

# Rigidity effects for antiferromagnetic thin films: a prototypical example

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## Abstract

We consider two-dimensional discrete thin films obtained from  $N$  layers of a triangular lattice, governed by an antiferromagnetic energy. By a dimension-reduction analysis we show that, in contrast with the “total frustration” of the triangular lattice, the overall behaviour of the thin film is described by a limit interfacial energy on functions taking  $2^N$  distinct parameters. In a sense, then the total frustration is recovered as  $N$  tends to infinity.

## 1 Introduction

We consider lattice energies defined on “spin functions” (i.e., functions  $u = \{u_i\}$  taking the only values  $-1$  or  $1$ ), of the form

$$-\sum_{i,j} c_{ij} u_i u_j, \quad (1)$$

where  $i, j$  are nodes of a (connected) portion of a lattice  $\mathcal{L}$  in  $\mathbb{R}^d$  and  $c_{ij}$  are interactions coefficients. In the case that  $c_{ij} \geq 0$  the system is called *ferromagnetic* and its ground states are the two constant states  $\pm 1$ . The overall behavior of the system when a large number of nodes are taken into account can then be described by a scaling procedure, by considering a scaling parameter  $\varepsilon > 0$ , a fixed parameter set  $\Omega$ , and the scaled energies (obtained from the previous ones by scaling and adding constants)

$$\sum_{i,j} \varepsilon^{d-1} c_{ij} (u_i - u_j)^2 \quad (2)$$

defined for  $i, j$  belonging to  $\mathcal{L} \cap \frac{1}{\varepsilon}\Omega$ . A discrete-to-continuum process allows to define an approximating continuum energy of interfacial type on  $\Omega$

$$\int_{\Omega \cap \partial\{u=1\}} \varphi(x, \nu_u) d\mathcal{H}^{d-1}(x),$$

where  $u : \Omega \rightarrow \{-1, 1\}$  is a macroscopic parameter (the *magnetization*) defined as the limit of piecewise-constant functions  $u^\varepsilon$  defined from spin functions  $\{u_i^\varepsilon\}$  as

$$u^\varepsilon(x) = u_{\lfloor x/\varepsilon \rfloor} \quad x \in \Omega$$

(up to some corrections close to  $\partial\Omega$ ). The *surface tension*  $\varphi$  depends on the orientation  $\nu_u$  of the interface between the two zones where  $u = 1$  or  $u = -1$ . In many cases it is also homogeneous, and is characterized by the *Wulff shape*; i.e., the characteristic shape of minimizers with given measure.

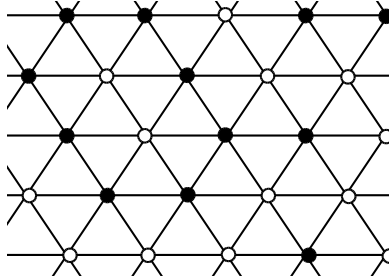


Figure 1: a ‘disordered’ minimizer in a portion of the triangular lattice (black and white dots represent  $-1$  and  $+1$  values, respectively)

If the system is *antiferromagnetic*; i.e.,  $c_{ij} \leq 0$ , or a mixture of ferromagnetic and antiferromagnetic interactions, in general ground states are *frustrated*. This means that the energies in (1) cannot be minimized for each single interaction (as pictured in Fig. 1), which, in the case of antiferromagnetic coefficients would imply that  $u_i = -u_j$ . The simplest case of frustration is when  $\mathcal{L}$  is a *triangular lattice* and we take  $c_{ij}$  different from zero only for *nearest-neighbours*, for which, for example  $c_{ij} = -1$ . In this case, ground states present no regularity and can arbitrarily mix the values  $u_i = 1$  and  $u_i = -1$ . For antiferromagnetic-ferromagnetic mixtures this is “generically” not the case in the square lattice if we have a small percentage of antiferromagnetic interactions [4]. In [6] examples are shown also of mixtures of nearest-neighbour ferromagnetic and antiferromagnetic interactions in the square lattice with a similar “total frustration”. This behaviour is not present in every system with antiferromagnetic interactions. Indeed, long-range antiferromagnetic interactions, also in the square lattice, may present a finite collection of striped or checkerboard-type ground states (see e.g. [8, 9]). Using the analysis of ground states, sometimes those systems can be described in a discrete-to-continuum fashion by a surface energy defined on partitions of the underlying reference set  $\Omega$  indexed by the different textures and modulated phases [6]. For a review on the subject we refer to [2].

In this paper we consider an example of thin films for spin energies. A discrete thin film is obtained by limiting the interactions to a  $N\varepsilon$ -neighbourhood of a  $d - 1$ -dimensional set  $\omega$  (as in [3, 5]). We then scale the energies accordingly, as

$$\sum_{i,j} \varepsilon^{d-2} c_{ij} (u_i - u_j)^2$$

(see [5]). In the simplest case of a “coordinate thin film”, when  $\omega$  is contained in  $\mathbb{R}^{d-1} \times \{0\}$  then the sum above may be considered as performed for  $i, j$  belonging to  $\mathcal{L} \cap (\frac{1}{\varepsilon}\omega \times [0, N])$ . The limit behaviour of these energies can be then described by a *dimensionally-reduced* energy of the form

$$\int_{\partial\{u=1\}} \varphi(x, \nu_u) d\mathcal{H}^{d-2}(x),$$

where the limit magnetization is interpreted as a function  $u : \omega \rightarrow \{-1, 1\}$  and the form of  $\varphi$  takes into account also optimization of the interactions in the “vertical” direction; i.e., in the  $d$ -coordinate (for an analog thin-film theory for bulk surface energies see [7]).

In our case, we consider  $d = 2$  and  $\mathcal{L} = \mathbb{T}$  the regular triangular lattice; i.e., the Bravais lattice generated by  $(1, 0)$  and  $(1/2, \sqrt{3}/2)$ . The nearest neighbours in  $\mathbb{T}$  are points at distance 1; i.e., differing by  $\pm(1, 0)$ ,  $\pm(1/2, \sqrt{3}/2)$ , or  $\pm(-1/2, \sqrt{3}/2)$ . For each  $N \in \mathbb{N}$ ,  $N > 0$ , we then consider the related discrete thin film composed of  $N$  layers with underlying set an interval  $I$ ; namely,

$$\Omega_{N,\varepsilon} = I \times [0, (N-1)\varepsilon\sqrt{3}/2].$$

If we consider nearest neighbour uniform anti-ferromagnetic interactions, the thin-film energy then simply reads

$$E_\varepsilon^N(u) = - \sum_{i,j} (u_i - u_j)^2, \quad (3)$$

where the sum  $i, j$  runs on nearest-neighbours in  $(\frac{1}{\varepsilon}I) \times [0, (N-1)\varepsilon\sqrt{3}/2]$ .

The simplest case is  $N = 1$ , when the underlying set  $\Omega_{0,\varepsilon}$  reduces to  $I \times \{0\}$ , which can be directly identified with  $I$ . The energy  $E_\varepsilon^1$  can then be seen as a “bulk” spin energy with underlying lattice  $\mathbb{Z}$ , and can be reduced to a ferromagnetic energy by adding the constant 4 in each interaction in order to make the sum positive; i.e., considering

$$E_\varepsilon^1(u) = - \sum_i ((u_i - u_{i-1})^2 - 4), \quad (4)$$

and by the change of variables  $v_i = (-1)^i u_i$  (see also [1]). Then the thin-film limit is defined on piecewise-constant functions  $v$  on  $I$  with values in  $\{-1, 1\}$  and is given by

$$F^1(v) = 4 \#(S(v)),$$

where  $S(v)$  is the discontinuity set of  $v$ . Note that the constant  $v = 1$  corresponds to taking  $u_i = (-1)^i$ , while the constant  $v = -1$  corresponds to  $u_i = (-1)^{i+1}$ , so that the two ground states in terms of  $v$  correspond to two variants of oscillating  $u$  (modulated phases).

As compared to the “total frustration” of the triangular lattice the case  $N = 1$  already hints that a dimensional-reduction process applied to this example of antiferromagnetic interactions may give a continuum limit taking into account only a finite number of parameters. However, this case seems oversimplified since no trace of the triangular geometry of the original lattice remains. In the rest of the paper we analyze the case  $N > 1$  to show how an  $N$ -dependent finite-parameter description holds.

## 2 Analysis of the thin-film limit

We first consider more in detail the case  $N = 2$ , which is pictured in Fig. 2. In the

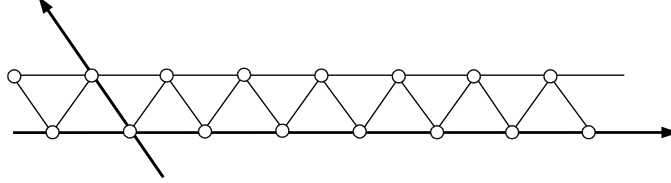


Figure 2: two-layer thin film with reference axes

notation above, the underlying thin film is

$$\Omega_{2,\varepsilon} = I \times [0, \varepsilon\sqrt{3}/2].$$

In order to simplify the notation we also introduce a non-orthogonal coordinate system as in figure, so that the points in the thin film are parameterized by

$$Z_2 := \{(n, m) : n \in \mathbb{Z}, m \in \{0, 1\}\}.$$

We can write the energy as a sum of terms of the form

$$-\left((u_{(n+1,0)} - u_{(n,0)})^2 + (u_{(n,1)} - u_{(n,0)})^2 + (u_{(n+1,1)} - u_{(n,1)})^2 + (u_{(n+1,1)} - u_{(n,0)})^2\right).$$

We may consider the case when the underlying interval is simply  $\mathbb{R}$ . In this case, we may sum on  $\mathbb{Z}$  after adding a constant and regrouping the interactions as follows to avoid  $+\infty - \infty$  indeterminate forms:

$$\begin{aligned} E_\varepsilon^2(u) &= - \sum_{n \in \mathbb{Z}} \left( (u_{(n+1,0)} - u_{(n,0)})^2 \right. \\ &\quad \left. + \frac{1}{2} (u_{(n+1,1)} - u_{(n,0)})^2 + \frac{1}{2} (u_{(n+1,1)} - u_{(n+1,0)})^2 - 6 \right) \\ &\quad - \sum_{n \in \mathbb{Z}} \left( (u_{(n+1,1)} - u_{(n,1)})^2 + \frac{1}{2} (u_{(n+1,1)} - u_{(n,0)})^2 + \frac{1}{2} (u_{(n,1)} - u_{(n,0)})^2 - 6 \right). \end{aligned} \quad (5)$$

In this way the energy is split in its contributions in each triangle. The first sum takes into account triangles with a side in the lower layer  $m = 1$  and the second sum takes into account triangles with a side in the upper layer  $m = 0$ . The factor  $1/2$  takes into account that non-horizontal sides belong to two neighbouring triangles. Note that not having alternate states on the horizontal (boundary) sides is more “costly” than on the others.

Note that the term

$$-\left((u_{(n+1,0)} - u_{(n,0)})^2 + \frac{1}{2}(u_{(n+1,1)} - u_{(n,0)})^2 + \frac{1}{2}(u_{(n+1,1)} - u_{(n+1,0)})^2 - 6\right)$$

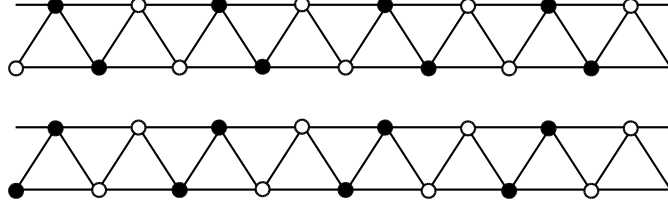


Figure 3: a picture of ground states, with black/white circles indicating  $-1/1$  values

is always non-negative, and it is zero only if

$$u_{(n+1,0)} \neq u_{(n,0)}.$$

In the same way, each term in the second sum is minimized only when  $u_{(n+1,1)} \neq u_{(n,1)}$ . This observation implies that ground states, with zero energy are all  $u$  that satisfy

$$\begin{aligned} u_{(n,0)} = (-1)^n \text{ for all } n & \quad \text{or} \quad u_{(n,0)} = (-1)^{n+1} \text{ for all } n, \\ u_{(n,1)} = (-1)^n \text{ for all } n & \quad \text{or} \quad u_{(n,1)} = (-1)^{n+1} \text{ for all } n; \end{aligned}$$

i.e., with alternating values of  $u$  on the two horizontal layers. Hence, we have four ground states determined by their values at  $n = 0$

$$(u_{(0,0)}, u_{(0,1)}) \in \{-1, 1\}^2 =: X_2.$$

For  $x \in X_2$  we define

$$v^x : Z_2 \rightarrow \{-1, 1\}$$

as the ground state with  $(v^x(0,0), v^x(0,1)) = x$ .

Note that the two ground states determined by  $\pm(1, 1)$  (or by  $\pm(-1, 1)$ , correspondingly), differ by a horizontal translation by  $(1, 0)$ , while those determined by  $(-1, 1)$  and  $(1, -1)$  are obtained by a reflection around a vertical line from  $(1, 1)$  and  $(-1, -1)$  (see Fig. 3).

Note that if  $u$  is a function with finite energy then there are a finite number of indices  $n$  such that  $u$  does not minimize the terms in the sum in (5). This implies that a sequence of functions with equibounded energy is precompact for the following notion of convergence.

The *discrete-to-continuum convergence* of a family of functions  $u^\varepsilon : Z_2 \rightarrow \{-1, 1\}$  to a function  $v : \mathbb{R} \rightarrow X_2$  with a finite number of points of discontinuity  $S(v) = \{t_1, \dots, t_K\}$  is defined by the requirement that, denoted by  $x_j$  ( $j = 0, \dots, K$ ) the constant value of  $v$  on  $(t_j, t_{j+1})$  (where  $t_0 = -\infty$  and  $t_{K+1} = +\infty$ ), for every  $\delta > 0$  if  $\varepsilon$  is small enough then  $u_n^\varepsilon$  is equal to the ground state  $v^{x_j}$  respectively for

$$\begin{aligned} -\frac{1}{\varepsilon\delta} < n < \frac{1}{\varepsilon}(t_1 - \delta) & \quad \text{if } j = 0 \\ \frac{1}{\varepsilon}(t_j + \delta) < n < \frac{1}{\varepsilon}(t_{j+1} - \delta) & \quad \text{if } j \in \{1, \dots, K-1\} \\ \frac{1}{\varepsilon}(t_K + \delta) < n < \frac{1}{\varepsilon\delta} & \quad \text{if } j = K. \end{aligned}$$

This convergence may be equally stated as the convergence of the auxiliary functions  $\tilde{u}_\varepsilon : \mathbb{R} \rightarrow V \cup \{(0, 0)\}$  defined by

$$\tilde{u}_\varepsilon(t) = \begin{cases} x & \text{if } u_j^\varepsilon = v^x \text{ on } \left\{ \left[ \frac{t}{\varepsilon} \right], \left[ \frac{t}{\varepsilon} \right] + 1 \right\} \times \{0, 1\} \\ (0, 0) & \text{otherwise} \end{cases}$$

in  $L^1_{\text{loc}}(\mathbb{R})$ . In the definition of the function  $\tilde{u}_\varepsilon$  we scale the domain by  $\varepsilon$  and identify the value on two consecutive triangles (i.e., on the vertices of a unit square in the parameterization on  $Z_2$ ) with the common parameter  $x \in X_2$  when the corresponding  $u^\varepsilon$  coincides with  $v^x$  on those triangles. This parameter  $x \in X_2$  is well defined except for a finite number of  $\lfloor \frac{t}{\varepsilon} \rfloor$ , so we may arbitrarily extend the definition by  $(0, 0)$  on the complement.

We may describe the limit behaviour of the energies  $E_\varepsilon^2$  as defined in (5) by exhibiting a  $\Gamma$ -limit with respect to the convergence above, of the form

$$F^2(v) = \sum_{t \in S(v)} \varphi(v(t^-), v(t^+)), \quad (6)$$

where  $t^\pm \in X_2$  are the left-hand and right-hand limit values of  $v$  at  $t$ . The energy function  $\varphi(x, x')$  is obtained by computing the optimal transition between two states  $v^x$  and  $v^{x'}$ .

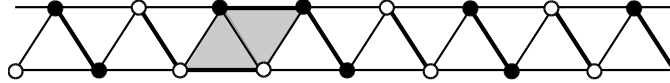


Figure 4: an optimal transition between  $(1, 1)$  and  $(-1, -1)$

The picture in Fig. 4 describes an optimal transition when  $x = (1, 1)$  and  $x' = (-1, -1)$ , or the converse. We may consider  $v(t) = x$  for  $t > 0$  and  $v(t) = x'$  for  $t < 0$  and  $u^\varepsilon \rightarrow v$ . In this case there must be some index  $n$  with a non-optimal interaction  $u^\varepsilon(n, 0) = u^\varepsilon(n + 1, 0)$  and some index  $n'$  with  $u^\varepsilon(n', 1) = u^\varepsilon(n' + 1, 1)$ . In the picture such a  $u^\varepsilon$  is shown, optimizing all other interactions. The thick lines correspond to frustrated interactions. Computing the energy of such  $u^\varepsilon$ , which amounts just to the contributions of the two triangles highlighted in the picture, we obtain the value  $\varphi((1, 1), (-1, -1)) = 4$ . The same argument and a vertical symmetry argument shows that  $\varphi((1, -1), (-1, 1))$  has the same value.

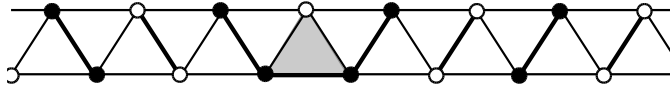


Figure 5: an optimal transitions between  $(1, 1)$  and  $(-1, 1)$

Similarly, in order to describes the optimal transition when  $x = (1, 1)$  and  $x' = (-1, 1)$  or the converse, we may remark that optimal  $u^\varepsilon$  must have  $u^\varepsilon(n, 0) = u^\varepsilon(n+1, 0)$

for some index  $n$ . In Fig. 5 we picture an optimal such  $u^\varepsilon$ , for which all interactions are optimal except one with  $u^\varepsilon(n, 0) = u^\varepsilon(n + 1, 0)$ . The corresponding computation gives  $\varphi((1, 1), (-1, 1)) = 2$ .

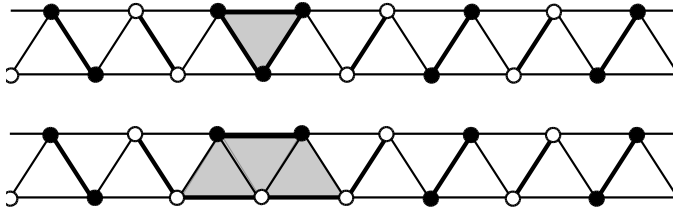


Figure 6: optimal transitions between  $(1, 1)$  and  $(1, -1)$

Finally, in the case  $x = (1, 1)$  and  $x' = (1, -1)$ , or the converse, we again note that optimal  $u^\varepsilon$  must have  $u^\varepsilon(n, 1) = u^\varepsilon(n + 1, 1)$  for some index  $n$ , but there are two equivalent optimal arrangements, whether  $u^\varepsilon(n, 0) = u^\varepsilon(n, 1)$  or  $u^\varepsilon(n, 0) \neq u^\varepsilon(n, 1)$ . These two cases are pictured in Fig. 6 and both give  $\varphi((1, 1), (-1, 1)) = 6$ . Note that

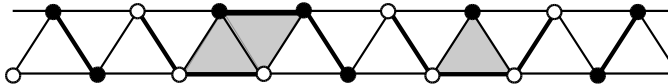


Figure 7: split optimal transitions between  $(1, 1)$  and  $(1, -1)$

another optimal arrangement is obtained e.g. by combining the transitions between  $(1, 1)$  and  $(-1, 1)$  and between  $(-1, 1)$  and  $(1, -1)$ . This corresponds to the lower case in Fig. 6 splitting the three non-optimal triangles into a pair with a common side and an isolated one (see Fig. 7). Analogously, the two joined triangles can be similarly split.

The  $\Gamma$ -limit result is finally obtained by superposing these constructions to obtain a recovery sequence for an arbitrary  $v$ .

Using a notation analogous to the one introduced above, we can now generalize this computation to a larger number of layers. For  $N > 2$  we will not compute the energy function  $\varphi$  as above, but focus on its definition and in particular on its domain.

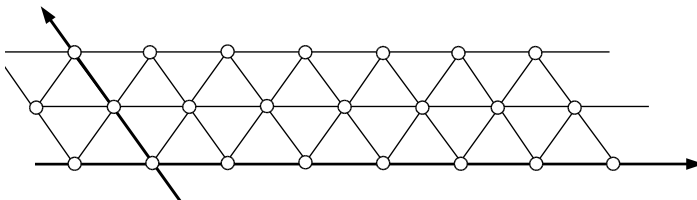


Figure 8: three-layer thin film with reference axes

We first consider the case  $N = 3$ , whose underlying thin film is pictured in Fig. 8 together with the reference axes. The corresponding reference set is

$$Z_3 := \{(n, m) : n \in \mathbb{Z}, m \in \{0, 1, 2\}\}.$$

We can again consider the antiferromagnetic energy as a sum of the contribution of each triangle. The difference with the case  $N = 1$  is that, while the energy of a triangle with a horizontal side on the top or bottom layer is as before, triangles with horizontal sides in the interior give an energy with a weight  $1/2$  for all sides. For example, we have the contribution

$$-\frac{1}{2} \left( (u_{(n+1,1)} - u_{(n,1)})^2 + (u_{(n,1)} - u_{(n,0)})^2 + (u_{(n+1,1)} - u_{(n,0)})^2 - 4 \right)$$

for triangles in the lower row of triangles and a side in the middle layer of points.

For every  $x = (x_0, x_1, x_2) \in \{\pm 1\}^3$  we denote the ground state given by

$$u^x(n, m) = x_m (-1)^n \text{ for all } n \in \mathbb{Z} \text{ and } m \in \{0, 1, 2\}.$$

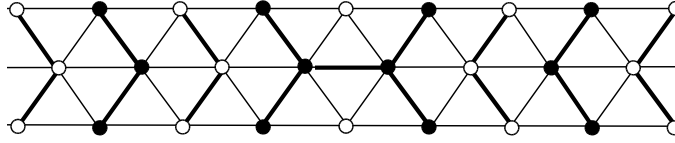


Figure 9: a non-periodic minimizer

Differently than the case  $N = 2$ , we note that a function  $u$  with zero energy is not necessarily one of those eight ground states, but may otherwise coincide with two of those for  $n \geq M$  and for  $n < -M$ , respectively, for some  $M \in \mathbb{N}$ . Such a case is pictured in Fig. 9. Note that all functions with zero energy must have alternating values for  $m = 0$  and  $m = 2$ . This implies that if, for example,  $u(n, 2) = u(n, 1)$  for some  $n$  then the value of  $u$  is determined for  $(n', 1)$  and  $(n', 2)$  for all  $n' \leq n$  as an alternating state. Similarly, if  $u(n-1, 0) = u(n, 1)$ . A symmetric argument also applies for minimizers which are determined for  $n' \geq n$ . This observation eventually implies that the one in Fig. 9 is the only non-periodic minimizer, up to translations.

As a consequence, we may define a convergence  $u^\varepsilon \rightarrow v$  analog to the case  $N = 2$ , where now  $v : \mathbb{R} \rightarrow X_3 := \{-1, 1\}^3$ . We may describe the  $\Gamma$ -limit as a thin-film limit  $F^3$  with the same form as (6), with  $\varphi(x, x')$  the optimal-transition energy. The observations above show that  $\varphi > 0$  except for

$$\varphi((1, -1, -1), (1, 1, -1)) = \varphi((-1, 1, 1), (-1 - 1, 1)) = 0.$$

Note that  $\varphi((1, -1, -1), (1, 1, -1)) \neq \varphi((1, 1, -1), (1, -1, -1))$  so that  $\varphi$  is not symmetric, and that the energy  $F^3$  is coercive even though its integrand is not strictly positive.

The two cases above carry the relevant information to treat the general case, which shows that the description of the thin-film limit needs a parameter space of increasing, but finite, cardinality; namely  $2^N$  where  $N$  is the number of layers. We briefly sketch the argument, which generalizes what has been noticed above.



We consider a minimizer  $u$ .

1) we first note that the upper layer must be alternating; i.e.,  $u(n+1, N) \neq u(n, N)$  for all  $n$

2) we either have  $u(n, N) \neq u(n-1, N-1)$  for all  $n$  or  $u(n_1, N) = u(n_1-1, N-1)$  for some  $n_1$ . In this case by minimality we have  $u(n_1, N-1) \neq u(n_1-1, N-1)$ . By Step 1 above we have  $u(n_1+1, N) = u(n_1, N-1)$ , so that we may proceed by induction and conclude that  $u(n, N) = u(n-1, N-1)$  for all  $n \geq n_1$ . Hence, either  $u$  is alternating on the  $N-1$ -th layer, or it is alternating for  $n < n_1$  and  $n > n_1$ .

3) proceeding in the same way we deduce that  $u$  is alternating in the  $(N-2)$ -th layer up to at most three indices (one less than  $n_1$ , one larger than  $n_1$ , and  $n_1$  itself). We note that, as in Step 2, for  $n > n_1$  there may exist a unique  $n_2$  such that  $u(n, N-1) = u(n-1, N-2)$  for  $n \geq n_2$  and  $u(n, N-1) \neq u(n-1, N-2)$  for  $n < n_2$ , but not the converse.

4) Proceeding by finite induction on the label of the layer, we deduce that  $n \mapsto u(n, k)$  is alternating for each  $k \in \{1, \dots, N\}$  up to a bounded number of  $n$ , with the bound independent of  $n$ . Moreover, in each interval of  $n$  where  $n \mapsto u(n, k)$  is alternating there may exist a unique  $\bar{n}$  such that  $u(n, k) = u(n-1, k-1)$  for  $n \geq \bar{n}$  and  $u(n, k) \neq u(n-1, k-1)$  for  $n < \bar{n}$ , but not the converse. Moreover,  $n \mapsto u(n, N)$  and  $n \mapsto u(n, 0)$  are alternating.

Note that this characterization also holds locally if we suppose that  $u$  has zero energy in an interval of  $n$ .

From this characterization, we deduce that if  $u^\varepsilon$  is a sequence with bounded energy, then it must coincide with an alternating state on each layer up to a finite number of indices. At this point we may proceed as above. The description in the general case is summarized in the conclusions below.

### 3 Conclusions

We consider an infinite thin film parameterized on the set

$$\mathbb{T}_{N,\varepsilon} = (\mathbb{R} \times [0, (N-1)\varepsilon\sqrt{3}/2]) \cap \varepsilon\mathbb{T},$$

where  $\mathbb{T}$  is a regular triangular lattice with one lattice vector  $(1, 0)$ , and the corresponding nearest-neighbour antiferromagnetic energy  $E_\varepsilon^N$ . In order to avoid indeterminate forms such energy is written as the sum of the contribution of each triangle of side-length  $\varepsilon$  contained in  $\mathbb{T}_{N,\varepsilon}$ , renormalized so that separately minimizing in each triangle gives zero energy. Note that the normalization is different if the triangle has one horizontal side on the upper or lower layer.

We have shown that there are  $2^N$  distinct ground states of  $E_1^N$ , which are two-periodic in the direction  $(1, 0)$ . On each of the layers such ground states are alternating, so that each of these ground states  $u^x$  can be parameterized by a point  $x$  in the set

$$Z_N := \{\pm 1\}^N.$$

We may define a compact convergence of discrete functions  $u^\varepsilon : \mathbb{T}_{N,\varepsilon} \rightarrow \{-1, 1\}$  to a function  $v : \mathbb{R} \rightarrow Z_N$  with a finite number of discontinuities, which highlights that,

up to a finite number of locations, a function  $u_\varepsilon$  with bounded energy  $E_\varepsilon^N$  coincides with a scaled version of the periodic minimizers.

With respect to this convergence the  $\Gamma$ -limit has the form

$$F^N(v) = \sum_{t \in S(v)} \varphi_N(v(t^-), v(t^+)),$$

where  $S(v)$  is the set of discontinuity points of  $v$ . The function  $\varphi_N : Z_N \times Z_N \rightarrow [0, +\infty)$  is an optimal-transition energy defined by

$$\varphi_N(x, x') = \min \left\{ E_1^N(u) : u = u^x \text{ on } \mathbb{T}_{N,1} \cap (-\infty, -M], \right. \\ \left. u = u^{x'} \text{ on } \mathbb{T}_{N,1} \cap [M, +\infty), M \in \mathbb{N} \right\}$$

(note that it suffices to take  $M = N$  since we have a bound by a test function for which only at most one column of  $N$  triangles is not optimal). The energy  $F^N$  is coercive; i.e., its finiteness implies a finite number of discontinuity points of  $v$ . Note that the description above also holds for thin films with  $\mathbb{R}$  substituted by a finite interval  $[a, b]$ , up to adding a boundary term. This extra term is not of interest since we focus on the number of limit parameters and not on the details of the energy.

The analysis above shows that the surface effects of the thin-film environment (i.e., the fact that ground states need to be alternating on the upper and lower layers due to the asymmetry of boundary sites) propagates inside the thin-film to limit the number of parameters needed to describe the limit. This rigidity effect “weakens” as the number of layers tends to infinity, as is testified by the (exponentially) diverging number of parameters. In a sense then, the “total frustration” of the triangular lattice can be seen as a limit behaviour as  $N \rightarrow +\infty$ .

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