definition of As a consequence, for \(b_n(x)\), we have that
\[
\zeta_n(x) - \zeta_n(x_0) = \log \left( \frac{|x_0 - x(n)|}{|x_0 - x(n_0)|} \right) \frac{1}{2\pi} \int_{\mathbb{R}^n} \rho_n h_j(z) e^{h_j(z)} e^{-h_j(z)} \left( 1 - e^{\rho_n h_j(z)} \right) \frac{1}{\|h_{j,n} - h_{j,0}\|_{L^\infty(M)}} \, dz
\]
(3.47) + \frac{1}{2} \sum_{j=1}^m \left( e^{\rho_n h_j(z)} (x_0 - x(n_0)) - e^{\rho_n h_j(z)} (x_0 - x(n_0)) \right)
= O(|x_0 - x(n)|) + o(1) + O(R^{-\frac{1}{2}} \log R).

Finally, by fixing a small constant \(r \in (0, d)\), and putting \(|x_0 - x(n_0)| = Re\frac{1}{2}\) and \(|x_0 - x(n_0)| = r\), then Lemma 3.2 and (3.30) imply that
\[
\zeta_n(x_0) = -b_{j,0} + o_R(1) + o(n_1), \quad \zeta_n(x_0) = -b_0 + o(n_1), \quad (4.48)
\]
where \(\lim_{R \to +\infty} o_R(1) = 0\) and \(\lim_{n \to +\infty} o(n_1) = 0\). As a consequence, since \(R > 0\) and \(r > 0\) are arbitrary, we see that (3.47)-(3.48) imply \(b_{j,0} = b_0\) for \(j = 1, \cdots, m\), in Case 2 as well. This fact concludes the proof of Lemma 3.4.  

4. Estimates via Pohozaev type identities

From now on, for a given function \(f(y, x)\), we shall use \(\partial\) and \(D\) to denote the partial derivatives with respect to \(y\) and \(x\) respectively. With a small abuse of notation, for a function \(f(x)\) we will use both \(\nabla\) and \(D\) to denote its gradient.

For \(j = 1, \cdots, m\), let
\[
\phi_{n,j}(y) = \frac{\rho_n}{m} (R(y, x(n)) - R(x(n_j), x(n_0))) + \frac{\rho_n}{m} \sum_{i \neq j} (G(y, x(n_j)) - G(x(n_j), x(n_0))),
\]
(4.1)
\[
\quad v_{n,j}(y) = \vec{1}^i(y) - \phi_{n,j}(y), \quad i = 1, 2.
\]
(4.2)
Recall the definition of \(\zeta_n\) given before (3.16). Our aim is to show that all \(b_{j,0} = 0\), see Lemma 3.2. We will start by showing that \(b_{j,0} = 0\). This is done by exploiting the following Pohozaev identity to derive a subtle estimate for \(\zeta_n\).

Lemma 4.1 ([47]). For any fixed \(r \in (0, \delta)\), it holds,
\[
\frac{1}{2} \int_{B_r(x)} \frac{r \rho_n h_j(\xi)}{||v_{n,j}^i - v_{n,j}^0||_{L^\infty(M)}} e^{\rho_n h_j(\xi) + \phi_{n,j} - e^{\rho_n h_j(\xi) + \phi_{n,j}}} \, d\sigma
= \int_{B_r(x)} \frac{r \rho_n h_j(\xi)}{||v_{n,j}^i - v_{n,j}^0||_{L^\infty(M)}} \left( e^{\rho_n h_j(\xi) + \phi_{n,j} - e^{\rho_n h_j(\xi) + \phi_{n,j}}} \right) < D(\log h_j + \phi_{n,j}, x - x(n_0)) > \, d\sigma.
\]
(4.3)

Proof. The identity (4.3) has been first obtained in [47]. We prove it for reader’s convenience. First of all, we observe that in local coordinates it holds,
\[
\{\Delta(v_{n,j}^i - v_{n,j}^0)\} = \{\nabla(v_{n,j}^i + v_{n,j}^0) \cdot (x - x(n))\} + \{\Delta(v_{n,j}^i + v_{n,j}^0)\} \{\nabla(v_{n,j}^i - v_{n,j}^0) \cdot (x - x(n))\}
= \text{div} \left\{ \nabla(v_{n,j}^i - v_{n,j}^0) \right\} \{\nabla(v_{n,j}^i + v_{n,j}^0) \cdot (x - x(n))\} + \text{div} \left\{ \nabla(v_{n,j}^i + v_{n,j}^0) \right\} \{\nabla(v_{n,j}^i - v_{n,j}^0) \cdot (x - x(n))\}
- \text{div} \left\{ \nabla(v_{n,j}^i - v_{n,j}^0) \right\} \{\nabla(v_{n,j}^i + v_{n,j}^0) \cdot (x - x(n))\}.
\] (4.4)

By the definition of \(v_{n,j}^i\), we also see that, for \(x \in B_r(x(n_j))\),
\[
\Delta(v_{n,j}^i - v_{n,j}^0) + \rho_n h_j(\xi) (e^{\rho_n h_j(\xi)} - e^{\rho_n h_j(\xi)}) = 0, \quad \text{and} \quad \Delta(v_{n,j}^i + v_{n,j}^0) + \rho_n h_j(\xi) (e^{\rho_n h_j(\xi)} + e^{\rho_n h_j(\xi)}) = 0.
\]
and then we find that,

\[
\{\Delta(v_{n,j}^{(1)} - v_{n,j}^{(2)})\} \{\nabla(v_{n,j}^{(1)} + v_{n,j}^{(2)} \cdot (x - z_{n,j}^{(1)})\} + \{\Delta(v_{n,j}^{(1)} + v_{n,j}^{(2)})\} \{\nabla(v_{n,j}^{(1)} - v_{n,j}^{(2)} \cdot (x - z_{n,j}^{(1)}))\}
\]

\[
= -\rho_n(e^{v_{n,j}^{(1)} + \phi_{n,j} + \log h_j - e^{v_{n,j}^{(2)} + \phi_{n,j} + \log h_j}}) \{\nabla(v_{n,j}^{(1)} + v_{n,j}^{(2)} \cdot (x - z_{n,j}^{(1)})\}
\]

\[
- \rho_n(e^{v_{n,j}^{(1)} + \phi_{n,j} + \log h_j + e^{v_{n,j}^{(2)} + \phi_{n,j} + \log h_j}}) \{\nabla(v_{n,j}^{(1)} - v_{n,j}^{(2)} \cdot (x - z_{n,j}^{(1)})\}
\]

\[
= -2\rho_n(e^{v_{n,j}^{(1)} + \phi_{n,j} + \log h_j} \{\nabla(v_{n,j}^{(1)} \cdot (x - z_{n,j}^{(1)}))\} + 2\rho_n(e^{v_{n,j}^{(2)} + \phi_{n,j} + \log h_j} \{\nabla(v_{n,j}^{(2)} \cdot (x - z_{n,j}^{(1)}))\}
\]

\[
= -\text{div} \left( 2\rho_n(e^{v_{n,j}^{(1)} + \phi_{n,j} + \log h_j} - e^{v_{n,j}^{(2)} + \phi_{n,j} + \log h_j})(x - z_{n,j}^{(1)}) \right)
\]

\[
+ 4\rho_n(e^{v_{n,j}^{(1)} + \phi_{n,j} + \log h_j} \{\nabla(v_{n,j}^{(1)} \cdot (x - z_{n,j}^{(1)}))\} + 2\rho_n(e^{v_{n,j}^{(2)} + \phi_{n,j} + \log h_j} - e^{v_{n,j}^{(2)} + \phi_{n,j} + \log h_j}) \{\nabla(\phi_{n,j} + \log h_j) \cdot (x - z_{n,j}^{(1)})\}\right).
\]

(4.5)

Clearly, since \(\zeta_n = \frac{v_{n,j}^{(1)} - v_{n,j}^{(2)}}{\|v_{n,j}^{(1)} - v_{n,j}^{(2)}\|_{L^\infty(M)}}\), then (4.4), (4.5) yield (4.3), as claimed.

Next we estimate both sides of (4.3). Recall the definition of \(A_{n,j}\) given in Lemma 3.3.

**Lemma 4.2.**

\[
\text{LHS of (4.3)} = -4\Delta A_{n,j} \frac{\rho_n e^{-\Lambda_n^{(1)}}}{\rho_n e^{-\Lambda_n^{(1)}}} \text{L} e^{G_j(q_j)} \int_{M_{\{\rho_n e^{-\Lambda_n^{(1)}} \}^2}} e^{G_j(x,q)} d\mu(x) + o(\varepsilon^{-\lambda_n^{(1)}})
\]

for fixed \(r \in (0, r_0)\) with \(r_0\) as defined in (2.3).

**Proof.** Next, let us denote by,

\[
\tilde{G}(x) = \frac{\rho_n}{m} \sum_{l=1}^m G(x, (x, x_{n,j}^{(l)})),
\]

(4.6)

so that, for \(x \in B_{2r_0}(z_{n,j}^{(1)}) \setminus \{z_{n,j}^{(1)}\}\), we have,

\[
\nabla(\tilde{G}(x) - \phi_{n,j}(x)) = -\frac{\rho_n}{2m \pi |x - z_{n,j}^{(1)}|^2} (x - z_{n,j}^{(1)}).
\]

(4.7)

In view of (2.19), (2.16), and (2.17), we conclude that,

\[
o(\varepsilon^{-\lambda_n^{(1)}}) = \nabla_{\delta} w_n = \nabla_M \left( v_{n,j}^{(1)} + \phi_{n,j} - \sum_{l=1}^m \frac{\rho_n}{m} G(x, x_{n,j}^{(l)}) \right) + O(\Lambda_n^{(1)} e^{-\lambda_n^{(1)}}) \quad \text{in } M \setminus \cup_{l=1}^m \text{B}_{r/2}(x_{n,j}^{(l)}),
\]

where \(\delta < \frac{r}{8}\). Therefore we find that,

\[
\nabla v_{n,j}^{(1)} = \nabla(\tilde{G} - \phi_{n,j}) + o(\varepsilon^{-\lambda_n^{(1)}}) \quad \text{in } \cup_{l=1}^m \text{B}_{2r_0}(x_{n,j}^{(l)}) \setminus \text{B}_{r/2}(x_{n,j}^{(l)}).
\]

(4.8)

As a consequence, letting \(v\) be the exterior unit normal, then (4.8) together with (4.7) and (2.17), imply that,

\[
\text{LHS of (4.3)} = \int_{\partial B_{r/2}(x_{n,j}^{(1)})} r < D(\tilde{G} - \phi_{n,j}, D\zeta_n) > d\sigma - 2 \int_{\partial B_{r/2}(x_{n,j}^{(1)})} r < v, D(\tilde{G} - \phi_{n,j}) > < v, D\zeta_n > d\sigma
\]

\[
+ o(\varepsilon^{-\lambda_n^{(1)}} \|D\zeta_n\|_{L^\infty(\partial B_{r/2}(x_{n,j}^{(1)}))})
\]

\[
= \int_{\partial B_{r/2}(x_{n,j}^{(1)})} \frac{\rho_n}{2m} < v, D\zeta_n > d\sigma + o(\varepsilon^{-\lambda_n^{(1)}} \|D\zeta_n\|_{L^\infty(\partial B_{r/2}(x_{n,j}^{(1)}))})
\]

\[
= \int_{\partial B_{r/2}(x_{n,j}^{(1)})} 4 < v, D\zeta_n > d\sigma + o(\varepsilon^{-\lambda_n^{(1)}} \|D\zeta_n\|_{L^\infty(\partial B_{r/2}(x_{n,j}^{(1)}))})
\]

(4.9)
By (4.9) and Lemma 3.3, we also see that,

\[
\text{LHS of (4.3)} = \int_{a_B(y_{a,J}^{(1)})} f_n^*(y) \nu(y) + o(e^{-\lambda_{n,J}^{(1)}}) + o(e^{-\lambda_{n,J}^{(1)}} \sum_{l=1}^m |A_{n,l}|). \tag{4.10}
\]

To estimate the right hand side of (4.10), we need a refined estimate about \(\zeta_n\) on \(\partial B_h(x_{a,l}^{(1)})\). So, by the Green representation formula with \(x \in \partial U^M(x_{a,l}^{(1)})\), we find that (see (3.23)),

\[
\zeta_n(x) - \int_M \zeta_n \nu = \int_M G(y, x) f_n^*(y) \nu(y) + \sum_{l=1}^m A_{n,l} G(x_{a,l}^{(1)}, x) + \sum_{l=1}^m \frac{2}{2} B_{n,l} T_s^{-1} \left( \partial_y G(y, x) \right)_{y=x_{a,l}^{(1)}} + \frac{1}{2} \sum_{l=1}^m \frac{2}{2} C_{n,l} T_s^{-1} \left( \partial_y^2 G(y, x) \right)_{y=x_{a,l}^{(1)}} + \sum_{l=1}^m \int_M \Psi_n(x) f_n^*(y) \nu(y) \tag*{where}
\]

\(A_{n,l} = \int_{M} f_n^*(y) \nu(y), \quad B_{n,l} = \int_{B_0(x_{a,l}^{(1)})} (y - x_{a,l}^{(1)}) \partial_y f_n^*(y) e^{\varphi_1} \nu(y) \)

\(C_{n,l,h} = \int_{B_0(x_{a,l}^{(1)})} (y - x_{a,l}^{(1)}) \partial_y^2 f_n^*(y) e^{\varphi_1} \nu(y) \)

\(\Psi_n(x, y) = G(y, x) - G(x_{a,l}^{(1)}, x) - T_s^{-1} \left( \partial_y G(y, x) \right)_{y=x_{a,l}^{(1)}} + \frac{1}{2} \sum_{l=1}^m \frac{2}{2} C_{n,l} T_s^{-1} \left( \partial_y^2 G(y, x) \right)_{y=x_{a,l}^{(1)}} + \frac{1}{2} \sum_{l=1}^m \frac{2}{2} C_{n,l} T_s^{-1} \left( \partial_y^2 G(y, x) \right)_{y=x_{a,l}^{(1)}} + \sum_{l=1}^m \int_M \Psi_n(x) f_n^*(y) \nu(y) \). \tag{4.11}

At this point, let us fix \(0 < \overline{\theta} < \frac{1}{2}\). By using Lemma 3.1 and Lemma 3.4, we find that,

\[
f_n^*(y) = \rho_n h e^{\varphi_1^{(1)}} (\zeta_n(y) + O(\|\hat{u}_n^{(1)} - \hat{u}_n^{(2)}\|_{L^\infty(M)}) = \rho_n h e^{\varphi_1^{(1)}}(-b_0 + o(1)), \tag{4.12}
\]

for any \(y \in M_j \setminus U^M_{\overline{\theta}}(x_{a,l}^{(1)})\), and, in view of (2.18) and (2.19),

\[
\hat{u}_n^{(1)}(y) - \sum_{l=1}^m f_{n,l}(y) x_{a,l}^{(1)} - \int_M \hat{u}_n^{(1)} \nu = o(e^{-\lambda_{n,l}^{(1)}}) \quad \text{for} \ y \in M_j \setminus U^M_{\overline{\theta}}(x_{a,l}^{(1)}). \tag{4.13}
\]

By (4.12)-(4.13), (2.12), and (2.16), we conclude that,

\[
f_n^*(y) = \rho_n h e^{\varphi_1^{(1)}} \sum_{l=1}^m f_{n,l}(y) x_{a,l}^{(1)} + \sum_{l=1}^m \hat{u}_n^{(1)} \nu(-b_0 + o(1)) \]

\[
= \rho_n h e^{\varphi_1^{(1)}} \sum_{l=1}^m f_{n,l}(y) x_{a,l}^{(1)} + \sum_{l=1}^m \hat{u}_n^{(1)} \nu(-b_0 + o(1)) \]

\[
= \frac{64 e^{-\lambda_{n,l}^{(1)}}}{\rho_n h} e^{\varphi_1^{(1)}}(-b_0 + o(1)) \quad \text{for} \ y \in M_j \setminus U^M_{\overline{\theta}}(x_{a,l}^{(1)}), \tag{4.14}
\]

where \(x_n = (x_{n,1}, x_{n,2}, \cdots, x_{n,m})\) and, \(\Phi_1(y, x_n) = \sum_{l=1}^m \log h(y, x_{a,l}^{(1)}) - G^{(1)}(y, x_{a,l}^{(1)}) + \log h(y) - \log h(x_{a,l}^{(1)})\).

On the other hand, by (2.9), we have, for \(y \in U^M_{\overline{\theta}}(x_{a,l}^{(1)})\),

\[
f_n^*(y) = \rho_n h e^{\varphi_1^{(1)}} (\zeta_n(y) + O(\|\hat{u}_n^{(1)} - \hat{u}_n^{(2)}\|_{L^\infty(M)}) = O \left( \frac{e^{\lambda_{n,l}^{(1)}}}{1 + e^{\lambda_{n,l}^{(1)}} \|y - x_{a,l}^{(1)}\|_2^2} \right) \leq O \left( \frac{e^{-\lambda_{n,l}^{(1)}}}{\|y - x_{a,l}^{(1)}\|} \right). \tag{4.15}
\]
Next, by (4.11), we have for \( y \in \mathcal{U}_r^M(x^{(1)}_{n_j}) \) and \( x \in \partial \mathcal{U}_r^M(x^{(1)}_{n_j}) \),

\[
\Psi_{n_j}(y, x) = O \left( \frac{|y - x^{(1)}_{n_j}|^3}{|x - x^{(1)}_{n_j}|^3} \right), \quad \text{and} \quad \langle \nabla_M \rangle_x \Psi_{n_j}(y, x) = O \left( \frac{|y - x^{(1)}_{n_j}|^3}{|x - x^{(1)}_{n_j}|^4} \right). \tag{4.16}
\]

Let us define,

\[
\overline{\zeta}_n(x) = \int_M \zeta_n d\mu + \sum_{j=1}^m A_{n_j} G(x^{(1)}_{n_j}, x) + \sum_{j=1}^m \sum_{l=1}^2 B_{n_j} T_l^{-1} \left( \partial_{\nu_l}^2 G(y, x) \bigg|_{y = x^{(1)}_{n_j}} \right) + \frac{1}{2} \sum_{l=1}^2 \sum_{h,k=1}^m C_{n_j} T_l^{-1} \left( \partial_{\nu_l}^2 G(y, x) \bigg|_{y = x^{(1)}_{n_j}} \right), \tag{4.17}
\]

so that, by (4.14)-(4.16), we conclude that, for \( x \in \partial \mathcal{U}_r^M(x^{(1)}_{n_j}) \), it holds,

\[
\zeta_n(x) - \overline{\zeta}_n(x) = \sum_{l=1}^m \left( \int_{M \setminus \mathcal{U}_r^M(x^{(1)}_{n_j})} \Psi_{n_j}(y, x) f_n(y) d\mu(y) + \int_{\partial \mathcal{U}_r^M(x^{(1)}_{n_j})} \Psi_{n_j}(y, x) f_n(y) d\mu(y) \right) = -b_0 \sum_{j=1}^m \int_{M \setminus \mathcal{U}_r^M(x^{(1)}_{n_j})} \frac{64e^{-\lambda_{n_j}}}{\rho_n h(x^{(1)}_{n_j})} \Psi_{n_j}(y, x) e^{\Phi_l(y,x_3)} d\mu(y) + O \left( \sum_{j=1}^m \int_{\partial \mathcal{U}_r^M(x^{(1)}_{n_j})} \frac{|y - x^{(1)}_{n_j}|^3}{|x - x^{(1)}_{n_j}|^3} |e^{-\lambda_{n_j}}| d\mu(y) \right) + o(e^{-\lambda_{n_j}}), \tag{4.18}
\]

At this point, let us set

\[
\zeta^*_n(x) = -b_0 \sum_{j=1}^m \int_{M \setminus \mathcal{U}_r^M(x^{(1)}_{n_j})} \frac{64e^{-\lambda_{n_j}}}{\rho_n h(x^{(1)}_{n_j})} \Psi_{n_j}(y, x) e^{\Phi_l(y,x)} d\mu(y), \tag{4.19}
\]

and then substitute (4.18) into (4.10), to obtain,

\[
\text{LHS of (4.3)} = \int_{\partial \mathcal{B}_r(x^{(1)}_{n_j})} 4 < \nu, D(\overline{\zeta}_n + \zeta^*_n)(x) > d\sigma(x) + o(e^{-\lambda_{n_j}/r}) + \sum_{j=1}^m |A_{n_j}| + o \left( \frac{\rho_n h(x^{(1)}_{n_j})}{r^2} \right) + o(e^{-\lambda_{n_j}}), \tag{4.20}
\]

for any \( \nu \in (0, \frac{1}{2}) \). To estimate the right hand side of (4.20), we note that for any pair of (smooth enough) functions \( u \) and \( v \), it holds,

\[
\Delta u \{ \nabla u \cdot (x - x^{(1)}_{n_j}) \} + \Delta v \{ \nabla u \cdot (x - x^{(1)}_{n_j}) \} = \text{div} \left( \nabla u (\nabla u \cdot (x - x^{(1)}_{n_j})) + \nabla v (\nabla u \cdot (x - x^{(1)}_{n_j})) - \nabla u \cdot \nabla v (x - x^{(1)}_{n_j}) \right). \tag{4.21}
\]

In view of (4.17) and (2.2), we also see that, for any fixed \( \tilde{\rho} \in (0, r) \),

\[
\Delta \overline{\zeta}_n(x) = \sum_{j=1}^m A_{n_j} = \int_M f_n^\ast d\mu = \int_M \rho_n h(\tilde{\rho} n^{(1)}_{x^{(1)}_{n_j}} - x^{(1)}_{n_j}) - \rho_n h(\tilde{\rho} n^{(2)}_{x^{(1)}_{n_j}}) d\mu = 0 \quad \text{for} \quad x \in \mathcal{U}_r^M(x^{(1)}_{n_j}) \setminus B_{\tilde{\rho}}(x^{(1)}_{n_j}), \tag{4.22}
\]

and, moreover, by using (4.6) and (4.1), we have,

\[
\Delta (\tilde{G} - \Phi_{n_j}) (x) = 0 \quad \text{for} \quad x \in B_r(x^{(1)}_{n_j}) \setminus B_{\tilde{\rho}}(x^{(1)}_{n_j}). \tag{4.23}
\]
By using (4.21)-(4.23) and (4.7), we conclude that,

$$0 = \int_{\partial B_\theta(z^{(1)}_n)} \left[ \Delta \nabla_n \{ \nabla (\tilde{G} - \phi_{n,j}) \cdot (\tilde{x} - x^{(1)}_{n,j}) \} + \Delta (\tilde{G} - \phi_{n,j}) \{ \nabla \nabla_n \cdot (\tilde{x} - x^{(1)}_{n,j}) \} \right] dz$$

$$= \int_{\partial B_\theta(z^{(1)}_n)} \left[ \frac{\partial \nabla_n}{\partial v} (\nabla (\tilde{G} - \phi_{n,j}) \cdot (\tilde{x} - x^{(1)}_{n,j})) + \frac{\partial (\tilde{G} - \phi_{n,j})}{\partial v} (\nabla \nabla_n \cdot (\tilde{x} - x^{(1)}_{n,j})) \right] \text{d}\sigma$$

$$= - \frac{\rho_n}{2\pi \text{m}} \int_{\partial B_\theta(z^{(1)}_n)} \frac{\partial \nabla_n}{\partial v} \text{d}\sigma,$$

and thus,

$$\int_{\partial B_\theta(z^{(1)}_n)} \frac{\partial \nabla_n}{\partial v} (x) \text{d}\sigma(x) = \int_{\partial B_\theta(z^{(1)}_n)} \frac{\partial \nabla_n}{\partial v} (x) \text{d}\sigma(x).$$

(4.24)

At this point, let us denote by $o_{\eta}(1)$ any quantity which converges to 0 as $\theta \to 0^+$, and then observe that,

$$4 \int_{\partial B_\theta(z^{(1)}_n)} < \nu, \sum_{l=1}^{m} A_{n,l} D_{n} G(x^{(1)}_{n,l}, x) > \text{d}\sigma(x)$$

$$= 4 A_{n,j} \int_{\partial B_\theta(z^{(1)}_n)} < \nu, D_{n} G(x^{(1)}_{n,j}, x) > \text{d}\sigma(x) + o_{\eta}(1) = -4 A_{n,j} + o_{\eta}(1).$$

(4.25)

By setting $z = \tilde{x} - x^{(1)}_{n,j}$ and since, $D_{n} D_{n} \log |z| = \frac{\delta_{n} |z|^2 - 2 \delta_{n} z \cdot \overline{z}}{|z|^4}$, then we find that,

$$\int_{\partial B_\theta(z^{(1)}_n)} < \nu, D_{n} \sum_{k=0}^{m} |z| \text{d}\sigma(z) = - \int_{\partial B_\theta(z^{(1)}_n)} \sum_{k=1}^{n} \psi_{k} \left( \frac{\delta_{n} |z|^2 - 2 \delta_{n} z \cdot \overline{z}}{|z|^4} \right) \text{d}\sigma(z) = 0.$$

(4.26)

We observe that, if $h = k$ then, $D_{k} \log |z| = \frac{z_{k}}{|z|^3}$, $D_{k} D_{k} \log |z| = - \frac{2z_{k}}{|z|^2} + \frac{4 \delta_{n} z_{k}}{|z|^4} + \frac{8 \delta_{n} z}{|z|^3}$, and thus,

$$\int_{\partial B_\theta(z^{(1)}_n)} < \nu, D_{n} \sum_{k=1}^{n} \psi_{k} \left( \frac{z_{k}}{|z|^3} \right) \text{d}\sigma(z) = 0.$$

(4.27)

If $h \neq k$, then, $D_{k} D_{k} \log |z| = - \frac{2z_{h} \delta_{h} + z_{h} \delta_{k}}{|z|^3} + \frac{8 \delta_{k} z}{|z|^3}$, which implies that,

$$\int_{\partial B_\theta(z^{(1)}_n)} < \nu, D_{n} \sum_{k=1}^{n} \psi_{k} \left( \frac{z_{h} \delta_{k} + z_{k} \delta_{h}}{|z|^3} \right) \text{d}\sigma(z) = 0.$$

(4.28)

From (4.24)-(4.28), we conclude that,

$$4 \int_{\partial B_\theta(z^{(1)}_n)} < \nu, D_{n} \nabla_n (x) > \text{d}\sigma(x) = -4 A_{n,j} + o_{\eta}(1).$$

(4.29)

Next we estimate the other term in (4.20), that is $4 \int_{\partial B_\theta(z^{(1)}_n)} < \nu, D_{n} \zeta^{*}_n (x) > \text{d}\sigma(x)$, where $\zeta^{*}_n$ is defined in (4.19).

Clearly we have,

$$D_{n} \zeta^{*}_n (y, x) = D_{n} \left( G(y, x) - G(z^{(1)}_n, x) - \frac{\partial y G(y, x)}{y = z^{(1)}_n} \right) + \left( \frac{\partial z^{(1)}_n}{\partial y} \right) \leq \frac{C}{\sqrt{\theta}}$$

for some constant $C > 0$,
which implies \( \int_{\partial \Omega_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} = o_{\nu}(1) \). Thus (4.26), (4.27), and (4.28), (4.30) imply that,

\[
4 \int_{\partial \Omega_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} = -4 \int_{\partial \Omega_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} - 4 \int_{\partial \Omega_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} + o_{\nu}(1) = 4 + o_{\nu}(1) \quad \text{for } y \in \Omega_{\nu} \setminus U_{\nu}^{(1)}, \quad \bar{y} = T_{\nu}(y) \text{ and } \bar{x} \in \partial \Omega_{\nu}^{(1)}.
\]

At this point, let us observe that \(-\Delta \Upsilon_{\nu}^{(1)}(y, \bar{y}) = \delta_{\nu} \) for \( y \in B_{r}(\Omega_{\nu}^{(1)}) \setminus \partial \Omega_{\nu}^{(1)} \) and let us choose \( u(x) = \Upsilon_{n_j}(y, \bar{y}) \) and \( v(\bar{x}) = \bar{G}(\bar{x}) - \phi_{n_j}(\bar{x}) \) in (4.21). Then we consider the following three cases:

(i) If \( y \in B_{r}(\Omega_{\nu}^{(1)}) \setminus \partial \Omega_{\nu}^{(1)} \), then, from (4.31) and (4.21), we obtain that,

\[
4 \int_{\partial \Omega_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} = 4 \int_{\partial \Omega_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} + o_{\nu}(1).
\]

(ii) If \( y \in \Omega_{\nu} \setminus U_{\nu}^{(1)}, \bar{y} = T_{\nu}(y) \), then we see from (4.31) and (4.21) that,

\[
4 \int_{\partial \Omega_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} = 4 \int_{\partial \Omega_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} + o_{\nu}(1).
\]

(iii) If \( y \in \Omega_{\nu} \) and \( x \in \partial \Omega_{\nu}^{(1)}, l \neq j \), then we have \( |D_{x} \Upsilon_{n_j}(y, \bar{y})| \leq C \) for some constant \( C > 0 \). So we conclude

\[
4 \int_{\partial \Omega_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} = 4 \int_{\partial \Omega_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} = o_{\nu}(1),
\]

and so, by (4.19) and (4.32)-(4.34), we finally conclude that,

\[
4 \int_{\partial \Omega_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} = -\sum_{j=1}^{\infty} 256 \rho_{\nu} h^{2s_j} T_{\nu}^{-1} \int_{\Omega_{\nu} \setminus U_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} \int_{\partial \Omega_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} = o_{\nu}(1).
\]

Finally, By (2.11) and (2.14), we see that,

\[
4 \int_{\partial \Omega_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} = -256 \rho_{\nu} h^{2s_j} T_{\nu}^{-1} \int_{\Omega_{\nu} \setminus U_{\nu}^{(1)}} \nu \cdot D \nabla G(y, \bar{y}) > d\mathbf{x} + o_{\nu}(1).
\]

Obviously (4.20), (4.29), and (4.35) conclude the proof of Lemma 4.2.

To estimate the right hand side of (4.3) of Lemma 4.1, we note that,

\[
f_{n}^{s}(x) = \frac{\rho_{\nu} h(x) e^{2s_j} - e^{2s_j}}{||D_{n_j}^{(1)} + v_{n_j}^{(2)}||_{L_{n}(M)}} = \frac{\rho_{\nu} h(x) e^{2s_j} - e^{2s_j}}{||\tilde{u}_{n_j}^{(1)} - \tilde{u}_{n_j}^{(2)}||_{L_{n}(M)}} = \rho_{\nu} h(x) e^{2s_j} (\zeta_{n} + o(1)),
\]

where we used (3.16) and (4.2). Then we will need the following estimate:
Lemma 4.3.

(i) \[ \int_{B_{r}^{(1)}(\xi)} r f_{n}^{*} e^{2\varphi_{j}} \, d\sigma = -\frac{128e^{-\lambda_{n,3}(1)}}{\rho_{n}(h(q_{j}))^{2}G_{j}(q_{j})} \frac{\pi}{r^{2}} \]

\[ -\frac{32\pi e^{-\lambda_{n,1}(1)}b_{0}h(q_{j})e^{G_{j}(q_{j})}}{\rho_{n}(h(q_{j}))^{2}G_{j}(q_{j})} (\Delta \log h(q_{j}) + 8\pi m - 2K(q_{j})) + O(re^{-\lambda_{n,1}(1)}) + o(e^{-\lambda_{n,1}(1)})/r^{2}. \]

(ii) \[ \sum_{j=1}^{m} \int_{U_{M}(x_{n,j})} f_{n}^{*} e^{2\varphi_{j}} d\mu(x) = \frac{64b_{0}e^{-\lambda_{n,1}(1)}}{\rho_{n}(h(q_{j}))^{2}G_{j}(q_{j})} \sum_{j=1}^{m} \int_{M \setminus U_{M}(q_{j})} h(q_{j}) e^{G_{j}(q_{j})} \rho_{1}(x,q_{j}) d\mu(x) + o(e^{-\lambda_{n,1}(1)}). \]

(iii) \[ \int_{B_{r}^{(1)}(\xi)} f_{n}^{*} e^{2\varphi_{j}} < D(\log h + \varphi_{n,j}) \lambda - \lambda_{n,j} > d\lambda. \]

\[ = \left[ 32\pi \left( \int_{M} \zeta_{n} d\mu - \left\| \|\zeta_{n}^{(2)}\|_{\infty}^{2} \right\|_{\infty}(\int_{M} \zeta_{n} d\mu)^{2} \right) (\Delta \log h(q_{j}) + 8\pi m - 2K(q_{j})) \right] \]

\[ \times h(q_{j}) e^{G_{j}(q_{j})} e^{-\lambda_{n,1}(1)} \left( \lambda_{n,1} + \log \left( \frac{\rho_{n}(h(q_{j}))^{2}G_{j}(q_{j})}{8h(q_{j}) e^{G_{j}(q_{j})}} - 2 + O(R^{-2}) \right) \right) \]

\[ + O(1) \left( \frac{|\log |e^{-\lambda_{n,j}(1)}|}{(\lambda_{n,j}(1))^{2}} \right) + O(\rho_{1}(x,q_{j})^{2}) + o(\rho_{1}(x,q_{j})) \log R \right] + O(1) \left( \frac{e^{-\lambda_{n,j}(1)}}{r^{2}} \right) \]

\[ + O(1) \left( \sum_{l=1}^{m} (\lambda_{n,j} + \lambda_{n,j}^{(1)}) \left( \frac{e^{-\lambda_{n,j}(1)}}{R} + e^{-\lambda_{n,j}(1)} (\lambda_{n,j} + |\log R|) \right) \right) \text{ for any } R > 1, \]

where \( O(1) \) here is used to denote any quantity uniformly bounded with respect to \( r, R \) and \( n \).

Proof. (i) We first observe that (1.4) and (4.14) imply that,

\[ \int_{\partial B_{r}^{(1)}(\xi)} r f_{n}^{*} e^{2\varphi_{j}} \, d\sigma = \int_{\partial B_{r}^{(1)}(\xi)} \frac{64e^{-\lambda_{n,j}(1)}(-b_{0} + o(1))e^{q_{j}^{*} + 2\varphi_{j}}}{\rho_{n} h(x_{n,j})^{2} |\xi - x_{n,j}^{(1)}|^{3}} \, d\sigma. \]

Since \( f_{q,j}(q_{j}) = 0, \nabla f_{q,j}(q_{j}) = 0, \varphi_{j}(x_{n,j}) = 0, \nabla \varphi_{j}(x_{n,j}) = 0, \) and in view of (2.14), we find that,

\[ f_{q,j}(\xi) + 2\varphi_{j}(\xi) = \frac{1}{2} < D^{2}_{\xi}(f_{q,j} + 2\varphi_{j}) \xi - x_{n,j}^{(1)}, \xi - x_{n,j}^{(1)} > + o(1) + O(|\xi - x_{n,j}^{(1)}|^{3}). \]
By (4.36) and (4.37), we obtain,

\[
\int_{\partial B_r(\bar{\xi}^{(1)}_{n,j})} r f_n e^{2\varphi} \, d\sigma = \int_{\partial B_r(\bar{\xi}^{(1)}_{n,j})} \frac{64 e^{-\lambda^{(1)}_{n,j}}}{\rho_n h(\bar{\xi}^{(1)}_{n,j}) |x - \bar{\xi}^{(1)}_{n,j}|^3} \times \left\{ b_0 \left( 1 + \frac{1}{2} - D_2^2(f_{q,j} + 2\varphi) \right) \right\} d\sigma
\]

\[
= \int_{\partial B_r(\bar{\xi}^{(1)}_{n,j})} \frac{64 e^{-\lambda^{(1)}_{n,j}} b_0 (1 + \frac{\Delta(f_{q,j}^2 + 2\varphi)(\bar{\xi}^{(1)}_{n,j})}{4}) |x - \bar{\xi}^{(1)}_{n,j}|^2}{\rho_n h(\bar{\xi}^{(1)}_{n,j}) |x - \bar{\xi}^{(1)}_{n,j}|^3} d\sigma + O(\rho_n^{-\lambda^{(1)}_{n,j}}) + \frac{o(\rho_n^{-\lambda^{(1)}_{n,j}})}{r^2}
\]

\[
= -\frac{128\pi}{\rho_n h(\bar{\xi}^{(1)}_{n,j}) r^2} \times \left\{ \frac{32\pi e^{-\lambda^{(1)}_{n,j}} b_0 (\Delta \log h(q_j) + 8\pi m - 2K(q_j))}{\rho_n h(\bar{\xi}^{(1)}_{n,j})} + O(\rho_n^{-\lambda^{(1)}_{n,j}}) + \frac{o(\rho_n^{-\lambda^{(1)}_{n,j}})}{r^2} \right\},
\]

where, in the last identity, we used (2.4), (2.5) and (2.14). By using (2.11) and (2.14), we find that,

\[
-\frac{128\pi e^{-\lambda^{(1)}_{n,j}} b_0 (\Delta \log h(q_j) + 8\pi m - 2K(q_j))}{\rho_n h(\bar{\xi}^{(1)}_{n,j}) r^2} - \frac{32\pi e^{-\lambda^{(1)}_{n,j}} b_0 (\Delta \log h(q_j) + 8\pi m - 2K(q_j))}{\rho_n h(\bar{\xi}^{(1)}_{n,j})}
\]

\[
= -\frac{32\pi e^{-\lambda^{(1)}_{n,j}} b_0 (\Delta \log h(q_j) + 8\pi m - 2K(q_j))}{\rho_n h(\bar{\xi}^{(1)}_{n,j})} e^{G^j_q(\bar{\xi}^{(1)}_{n,j})} + O(\rho_n^{-\lambda^{(1)}_{n,j}}) + \frac{o(\rho_n^{-\lambda^{(1)}_{n,j}})}{r^2},
\]

which proves (i).

(ii) We note that \( \int_M f_n^* \, d\mu = 0 \), and thus,

\[
\sum_{j=1}^m \int_{U^{(1)}_{n,j}} f_n^* \, d\mu = \sum_{j=1}^m \int_{M \setminus U^{(1)}_{n,j}} f_n^* \, d\mu(x).
\]

By (4.14), (2.11), (2.16) and (2.7), we see that

\[
-\sum_{j=1}^m \int_{M \setminus U^{(1)}_{n,j}} f_n^* \, d\mu(x) = \sum_{j=1}^m \int_{M \setminus U^{(1)}_{n,j}} \frac{64 b_0 e^{-\lambda^{(1)}_{n,j}}}{\rho_n h(\bar{\xi}^{(1)}_{n,j})} \varphi_{j}(x,a) \, d\mu(x) + o(\rho_n^{-\lambda^{(1)}_{n,j}})
\]

\[
= \sum_{j=1}^m \int_{M \setminus U^{(1)}_{n,j}} \frac{64 b_0 e^{-\lambda^{(1)}_{n,j}}}{\rho_n h(\bar{\xi}^{(1)}_{n,j})} e^{G_j^q(\bar{\xi}^{(1)}_{n,j})} \varphi_{j}(x,a) \, d\mu(x) + o(\rho_n^{-\lambda^{(1)}_{n,j}})
\]

\[
= \frac{64 b_0 e^{-\lambda^{(1)}_{n,j}}}{\rho_n h(q_j)} \sum_{j=1}^m \int_{M \setminus U^{(1)}_{n,j}} h(q_j) e^{G_j^q(\bar{\xi}^{(1)}_{n,j})} \varphi_{j}(x,a) \, d\mu(x) + o(\rho_n^{-\lambda^{(1)}_{n,j}}).
\]

Clearly (4.38) and (4.39) prove (ii).

(iii) Let us recall the definition of \( \phi_{n,j} \) and \( G_j^q \) from (4.1) and (1.2).

By (2.13) and (2.17), we find that \( D(\log h_j + \phi_{n,j})(\bar{\xi}^{(1)}_{n,j}) = O(\lambda^{(1)}_{n,j} e^{-\lambda^{(1)}_{n,j}}) \), which readily implies that,

\[
D(\log h_j + \phi_{n,j})(\bar{x}) = D(\log h_j + \phi_{n,j})(\bar{x}^{(1)}_{n,j}) + <D^2(\log h_j + \phi_{n,j})(\bar{x}^{(1)}_{n,j})\bar{x} - \bar{x}^{(1)}_{n,j} > + O((\bar{x} - \bar{x}^{(1)}_{n,j})^2)
\]

\[
= <D^2(\log h_j + \phi_{n,j})(\bar{x}^{(1)}_{n,j})\bar{x} - \bar{x}^{(1)}_{n,j} > + O(\lambda^{(1)}_{n,j} e^{-\lambda^{(1)}_{n,j}}) + O((\bar{x} - \bar{x}^{(1)}_{n,j})^2).
\]
By using (2.8), (2.9) and the scaling $\bar{z} = e^\frac{\lambda_{n,j}^{(1)}}{2} z + x_n^{(1)}$, recalling the notation of $\mathcal{F}$ introduced before Lemma 3.4, we find that,

\[
\int_{B_r(x_n^{(1)}, x)} \rho_n h_j(x_n^{(1)}) e^\lambda_{n,j}^{(1)} \left( \frac{\|\bar{u}_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)}^2}{2} \right) \beta_{n,j} + O(\|\bar{u}_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)}) \\
	imes < D(\log h_j + \phi_{n,j})(x_n^{(1)}), (x_n^{(1)} - \bar{z}_n^{(1)}) > \, \, dx
\]

\[
= \int_{B_{r}(x_n^{(1)}, x)} \rho_n h_j(x_n^{(1)}) e^\lambda_{n,j}^{(1)} \left( \frac{\|\bar{u}_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)}^2}{2} \right) \beta_{n,j} + O(\|\bar{u}_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)}) \\
	imes < D^2(\log h_j + \phi_{n,j})(x_n^{(1)}), (x_n^{(1)} - \bar{z}_n^{(1)}) > + O(\|\bar{z}_n^{(1)} - \bar{z}_n^{(1)}\|_{L^\infty(M)}^2, \|\bar{z}_n^{(1)} - \bar{z}_n^{(1)}\|_{L^\infty(M)}^2) \, \, dx
\]

\[
= \int_{B_{r}(x_n^{(1)}, x)} \rho_n h_j(x_n^{(1)}) e^\lambda_{n,j}^{(1)} \left( \frac{\|\bar{u}_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)}^2}{2} \right) \beta_{n,j} + O(\|\bar{u}_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)})
\]

By using (4.40) together with (2.9), (2.7), and Lemma 3.4, we conclude that,

\[
K_{n,j,r} = \int_{B_{r}(x_n^{(1)}, x)} \rho_n h_j(x_n^{(1)}) e^\lambda_{n,j}^{(1)} \left( \frac{\|\bar{u}_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)}^2}{2} \right) \beta_{n,j} + O(\|\bar{u}_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)}^2, \|\bar{z}_n^{(1)} - \bar{z}_n^{(1)}\|_{L^\infty(M)}^2, \|\bar{z}_n^{(1)} - \bar{z}_n^{(1)}\|_{L^\infty(M)}^2)
\]

\[
= \int_{B_{r}(x_n^{(1)}, x)} \rho_n h_j(x_n^{(1)}) e^\lambda_{n,j}^{(1)} \left( \frac{\|\bar{u}_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)}^2}{2} \right) \beta_{n,j} + O(\|\bar{u}_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)})
\]

Since $\int \frac{(1-r^2)^2}{1+r^2} \, dr = \frac{1}{2} \left( \frac{3\pi^2}{8} - \log(1+r^2) \right) + C$, then, for any fixed and large $R > 0$, by Lemma 3.2 and Lemma 3.4, we see that,

\[
\int_{B_R(0)} \rho_n h_j(x_n^{(1)}) e^\lambda_{n,j}^{(1)} \left( \frac{\|\bar{u}_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)}^2}{2} \right) \beta_{n,j} + O(\|\bar{u}_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)})
\]

\[
= \int_{B_R(0)} \rho_n h_j(x_n^{(1)}) e^\lambda_{n,j}^{(1)} \left( \frac{\|\bar{u}_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)}^2}{2} \right) \beta_{n,j} + o(1)
\]

\[
< D^2(\log h_j + \phi_{n,j})(x_n^{(1)}), (x_n^{(1)} - \bar{z}_n^{(1)}) > + O(\|\bar{z}_n^{(1)} - \bar{z}_n^{(1)}\|_{L^\infty(M)}^2, \|\bar{z}_n^{(1)} - \bar{z}_n^{(1)}\|_{L^\infty(M)}^2)
\]
In view of (3.22), we also see that if
\[
\int \frac{\rho_{n} h_{j}(\xi_{n,j}^{(1)}) b_{j,0} \left( \frac{1 - \rho_{n} h_{j}(\xi_{n,j}^{(1)})}{1 + \rho_{n} h_{j}(\xi_{n,j}^{(1)}) |z|^2} \right)}{\Delta(\log h_{j} + \phi_{n,j}) (\xi_{n,j}^{(1)}) |z|^2 e^{-\lambda_{n,j}^{(1)}}} \, dz + o(e^{-\lambda_{n,j}^{(1)}} | \log R |)
\]
\[
= 2(1 + \frac{\rho_{n} h_{j}(\xi_{n,j}^{(1)})}{R} |z|^2)^2
\]
\[
= 64 \pi b_{0} \Delta(\log h_{j} + \phi_{n,j}) (\xi_{n,j}^{(1)}) e^{-\lambda_{n,j}^{(1)}} (\int_{0}^{R} \frac{\rho_{n} h_{j}(\xi_{n,j}^{(1)})}{(1 + \rho_{n} h_{j}(\xi_{n,j}^{(1)}) |z|^2)^2} ds + o(e^{-\lambda_{n,j}^{(1)}} | \log R |)
\]
\[
= 32 \pi b_{0} \Delta(\log h_{j} + \phi_{n,j}) (\xi_{n,j}^{(1)}) e^{-\lambda_{n,j}^{(1)}} (\frac{\rho_{n} h_{j}(\xi_{n,j}^{(1)})}{R^{2}} (1 + \frac{\rho_{n} h_{j}(\xi_{n,j}^{(1)})}{4} R^{2}) - \log(1 + \frac{\rho_{n} h_{j}(\xi_{n,j}^{(1)})}{8} R^{2}) + o(e^{-\lambda_{n,j}^{(1)}} | \log R |)
\]

Next, let us observe that,
\[
\int \frac{r^{2}}{(1 + r^{2})^{2}} \, dr = \frac{1}{2} \left( \frac{1}{1 + r^{2}} + \log(1 + r^{2}) \right) + C.
\]

In view of (3.22), we also see that if |z| ≥ R, then it holds,
\[
\bar{\xi}_{n}(z) = \int_{M} \bar{\xi}_{n} d\mu + O(1) \left( \sum_{i=1}^{m} (|A_{n,j}| + e^{-\frac{\lambda_{n,j}^{(1)}}{|z|}}) \left( \frac{\lambda_{n,j}^{(1)}}{|z|} \right) + 1 \right), \text{ and thus (4.42)}
\]
\[
(\bar{\xi}_{n}(z))^{2} = \left( \int_{M} \bar{\xi}_{n} d\mu \right)^{2} + O(1) \left( \sum_{i=1}^{m} (|A_{n,j}| + e^{-\frac{\lambda_{n,j}^{(1)}}{|z|}}) \left( \frac{\lambda_{n,j}^{(1)}}{|z|} \right) + 1 \right).
\]

In view of (4.42), we also find that,
\[
\int_{B_{A_{n,j}^{(1)}}(0) \setminus B_{R}(0)} \frac{\rho_{n} h_{j}(\xi_{n,j}^{(1)}) (\bar{\xi}_{n} - \frac{\|\bar{\xi}_{n} - \bar{\xi}_{n}^{(1)}\|_{L^{\infty}(M)}}{2} (\bar{\xi}_{n}^{(1)}))}{(1 + \frac{\rho_{n} h_{j}(\xi_{n,j}^{(1)})}{|z|^2})^2} < D^{2}(\log h_{j} + \phi_{n,j}) (\xi_{n,j}^{(1)}) z, z > e^{-\lambda_{n,j}^{(1)}} dz
\]
\[
= \int_{B_{A_{n,j}^{(1)}}(0) \setminus B_{R}(0)} \frac{\rho_{n} h_{j}(\xi_{n,j}^{(1)}) (\int_{M} \bar{\xi}_{n} d\mu - \frac{\|\bar{\xi}_{n} - \bar{\xi}_{n}^{(1)}\|_{L^{\infty}(M)}}{2} (\int_{M} \bar{\xi}_{n} d\mu))}{(1 + \frac{\rho_{n} h_{j}(\xi_{n,j}^{(1)})}{|z|^2})^2} < D^{2}(\log h_{j} + \phi_{n,j}) (\xi_{n,j}^{(1)}) z, z > e^{-\lambda_{n,j}^{(1)}} dz
\]
\[
+ \int_{B_{A_{n,j}^{(1)}}(0) \setminus B_{R}(0)} \frac{O(1) \left( \sum_{i=1}^{m} (|A_{n,j}| + e^{-\frac{\lambda_{n,j}^{(1)}}{|z|}}) \left( \frac{\lambda_{n,j}^{(1)}}{|z|} \right) + 1 \right) e^{-\lambda_{n,j}^{(1)} |z|^2}}{(1 + \frac{\rho_{n} h_{j}(\xi_{n,j}^{(1)})}{8} |z|^2)^2} dz =: (I) + (II).
\]

It is easy to see that,
\[
(II) = O(1) \left( \sum_{i=1}^{m} (|A_{n,j}| + e^{-\frac{\lambda_{n,j}^{(1)}}{|z|}}) \left( \frac{e^{-\lambda_{n,j}^{(1)}}}{R} + e^{-\lambda_{n,j}^{(1)}} (\lambda_{n,j}^{(1)} + | \log r |) \right).
\]
Since $3_n := \int_M \zeta_n d\mu - \frac{\|\nu_n(1) - \nu_n(2)\|_{L^2(M)}}{2} (\int_M \zeta_n d\mu)^2$ is constant, we also conclude from (4.41) that,

\[
(1) = \frac{64\pi 3_n \Delta (\log h_j + \phi_{n,j}) (\mathbf{X}_n^{(1)})}{\rho_n h_j (\mathbf{X}_n^{(1)})} e^{-\lambda_n^{(1)}} \int \frac{\rho_n h_j (\mathbf{X}_n^{(1)})}{(1 + s^2)^2} ds
\]

\[
= -\frac{32\pi 3_n \Delta (\log h_j + \phi_{n,j}) (\mathbf{X}_n^{(1)})}{\rho_n h_j (\mathbf{X}_n^{(1)})} e^{-\lambda_n^{(1)}}
\times \left[ \frac{1}{1 + \frac{\rho_n h_j (\mathbf{X}_n^{(1)})}{8} R^2} + \log\left(1 + \frac{\rho_n h_j (\mathbf{X}_n^{(1)})}{8} R^2\right) - \lambda_n^{(1)} - \log\left(\frac{\rho_n h_j (\mathbf{X}_n^{(1)})}{8} R^2\right) \right] + O\left(\frac{e^{-2\lambda_n^{(1)}}}{r^2}\right)
\]

(4.43)

By Lemma 3.4, it is easy to check that,

\[
\int_M \zeta_n d\mu = -b_0 + o(1).
\]

(4.44)

From (4.40)-(4.43) and (4.44), we obtain

\[
\int_{B_r(x_n^{(1)})} f_n^* < D (\log h_j + \phi_{n,j}, x - x_n^{(1)}) > e^{2\phi_j} dx
\]

\[
\frac{32\pi 3_n \Delta (\log h_j + \phi_{n,j}) (\mathbf{X}_n^{(1)})}{\rho_n h_j (\mathbf{X}_n^{(1)})} e^{-\lambda_n^{(1)}} (\lambda_n^{(1)} + \log\left(\frac{\rho_n h_j (\mathbf{X}_n^{(1)})}{8} R^2\right)) - 2 + O(R^{-2})
\]

\[
+ O(1) \left( \log r e^{-\lambda_n^{(1)}} (\lambda_n^{(1)})^2 \right) + O(1) \left( e^{-\lambda_n^{(1)}} + o(e^{-\lambda_n^{(1)}}) \right) \log R | + O(1) \left( \frac{e^{-2\lambda_n^{(1)}}}{r^2} \right)
\]

\[
+ O(1) \left( \sum_{i=1}^m (|A_{n,i}| + e^{-\lambda_n^{(1)}} (\lambda_n^{(1)}) + |\log r|) \right).
\]

Finally, since (2.4),(2.5), (2.14) and (2.17) imply that,

\[
\Delta (\log h_j + \phi_{n,j}) (\mathbf{X}_n^{(1)}) = \Delta \log h(q_j) + 8\pi m - 2K(q_j) + O(\lambda_n^{(1)} e^{-\lambda_n^{(1)}}),
\]

then (2.11) and (2.14), show that,

\[
\int_{B_r(x_n^{(1)})} f_n^* < D (\log h_j + \phi_{n,j}, x - x_n^{(1)}) > e^{2\phi_j} dx
\]

\[
\frac{32\pi 3_n (\Delta \log h(q_j) + 8\pi m - 2K(q_j)) h(q_j) e^{\mathbf{G}_j(q_j)}}{\rho_n (h(q_j)) e^{\mathbf{G}_j(q_j)}} e^{-\lambda_n^{(1)}} (\lambda_n^{(1)} + \log\left(\frac{\rho_n (h(q_j)) e^{\mathbf{G}_j(q_j)}}{8 h(q_j) e^{\mathbf{G}_j(q_j)}} (\lambda_n^{(1)})^2\right)) - 2 + O(R^{-2})
\]

\[
+ O(1) \left( \log r e^{-\lambda_n^{(1)}} (\lambda_n^{(1)})^2 \right) + O(1) \left( e^{-\lambda_n^{(1)}} + o(e^{-\lambda_n^{(1)}}) \right) \log R | + O(1) \left( \frac{e^{-2\lambda_n^{(1)}}}{r^2} \right)
\]

\[
+ O(1) \left( \sum_{i=1}^m (|A_{n,i}| + e^{-\lambda_n^{(1)}} (\lambda_n^{(1)}) + |\log r|) \right).
\]

(4.45)
The estimate \((4.45)\) readily yields \((iii)\), as claimed. This fact concludes the proof of Lemma 4.3.

\[
\square
\]

**Lemma 4.4.**

(i) \(A_{n,j} = \int_{M_j} f_n^*(y) \, d\mu(y) = o(e^{-\frac{\lambda_{n,1}}{r^2}})\).

(ii) \(b_0 = 0\) and in particular \(b_{j,0} = 0\), \(j = 1, \cdots, m\).

**Proof.** (i) By \((4.3)\), Lemmas 4.2-4.3 and \((2.10)\), we see that,

\[-4A_{n,j} + O(\lambda_{n,1}^j e^{-\lambda_{n,1}^j}) + o(e^{-\frac{\lambda_{n,1}^j}{r^2}}) \sum_{i=1}^{m} |A_{n,j}| + o(e^{-\lambda_{n,1}^j}) = -2A_{n,j} + O(\lambda_{n,1}^j e^{-\lambda_{n,1}^j}) + O(e^{-\frac{\lambda_{n,1}^j}{r^2}}) + o(e^{-\lambda_{n,1}^j} \log R),\]

which proves \((i)\).

(ii) For any \(r > 0\), let

\[r_j = r \sqrt{8h(q_j)G_j(q_j)}\quad \text{for} \quad j = 1, \cdots, m.\] (4.46)

By \((4.3)\), Lemmas 4.2-4.3 and \((i)\), we have for any \(r \in (0, 1)\) and \(R > 1\),

\[
\sum_{j=1}^{m} \left[ -4A_{n,j} - \frac{256b_0 e^{-\lambda_{n,1}^j} h(q_j) e^{G_j^*(q_j)}}{\rho_n(h(q_j))} \frac{1}{2} \int_{M_j \setminus U_q^n(q_j)} e^{\Phi_j(y,q)} \, d\mu(y) + o(e^{-\frac{\lambda_{n,1}^j}{r^2}}) \sum_{i=1}^{m} |A_{n,j}| + o(e^{-\lambda_{n,1}^j}) \right]
\]

\[
= \sum_{j=1}^{m} \left[ -\frac{128e^{-\lambda_{n,1}^j} b_0 h(q_j) e^{G_j^*(q_j)}}{\rho_n(h(q_j))} \frac{\pi}{r_j^2} \int_{M_j \setminus U_q^n(q_j)} h(q_j) e^{G_j^*(q_j)} e^{\Phi_j(x,q)} \, d\mu(x) \right]
\]

\[
- \frac{32\pi}{\rho_n(h(q_j))} \left( \int_{M_j \setminus U_q^n(q_j)} e^{\Phi_j(x,q)} \, d\mu(x) \right) \left( \frac{\int_{M_j \setminus U_q^n(q_j)} e^{\Phi_j(x,q)} \, d\mu(x)}{2} \right) \left( \frac{\int_{M_j \setminus U_q^n(q_j)} e^{\Phi_j(x,q)} \, d\mu(x)}{2} \right) \left( \frac{\int_{M_j \setminus U_q^n(q_j)} e^{\Phi_j(x,q)} \, d\mu(x)}{2} \right)
\]

\[
\times \left( \lambda_{n,1}^j + \log \left( \frac{\rho_n(h(q_j))}{8h(q_j) e^{G_j^*(q_j)}} \frac{1}{r_j^2} \right) - 2 \right) + O(e^{-\lambda_{n,1}^j}(r + R^{-1}) + o(e^{-\lambda_{n,1}^j})(\log r + \log R)),
\]

where we used \(O(1)\) to denote any quantity uniformly bounded with respect to \(r, R\) and \(n\). Then, in view of \((i)\), \(\sum_{j=1}^{m} A_{n,j} = 0\) and by the definition of \(f(q)\) we see that,

\[-\frac{256b_0 e^{-\lambda_{n,1}^j}}{\rho_n(h(q_j))} \sum_{j=1}^{m} h(q_j) e^{G_j^*(q_j)} \int_{M_j \setminus U_q^n(q_j)} e^{\Phi_j(y,q)} \, d\mu(y)\]

\[
= \frac{128\pi}{\rho_n(h(q_j))} \left( \frac{\int_{M_j \setminus U_q^n(q_j)} e^{\Phi_j(x,q)} \, d\mu(x)}{2} \right) \left( \frac{\int_{M_j \setminus U_q^n(q_j)} e^{\Phi_j(x,q)} \, d\mu(x)}{2} \right) \left( \frac{\int_{M_j \setminus U_q^n(q_j)} e^{\Phi_j(x,q)} \, d\mu(x)}{2} \right)
\]

\[
+ o(e^{-\lambda_{n,1}^j}(r + R^{-1}) + o(e^{-\lambda_{n,1}^j})(\log r + \log R)).
\]

At this point we consider two cases:

**Case 1.** \(f(q) \neq 0\).

By \((4.44)\), \((4.47)\) and in view of Lemma 3.1 we see that, \(f(q)(b_0 + o(1)) = o(1)\). Therefore, since \(f(q) \neq 0\), then \(b_0 = 0\).

**Case 2.** \(f(q) = 0\) and \(D(q) \neq 0\).

Since \(f(q) = 0\), then, by using the definition of \(D(q)\) and \((4.47)\), we conclude that,

\[
(D(q) + o_r(1)) b_0 e^{-\lambda_{n,1}^j} = O(e^{-\lambda_{n,1}^j}(r + R^{-1}) + o(e^{-\lambda_{n,1}^j})(\log r + \log R)).
\]
Since \( r > 0 \) and \( R > 0 \) are arbitrary, then \( D(q) \neq 0 \) implies \( b_0 = 0 \).

At this point, by \( b_0 = 0 \), Lemma 3.4 shows that \( b_{j,0} = 0 = j, 1, \cdots , m \). This fact concludes the proof of (ii). \( \square \)

Next we prove that \( b_{j,1} = b_{j,2} = 0 \) by exploiting the following Pohozaev identity.

**Lemma 4.5** ([47]). We have for \( i = 1, 2 \) and fixed small \( r > 0 \),

\[
\int_{\partial B_r(z_n^{(i)})} \phi_n \frac{1}{\|D_{n,j} \phi_n\|_\infty} \left( \frac{x - z_n^{(i)}}{\|x - z_n^{(i)}\|} \right) dx = - \int_{\partial B_r(z_n^{(i)})} \rho_n h_j(x) \frac{1}{\|D_{n,j} \phi_n\|_\infty} \left( \frac{x - z_n^{(i)}}{\|x - z_n^{(i)}\|} \right) dx + \int_{B_r(z_n^{(i)})} \rho_n h_j(x) \frac{1}{\|D_{n,j} \phi_n\|_\infty} \left( \frac{x - z_n^{(i)}}{\|x - z_n^{(i)}\|} \right) dx.
\]

**Proof.** The identity (4.48) has been obtained in [47]. We prove it here for reader’s convenience. We first observe that,

\[
\Delta \phi_n^{(i)} + \rho_n h_j \phi_n^{(i)} = 0 \text{ in } B_r(z_n^{(i)}), \text{ and}
\]

\[
\text{div} \left( \nabla \phi_n^{(i)} + \nabla \phi_n^{(j)} \right) = \Delta \phi_n^{(i)} + \Delta \phi_n^{(j)}.
\]

Therefore we find that,

\[
\text{div} \left( -2 \rho_n h_j(x) \frac{1}{\|D_{n,j} \phi_n\|_\infty} \left( \frac{x - z_n^{(i)}}{\|x - z_n^{(i)}\|} \right) dx \right) + 2 \rho_n h_j(x) \frac{1}{\|D_{n,j} \phi_n\|_\infty} \left( \frac{x - z_n^{(i)}}{\|x - z_n^{(i)}\|} \right) dx = \text{div} \left( 2 \nabla \phi_n^{(i)} + 2 \nabla \phi_n^{(j)} \right).
\]

which proves Lemma 4.5. \( \square \)

Next, we shall estimate the left and right hand side of the identity (4.48).

**Lemma 4.6.**

\[
\text{(RHS) of (4.48)} = \hat{B}_j \left( \sum_{h=1}^2 D_{n,j}^2 (\phi_n + \log h_j)(z_n^{(i)}) e^{\frac{\lambda_1}{2} b_{j,h}} + o(e^{-\frac{\lambda_1}{2}}), \right), \text{ where } \hat{B}_j = 4 \frac{8}{\rho_n h_j(x_n^{(i)})} \int_{R^2} \frac{|z|^2}{(1 + |z|^2)^3} dz.
\]

**Proof.** First of all, in view of (2.10), we find that,

\[
\int_{\partial B_r(z_n^{(i)})} \rho_n h_j(x) \frac{1}{\|D_{n,j} \phi_n\|_\infty} \left( \frac{x - z_n^{(i)}}{\|x - z_n^{(i)}\|} \right) dx = \int_{\partial B_r(z_n^{(i)})} \rho_n h_j(x) \phi_n^{(i)}(z_n + o(1)) \frac{1}{\|x - z_n^{(i)}\|} dx = O(e^{-\lambda_1}).
\]

(4.49)
while by (2.8), we also see that,
\[
\int_{B_r(z_0)} \frac{\rho_n h_j(x) (e^{\phi_n} - e^{\phi_0})}{\| \tilde{u}_n \|_{L^\infty(M)}} D_1(\phi_{n,j} + \log h_j) d\mathcal{X} = \int_{B_r(z_0)} \frac{\rho_n h_j(x) (e^{\phi_n} - e^{\phi_0})}{\| \tilde{u}_n \|_{L^\infty(M)}} (\xi_n + o(1)) D_1(\phi_{n,j} + \log h_j) d\mathcal{X}
\]
\[
= \int_{B_r(z_0)} \rho_n h_j(x) e^{\phi_n} e^{\phi_0} + G_j(z) - G_j^*(z_0) \left( \xi_n + o(1) \right) + \sum_{h=1}^2 D_h^2 (\phi_{n,j} + \log h_j) (\xi_n - \xi_n) h + O(|x - \xi_n|^2) \right) d\mathcal{X}
\]
(4.50)

Next, since $q$ is a critical point of $f_m$, then by using (2.17), (2.16), and (2.14), we find that,
\[
D_1(\phi_{n,j} + \log h_j) (\xi_n) = D_1(G_j^* + \log h_j) (\xi_n) + O(\lambda_j e^{-\lambda_j}) = O(\lambda_j e^{-\lambda_j}).
\]
(4.51)

By using (4.50), (4.51), and (2.9), we have,
\[
\int_{B_r(z_0)} \frac{\rho_n h_j(x) (e^{\phi_n} - e^{\phi_0})}{\| \tilde{u}_n \|_{L^\infty(M)}} D_1(\phi_{n,j} + \log h_j) d\mathcal{X}
\]
\[
= \int_{B_r(z_0)} \frac{\rho_n h_j(x) e^{\phi_n} e^{\phi_0} + G_j(z) - G_j^*(z_0)}{8} (1 + \rho_h \| \tilde{u}_n \|_{L^\infty(M)}) (|z|^2)^2 + \sum_{h=1}^2 D_h^2 (\phi_{n,j} + \log h_j) (\xi_n) e^{-\lambda_j} b_{j,h} + o(e^{-\lambda_j}),
\]
and then, in view of Lemma 3.2 and Lemma 4.4, we conclude that,
\[
\int_{B_r(z_0)} \frac{\rho_n h_j(x) (e^{\phi_n} - e^{\phi_0})}{\| \tilde{u}_n \|_{L^\infty(M)}} D_1(\phi_{n,j} + \log h_j) d\mathcal{X}
\]
\[
= \int_{B_r(z_0)} \frac{\rho_n h_j(x) e^{\phi_n} e^{\phi_0} + G_j(z) - G_j^*(z_0)}{8} (1 + \rho_h \| \tilde{u}_n \|_{L^\infty(M)}) (|z|^2)^2 + \sum_{h=1}^2 D_h^2 (\phi_{n,j} + \log h_j) (\xi_n) e^{-\lambda_j} b_{j,h} + o(e^{-\lambda_j}),
\]
(4.52)

Clearly (4.49) and (4.52) conclude the proof of Lemma 4.6. \hfill \square

**Lemma 4.7.**

(LHS) of (4.48) = $-8\pi \left[ \sum_{i<j} e^{-\frac{\lambda_i}{2}} D_1 G_{i,j}^*(x_n) + e^{-\frac{\lambda_i}{2}} D_1 \sum_{h=1}^2 \partial_{y_i} R(y, x) \right]_{y=x_n, b_{j,h}} B_i + o(e^{-\lambda_i}).$

where $G_{i,j}^*(x) = \sum_{i=1}^2 \partial_{y_i} G(y, x) \bigg|_{y=x_n, b_{j,h}} b_{j,h}.$
Proof. By the definition of \( G_{n,j}^* \), we have for any \( \theta \in (0, r) \), \( \Delta G_{n,j} = 0 \) in \( B_r(\underline{x}_{n,j}) \setminus B_\theta(\underline{x}_{n,j}) \).

Then for \( x \in B_r(\underline{x}_{n,j}) \setminus B_\theta(\underline{x}_{n,j}) \), and setting \( e_i = \frac{x_i}{|x|}, i = 1, 2 \), we have,

\[
0 = \Delta G_{n,j}^* D_1 \log \frac{1}{|x - \underline{x}_{n,j}|} + \Delta \log \frac{1}{|x - \underline{x}_{n,j}|} D_2 G_{n,j}^*
\]

\[
= \text{div} \left( \nabla G_{n,j}^* D_1 \log \frac{1}{|x - \underline{x}_{n,j}|} + \nabla \log \frac{1}{|x - \underline{x}_{n,j}|} D_2 G_{n,j}^* - \nabla G_{n,j}^* \cdot \nabla \log \frac{1}{|x - \underline{x}_{n,j}|} e_i \right),
\]

which readily implies that,

\[
\int_{\partial \overline{B}_\theta(\underline{x}_{n,j})} \frac{\nabla_i G_{n,j}^*}{|x - \underline{x}_{n,j}|} \, d\sigma = \int_{\partial B_\theta(\underline{x}_{n,j})} \frac{\nabla_i G_{n,j}^*}{|x - \underline{x}_{n,j}|} \, d\sigma.
\]

In view of Lemma 3.3 and Lemma 4.4, we also have,

\[
\zeta_n(x) - \int_M \zeta_n \, d\mu = \sum_{i=1}^m e^{-\frac{\lambda_n(1)}{m}} G_{n,j}(x) + o(e^{-\frac{\lambda_n(1)}{m}}) \quad \text{in} \quad C^1(B_{\rho}(\underline{x}_{n,j}) \setminus B_{\rho}(\underline{x}_{n,j})).
\]

By using (2.16)-(2.17), we find \( \rho_{n,j} = \rho_{n,j} + O(\lambda_{n,j}(1) e^{-\lambda_{n,j}(1)}) = 8\pi + O(\lambda_{n,j}(1) e^{-\lambda_{n,j}(1)}) \), which, together with (2.18), (2.19), and (3.3), implies that,

\[
D_{\rho}(\underline{x}_{n,j}) = D(\tilde{u}_{n} - \Phi_{n,j}) = D\left(\tilde{u}_{n} - \tilde{\rho}_{n,m} R(x, \underline{x}_{n,j}) - \tilde{\rho}_{n,m} \sum_{l \neq j} G(x, \underline{x}_{n,j})\right)
\]

\[
= D\left(\tilde{u}_{n} - \frac{\rho_{n}}{m} \sum_{l=1}^m G(x, \underline{x}_{n,j}) - \frac{\rho_{n}}{m} \log |x - \underline{x}_{n,j}|\right) = D\omega_n - 4 \frac{(x - \underline{x}_{n,j})}{|x - \underline{x}_{n,j}|^2} + o(e^{-\frac{\lambda_n(1)}{m}})
\]

\[
= -4 \frac{(x - \underline{x}_{n,j})}{|x - \underline{x}_{n,j}|^2} + o(e^{-\frac{\lambda_n(1)}{m}}) \quad \text{in} \quad C^1(B_{\rho}(\underline{x}_{n,j}) \setminus B_{\rho}(\underline{x}_{n,j})).
\]

Next, since \( D_1 D_2 (\log |z|) = \frac{6|z|^2 - 2z \bar{z} \partial |z|}{|z|^3} \), we see that,

\[
\int_{\partial B_{\rho}(\underline{x}_{n,j})} \frac{\nabla_i G_{n,j}^*}{|x - \underline{x}_{n,j}|} \, d\sigma = 2\pi D_1 \sum_{l=1}^m \partial \omega_n \left. \frac{\partial \omega_n}{\partial y_i} \right|_{y = \underline{x}_{n,j}} b_{ij} B_j + o(1),
\]

which, together with (4.54), (4.55), and (4.53), implies that, for any \( \theta \in (0, r) \),

\[
(\text{LHS}) \quad \text{of} \quad (4.48)
\]

\[
= -4 \int_{\partial B_{\rho}(\underline{x}_{n,j})} \sum_{l=1}^m e^{-\frac{\lambda_{n,j}(1)}{m}} \frac{\nabla_i G_{n,j}^*}{|x - \underline{x}_{n,j}|} \, d\sigma + o(e^{-\frac{\lambda_n(1)}{m}})
\]

\[
= -8\pi \left[ \sum_{l \neq j} e^{-\frac{\lambda_{n,j}(1)}{m}} D_1 G_{n,j}(\underline{x}_{n,j}) + e^{-\frac{\lambda_{n,j}(1)}{m}} D_1 \sum_{l=1}^m \partial \omega_n \left. \frac{\partial \omega_n}{\partial y_i} \right|_{y = \underline{x}_{n,j}} b_{ij} B_j \right] + o(e^{-\frac{\lambda_n(1)}{m}}) + o(1) e^{-\frac{\lambda_n(1)}{m}},
\]

which proves Lemma 4.7. \( \square \)

Finally, we have the following.

**Lemma 4.8.** \( b_{j,1} = b_{j,2} = 0, j = 1, \ldots, m. \)
Proof. Obviously Lemma 4.5, Lemma 4.6, and Lemma 4.7 together imply, for $i = 1, 2$,

\[
\mathcal{B}_i \sum_{h=1}^{2} (D_{m^2}^s(\varphi_{n_i}) + \log h_i)(\partial_{m_i}) b_j k \epsilon^{\Delta_i(1)}_{1/2} = -8\pi \sum_{j \neq i} e^{-\epsilon^{\Delta_i(1)}_{1/2} D_i G_{i,j}^s} e^{\epsilon^{\Delta_i(1)}_{1/2} D_i} \sum_{h=1}^{2} \partial x R(\varphi, y)\bigg|_{y = y = \mathcal{B}_i B_j} + o\left(e^{-\epsilon^{\Delta_i(1)}_{1/2}}\right)
\]

\[
= -8\pi \sum_{j \neq i} e^{-\epsilon^{\Delta_i(1)}_{1/2} D_i G_{i,j}^s} e^{\epsilon^{\Delta_i(1)}_{1/2} D_i} \sum_{h=1}^{2} \partial x R(\varphi, y)\bigg|_{y = y = \mathcal{B}_i B_j} + o\left(e^{-\epsilon^{\Delta_i(1)}_{1/2}}\right).
\]

(4.56)

Set $\bar{b} = (\bar{b}_{i,1} \bar{B}_1, \cdots, \bar{b}_{i,m} \bar{B}_m)$, where $\bar{b}_{i,j} = \lim_{n \to +\infty} (e^{-\epsilon^{\Delta_i(1)}_{1/2} b_{i,j}})$. Then, by using (2.14) and passing to the limit as $n \to +\infty$, we conclude from (4.56) that, $D^2 f_m(q_1, q_2, \cdots, q_m) \cdot \bar{b} = 0$, where $f_m(q_1, q_2, \cdots, q_m)$ is a suitably defined local expression of $f_m(x_1, \cdots, x_m)$. By using the fact that the rank of isothermal maps is always maximum, together with the non degeneracy assumption $\det(D^2 f_m(q)) \neq 0$, we conclude that $b_{1,1} = b_{i,j} = 0, j = 1, \cdots, m$. □

Proof of Theorem 1.1. Let $x_n^*$ be a maximum point of $\zeta_n$, then we have,

\[
|\zeta_n(x_n^*)| = 1.
\]

(4.57)

Therefore, in view of Lemma 3.4 and Lemma 4.4, we find that, $\lim_{n \to +\infty} x_n^* = q_j$, for some $j$. Moreover, by Lemma 4.4 and Lemma 4.8, it holds

\[
\lim_{n \to +\infty} \epsilon^{\Delta_i(1)}_{1/2} s_n = +\infty, \quad \text{where} \quad s_n = |x_n^* - \mathbf{x}_n^{(1)}|.
\]

(4.58)

Setting $\bar{\zeta}_n(\mathbf{x}) = \zeta_n(s_n \mathbf{x} + \mathbf{x}_n^{(1)})$, then (3.17) and (2.9) imply that $\bar{\zeta}_n$ satisfies,

\[
0 = \Delta \bar{\zeta}_n + \rho_n h_n^2 \varphi(\mathbf{x}_n^{(1)}) \epsilon^{\Delta_n(1)}_{1/2} \zeta_n(s_n \mathbf{x} + \mathbf{x}_n^{(1)}) \bar{\zeta}_n = \Delta \bar{\zeta}_n + \frac{\rho_n h_n^2 \varphi(\mathbf{x}_n^{(1)}) s_n^2 \epsilon^{\Delta_n(1)}_{1/2} (1 + O(s_n \mathbf{x})) + o(1))}{(1 + \rho_n h_n^2 \varphi(\mathbf{x}_n^{(1)}) s_n^2 \epsilon^{\Delta_n(1)}_{1/2})^2} \zeta_n.
\]

On the other side, by (4.57), we also have,

\[
|\bar{\zeta}_n(s_n \mathbf{x} + \mathbf{x}_n^{(1)})| = |\mathcal{A}_n(\mathbf{x}_n^*)| = 1.
\]

(4.59)

In view of (4.58) and $|\zeta_n| \leq 1$ we see that $\bar{\zeta}_n \to \bar{\zeta}_0$ on any compact subset of $\mathbb{R}^2 \backslash \{0\}$, where $\bar{\zeta}_0$ satisfies $\Delta \bar{\zeta}_0 = 0$ in $\mathbb{R}^2 \backslash \{0\}$. Since $|\zeta_n| \leq 1$, we have $\Delta \bar{\zeta}_0 = 0$ in $\mathbb{R}^2$, whence $\bar{\zeta}_0$ is a constant. At this point, since $\epsilon^{\Delta_n(1)}_{1/2} s_n = 1$ and in view of (4.59), we find that, $\bar{\zeta}_0 = 0$ or $\bar{\zeta}_0 = -1$. As a consequence we conclude that, $|\zeta_n(\mathbf{x})| \geq 1/2$ if $s_n \leq |\mathbf{x} - \mathbf{x}_n^{(1)}| \leq 2s_n$, which contradicts (3.34), (3.47), and (3.48) since $e^{-\epsilon^{\Delta_n(1)}_{1/2}} \ll s_n \lim_{n \to +\infty} s_n = 0$, and $b_0 = b_{j,0} = 0$. This fact concludes the proof of Theorem 1.1. □

5. The Dirichlet problem

Let $\Omega$ be an open and bounded two dimensional domain, $\Omega \subset \mathbb{R}^2$. As in [21], we say that $\Omega$ is regular if its boundary $\partial \Omega$ is of class $C^2$ but for a finite number of points $\{Q_1, \cdots, Q_{N_0}\} \subset \partial \Omega$ such that the following conditions holds at each $Q_j$.

(i) The inner angle $\theta_j$ of $\partial \Omega$ at $Q_j$ satisfies $0 < \theta_j < 2\pi$;

(ii) At each $Q_j$ there is an univalent conformal map from $B_{\delta}(Q_j) \cap \overline{\Omega}$ to the complex plane $\mathbb{C}$ such that $\partial \Omega \cap B_{\delta}(Q_j)$ is mapped to a $C^2$ curve.

Obviously any non degenerate polygon is regular according to this definition.
In this section we are concerned with the uniqueness result for the mean field equation with Dirichlet boundary conditions on regular domains,

$$\Delta u_n + \rho_n \frac{h(x)e^{nu_n(x)}}{\int_\Omega h e^{nu_n} dx} = 0 \quad \text{in} \quad \Omega, \quad u_n = 0 \quad \text{on} \quad \partial \Omega, \quad (5.1)$$

where $h(x) = h_0(x) e^{-4\pi \sum_{i=1}^{N} a_i G_{i}(x,p_j)} \geq 0$, $p_j$ are distinct points in $\Omega$, $a_j > -1$, $h_0 > 0$, $h_0 \in C^{2,\sigma}(\Omega)$, and $G_{i}$ is the Green function uniquely defined as follows, $-\Delta G\!\!\!_{i}(x, p) = \delta_p$ in $\Omega$, $G\!\!\!_{i}(x, p) = 0$ on $\partial \Omega$.

**Definition 5.1.** Let $u_n$ be a sequence of solutions of (5.1). We say that $u_n$ blows up at the points $q_j \notin \{p_1, \ldots, p_N\}$, $j = 1, \ldots, m$ if \( \int_{\lambda_j}^{\Omega} \frac{h(x) e^{nu_n(x)}}{\int_\Omega h e^{nu_n} dx} dx \to 8 \pi \sum_{j=1}^{m} \delta_{q_j}, \) weakly in the sense of measure in $\Omega$.

Let $R_{\Omega}(x, y) = \frac{1}{\pi n} \log |x - y| + G\!\!\!_{i}(x, y)$, be the regular part of $G\!\!\!_{i}(x, y)$. For $q = (q_1, \ldots, q_m) \in \Omega \times \cdots \times \Omega$, we denote by $G\!\!\!_{i}(x, q) = 8 \pi R\!\!\!_{\Omega}(x, q) + 8 \pi \sum_{j \neq q} G\!\!\!_{i}(x, q_j)$, and $l_{\Omega}(q) = \sum_{j = 1}^{m} [\Delta \log h(q_j)] h(q_j) e^{G\!\!\!_{i}(q)}$.

If $m \geq 2$, let us fix a constant $r_0 (0, \frac{1}{4})$, and a family of open sets $\Omega_j$ satisfying, $\Omega_j \cap \Omega = \emptyset$ if $l \neq j$, $\bigcup_{j = 1}^{m} \Omega_j = \Omega$, $B_{r_0}(q) \in \Omega_j$, $j = 1, \ldots, m$. Then let us define,

$$D_{\Omega}(q) = \lim_{r \to 0} \left[ \sum_{j = 1}^{m} \int_{\Omega_j \setminus B_{r}(q)} e^{G\!\!\!_{i}(q)} \left( \int_{\Omega \setminus B_{r}(q)} e^{\sum_{j = 1}^{m} \Delta \log h(q_j) G\!\!\!_{i}(q_j) + \log h(q_j)} \right) dx \right],$$

where $r_j = r \sqrt{8 \pi h(q_j) e^{G\!\!\!_{i}(q)}}$ and $\Omega_1 \equiv \Omega$ if $m = 1$. For $(x_1, \ldots, x_m) \in \Omega \times \cdots \Omega$, we also define,

$$f_{m,\Omega}(x_1, x_2, \ldots, x_m) = \sum_{j = 1}^{m} \left( \log h(x_j) \right) + 4 \pi R\!\!\!_{\Omega}(x_j, x_j) + 4 \pi \sum_{i \neq j} G\!\!\!_{i}(x_i, x_j).$$

Of course, even in this situation we first need to derive the following improvement of Theorem 6.2 in [24].

**Theorem 5.1.** Let $u_n$ be a sequence of solutions of (5.1) which blows up at the points $q_j \notin \{p_1, \ldots, p_N\}$, $j = 1, \ldots, m$, $\delta > 0$ be a fixed constant and $\lambda_{n,j} = \max_{B_{r}(q)} \left( u_n - \log \left( \int_{\Omega} h e^{nu_n} \right) \right)$ for $j = 1, \ldots, m$.

Then, for any $n$ large enough, the following estimate holds,

$$\rho_n - 8 \pi n = \frac{2 \ell_{\Omega}(q) e^{-\lambda_{n,1}}}{m h^2(q_1) e^{G\!\!\!_{i}(q_1)}} \left( \lambda_{n,1} + \log \rho_n h^2(q_1) e^{G\!\!\!_{i}(q_1)} \phi^2 - 2 \right) + \frac{8 e^{-\lambda_{n,1}}}{h^2(q_1) e^{G\!\!\!_{i}(q_1) \pi m}} \left( D_{\Omega}(q) + O(\phi^2) \right)$$

$$+ O(\lambda_{n,1}^2 e^{-\frac{1}{2} \lambda_{n,1}} + O(e^{-(\frac{1}{2} + \phi)\lambda_{n,1}}), \) \quad \text{where} \quad \sigma > 0 \quad \text{is defined by} \quad h_0 \in C^{2,\sigma}(M).$$

Then we have,

**Theorem 5.2.** Let $u_n^{(1)}$ and $u_n^{(2)}$ be two sequence of solutions of (5.1), with $\rho_n^{(1)} = \rho_n = \rho_n^{(2)}$ and blowing up at the points $q_j \notin \{p_1, \ldots, p_N\}$, $j = 1, \ldots, m$, where $q = (q_1, \ldots, q_m)$ is a critical point of $f_{m,\Omega}$ and det($D^2 f_{m,\Omega}(q_1)$) $\neq 0$. Assume that, either,

1. $\ell_{\Omega}(q) \neq 0$, or,
2. $\ell_{\Omega}(q) = 0$ and $D_{\Omega}(q) \neq 0$.

Then there exists $n_0 \geq 1$ such that $u_n^{(1)} = u_n^{(2)}$ for all $n \geq n_0$.

Proof of Theorems 5.1 and 5.2. The proof of Theorems 5.1 and 5.2 can be worked out by a step by step adaptation of the one of Theorems 1.3 and 1.1 with minor changes. Actually the arguments are somehow easier in this case, since we don’t need to pass to local isothermal coordinates around each blow up point. In particular it is readily seen that the subtle part of the estimates obtained in section 3 and 4 relies on the local estimates for blow up solutions of (1.1) listed in section 2. The corresponding estimates for the Dirichlet problem was already obtained in [24] and have the same form just with minor changes, as for example concerning the fact that here we have $q_j \equiv 0$ and $K \equiv 0$. Actually the estimates about the Dirichlet problem in [24] are worked out with $a_j = 0, j = 1, \ldots, N$ but since $\{q_1, \ldots, q_m\} \cap \{p_1, \ldots, p_N\} = \emptyset$, then it is straightforward to check that they still hold as they stand possibly with few changes about the regularity of solutions, see also Remark 1.4. We refer the reader to [24] for more details concerning this point. Actually, by our regularity assumption about $\delta \Omega$, it can be shown by a moving plane argument (see [21]) that
Therefore, by using (2.14), (2.16) and (6.2), we conclude that, 

\[
\rho_n = \rho_n \int_M h e^{\varphi_n} d\mu = \rho_n \left( \int_{M \setminus \bigcup_{j=1}^m U_j'(q_j)} + \int_{M \setminus \bigcup_{j=1}^m U_j'(q_j)} \right) h e^{\varphi_n} d\mu = \sum_{j=1}^m \rho_{n,j} + \int_{M \setminus \bigcup_{j=1}^m U_j'(q_j)} h e^{\varphi_n} d\mu. 
\]  

(6.1)

\textbf{Step 1.} In step 1 we provide and estimate about \( \int_{M \setminus \bigcup_{j=1}^m U_j'(q_j)} h e^{\varphi_n} d\mu \).

In view of (2.12), (2.18) and (2.19), we see that,

\[
\rho_n \int_{M \setminus \bigcup_{j=1}^m U_j'(q_j)} h e^{\varphi_n} d\mu = \rho_n \sum_{j=1}^m \int_{M \setminus U_j'(q_j)} h e^{\varphi_n} d\mu + \int_{M \setminus \bigcup_{j=1}^m U_j'(q_j)} \rho_n G(x, x_n) + f_M \varphi_n d\mu \\
= \rho_n \sum_{j=1}^m \int_{M \setminus U_j'(q_j)} h e^{\varphi_n}(1 + o(e^{-\frac{\lambda_n x_j}{2}})) d\mu + \int_{M \setminus \bigcup_{j=1}^m U_j'(q_j)} \rho_n G(x, x_n) - \lambda_n x_j - 2 \log \left| x \right| - G^*(x) (1 + o(e^{-\frac{\lambda_n x_j}{2}})) d\mu. 
\]  

(6.2)

Therefore, by using (2.14), (2.16) and (6.2), we conclude that,

\[
\rho_n \int_{M \setminus \bigcup_{j=1}^m U_j'(q_j)} h e^{\varphi_n} d\mu = \sum_{j=1}^m \frac{64 e^{-\lambda_n x_j}}{h(q_j) \rho_n} \int_{M \setminus U_j'(q_j)} e^{\Phi_j(x, q_j)} (1 + o(e^{-\frac{\lambda_n x_j}{2}})) d\mu \\
= \sum_{j=1}^m \frac{64 e^{-\lambda_n x_j}}{h(q_j) \rho_n} e^{G^*_j(q_j)} (1 + O(e^{-\frac{\lambda_n x_j}{2}})) \int_{M \setminus \bigcup_{j=1}^m U_j'(q_j)} e^{\Phi_j(x, q_j)} (1 + o(e^{-\frac{\lambda_n x_j}{2}})) d\mu \\
= \sum_{j=1}^m \frac{64 e^{-\lambda_n x_j}}{h(q_j) \rho_n} e^{G^*_j(q_j)} \int_{M \setminus \bigcup_{j=1}^m U_j'(q_j)} e^{\Phi_j(x, q_j)} d\mu + O(e^{-\frac{33 \lambda_n x_j}{4}}).
\]  

(6.3)

\textbf{Step 2.} In this step we provide an estimate about \( \rho_{n,j} \) (see (2.15)).

First of all let us set,

\[
h_j(x) = \frac{h(x) e^{\lambda_n x_j}}{e^{\Phi_j(x, q_j)}}. 
\]  

(6.4)

By using (2.8) and setting \( \tau_{n,j} = e^{\frac{\lambda_n x_j}{2}} \) and

\[
I_{n,j} = \int_{B_j(x)} \frac{\rho_n h_j(x, x_n_j) e^{\lambda_n x_j} (e^{G^*_j(x)} - G^*_j(x_n_j) + \log h(x) - \log h_j(x) + \eta_n(x) - 1)}{(1 + \frac{\rho_n h_j(x, x_n_j)}{8} e^{\lambda_n x_j} (x - x_{n,j,x_n_j})^2)} d\omega.
\]
we find that,

\[ \rho_{n,j} = \int_{\mathcal{B}} \rho_n h e^{\delta \theta_n} \, d\mu = \int_{\mathcal{B}} \rho_n h \frac{e^{\delta \lambda_{n,j}}}{\rho_n h} \rho_n h_{\mathcal{B}}(q_j) \frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{\frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{8} e^{\lambda_{n,j}} |x - \mathcal{X}_{n,j} |^2} \, dx \]

\[ = \int_{\mathcal{B}} \rho_{n,j} \frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{\frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{8} e^{\lambda_{n,j}} |x - \mathcal{X}_{n,j} |^2} \, dx + I_{n,j}^* \]

\[ = \int_{\mathcal{B}} \frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{\frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{8} e^{\lambda_{n,j}} |x - \mathcal{X}_{n,j} |^2} \, dx + I_{n,j}^* \]

(6.6)

On the other side, by (2.7) and (2.14), we see that,

\[ = 8\pi \int_{\mathcal{B}} \frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{\frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{8} e^{\lambda_{n,j}} |x - \mathcal{X}_{n,j} |^2} \, dx + I_{n,j}^* \]

(6.7)

where we used the identity, \( \int_{\mathcal{B}} \frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{\frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{8} e^{\lambda_{n,j}} |x - \mathcal{X}_{n,j} |^2} \, dx \)

\[ = \frac{8}{\frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{8} e^{\lambda_{n,j}} |x - \mathcal{X}_{n,j} |^2} \int_{\mathcal{B}} \frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{\frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{8} e^{\lambda_{n,j}} |x - \mathcal{X}_{n,j} |^2} \, dx \]

\[ = 8\pi \int_{\mathcal{B}} \frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{\frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{8} e^{\lambda_{n,j}} |x - \mathcal{X}_{n,j} |^2} \, dx + I_{n,j}^* \]

(6.8)

The estimate of the term \( I_{n,j}^* \) is more delicate. Toward this goal we have to work out a refined version of an argument first introduced in [24].

First of all, let us recall that (see (2.8)),

\[ \eta_{n,j}(x) = \frac{\rho_n h_{\mathcal{B}}(q_j)}{\rho_n h_{\mathcal{B}}(q_j)} \frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{8} e^{\lambda_{n,j}} (\mathcal{X}_{n,j} - \mathcal{X}_{n,j} |^2) \]

Then, in view of (2.5), for \( x \in B_2(q_j) \) we have,

\[ \Delta \eta_{n,j} = -\rho_n h (e^{2\theta_n} - 1) + \frac{\rho_n h_{\mathcal{B}}(q_j)}{\rho_n h_{\mathcal{B}}(q_j)} \frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{8} e^{\lambda_{n,j}} (\mathcal{X}_{n,j} - \mathcal{X}_{n,j} |^2) \]

(6.9)

which immediately implies that,

\[ I_{n,j}^* = -\int_{\partial B_{\mathcal{B}}(q_j)} \frac{\partial \eta_{n,j}}{\partial \nu} \, d\sigma - \int_{B_2(q_j)} (8\pi m - \rho_n) e^{2\psi} \, dx \]

(6.10)

Next we obtain an estimate about \( \int_{B_2(q_j)} \frac{\partial \eta_{n,j}}{\partial \nu} \, d\sigma \). Let us define,

\[ A_{n,j}(x) = \frac{\rho_n h_{\mathcal{B}}(q_j)}{\rho_n h_{\mathcal{B}}(q_j)} \frac{1 + \rho_n h_{\mathcal{B}}(q_j)}{8} e^{\lambda_{n,j}} (\mathcal{X}_{n,j} - \mathcal{X}_{n,j} |^2) \]


\[ B_{n,j}(\mathbf{x}) = \rho_n h_j(\mathbf{x}) e^{\lambda_n j(\mathbf{G}_j(\mathbf{x}) - \mathbf{G}_j(\mathbf{x}_n)) + \log h_j(\mathbf{x}) - \log h_j(\mathbf{x}_n) + \eta_{n,j}(\mathbf{x})} - 1 - \eta_{n,j}(\mathbf{x}) \], \quad \text{and}

\[ \psi_{n,j}(\mathbf{x}) = \frac{1 - \frac{\rho_n h_j(\mathbf{x}) e^{\lambda_n j}}{1 + \frac{\rho_n h_j(\mathbf{x}) e^{\lambda_n j}}{|\mathbf{x} - \mathbf{x}_{n,j,s}|^2}^2} \psi_{n,j} = 0. \]

which satisfies \( \Delta \psi_{n,j} + \frac{\rho_n h_j(\mathbf{x}) e^{\lambda_n j}}{(1 + \frac{\rho_n h_j(\mathbf{x}) e^{\lambda_n j}}{|\mathbf{x} - \mathbf{x}_{n,j,s}|^2}^2)^2} \psi_{n,j} = 0. \)

In view of (6.9) and integrating by parts, we find that, \( \int B_{n,j}(\mathbf{q}) \left( \psi_{n,j,\eta} - \eta_{n,j} \psi_{n,j} \right) d\sigma = \int B_{n,j}(\mathbf{q}) \left( \psi_{n,j} \Delta \eta_{n,j} - \eta_{n,j} \Delta \psi_{n,j} \right) d\mathbf{x} \)

\[ = \int B_{n,j}(\mathbf{q}) \left( - \psi_{n,j} A_{n,j} - \psi_{n,j}(8\pi m - \rho_n) e^{2\varphi_j} + \frac{\eta_{n,j} \eta_{n,j} \rho_n h_j(\mathbf{x}) e^{\lambda_n j}}{(1 + \frac{\rho_n h_j(\mathbf{x}) e^{\lambda_n j}}{|\mathbf{x} - \mathbf{x}_{n,j,s}|^2}^2)^2} \right) d\mathbf{x} \]

In the same time, for \( \mathbf{x} \in \partial B_d(\mathbf{q}) \), we have,

\[ \psi_{n,j}(\mathbf{x}) = -1 + \frac{2}{1 + \frac{\rho_n h_j(\mathbf{x}) e^{\lambda_n j}}{|\mathbf{x} - \mathbf{x}_{n,j,s}|^2}^2} = -1 + O(e^{-\lambda_{n,j}}), \quad \nabla \psi_{n,j} = O(e^{-\lambda_{n,j}}). \]

In view of (2.9) and [24, Lemma 4.1], we also have for \( \mathbf{x} \in \partial B_d(\mathbf{q}) \),

\[ |\eta_{n,j}| + |\nabla \eta_{n,j}| = O(\lambda_{n,j} e^{-\lambda_{n,j}}). \]

At this point, by using (6.12)-(6.14) and (2.17), we see that,

\[ - \int \frac{\partial \eta_{n,j}}{\partial \mathbf{v}} d\sigma = \int B_{n,j}(\mathbf{q}) \left( 1 - \frac{2}{1 + \frac{\rho_n h_j(\mathbf{x}) e^{\lambda_n j}}{|\mathbf{x} - \mathbf{x}_{n,j,s}|^2}^2} \right) (8\pi m - \rho_n) e^{2\varphi_j} - \psi_{n,j} B_{n,j}(\mathbf{x}) d\mathbf{x} + O(\lambda_{n,j} e^{-2\lambda_{n,j}}) \]

\[ = \int B_{n,j}(\mathbf{q}) (8\pi m - \rho_n) e^{2\varphi_j} d\mathbf{x} - \int B_{n,j}(\mathbf{q}) \psi_{n,j} B_{n,j}(\mathbf{x}) d\mathbf{x} + O((8\pi m - \rho_n \lambda_{n,j} e^{-\lambda_{n,j}}) + O(\lambda_{n,j} e^{-2\lambda_{n,j}}) \]

\[ = \int B_{n,j}(\mathbf{q}) (8\pi m - \rho_n) e^{2\varphi_j} d\mathbf{x} - \int B_{n,j}(\mathbf{q}) \psi_{n,j} B_{n,j}(\mathbf{x}) d\mathbf{x} + O(\lambda_{n,j} e^{-2\lambda_{n,j}}). \]

Next, let us observe that, since \( h \in C^{2,\alpha}(M) \) and in view of (2.7), (2.9) and (2.13), for \( \mathbf{x} \in B_d(\mathbf{q}) \) we have,

\[ e^{G_j(\mathbf{x})} - e^{G_j(\mathbf{x}_n)} + \log h_j(\mathbf{x}) - \log h_j(\mathbf{x}_n) + \eta_{n,j}(\mathbf{x}) - 1 - \eta_{n,j}(\mathbf{x}) \]

\[ \nabla \left( e^{G_j(\mathbf{x})} + \log h_j(\mathbf{x}) \right) \left|_{\mathbf{x} = \mathbf{x}_{n,j,s}} \cdot (\mathbf{x} - \mathbf{x}_{n,j,s}) + \frac{1}{2} \sum_{1 \leq k, l \leq 2} \nabla_{k=1, l=1}^2 \left( e^{G_j(\mathbf{x})} + \log h_j(\mathbf{x}) \right) \left|_{\mathbf{x} = \mathbf{x}_{n,j,s}} (\mathbf{x} - \mathbf{x}_{n,j,s}) \cdot (\mathbf{x} - \mathbf{x}_{n,j,s}) \right) \]

\[ + O(|\mathbf{x} - \mathbf{x}_{n,j,s}|^{2+\alpha}) + O(\lambda_{n,j} e^{-2\lambda_{n,j}}) + O(\lambda_{n,j}^2 e^{-\lambda_{n,j}}) \]

Clearly (2.7) and (2.14) imply that,

\[ |B_d(\mathbf{q}) \setminus B_d(\mathbf{x}_{n,j,s})| + |B_d(\mathbf{x}_{n,j,s}) \setminus B_d(\mathbf{q})| = O(|\mathbf{x}_{n,j,s} - \mathbf{q}|) = O(\lambda_{n,j} e^{-\lambda_{n,j}}). \]

At this point we use (6.15), (6.16), and (6.17), to conclude that,

\[ \int B_d(\mathbf{q}) \psi_{n,j} B_{n,j}(\mathbf{x}) d\mathbf{x} \]

\[ = \int B_d(\mathbf{x}_{n,j,s}) \psi_{n,j} \frac{\rho_n h_j(\mathbf{x}_{n,j,s}) e^{\lambda_n j}}{(1 + \frac{\rho_n h_j(\mathbf{x}) e^{\lambda_n j}}{|\mathbf{x} - \mathbf{x}_{n,j,s}|^2}^2)^2} \left( \nabla \left( e^{G_j(\mathbf{x})} + \log h_j(\mathbf{x}) \right) \right) \left|_{\mathbf{x} = \mathbf{x}_{n,j,s}} \cdot (\mathbf{x} - \mathbf{x}_{n,j,s}) \right) \]

\[ + \frac{1}{2} \sum_{1 \leq k, l \leq 2} \nabla_{k=1, l=1}^2 \left( e^{G_j(\mathbf{x})} + \log h_j(\mathbf{x}) \right) \left|_{\mathbf{x} = \mathbf{x}_{n,j,s}} (\mathbf{x} - \mathbf{x}_{n,j,s}) \cdot (\mathbf{x} - \mathbf{x}_{n,j,s}) \right) \]
where in the last equality we used (1.2), (2.4), (2.5) and \( \phi_j(x_j) = 0 \). Therefore, by using (6.10), (6.15) and (6.18), we conclude that,

\[
\begin{align*}
I_{n,j}^* &= -\frac{32\pi(\Delta \log h(x_{n,j}) - 2K(x_{n,j}) + 8\pi m)e^{-\lambda_{n,j}}}{\rho_n h(x_{n,j})} \int_0^1 \sqrt{\frac{\rho_n h(x_{n,j})}{8}} \frac{\lambda_{n,j}}{r(1 - r^2)} \frac{r^3(1 - r^2)}{(1 + r^2)^3} \rho_n h(x_{n,j}) \delta^2 - 2) \\
&+ O(e^{-\lambda_{n,j}\delta^2}) + O(\lambda_{n,j}^2 e^{-\frac{2}{3}\lambda_{n,j}}) + O(e^{-1+\frac{2}{3}\lambda_{n,j}}),
\end{align*}
\]

where we used the identity for large \( R \gg 1 \),

\[
\int_0^R \frac{r^3(1 - r^2)}{(1 + r^2)^3} \, dr = \frac{2R^4 + R^2 - 1}{2(R^2 + 1)^2} = -\log(R^2 + 1) = -\log R + O(R^{-2}).
\]

By using (2.11) and (2.14) we can write this estimate in the following form,

\[
\begin{align*}
I_{n,j}^* &= \left\{ \frac{16\pi h(q_j)e^{G_j(q_j)}(\Delta \log h(q_j) - 2K(q_j) + 8\pi m)e^{-\lambda_{n,1}}}{\rho_n h^2(q_j)e^{G_j(q_j)}} \left( \lambda_{n,1} + \log \frac{\rho_n h^2(q_j)e^{G_j(q_j)}}{8h(q_j)e^{G_j(q_j)}} \delta^2 - 2 \right) \right\} \\
&+ O(e^{-\lambda_{n,1}\delta^2}) + O(\lambda_{n,1}^2 e^{-\frac{2}{3}\lambda_{n,1}}) + O(e^{-1+\frac{2}{3}\lambda_{n,1}}),
\end{align*}
\]

and eventually use it with (6.6) and (6.8) to obtain that,

\[
\begin{align*}
\rho_{n,j} &= 8\pi - \frac{64\pi h(q_j)e^{G_j(q_j)}e^{-\lambda_{n,1}}}{\rho_n h^2(q_j)e^{G_j(q_j)}} + O(e^{-\lambda_{n,1}\delta^2}) + O(\lambda_{n,1}^2 e^{-\frac{2}{3}\lambda_{n,1}}) + O(e^{-1+\frac{2}{3}\lambda_{n,1}}) \\
&+ \left\{ \frac{16\pi h(q_j)e^{G_j(q_j)}(\Delta \log h(q_j) - 2K(q_j) + 8\pi m)e^{-\lambda_{n,1}}}{\rho_n h^2(q_j)e^{G_j(q_j)}} \left( \lambda_{n,1} + \log \frac{\rho_n h^2(q_j)e^{G_j(q_j)}}{8h(q_j)e^{G_j(q_j)}} \delta^2 - 2 \right) \right\}.
\end{align*}
\]
Step 3. In view of (6.1), (6.4) and (6.22), we find that,

\[
\rho_n = \sum_{j=1}^{m} \rho_{n,j} + \rho_n \int_{M \setminus \cup_{j=1}^{m} U_j(q_j)} h e^{\theta_n} d\mu
\]

\[
= 8\pi m + \sum_{j=1}^{m} \left\{ \frac{16\pi n h(q_j) e^{G_j}(\Delta \log h(q_j) - 2K(q_j) + 8\pi m) e^{-\lambda_{n,1}}}{\rho_n h^2(q_j) e^{G_j}(q_j)} \left( \lambda_{n,1} + \log \frac{\rho_n h^2(q_j) e^{G_j}(q_j) \delta^2}{8h(q_j) e^{G_j}(q_j)} - 2 \right) \right\}
\]

\[
+ \sum_{j=1}^{m} 64 e^{-\lambda_{n,1}} h(q_j) e^{G_j}(q_j) \int_{U_j(q_j)} \Phi_{(x,q)} d\mu - \sum_{j=1}^{m} \frac{64\pi n h(q_j) e^{G_j}(q_j) e^{-\lambda_{n,1}}}{\rho_n h^2(q_j) e^{G_j}(q_j) \delta^2}
\]

\[
+ O(e^{-\lambda_{n,1}} \delta^\nu) + O(\lambda_{n,1}^2 e^{-1/2\lambda_{n,1}}) + O(e^{-(1+\frac{2}{3})\lambda_{n,1}}),
\]

where we used (2.10).

By using (6.23), (2.17) and the definition of \( \ell(q) \), we see that,

\[
\rho_n - 8\pi m = \frac{2\ell(q) e^{-\lambda_{n,1}}}{mh^2(q_1) e^{G_j(q_1)}} \left( \lambda_{n,1} + \log \frac{\rho_n h^2(q_1) e^{G_j(q_1)} \delta^2}{2} \right)
\]

\[
- \sum_{j=1}^{m} \frac{2h(q_j) e^{G_j(q_j)} (\Delta \log h(q_j) - 2K(q_j) + 8\pi m) e^{-\lambda_{n,1}}}{mh^2(q_j) e^{G_j(q_j)}} \left( \log 8h(q_j) e^{G_j(q_j)} \right)
\]

\[
+ \sum_{j=1}^{m} \frac{8e^{-\lambda_{n,1}} h(q_j) e^{G_j(q_j)}}{h^2(q_j) e^{G_j(q_j)}} \left( \int_{M \setminus \cup_{j=1}^{m} U_j(q_j)} \Phi_{(x,q)} d\mu - \frac{\pi}{\delta^2} + O(\delta^\nu) \right) + O(\lambda_{n,1}^2 e^{-1/2\lambda_{n,1}}) + O(e^{-(1+\frac{2}{3})\lambda_{n,1}}).
\]

For small \( r > 0 \), let \( r_j = r \sqrt{8h(q_j) e^{G_j(q_j)}} \) and observe that,

\[
\sum_{j=1}^{m} h(q_j) e^{G_j(q_j)} \left( \int_{M \setminus \cup_{j=1}^{m} U_j(q_j)} \Phi_{(x,q)} d\mu \right)
\]

\[
= \sum_{j=1}^{m} h(q_j) e^{G_j(q_j)} \left( \int_{M \setminus \cup_{j=1}^{m} U_j(q_j)} \Phi_{(x,q)} d\mu - \int_{B_j(q_j) \setminus B_{r_j}(q_j)} \frac{G_j(x) - G_j(q_j) + \log h(x) - \log h(q_j) + 2\varphi_1(x)}{|x - q_j|^4} d\mu \right).
\]

Since \( \nabla f_m(q) = 0 \) and \( \varphi_j(x_{nj}) = \nabla \varphi_j(x_{nj}) = 0 \), for \( x \in B_j(q_j) \setminus B_{r_j}(q_j) \), we see from (2.14) that,

\[
G_j(x) - G_j(q_j) + \log h(x) - \log h(q_j) + 2\varphi_1(x)
\]

\[
= 2\varphi_1(q_j) + \left( \nabla \varphi_1 f_m(q) + \nabla^2 \varphi_1(q_j) \cdot (x - q_j) \right) + \frac{1}{2} \sum_{1 \leq i, k \leq 2} (\nabla^2 \varphi_1 f_m(q) + \nabla^2 \varphi_1(q_j)) (|x_k - q_k| - |x_i - q_i|) (|x_i - q_i| - |x_k - q_k|)
\]

\[
+ O(|x - q_j|^3) = \frac{1}{2} \sum_{1 \leq i, k \leq 2} (\nabla^2 \varphi_1 f_m(q) + \nabla^2 \varphi_1(q_j)) (|x_k - q_k| - |x_i - q_i|) (|x_i - q_i| - |x_k - q_k|) + O(|x - q_j|^3) + O(\lambda_{n,1} e^{-\lambda_{n,1}}),
\]

which implies that,

\[
\int_{B_j(q_j) \setminus B_{r_j}(q_j)} \frac{e^{G_j(x) - G_j(q_j) + \log h(x) - \log h(q_j) + 2\varphi_1(x)}}{|x - q_j|^4} d\mu
\]

\[
= \int_{B_j(q_j) \setminus B_{r_j}(q_j)} \frac{1 + \frac{(\Delta \varphi_1 f_m(q) + \Delta \varphi_1(q_j))}{4} |x - q_j|^2 + O(|x - q_j|^3) + O(\lambda_{n,1} e^{-\lambda_{n,1}})}{|x - q_j|^4} d\mu
\]

\[
= \int_{B_j(q_j) \setminus B_{r_j}(q_j)} \frac{1 + \frac{(\Delta \log q_j) + 8\pi m - 2K(q_j)}{4} |x - q_j|^2 + O(\delta) + O(\lambda_{n,1} e^{-\lambda_{n,1}})}{|x - q_j|^4} d\mu
\]

\[
= \frac{\pi}{\delta^2} + \frac{\pi}{\delta^2} (\Delta \log q_j) + 8\pi m - 2K(q_j)
\]

\[
(\log \delta - \log r_j) + O(\delta) + O(\lambda_{n,1} e^{-\lambda_{n,1}}),
\]

where we used (2.4), (2.5).
In view of (6.24), (6.25), (6.27), we obtain
\[
\rho_n - 8\pi m n \left( \frac{2f(q)e^{-\lambda_{n,1}}}{mh^2(q_1)e^{G_1(q_1)}} \left( \lambda_{n,1} + \log \rho_n h^2(q_1)e^{G_1(q_1)} r^2 - 2 \right) + \frac{8e^{-\lambda_{n,1}}}{h^2(q_1)e^{G_1(q_1)}} \pi m \left( \int_{M_j \setminus S_j^*} e^{\Phi_j(x,a)} d\mu - \frac{\pi}{r_j} + O(\delta^r) \right) + O(\lambda_{n,1}^2 e^{-\frac{3}{2} \lambda_{n,1}}) + O(e^{-2 \lambda_{n,1}}) \right) \right) \quad \forall r > 0,
\]
where we used the explicit form of $r_j$ to cancel out the second line of (6.28). Therefore, we conclude that
\[
\rho_n - 8\pi m = \frac{2f(q)e^{-\lambda_{n,1}}}{mh^2(q_1)e^{G_1(q_1)}} \left( \lambda_{n,1} + \log \rho_n h^2(q_1)e^{G_1(q_1)} r^2 - 2 \right) + \frac{8e^{-\lambda_{n,1}}}{h^2(q_1)e^{G_1(q_1)}} \pi m \left( D(q) + O(\delta^r) \right)
\]
which is just the estimate (1.8).

\[\square\]

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One-dimensional symmetry of periodic minimizers for a mean field equation
Uniqueness of solutions for a mean field equation on torus


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