Design of lattice surface energies

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Abstract

We provide a general framework for the design of surface energies on lattices. We prove sharp bounds for the homogenization of discrete systems describing mixtures of ferromagnetic interactions by constructing optimal microgeometries, and we prove a localization principle which allows to reduce to the periodic setting in the general nonperiodic case.

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1 Introduction

The optimization of the design of structures can sometimes be viewed as a variational problem where we minimize or maximize a cost or a compliance subjected to design constraints. A typical example is the shape optimization of conducting or elastic structures composed of a prescribed amount of a certain number of materials and with given boundary conditions or loads. In that case the existence of an optimal shape is not guaranteed, and a relaxed formulation must be introduced that takes into account the possibility of fine mixtures. The homogenization method as presented for example in the book by Allaire [4] can be regarded as subdividing the problem into the description of all possible materials obtained as mixtures, and subsequently optimize in the enlarged class of homogenized materials that satisfy the corresponding relaxed design constraint.

In this paper we extend the homogenization method to the optimal design of networks for surface energies. From a standpoint of Statistical Mechanics, the object of our study are mixtures of ferromagnetic interactions under the constraint that interaction coefficients (bonds) may only take values in a fixed set of parameters. We consider energies of Ising type defined on a cubic lattice, of the form

$$E(u) = \frac{1}{4} \sum_{\xi \in V} \sum_{i \in \mathbb{Z}^d} c_{i,\xi} (u_i - u_{i+\xi})^2, \tag{1}$$

where u_i is a *spin function*; i.e., a function that may only take the values -1 and 1, and $V \subset \mathbb{Z}^d$ is a finite set containing the standard orthonormal basis $\{e_1, \ldots, e_d\}$. In the energy we only consider pairs i, j with $\xi = j - i \in V$, so that V describes the range of interactions. Such pairs (i, j) (or equivalently (i, ξ)) will be called *bonds*. The non-negative coefficients $c_{i,\xi}$, representing the strength of the interaction (or bond strength) at the point i in direction ξ, will satisfy some design constraint, which will be at the core of our analysis and will be described below. We assume $c_{i,\xi} \geq c > 0$ for $\xi \in \{e_1, \ldots, e_d\}$ in order that the energies be equicoercive with respect to the strong L^1-convergence, and we prefer to write the interactions in the form $c_{i,\xi}(u_i - u_{i+\xi})^2$ than that in the form $-c_{i,\xi} u_i u_{i+\xi}$ (equivalent, up to a scaling factor), which is more customary in Statistical Mechanics, since in this way the energy of ground states is normalized to 0 and we avoid possible indeterminate forms in the case of infinite domains. In order to describe the surface energy corresponding to such a system we follow a discrete-to-continuum approach as in [19, 3] (see also [21, 12]): we scale the energies introducing a parameter ε and defining

$$E_\varepsilon(u) = \frac{1}{4} \sum_{\xi \in V} \sum_{i \in \mathbb{Z}^d} \varepsilon^{d-1} c_{i,\xi} (u_i - u_{i+\xi})^2, \tag{2}$$

where now the functions u are considered as defined on $\varepsilon\mathbb{Z}^d$, with u_i the value at εi. By identifying each u with the corresponding piecewise-constant interpolation on $\varepsilon\mathbb{Z}^d$ such energies can be considered as defined in a Lebesgue space, where they are equicoercive, so that their Γ-limit can be used as a continuum approximation in the description of the corresponding minimum problem [10].

Since in our optimal-design problem we have to take into account the possibility of locally varying the arrangement of the interactions we further introduce a dependence on ε on the coefficients, and consider energies

$$E_\varepsilon(u) = \frac{1}{4} \sum_{\xi \in V} \sum_{i \in \mathbb{Z}^d} \varepsilon^{d-1} c^\varepsilon_{i,\xi} (u_i - u_{i+\xi})^2. \tag{3}$$

Compactness theorems (see [3]) ensure that the Γ-limit of such energies exists up to subsequences, is finite on functions in $BV_{\rm loc}(\mathbb{R}^d; \{-1, 1\})$, and takes the form of a surface energy

$$F(u) = \int_{\partial^*\{u=1\}} \varphi(x, \nu_u) \, d\mathcal{H}^{d-1}, \tag{4}$$

where $\partial^*\{u = 1\}$ is the reduced boundary of $\{u = 1\}$ and ν_u is its inner normal.

The optimal-design constraint that we consider is that for fixed $\xi \in V$ the bond strengths $c_{i,\xi}$ may take two positive values α_ξ and β_ξ depending on ξ with

$$\alpha_\xi < \beta_\xi. \tag{5}$$

The simplest case is that of nearest neighbours, when $V = \{e_1, \dots, e_d\}$ is the canonical basis of $\mathbb{R}^d$ and the strength of the bonds is independent of the direction: $\alpha_{e_j} = \alpha$, $\beta_{\varepsilon_j} = \beta$. In that case we are mixing two types of connections in a cubic lattice. A simplified description of the two-dimensional setting for nearest neighbour-interactions can be found in [17].

The first step in the homogenization method is to consider all possible φ in the periodic-bond setting; that is, when $i \mapsto c_{i,\xi}$ are periodic, in which case φ is independent of x. We denote such φ, extended to $\mathbb{R}^d$ by positive homogeneity of degree one, as the *homogenized surface tension* of the system $\{c_{i,\xi}\}$. The description of such φ with fixed *volume fraction* (proportion) θ of β-type bonds is what is usually referred to as a *G-closure* problem, with a terminology borrowed from elliptic homogenization [4, 24]. We show that all possible such φ are the (positively homogeneous of degree one) symmetric convex functions such that

$$\sum_{\xi \in V} \alpha_\xi |\langle \nu, \xi \rangle| \le \varphi(\nu) \le \sum_{\xi \in V} (\beta_\xi \theta_\xi + (1 - \theta_\xi)\alpha_\xi) |\langle \nu, \xi \rangle|, \tag{6}$$

where $\theta_\xi \in [0, 1]$ is the *volume fraction* of the interaction coefficients which accounts only for points i interacting with points $i + \xi$, so that

$$\frac{1}{\#V} \sum_{\xi \in V} \theta_\xi = \theta.$$

Note that, strictly speaking, such a description makes sense only for θ a rational number. We denote by $\mathbf{H}(\theta)$ the family of all φ as above satisfying (6). If θ is not a rational number, then the elements of $\mathbf{H}(\theta)$ are regarded as approximated by elements of $\mathbf{H}(\theta_h)$ with $\theta_h \to \theta$ and rational.

In dimension two we may compare this G-closure problem with a continuous analog on curves in $\mathbb{R}^2$, which consists in the determination of optimal bounds for Finsler metrics obtained from the homogenization of periodic Riemannian metrics (see [1, 15, 13]) of the form

$$\int_a^b a\Big(\frac{u(t)}{\varepsilon}\Big) |u'|^2 dt,$$

and $a(u)$ is a periodic function in $\mathbb{R}^d$ taking only the values α and β. Even though curves and boundaries of sets have some topological differences, it has been shown in [19], that in the periodic setting the homogenized energy densities can be computed by optimal paths (curves) on the dual lattice. The problem on curves has been studied in [23], where it is shown that homogenized metrics satisfy

$$\alpha |\nu| \le \varphi(\nu) \le (\theta\beta + (1 - \theta)\alpha)|\nu|,$$

but the optimality of such bounds is not proved. That result provides bounds also for the 'dual' equivalent formulation in dimension 2 of the homogenization of periodic perimeter functionals of the form

$$\int_{\partial A} a\Big(\frac{x}{\varepsilon}\Big)\, d\mathcal{H}^{d-1}$$

with the same type of a as above (see [6, 8]). The corresponding φ in this case can be interpreted as the surface tension of the homogenized perimeter functional.

The discrete setting allows to give a (relatively) easy description of the optimal bounds in a way similar to the treatment of mixtures of linearly elastic discrete structures [16]. The bounds obtained by sections and by averages in the elastic case have as counterpart *bounds by projection*, where the homogenized surface tension is estimated from below by considering the minimal value of the coefficient on each section, and *bounds by averaging*, where coefficients on a section are substituted with their average. The discrete setting allows to construct (almost-)optimal periodic geometries, which optimize one type or the other of the bounds in every direction ξ at the same time. Since the constructions of optimal geometries for fixed ξ may overlap, some extra care must be used to make sure that they are compatible. This is done by a separation-of-scales argument.

The homogenization method is completed by proving that we may always locally reduce to the case of periodic coefficients. More precisely, we note that, up to subsequences, in the general non-periodic setting of energies as in (3), we may define a *continuum local volume fraction* $\theta = \theta(x)$ describing the local percentage of β-type bonds, as the average of the densities $\theta_\xi(x)$ of the weak*-limit of the measures

$$\mu^\xi_\varepsilon = \sum_{i: c^\varepsilon_{i,\xi} = \beta_\xi} \delta_{\varepsilon i}.$$

We then prove a *localization principle*, similar to the one for quadratic gradient energies in the Sobolev space setting stated by Dal Maso and Kohn (see [25, 11]). In our case, this amounts to proving that all φ that we may obtain in (4) are exactly those such that, upon suitably choosing their representative,

$$\varphi(x,\cdot) \in \mathbf{H}(\theta(x)) \tag{7}$$

for almost all x. Conversely, every lower-semicontinuous energy F as in (4) with a surface energy density φ such that (7) holds for almost every x is the Γ-limit of an Ising energy with coefficients $\{c^\varepsilon_{i,\xi}\}$ with continuum local volume fraction θ. This localization result turns out much more complex than the one in the elliptic case both because surface energies are not characterized by a single cell-problem formula and because their values must be characterized along $d-1$-hypersurfaces for $\mathcal{H}^{d-1}$-almost all values of x.

The paper is organized as follows: in Section 2 we fix some notation and introduce the general setting of the problem. In Section 3 we prove Theorem 5, which shows the optimality of the bounds in the periodic case. The proof holds with a direct construction when the target energy density is in a dense class of crystalline energy densities, and it is proved by a multi-scale approximation in the general case. It is interesting to note that, in order to recover a system of discrete interactions, it is convenient to interpret homogenized surface energy densities in a $W^{1,1}$ setting, where the extension gives a convex integrand. In Section 4 we prove the localization principle, which is subdivided in Theorems 18 and 19. Their proof is rather technical and makes use of representation and blow-up arguments. In particular, in order to recover (7) we use the results in [18], which provide a blow-up formula for the limit energy density at all points.

2 Notation and setting

For $x \in \mathbb{R}^d$, $\nu \in S^{d-1}$ and $\xi \in \mathbb{R}^d$, we define the half spaces $H^\pm_\nu(x) = \{y \in \mathbb{R}^d : \pm\langle y-x, \nu\rangle > 0\}$, the hyperplane $\Pi_\nu(x) = \{y \in \mathbb{R}^d : \langle y - x, \nu\rangle = 0\}$ and the strip $\Pi^\xi_\nu(x) = \{y \in \mathbb{R}^d : 0 \leq \langle y - x, \nu\rangle < \langle \xi, \nu\rangle\}$. In the notation we omit the dependence on x if $x = 0$; e.g., we write $H^\pm_\nu$ for $H^\pm_\nu(0)$, and similar. We define $u_{x,\nu} : \mathbb{R}^d \to \{\pm 1\}$ by

$$u_{x,\nu}(z) = \begin{cases} 1 & x \in H^+_\nu(x) \\ -1 & x \in H^-_\nu(x). \end{cases} \tag{8}$$

If $x = 0$ we use the notation and $u_\nu(z) = u_{0,\nu}(z)$.

For $R > 0$ we denote by $B_R(x) = \{y \in \mathbb{R}^d : |y - x| < R\}$ the open ball with radius R centred in x. If $x = 0$ we write B_R for $B_R(0)$. Furthermore, we set $B^\pm_{R,\nu} = H^\pm_\nu \cap B_R$ the half open ball of radius R and centre 0 contained in $H^\pm_\nu$. We denote by w_{d-1} the $(d-1)$-dimensional measure of the $(d-1)$-dimensional unit ball. Q denotes the d-dimensional unit open cube centred at 0; i.e., $Q = \{x \in \mathbb{R}^d : |\langle x, e_i\rangle| < \frac{1}{2},$ for all $i \in \{1, \cdots, d\}\}$, whereas we denote by $Q(x_0)$ the cube centred at x_0; i.e., $Q(x_0) = x_0 + Q$. Let $R_\nu \in SO(d)$ be a rotation such that $R(e_n) = \nu$. We denote by $Q^\nu = \{R_\nu(x) : x \in Q\}$ a unit cube with sides either parallel or orthogonal to ν, $Q^\nu(x_0) = x_0 + Q^\nu$ a unit cube centred at x_0 with sides either parallel or orthogonal to ν, and $Q^\nu_\rho(x_0) = \rho Q^\nu + x_0$ a cube centred at x_0 with side lengths $\rho > 0$ and sides either parallel or orthogonal to ν.

Given A open bounded with Lipschitz boundary and $u \in BV(A)$ we set $\mathrm{tr}(u) \in L^1(\partial(A))$ the inner trace of the function u on the boundary of A. We say that $\nu \in S^{d-1}$ is rational if there exists $\lambda \in \mathbb{R}$ such that $\lambda\nu \in \mathbb{Z}^d$.

We define by $D_1(\mathbb{R}^d)$ the space of functions that are convex, even, and positively homogeneous of degree 1; i.e., all functions $\varphi : \mathbb{R}^d \to [0, +\infty)$ that satisfy $\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda \geq 0$ and $x \in \mathbb{R}^d$. Furthermore we define $\mathrm{d} : D_1(\mathbb{R}^d) \times D_1(\mathbb{R}^d) \to [0, +\infty)$ as

$$\mathrm{d}(\varphi, \psi) = \sup\left\{|\varphi(\nu) - \psi(\nu)| : \nu \in S^{d-1}\right\}. \tag{9}$$

Note that $D_1(\mathbb{R}^d)$ endowed with the metric d is a metric space. We denote by

$$B_r(\varphi) = \{\varphi' \in D_1(\mathbb{R}^d) : \mathrm{d}(\varphi, \varphi') < r\}.$$

2.1 Spin functions and lattice energies

In what follows Ω will denote a bounded open set of $\mathbb{R}^d$ with Lipschitz boundary. We denote by $\mathcal{A}(\Omega)$ the set of all open subsets contained in Ω. Given $T \subset \mathbb{R}$ and $\varepsilon > 0$ we define the set of functions

$$\mathcal{PC}_\varepsilon(\Omega, T) := \{u : \varepsilon\mathbb{Z}^d \cap \Omega \to T\}.$$

We omit the dependence on T when $T = \mathbb{R}$; i.e., $\mathcal{PC}_\varepsilon(\Omega) = \mathcal{PC}_\varepsilon(\Omega, \mathbb{R})$, as well as when $\varepsilon = 1$; i.e., $\mathcal{PC}_1(\Omega, T) = \mathcal{PC}(\Omega, T)$. In order to carry out our analysis embedding all spaces in a common framework it is convenient to regard $\mathcal{PC}_\varepsilon(\Omega, \{\pm 1\})$ as a subset of $L^1(\Omega)$. To that end, we will identify a function $u \in \mathcal{PC}_\varepsilon(\Omega, \{\pm 1\})$ with its piecewise-constant interpolation on the ε-cubes centered in the lattice. That interpolation will still be denoted by u, and, precisely, is defined by setting $u(z) = 0$ if $z \in \varepsilon\mathbb{Z}^d \setminus \Omega$ and $u(x) = u(z_x^\varepsilon)$, where z_x^ε is the closest point in $\varepsilon\mathbb{Z}^d$ to x (which is uniquely defined up to a set of zero measure). Other similar interpolations could be taken into account, actually not affecting our asymptotic analysis.

Let $V \subset \mathbb{Z}^d$ be a finite set containing the standard orthonormal basis $\{e_j\}_{j=1}^d$. For every $\varepsilon > 0$ we consider a system of long-range (but finite range) interactions with coefficients $c_{i,\xi}^\varepsilon \geq 0$ for $i \in \mathbb{Z}^d$ and $\xi \in V$. The corresponding *ferromagnetic spin energy* $F_\varepsilon : \mathcal{PC}(\mathbb{R}^d; \{\pm 1\}) \to [0, +\infty]$ is given by

$$F_\varepsilon(u) = \frac{1}{4} \sum_{i \in \mathbb{Z}^d} \sum_{\xi \in V} c_{i,\xi}^\varepsilon (u_i - u_{i+\xi})^2, \tag{10}$$

where $u_i = u(i)$ and $\frac{1}{4}$ is a normalization factor. Such energies correspond to inhomogeneous surface energies in the continuum.

Definition 1. *Let $\{c_{i,\xi}^\varepsilon\}, i \in \mathbb{Z}^d, \xi \in V, \varepsilon > 0$ be coefficients as above such that*

$$c_{i,e_j}^\varepsilon \geq c > 0 \text{ for all } i \in \mathbb{Z}^d, \xi \in V, \varepsilon > 0.$$

Then, we define the macroscopic energy density *of $\{c_{i,\xi}^\varepsilon\}$ as the function $\varphi : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ such that for all $x \in \mathbb{R}^d$, $\varphi(x, \cdot)$ is positively one homogeneous of degree one and for all $\nu \in S^{d-1}$ and $x \in \mathbb{R}^d$ we have*

$$\varphi(x, \nu) = \limsup_{\rho \to 0} \limsup_{\varepsilon \to 0} \frac{1}{4 w_{d-1} \rho^{d-1}} \inf \Big\{ \sum_{i \in (B_\rho(x))_\varepsilon} \sum_{\xi \in V} \varepsilon^{d-1} c_{i,\xi}^\varepsilon (u_i - u_{i+\xi})^2 : \tag{11}$$

$$u \in \mathcal{PC}_\varepsilon(\mathbb{R}^d, \{\pm 1\}), u_i = u_{x,\nu}(i), i \notin (B_\rho(x))_\varepsilon \Big\},$$

where $(B_\rho(x))_\varepsilon = \mathbb{Z}^d \cap \left(\frac{1}{\varepsilon}B_\rho(x)\right)$ *and* $u_i = u(\varepsilon i)$.

Remark 2. The definition above can be interpreted in terms of a passage from a discrete to a continuum description as follows. We consider the scaled energies on Ω

$$E_\varepsilon(u) = \frac{1}{4}\sum_{i,i+\varepsilon\xi\in\Omega_\varepsilon}\sum_{\xi\in V}\varepsilon^{d-1}c^\varepsilon_{i,\xi}(u_i - u_{i+\xi})^2 \tag{12}$$

where $u \in \mathcal{PC}_\varepsilon(\Omega;\{\pm1\})$, $u_i = u(\varepsilon i)$ and $\Omega_\varepsilon = \mathbb{Z}^d \cap \left(\frac{1}{\varepsilon}\Omega\right)$. Identifying u with its piecewise constant interpolation, we can regard these energies defined on $L^1(\Omega)$. Their Γ-limit in that space, if it exists, is finite only on $BV(\Omega,\{\pm1\})$, where it has the form

$$E_\varphi(u) = \int_{\partial^*\{u=1\}\cap\Omega}\varphi(x,\nu_u(x))\mathrm{d}\mathcal{H}^{d-1}$$

with φ as above ([3, 19]).

3 Bounds for systems of periodic interactions

In this section we will consider the case where $\{c^\varepsilon_{i,\xi}\} = \{c_{i,\xi}\}$ is indepedent of ε and it is periodic; i.e., there exists $T\in\mathbb{N}$ such that we have

$$c_{(i+Te_j),\xi} = c_{i,\xi} \text{ for all } j\in\{1,\cdots,d\}, i\in\mathbb{Z}^d \text{ for all } \xi\in V.$$

Furthermore, we will consider interaction coefficients satisfying the *design constraint*

$$c_{i,\xi}\in\{\alpha_\xi,\beta_\xi\} \text{ with } 0<\alpha_\xi<\beta_\xi \tag{13}$$

with $\alpha = (\alpha_\xi)_{\xi\in V}$ and $\beta = (\beta_\xi)_{\xi\in V}$ fixed.

Remark 3. Let $\{c_{i,\xi} : i\in\mathbb{Z}^d, \xi\in V\}$ be periodic coefficients. If we consider $\{c^\varepsilon_{i,\xi}\} = \{c_{i,\xi}\}$ in Definition 1 then the *macroscopic energy density* of $\{c^\varepsilon_{i,\xi}\}$ reduces to the *homogenized energy density* of $\{c_{i,\xi}\}$; namely, to the convex positively homogeneous function of degree one $\varphi:\mathbb{R}^d\to[0,+\infty)$ such that for all $\nu\in S^{d-1}$ we have

$$\begin{aligned}\varphi(\nu) = \lim_{R\to+\infty}\frac{1}{4w_{d-1}R^{d-1}}\inf\Big\{\sum_{i\in\mathbb{Z}^d\cap B_R}\sum_{\xi\in V}c_{i,\xi}(u_i-u_{i+\xi})^2 : u\in\mathcal{PC}(\mathbb{R}^d,\{\pm1\}),\\ u(i) = u_{0,\nu}(i), i\notin B_R\Big\}.\end{aligned} \tag{14}$$

This formula has been proved in [3].

For any such periodic coefficients we define the *volume fraction* of β_ξ-bonds and the *total volume fraction of β-bonds*, respectively, as

$$
\begin{aligned}
\theta_\xi(\{c_{i,\xi}\}) &= \frac{1}{T^d}\#\{i\in\mathbb{Z}^d : i\in[0,T)^d, c_{i,\xi}=\beta_\xi\},\\
\theta(\{c_{i,\xi}\}) &= \frac{1}{\#V}\sum_{\xi\in V}\theta_\xi(\{c_{i,\xi}\}).
\end{aligned}
\tag{15}
$$

Definition 4. *Let $\theta\in[0,1]$. The set of* homogenized energy densities of mixtures of α and β bonds corresponding to V, with volume fraction θ (of β bonds) *is defined as*

$$
\begin{aligned}
\boldsymbol{H}_{\alpha,\beta,V}(\theta) = \big\{\varphi:\mathbb{R}^d\to[0,+\infty) :\ & \textit{there exist } \theta^k\to\theta, \varphi^k\to\varphi \textit{ and } \{c^k_{i,\xi}\},\\
& \textit{with } \theta(\{c^k_{i,\xi}\})=\theta^k \textit{ and } \varphi^k \textit{ homogenized energy density of } \{c^k_{i,\xi}\}\big\},
\end{aligned}
\tag{16}
$$

where the convergence $\varphi_k\to\varphi$ is defined through the distance d *defined in* (9)*.*

The following theorem completely characterizes the set $\mathbf{H}_{\alpha,\beta,V}(\theta)$.

Theorem 5 (Optimal bounds)**.** *The elements of the set $\boldsymbol{H}_{\alpha,\beta,V}(\theta)$ are all the even and convex positively homogeneous functions of degree one $\varphi:\mathbb{R}^d\to[0,+\infty)$ such that*

$$
\sum_{\xi\in V}\alpha_\xi|\langle\nu,\xi\rangle|\le\varphi(\nu)\le\sum_{\xi\in V}(\theta_\xi\beta_\xi+(1-\theta_\xi)\alpha_\xi)|\langle\nu,\xi\rangle|
\tag{17}
$$

for some $\theta_\xi\in[0,1]$ for $\xi\in V$, such that

$$
\frac{1}{\#V}\sum_{\xi\in V}\theta_\xi=\theta.
\tag{18}
$$

Remark 6. Note that the lower bound for functions in $\mathbf{H}_{\alpha,\beta,V}(\theta)$ is independent of β. This bound follows by a comparison argument from [2] in the case for nearest-neighbour spin systems. The more refined argument in Proposition 9 will give the optimality of the lower bound in the general case.

Note that in the case $\theta=1$ we have all functions satisfying the trivial bounds

$$
\sum_{\xi\in V}\alpha_\xi|\langle\nu,\xi\rangle|\le\varphi(\nu)\le\sum_{\xi\in V}\beta_\xi|\langle\nu,\xi\rangle|,
\tag{19}
$$

and the lower bound is sharp even though we have a zero density of α-bonds. This is due to the fact that in that case by considering $\theta^k\to 1$ we allow a volume fraction of α bonds, which is vanishing but not zero, and is sufficient to allow for all such φ.

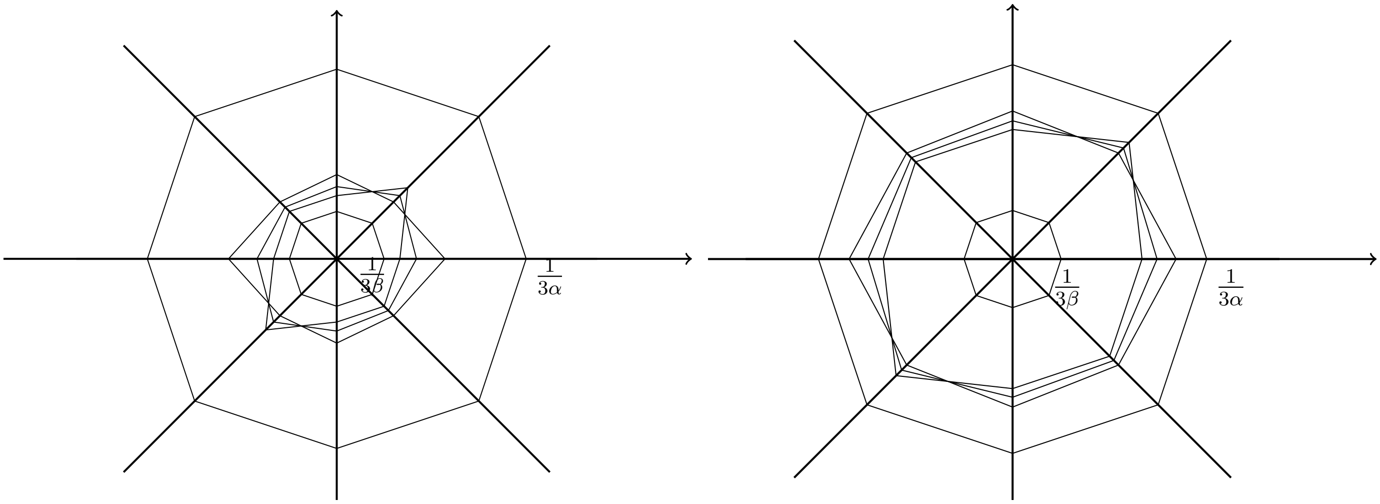

Figure 1: Level sets of φ in the case $\theta \geq \frac{1}{2}$ and $\theta < \frac{1}{2}$, respectively

Example 7. We consider the two-dimensional case with nearest and next-to-nearest neighbour interactions; i.e., choosing $V = \{e_1, e_2, e_1 + e_2 =: v_1, e_1 - e_2 =: v_2\}$. Theorem 5 states that if $\varphi \in \mathbf{H}_{\alpha,\beta,V}(\theta)$, then

$$\alpha(|\nu_1| + |\nu_2| + |\langle \nu, v_1\rangle| + |\langle \nu, v_2\rangle|) \leq \varphi(\nu) \leq c_{1,1}|\nu_1| + c_{1,2}|\nu_2| + c_{2,1}|\langle \nu, v_1\rangle| + c_{2,2}|\langle \nu, v_2\rangle|,$$

where

$$c_{1,1} + c_{1,2} = \theta_1\beta + (1 - \theta_1)\alpha, \qquad c_{2,1} + c_{2,2} = \theta_2\beta + (1 - \theta_2)\alpha,$$

and $\theta_1 + \theta_2 = 2\theta$. These bounds are pictured in terms of the sublevel sets $\{\varphi \leq 1\}$ on the left-hand side of Fig. 1 for some $\theta \geq \frac{1}{2}$ and on the right-hand side of Fig. 1 for some $\theta < \frac{1}{2}$. Such sublevelsets are convex and symmetric with respect to the origin. Note that by the trivial bounds such sublevelsets are contained in the sublevelset of φ with all the coefficients equal to α and contain the sublevelset of φ with all the coefficients equal to β. These are the smallest and the largest regular octagons, respectively, in the figures. Furthermore, since $\varphi \in \mathbf{H}_{\alpha,\beta,V}(\theta)$ satisfies the upper bound by averaging (see Proposition 8 below), they contain at least one of the even octagons whose vertices lie on the straight lines with coordinates

$$
\begin{aligned}
x_1 &= \pm\frac{1}{2(\theta_1+\theta_{2,1})(\beta-\alpha)+4\alpha}, && x_2 = x_1\\
x_1 &= \pm\frac{1}{(2\theta_2+\theta_{1,1})(\beta-\alpha)+3\alpha}, && x_2 = 0\\
x_1 &= 0, && x_2 = \pm\frac{1}{(2\theta_2+\theta_{1,2})(\beta-\alpha)+3\alpha}\\
x_1 &= \pm\frac{1}{2(\theta_1+\theta_{2,2})(\beta-\alpha)+4\alpha}, && x_2 = -x_1,
\end{aligned}
$$

where $\theta_{1,1},\theta_{1,2}$ are the vertical and horizontal volume fractions of the nearest-neighbor bonds satisfying $\theta_{1,1}+\theta_{1,2}=2\theta_1$ and $\theta_{2,1},\theta_{2,2}$ are the volume fractions corresponding to v_2 and v_1 respectively and satisfying $\theta_{2,1}+\theta_{2,2}=2\theta_2$. Those are represented by the thick lines.

3.1 Derivation of bounds

We now derive the bounds of Theorem 5 by using the following proposition.

Proposition 8 (Bounds by averaging)**.** *Let φ be the homogenized energy density of $\{c_{i,\xi}\}$ as in* (14)*. Then we have*

$$\varphi(\nu)\le\sum_{\xi\in V}(\theta_\xi\beta_\xi+(1-\theta_\xi)\alpha_\xi)|\langle\nu,\xi\rangle|. \tag{20}$$

Proof. The proof is obtained by constructing a suitable competitor $u\in\mathcal{PC}(\mathbb{R}^d,\{\pm1\})$, $u(i)=u_\nu(i), i\notin B_R$ in the definition of (14). To this end, we define for $j\in[0,T)^d\cap\mathbb{Z}^d$ the function $u_j\in\mathcal{PC}(\mathbb{R}^d,\{\pm1\})$ by

$$u_j(i)=\begin{cases}u_\nu(i) & \text{if } i\notin B_R\\ u_{j,\nu}(i) & \text{if } i\in B_R,\end{cases}$$

where $u_{j,\nu},u_\nu$ are given by (8).

Let $j_0\in[0,T)^d\cap\mathbb{Z}^d$ be such that

$$\frac14\sum_{\xi\in V}\sum_{i\in\mathbb{Z}^d\cap B_R}c_{i,\xi}((u_{j_0})_i-(u_{j_0})_{i+\xi})^2\le\frac{1}{T^d}\frac14\sum_{j\in[0,T)^d\cap\mathbb{Z}^d}\sum_{\xi\in V}\sum_{i\in\mathbb{Z}^d\cap B_R}c_{i,\xi}((u_j)_i-(u_j)_{i+\xi})^2.$$

We have

$$
\begin{aligned}
&\frac{1}{T^d}\frac{1}{4}\sum_{j\in[0,T)^d\cap\mathbb{Z}^d}\sum_{\xi\in V}\sum_{i\in\mathbb{Z}^d\cap B_R} c_{i,\xi}((u_j)_i-(u_j)_{i+\xi})^2\\
&= \frac{1}{T^d}\sum_{j\in[0,T)^d\cap\mathbb{Z}^d}\sum_{\xi\in V}\sum_{i\in\Pi^\xi_\nu(j)\cap B_R\cap\mathbb{Z}^d} c_{i,\xi}\\
&= \frac{1}{T^d}\sum_{\xi\in V}\sum_{i\in[0,T)^d\cap\mathbb{Z}^d}\sum_{j\in[0,T)^d\cap\mathbb{Z}^d} c_{i,\xi}\#\{i'\in(T\mathbb{Z}^d+i)\cap\Pi^\xi_\nu(j)\cap B_R\}\\
&\le \frac{1}{T^d}\sum_{\xi\in V}\sum_{i\in[0,T)^d\cap\mathbb{Z}^d} c_{i,\xi}w_{d-1}R^{d-1}|\langle\nu,\xi\rangle|+o(R^{d-1})\\
&= w_{d-1}R^{d-1}\sum_{\xi\in V}(\theta_\xi\beta_\xi+(1-\theta_\xi)\alpha_\xi)|\langle\nu,\xi\rangle|+o(R^{d-1}).
\end{aligned}
$$

The last equality follows from splitting the sum into the two sets where $c_{i,\xi}$ equals α_ξ or β_ξ respectively. Since u_{j_0} is admissible in the definition of (14), dividing by $w_{d-1}R^{d-1}$ and letting $R\to+\infty$ yields the claim. $\square$

Proposition 8 together with the trivial bound from below gives the bounds in the statement of Theorem 5. In the following section we prove their optimality.

3.2 Optimality of bounds

The construction of optimal bond systems for the upper bound relies on the observation that, when we have a periodic system of bonds, we may give a lower bound for $\varphi(\nu)$ by a "projection" argument. In order to illustrate this argument we may consider the case when ν is one of the vectors of the standard orthonormal basis and we only have nearest-neighbour interactions; i.e., $V=\{e_1,\dots,e_d\}$. In this case a lower bound for $\varphi(\nu)$ is obtained by considering the minimal value of the bond strengths in each lattice section parallel to e_k. The optimality of the upper bound is obtained by exhibiting periodic systems for which this estimate is sharp. This can be done for a class of φ with polyhedral sublevel sets. Such φ are then used to approximate arbitrary functions satisfying the bounds by means of a multi-scale construction in the continuum. A diagonal argument allows to construct periodic system of bonds as in the definition of $\mathbf{H}_{\alpha,\beta,V}(\theta)$. The proof of their optimality is finally achieved by using an extension of the energies to convex functionals in $W^{1,1}$, where a more handy cell-problem formula can be used.

3.2.1 A projection argument

The projection argument outlined above, when extended to general sets of interactions V and directions ν, needs some notation, which we now introduce. If $\Xi=\{\xi_1,\cdots,\xi_d\}\subset\mathbb{Z}^d$

is an orthogonal basis and $z \in \mathbb{Z}^d$ we define

$$\mathcal{L}_z(\Xi) := \Big\{ i \in \mathbb{R}^d : i = z + \sum_{k=1}^{d} \lambda_k \xi_k ; \lambda_k \in \mathbb{Z} \Big\}, \quad \mathcal{L}_z^T(\Xi) = z + T\mathcal{L}_0(\Xi).$$

For $j = 1, \cdots, d$ we set

$$\mathcal{L}_{z,j}(\Xi) := \Big\{ i \in \mathbb{R}^d : i = z + \sum_{k=1}^{d} \lambda_k \xi_k ; \lambda_k \in \mathbb{Z}, \lambda_j = 0 \Big\}, \quad \mathcal{L}_{z,j}^T(\Xi) = z + T\mathcal{L}_{0,j}(\Xi),$$

which represent the projection of the lattices onto the plane orthogonal to ξ_j. We define

$$P_z(\Xi) := \Big\{ i \in \mathbb{R}^d : i = z + \sum_{j=1}^{d} \lambda_j \xi_j ; \lambda_j \in [0,1) \Big\}$$

the fundamental parallelepiped spanned by the vectors Ξ translated to the point z, and

$$P_z^T(\Xi) := \Big\{ i \in \mathbb{R}^d : i = z + T \sum_{j=1}^{d} \lambda_j \xi_j ; \lambda_j \in [0,1) \Big\}.$$

If $\{c_{i,\xi_j} : i \in \mathcal{L}_z(\Xi), j \in \{1, \cdots, d\}\}$ is a system of interactions in the lattice $\mathcal{L}_z(\Xi)$ with $c_{i,\xi_j} \geq 0$ and T-periodic; i.e., such that $c_{i+T\xi_j,\xi_k} = c_{i,\xi_k}$ for all $j,k \in \{1, \cdots, d\}$, we define

$$\begin{aligned} \varphi_{\Xi,z}(\nu) = \limsup_{R \to +\infty} \frac{1}{4w_{d-1}R^{d-1}} \inf \Big\{ & \sum_{i \in \mathcal{L}_z(\Xi) \cap B_R} \sum_{j=1}^{d} c_{i,\xi_j} (u_i - u_{i+\xi_j})^2 ; \\ & u : \mathcal{L}_z(\Xi) \to \{\pm 1\}, u(i) = u_{0,\nu}(i), i \notin B_R \Big\} \end{aligned} \tag{21}$$

(in the notation of $\varphi_{\Xi,z}$ we omit the dependence on $\{c_{i,\xi_j}\}$). By regrouping the interactions $\{c_{i,\xi}\}$ of energy (10) on sublattices $\mathcal{L}_{z,j}(\Xi)$ we will use (21) to obtain a lower bound in (14). Note that possibly one has to set $c_{i,\xi_j} = 0$ for some $j \in \{1, \cdots, d\}$ if $\xi_j \notin V$.

Note that if $c_{i,\xi}$ is T-periodic along the coordinate directions, then for every $\eta \in \mathbb{Z}^d$ there exists $T' = T'_\eta$ such that $c_{i+T'\eta,\xi} = c_{i,\xi}$.

Proposition 9 (Bounds by projection)**.** *Let $\Xi = \{\xi_1, \cdots, \xi_d\}$ be an orthogonal basis, $z \in \mathbb{Z}^d$ and $\{c_{i,\xi_j} : i \in \mathcal{L}_z(\Xi), j \in \{1, \cdots, d\}\}$ be non-negative coefficients. Let $\varphi_{\Xi,z} : \mathbb{R}^d \to [0, +\infty)$ be the even convex positively homogeneous function of degree one given by* (21)*. Then*

$$\varphi_{\Xi,z}(\nu) \geq \sum_{j=1}^{d} c_j^p |\langle \nu, \xi_j \rangle| \tag{22}$$

where

$$c_j^p = \frac{1}{T^{d-1}|P_z(\Xi)|} \sum_{k\in\mathcal{L}_{z,j}(\Xi)\cap P_z^T(\Xi)} \min\{c_{i,\xi_j} : i-k = \lambda\xi_j \textit{ for some } \lambda\in\mathbb{Z}\} \tag{23}$$

(the letter p in c_j^p stands for projection*).*

Proof. Let $u : \mathcal{L}_z \to \{\pm 1\}$ be such that $u(i) = u_\nu(i)$ for all $i \notin B_R$. Set for $j = 1, \cdots, N$

$$I_j := \Big\{ i \in \mathcal{L}_{z,j}(\Xi) : \{i + t\xi_j : t\in\mathbb{R}\} \cap B^{\pm}_{R,\nu} \neq \emptyset \Big\}.$$

Noting that for all $k \in I_j$ there exists at least one $i \in \{k + \lambda\xi : \lambda \in \mathbb{Z}\} \cap B_R$, such that $u_i \neq u_{i+\xi}$, we have

$$\begin{aligned}
& \frac{1}{4}\sum_{i\in\mathcal{L}_z\cap B_R}\sum_{j=1}^{d} c_{i,\xi_j}(u_i - u_{i+\xi_j})^2 \\
\geq\ & \sum_{j=1}^{d}\sum_{k\in I_j} \min\{c_{i,\xi_j} : i-k=\lambda\xi_j \text{ for some } \lambda\in\mathbb{Z}\} \\
\geq\ & \sum_{j=1}^{d}\sum_{k\in\mathcal{L}_{z,j}(\Xi)\cap P_z^T(\Xi)} \min\{c_{i,\xi_j} : i-k=\lambda\xi_j \text{ for some } \lambda\in\mathbb{Z}\}\#(T\mathcal{L}_{k,j}(\Xi)\cap I_j) \\
\geq\ & \sum_{j=1}^{d}\sum_{k\in\mathcal{L}_{z,j}(\Xi)\cap P_z^T(\Xi)} \min\{c_{i,\xi_j} : i-k=\lambda\xi_j \text{ for some } \lambda\in\mathbb{Z}\}\frac{|\langle\nu,\xi_j\rangle|}{|P_{z,j}(\Xi)|}\frac{w_{d-1}R^{d-1}}{T^{d-1}} + o(R^{d-1}).
\end{aligned}$$

Note that we have used the fact that

$$\begin{aligned}
\#(T\mathcal{L}_{k,j}\cap I_j) &= \frac{\mathcal{H}^{d-1}\Big((\Pi_\nu\cap B_R) \text{ projected onto } z + \Pi_{\frac{\xi_j}{||\xi_j||}}\Big)}{\mathcal{H}^{d-1}\Big(P_z^T(\Xi)\cap\Big(z+\Pi_{\frac{\xi_j}{||\xi_j||}}\Big)\Big)} \\
&= \frac{\frac{1}{||\xi_j||}|\langle\nu,\xi_j\rangle|}{\Pi_{\substack{i=1\\ i\neq j}}^{d}||\xi_i||}\frac{w_{d-1}R^{d-1}}{T^{d-1}} + o(R^{d-1}).
\end{aligned}$$

Taking the infimum over $u : \mathcal{L}_z \to \{\pm 1\}$ such that $u(i) = u_\nu(i), i \notin B_R$, dividing by $w_{d-1}R^{d-1}$ and letting $R \to +\infty$ we obtain the claim. ☐

We now use Proposition 9 to prove the optimality of bounds for the special case of crystalline ψ, from which the general result will be deduced by approximation.

Proposition 10. *Let V be as in Theorem 5 and let*

$$\psi(\nu) = \sum_{\xi\in V} c_\xi |\langle \nu, \xi\rangle|$$

with coefficients $\alpha_\xi \leq c_\xi \leq \beta_\xi, \xi \in V$ such that

$$c_\xi = t_\xi \beta_\xi + (1-t_\xi)\alpha_\xi \leq \theta_\xi\beta_\xi + (1-\theta_\xi)\alpha_\xi,$$

where $\theta_\xi, t_\xi \in (0,1)\cap\mathbb{Q}, \xi \in V$. Then there exist $T \in \mathbb{N}$ and $\{c_{i,\xi}\}$ T-periodic with $\theta_\xi(\{c_{i,\xi}\}) = \theta_\xi$ such that ψ is the homogenized energy density *of $\{c_{i,\xi}\}$. In particular if θ satisfies* (18) *with $\theta_\xi \in (0,1)$ then $\psi \in \boldsymbol{H}_{\alpha,\beta,V}(\theta)$.*

Proof. We construct $\{c_{i,\xi}\}$ with some period $T \in \mathbb{N}$ and

$$\theta_\xi(\{c_{i,\xi}\}) = \theta_\xi \text{ for all } \xi \in V$$

by defining the bonds separately for each direction of interaction $\xi \in V$. Note that if $\{c_{i,\xi_j} : i \in \mathcal{L}_z(\Xi), j \in \{1,\cdots,d\}\}$ are T_Ξ-periodic for each Ξ corresponding to $\xi \in V$, then there exists $T \in \mathbb{N}$ such that $\{c_{i,\xi}\}$ are T-periodic.

For $\xi \in V$, let $\Xi = \{\xi_1,\cdots,\xi_d = \xi\} \subset \mathbb{Z}^d$ be an orthogonal basis and let $z \in \mathbb{Z}^d \cap P_0(\Xi)$. For $T \in \mathbb{N}$ and $\nu = \frac{v}{||v||}$ for some $v \in V$, we set

$$A^z_\nu(T,\Xi) := \Big\{ i \in \mathcal{L}_z(\Xi) : \Big(\{i + t\xi : t \in [0,1)\}\Big) \cap \Big(\bigcup_{j\in\mathcal{L}^T_0(\Xi)} (j + \Pi_\nu)\Big) \neq \emptyset \Big\}.$$

This is the minimal T-periodic set of points in the lattice $\mathcal{L}_z(\Xi)$ interacting in direction ξ when we use u_ν as a test function; i.e., such that $u_\nu(i) \neq u_\nu(i+\xi)$. Note that for $i \in \mathcal{L}_{z,d}(\Xi)$ we have

$$\#\Big(A^z_\nu(T,\Xi) \cap \{i + t\xi : t \in \mathbb{R}\} \cap P^T_z(\Xi)\Big) \leq C(\nu,\xi)$$

for all $\langle\nu,\xi\rangle \neq 0$, $\nu \in S^{d-1}$ rational, and $\xi \in \mathbb{Z}^d$. In fact, if ν is rational and Ξ is the standard orthonormal basis we have that for $\{\nu,\nu_1,\ldots,\nu_{d-1}\}$ with $\nu_i \in \mathbb{Z}^d$ (which can be chosen satisfying this condition since ν is rational) one can choose $C(\nu,\xi) \leq \Pi_{i=1}^{d-1}||\nu_i||_1$ and the general case can be reduced to this one by a change of coordinate which preserves the rationality of ν. Choose $T \in \mathbb{N}$ such that $T^d\theta_\xi \in \mathbb{N}$, $T^{d-1}(1-t_\xi) = N_\xi \in \mathbb{N}$ and

$$(1-t_\xi) \sum_{\substack{v\in V\\ \langle v,\xi\rangle\neq 0}} C\Big(\frac{v}{||v||},\xi\Big) \leq T(1-\theta_\xi) \tag{24}$$

for all $\xi \in V$. Choose $A_\xi \subset \mathcal{L}_{z,d}(\Xi) \cap P_z^T(\Xi)$ such that $\# A_\xi = N_\xi$. We define

$$c_{i,\xi} = \begin{cases} \alpha_\xi & i = \lambda\xi + i', \lambda \in \mathbb{Z}, i' \in A_\xi, i \in A_\nu^z(T, \Xi) \\ \beta_\xi & i = \lambda\xi + i', \lambda \in \mathbb{Z}, i' \in (\mathcal{L}_{z,d}(\Xi) \cap P_z^T(\Xi)) \setminus A_\xi, \end{cases}$$

and $c_{i,\xi}$ arbitrarily equal to either α_ξ or β_ξ for other indices i, only subject to the total constraint that $\theta_\xi(\{c_{i,\xi}\}) = \theta_\xi$. This is possible, since, due to (24), we have

$$\begin{aligned} \#\Big\{ i = \lambda\xi + i', \lambda \in \mathbb{Z}, i' \in A_\xi, i \in A_\nu^z(T, \Xi) \Big\} &\leq N_\xi \sum_{\substack{v \in V \\ \langle v, \xi \rangle \neq 0}} C\Big(\frac{v}{||v||}, \xi\Big) \\ &= T^{d-1}(1 - t_\xi) \sum_{\substack{v \in V \\ \langle v, \xi \rangle \neq 0}} C\Big(\frac{v}{||v||}, \xi\Big) \leq (1 - \theta_\xi) T^d \end{aligned}$$

and

$$\#\Big\{ i = \lambda\xi + i', \lambda \in \mathbb{Z}, i' \in (\mathcal{L}_{z,d}(\Xi) \cap P_z^T(\Xi)) \setminus A_\xi \Big\} = (T^{d-1} - N_\xi) T = t_\xi T^d \leq \theta_\xi T^d.$$

With this choice of $c_{i,\xi}$ we have

$$\min\{c_{i,\xi} : i = \lambda\xi + k \text{ for some } \lambda \in \mathbb{Z}\} = \begin{cases} \alpha_\xi & \text{if } k \in A_\xi \\ \beta_\xi & \text{if } k \in (\mathcal{L}_{z,d}(\Xi) \cap P_z^T(\Xi)) \setminus A_\xi. \end{cases}$$

Hence, Proposition 9 yields that the homogenized energy density φ of $\{c_{i,\xi}\}$ satisfies

$$\begin{aligned} \varphi(\nu) &\geq \sum_{\xi \in V} \sum_{z \in P_z(\Xi)} \frac{1}{|P_0(\Xi)|} (t_\xi \beta_\xi + (1 - t_\xi)\alpha_\xi) |\langle \nu, \xi \rangle| \\ &\geq \sum_{\xi \in V} (t_\xi \beta_\xi + (1 - t_\xi)\alpha_\xi) |\langle \nu, \xi \rangle| = \psi(\nu) \end{aligned}$$

as desired. To give an upper bound, let $v \in V$ and set $\nu = \frac{v}{||v||} \in S^{d-1}$. Testing (14) with u_ν we have that

$$\begin{aligned} \varphi(\nu) &\leq \lim_{R \to \infty} \frac{1}{w_{d-1} R^{d-1}} \sum_{\xi \in V} \sum_{i \in \mathbb{Z}^d \cap B_R} c_{i,\xi} ((u_\nu)_i - (u_\nu)_{i+\xi})^2 \\ &\leq \sum_{\xi \in V} (t_\xi \beta_\xi + (1 - t_\xi)\alpha_\xi) |\langle \nu, \xi \rangle| = \psi(\nu). \end{aligned}$$

Noting that $\varphi \in D_1(\mathbb{R}^d)$, $\varphi(v) \leq \psi(\nu)$ for all $v \in V$, and

$$\psi(\nu) = \sup\{g(\nu) : g \in D_1(\mathbb{R}^d),\, g(v) \leq \psi(v) \text{ for all } v \in V\}$$

the desired equality equality follows. $\square$

3.2.2 Construction of bond systems by a multi-scale argument

In this section we show that for any φ satisfying the bounds of Theorem 5 its associated surface energy E_φ as in Remark 2 is the Γ-limit of energies of the type (10) where we have that the period T^ε of the interaction coefficients $\{c^\varepsilon_{i,\xi}\}$ of the approximating energies diverges.

The construction of the bond systems $\{c^\varepsilon_{i,\xi}\}$ follows a multi-scale argument, which we briefly outline (note that in this description we use a slightly different notation with respect to that used in the rest of the section):

1) thanks to its convexity, the target energy φ can be approximated by energies φ_k with polyhedral sublevel sets whose vertices are rational directions $\{\nu_1,\dots,\nu_k\}$;

2) the homogeneous anisotropic $d-1$-dimensional perimeter with energy function φ_k can be obtained by homogenization of periodic perimeter energies with integrand $f_k(y,\nu)$ periodic in y. These energies are equal to φ_k on a system of hyperplanes orthogonal to the directions $\nu_1,\dots,\nu_k$ and to a function ψ larger than φ and of the form as in Proposition 10 outside those hyperplanes;

3) the same energy functions $f_k(y,\nu)$ can be modified on the system of hyperplanes without changing the corresponding homogenized functional so that at each $f_k(y,\cdot)$ has the form of a function ψ_j as in Proposition 10, depending only on the label j (corresponding to ν_j) of the corresponding hyperplane;

4) we fix $\eta>0$ and consider functions f^η_k obtained by modifying f_k by setting $f^\eta_k(y,\cdot)=\psi_j$ on a η-neighbourhood of the hyperplanes orthogonal to ν_j (some care must be used in the intersection of these stripes). We note that the inhomogeneous perimeter corresponding to f^η_k converges to that corresponding to f_k as $\eta\to 0$;

5) we consider a scale $\varepsilon << \eta$ and periodic systems on a lattice of spacing ε whose homogenized energy density is the corresponding ψ or ψ_j in each zone where f^η_k is constant, so that, letting $\varepsilon\to 0$, the Γ-limit of this system is the inhomogeneous perimeter in the previous step;

6) by a diagonal argument on ε,η and k we obtain an approximation of φ.

Note that this convergence result is not sufficient to prove the optimality of the bounds. In the next section we will use the existence of such an approximating sequence of energies to deduce the convergence of the homogenized energy densities φ^ε corresponding to $\{c^\varepsilon_{i,\xi}\}$ with $\varepsilon>0$ fixed to the target energy density φ as $\varepsilon\to 0$.

Theorem 11. *Let $\varphi:\mathbb{R}^d\to[0,+\infty)$ be convex, even, positively 1-homogeneous and such that*

$$\sum_{\xi\in V}\alpha_\xi|\langle\nu,\xi\rangle|\le\varphi(\nu)\le\sum_{\xi\in V}(\theta_\xi\beta_\xi+(1-\theta_\xi)\alpha_\xi)|\langle\nu,\xi\rangle|, \tag{25}$$

with $\theta_\xi\in[0,1]$ for all $\xi\in V$. Then there exist $\{c^\varepsilon_{i,\xi}\}$, T_ε-periodic, with T_ε of order $\frac{1}{\varepsilon}$, such that $\theta_\xi(\{c^\varepsilon_{i,\xi}\})\to\theta_\xi$ for all $\xi\in V$ and as $\varepsilon\to 0$ the family of functionals E_ε :

$L^1(\Omega) \to [0, +\infty]$ *given by* (12) Γ*-converges with respect to the strong* $L^1(\Omega)$*-topology to the functional* $E_\varphi : L^1(\Omega) \to [0, +\infty]$ *given by*

$$E_\varphi(u) = \begin{cases} \displaystyle\int_{\partial^*\{u=1\}} \varphi(\nu_u(x)) \mathrm{d}\mathcal{H}^{d-1} & u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \textit{otherwise.} \end{cases}$$

Proof. Step 1. We may suppose that

$$\sum_{\xi \in V} \alpha_\xi |\langle \nu, \xi \rangle| < \varphi(\nu) < \sum_{\xi \in V} c_\xi |\langle \nu, \xi \rangle| = \sum_{\xi \in V} (\theta_\xi \beta_\xi + (1 - \theta_\xi)\alpha_\xi) |\langle \nu, \xi \rangle| \tag{26}$$

for some $\theta_\xi \in (0, 1]$ for each $\xi \in V$ satisfying (18) for some $\theta \in (0, 1]$. Indeed if we have an equality in place of the second inequality in (26) we can find φ_k satisfying (26) and $\varphi_k \to \varphi$ monotonically. This can be done by defining $\varphi_k : \mathbb{R}^d \to [0, +\infty)$ by

$$\varphi_k(\nu) = \sum_{\xi \in V} \Big(c_\xi - \frac{1}{k} \Big) |\langle \nu, \xi \rangle|.$$

As for the first inequality in (26), since $\theta_\xi > 0$ for all $\xi \in V$ we have that

$$\sum_{\xi \in V} \alpha_\xi |\langle \nu, \xi \rangle| < \varphi_k(\nu)$$

for k large enough. Hence by the monotone convergence theorem we have that $E_{\varphi_k}(u) \to E_\varphi(u)$ monotonically for all $u \in BV(\Omega, \{\pm 1\})$, therefore by [[22], Proposition 5.4 and 5.7] we have that E_{φ_k} Γ-converge to E_φ.

Additionally, we can assume that φ is crystalline and the vertices of the set $\{\varphi \leq 1\}$ correspond to rational directions and contain the directions V; i.e., there exists $N \in \mathbb{N}, N \geq \#V$ and $c_j \geq 0$ such that

$$\varphi(\nu) = \sum_{j=1}^{N} c_j |\langle \nu, \nu_j \rangle|, \tag{27}$$

and for all $\xi \in V$ there exists $k \in \{1, \cdots, \#V\}$ such that $\lambda_k \nu_k = \xi$. Note that this is possible due to an approximation argument that still maintains the bound (26). Such an argument is analogous to the one used for the approximation of Finsler metrics in [13].

Step 2. For every φ satisfying (27) and (26) the functionals $E_\varepsilon : BV(\Omega; \{\pm 1\}) \to [0, +\infty)$ given by

$$E_\varepsilon(u) = \int_{\partial^*\{u=1\}} f\Big(\frac{x}{\varepsilon}, \nu_u(x)\Big) \mathrm{d}\mathcal{H}^{d-1}, \tag{28}$$

where

$$f(y,\nu) = \begin{cases} \varphi(\nu_k) & \text{if } y \in \Pi_{\nu_k} + \mathbb{Z}^d, k = 1, \cdots, N \\ \sum_{\xi \in V} c_\xi |\langle \nu, \xi \rangle| & \text{otherwise,} \end{cases} \tag{29}$$

Γ-converge to E_φ with respect to the strong $L^1(\Omega)$-topology.

In fact, by [8] we have that E_ε Γ-converges as $\varepsilon \to 0$ to $E : BV(\Omega; \{\pm 1\}) \to [0, \infty)$ defined by

$$E(u) = \int_{\partial^*\{u=1\}} f_{\text{hom}}(\nu_u(x)) \mathrm{d}\mathcal{H}^{d-1},$$

where

$$f_{\text{hom}}(\nu) = \lim_{T \to \infty} \frac{1}{T^{d-1}} \inf \Big\{ \int_{\partial^*\{u=1\} \cap TQ_\nu} f(x, \nu_u(x)) \mathrm{d}\mathcal{H}^{d-1} : u \in BV(TQ_\nu; \{\pm 1\}), \\ u = u_\nu \text{ on } \partial TQ_\nu \Big\}.$$

We now prove that $f_{\text{hom}}(\nu) = \varphi(\nu)$.

We first show that $f_{\text{hom}}(\nu) \leq \varphi(\nu)$. Let $u \in BV(TQ, \{\pm 1\})$ be such that $u = u_\nu$ on ∂TQ_ν and define $A_j = \Pi_{\nu_j} + \mathbb{Z}^d$. We know that $\nu_j = \pm \nu_u \, \mathcal{H}^{d-1}$-almost everywhere on $\partial^*\{u = 1\} \cap A_j$ and $\varphi(\nu) = \varphi(-\nu)$. Hence, we get

$$\begin{aligned} &\int_{\partial^*\{u=1\} \cap TQ_\nu} f(x, \nu_u(x)) \mathrm{d}\mathcal{H}^{d-1} \\ &= \sum_{j=1}^N \int_{\partial^*\{u=1\} \cap A_j \cap TQ_\nu} f(x, \nu_u(x)) \mathrm{d}\mathcal{H}^{d-1} + \int_{(\partial^*\{u=1\} \setminus A_j) \cap TQ_\nu} f(x, \nu_u(x)) \mathrm{d}\mathcal{H}^{d-1} \\ &\geq \sum_{j=1}^N \int_{\partial^*\{u=1\} \cap A_j \cap TQ_\nu} \varphi(\nu_u(x)) \mathrm{d}\mathcal{H}^{d-1} + \int_{(\partial^*\{u=1\} \setminus A_j) \cap TQ_\nu} \varphi(\nu_u(x)) \mathrm{d}\mathcal{H}^{d-1} \\ &= \int_{\partial^*\{u=1\} \cap TQ_\nu} \varphi(\nu_u(x)) \mathrm{d}\mathcal{H}^{d-1} \\ &\geq T^{d-1} \varphi(\nu), \end{aligned}$$

where the last inequality follows from BV-ellipticity (see [6]) and a scaling argument. Thus by the definition of f_{hom}^N we get $f_{\text{hom}}(\nu) \geq \varphi(\nu)$.

Now we deal with the inequality $f_{\text{hom}} \leq \varphi$. We have for every $j = 1, \cdots, N$

$$\int_{\Pi_{\nu_j} \cap TQ_{\nu j}} f(x, \nu_u(x)) \mathrm{d}\mathcal{H}^{d-1} = T^{d-1} \varphi(\nu_j),$$

Now since φ is the greatest convex, even positively 1-homogeneous function g such that $g(\nu_j) \leq \varphi(\nu_j)$ for all $j = 1, \cdots, N$ we have that $f_{\text{hom}}(\nu) \leq \varphi(\nu)$ for all $\nu \in S^{d-1}$.

Step 3. Note that for every $k \in \mathbb{N}$ we can write

$$\varphi(\nu_k) = \sum_{\xi \in V} (t_\xi \beta_\xi + (1 - t_\xi)\alpha_\xi)|\langle \nu_k, \xi \rangle| = \sum_{\xi \in V} c_\xi^k |\langle \nu_k, \xi \rangle|, \tag{30}$$

with $\alpha_\xi < c_\xi^k < c_\xi$. Hence, in place of (31) we can consider equivalently

$$f(y, \nu) = \begin{cases} \displaystyle\sum_{\xi \in V} c_\xi^k |\langle \nu, \xi \rangle| & \text{if } y \in \Pi_{\nu_k} + \mathbb{Z}^d, k = 1, \cdots, N \\ \displaystyle\sum_{\xi \in V} c_\xi |\langle \nu, \xi \rangle| & \text{otherwise.} \end{cases} \tag{31}$$

Indeed, for $y \in \Pi_{\nu_k} + \mathbb{Z}$ only $\nu = \nu_k$ is accounted for for $\mathcal{H}^{d-1}$-almost every y.

Every functional of the form (28) can be approximated by a functional $E_{\delta,\varepsilon} : BV(\Omega, \{\pm 1\}) \to [0, +\infty)$ of the form

$$E_{\delta,\varepsilon}(u) = \int_{\partial^* \{u=1\}} f_\delta\Big(\frac{x}{\varepsilon}, \nu_u(x)\Big) \mathrm{d}\mathcal{H}^{d-1} \tag{32}$$

where for $\delta > 0$ $f_\delta : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ is defined by

$$f_\delta(y, \nu) = \begin{cases} \displaystyle\sum_{\xi \in V} c_\xi^k |\langle \nu, \xi \rangle| & \text{if } y \in A_{k,\delta}, y \notin A_{j,\delta} \text{ for all } j \neq k, k = 1, \cdots, N \\ \displaystyle\sum_{\xi \in V} \alpha_\xi |\langle \nu, \xi \rangle| & \text{if } y \in A_{k,\delta} \cap A_{j,\delta} \text{ for some } j, k \in \{1, \cdots, N\}, j \neq k \\ \displaystyle\sum_{\xi \in V} c_\xi |\langle \nu, \xi \rangle| & \text{otherwise,} \end{cases}$$

where $A_{j,\delta} = \{y \in \mathbb{R}^d : \mathrm{dist}_\infty(y, \Pi_{\nu_j} + \mathbb{Z}^d) \leq \delta\}$.

In fact f_δ increasingly converges to f as $\delta \to 0$ on $\mathbb{R}^d \setminus \mathcal{N}$, where

$$\mathcal{N} = \bigcup_{j,k \in \{1, \cdots, N\},\ j \neq k} \Big((\Pi_{\nu_k} + \mathbb{Z}^d) \cap (\Pi_{\nu_j} + \mathbb{Z}^d) \Big)$$

and $\mathcal{H}^{d-1}(\mathcal{N}) = 0$. Hence, by the Monotone Convergence Theorem and by [[22],Proposition 5.4] the claim follows.

Step 4. Every functional of the form (32) can be approximated by a functional $E_{\eta,\delta,\varepsilon} : BV(\Omega, \{\pm 1\}) \to [0, +\infty)$ of the form

$$E_{\eta,\delta,\varepsilon}(u) = \int_{\partial^* \{u=1\}} f_{\eta,\delta}\Big(\frac{x}{\varepsilon}, \nu_u(x)\Big) \mathrm{d}\mathcal{H}^{d-1}, \tag{33}$$

where for $\eta, \delta > 0$ $f_{\eta,\delta} : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ is defined by

$$f_{\eta,\delta}(y,\nu) = \begin{cases} \displaystyle\sum_{\xi\in V} c^k_\xi |\langle \nu, \xi\rangle| & \text{if } y \in A_{k,\delta}, y \notin A_{j,\delta} \text{ for all } j \neq k, k = 1, \cdots, N \\ \displaystyle\sum_{\xi\in V} \alpha_\xi |\langle \nu, \xi\rangle| & \text{if } y \in A_{k,\delta} \cap A_{j,\delta} \text{ for some } j, k \in \{1, \cdots, N\}, j \neq k \\ \displaystyle\sum_{\xi\in V} \beta_\xi |\langle \nu, \xi\rangle| & \text{if } y \in A_{k,\delta+\eta} \setminus A_{k,\delta}, y \notin A_{j,\delta} \text{ for all } j \neq k, k = 1, \cdots, N \\ \displaystyle\sum_{\xi\in V} c_\xi |\langle \nu, \xi\rangle| & \text{otherwise.} \end{cases}$$

In fact $f_{\eta,\delta}$ decreasingly converges to f_δ as $\eta \to 0$. Hence, by the Monotone Convergence Theorem and by [[22],Proposition 5.4] the claim follows.

Step 5. Every functional of the form (33) can be approximated by a functional $E_{n,\eta,\delta,\varepsilon,N} : BV(\Omega, \{\pm 1\}) \to [0, +\infty]$ of the form

$$E_{n,\eta,\delta,\varepsilon}(u) = \frac{1}{4} \sum_{\xi\in V} \sum_{i\in\Omega_{1/n}} \frac{1}{n^{d-1}} c^{n,\eta,\delta,\varepsilon}_{i,\xi} (u_{\frac{i}{n}} - u_{\frac{i+\xi}{n}})^2, \tag{34}$$

where $c^{n,\eta,\delta,\varepsilon}_{i,\xi}$ is n-periodic,

$$|\theta(\{c^{n,\eta,\delta,\varepsilon}_{i,\xi}\}) - \theta_\xi| \leq C(\varepsilon, N)\eta \text{ for all } \xi \in V, \tag{35}$$

and such that $E_{n,\eta,\delta,\varepsilon}$ Γ-converges to $E_{\eta,\delta,\varepsilon}$ as $n \to +\infty$.

By Proposition 10 there exist $\{c^k_{i,\xi}\}$ for $k = 1, \cdots, N$, $c^\alpha_{i,\xi}, c^\beta_{i,\xi}, c^0_{i,\xi}$ T-periodic for some $T \in \mathbb{N}$ such that

$$\theta(\{c^m_{i,\xi}\}) = \theta_\xi \text{ for all } m \in \{\alpha, 0, 1, \cdots, N\}, \quad \theta(\{c^\beta_{i,\xi}\}) = 1,$$

and, for each $m \in \{\alpha, \beta, 0, 1, \cdots, N\}$, $E^m_n : BV(\Omega, \{\pm 1\}) \to [0, +\infty]$, defined by

$$E^m_n(u) = \frac{1}{4} \sum_{\xi\in V} \sum_{i\in\Omega_{1/n}} \frac{1}{n^{d-1}} c^m_{i,\xi} (u_{\frac{i}{n}} - u_{\frac{i+\xi}{n}})^2,$$

Γ-converge with respect to the strong $L^1(\Omega)$-topology to $E^m : BV(\Omega, \{\pm 1\}) \to [0, +\infty)$ defined by

$$E^m(u) = \int_{\partial^*\{u=1\}} \sum_{\xi\in V} c^m_\xi |\langle \nu, \xi\rangle| \mathrm{d}\mathcal{H}^{d-1}, \tag{36}$$

where $c_\xi^\alpha=\alpha_\xi, c_\xi^\beta=\beta_\xi$ and $c_\xi^0=c_\xi$. We define $c_{i,\xi}^{n,\eta,\delta,\varepsilon}$ in $[-\frac{n}{2},\frac{n}{2})^d$ by

$$c_{i,\xi}^{n,\eta,\delta,\varepsilon}=\begin{cases} c_{i,\xi}^k & \text{if } i\in(A_{k,\delta})_{\frac{1}{n}}, i\notin(A_{j,\delta})_{\frac{1}{n}} \text{ for all } j\neq k, k=1,\cdots,N\\ c_{i,\xi}^\alpha & \text{if } i\in(A_{k,\delta}\cap A_{j,\delta})_{\frac{1}{n}} \text{ for some } j,k\in\{1,\cdots,N\}, j\neq k\\ c_{i,\xi}^\beta & \text{if } i\in(A_{k,\delta+\eta}\setminus A_{k,\delta})_{\frac{1}{n}}, i\notin(A_{j,\delta})_{\frac{1}{n}} \text{ for all } j\neq k, k=1,\cdots,N\\ c_{i,\xi}^0 & \text{otherwise,}\end{cases}\tag{37}$$

and extend it n-periodically. Now (35) holds, since

$$\begin{aligned}\theta_\xi\leq\theta(\{c_{i,\xi}^{n,\eta,\delta,\varepsilon}\})\leq\frac{1}{n^2}\Bigg(\#\Big\{i\in\mathbb{Z}^d\cap\Big(\Big[-\frac{n}{2},\frac{n}{2}\Big)^d\setminus\Big(A_{k,\delta+\eta}\setminus A_{k,\delta}\Big)_{\frac{1}{n}}\Big):c_{i,\xi}^{n,\eta,\delta,\varepsilon}=\beta_\xi\Big\}\\ +\#\Big\{i\in\mathbb{Z}^d\cap(A_{k,\delta+\eta}\setminus A_{k,\delta})_{\frac{1}{n}}:c_{i,\xi}^{n,\eta,\delta,\varepsilon}=\beta_\xi\Big\}\Bigg)\\ \leq\theta_\xi+C(d)\frac{N}{\varepsilon^d}\eta.\end{aligned}$$

It remains to show that $E_{n,\eta,\delta,\varepsilon}$ Γ-converges to $E_{\eta,\delta,\varepsilon}$ as $n\to+\infty$.

We set

$$E=\Gamma\text{-}\lim_{n\to+\infty}E_{n,\eta,\delta,\varepsilon},$$

which exists up to subsequences. By [[3],Theorem 4.2] we know that for all $(u,A)\in BV(\Omega,\{\pm1\})\times\mathcal{A}(\Omega)$ we have that

$$E(u,A)=\int_{\partial^*\{u=1\}\cap A}\varphi'(x,\nu_u(x))\mathrm{d}\mathcal{H}^{d-1}$$

for some $\varphi':\Omega\times\mathbb{R}^d\to[0,+\infty)$, where $E(\cdot,\cdot)$ is the localized version of $E(\cdot)$ obtained as the Γ-limit of the corresponding restriction of $E_{n,\eta,\delta,\varepsilon}$. Fix such $u\in BV(\Omega,\{\pm1\})$. We are done if we prove $\varphi'(x,\nu)=f_{\eta,\delta}(\frac{x}{\varepsilon},\nu)$ for $\mathcal{H}^{d-1}$-almost every $x\in\partial^*\{u=1\}$. We know that for $\mathcal{H}^{d-1}$-almost every $x\in\partial^*\{u=1\}$ by the Radon-Nikodym Theorem we have

$$\varphi'(x,\nu)=\lim_{\rho\to0}\frac{1}{\rho^{d-1}}E(u,Q_\rho^\nu(x)).$$

Now, for $\rho>0$ fixed, let $u_{n,\rho}\to u$ in $L^1(Q_\rho^\nu(x))$ as $n\to\infty$ and be such that

$$\lim_{n\to\infty}E_{n,\eta,\delta,\varepsilon}(u_{n,\rho},Q_\rho^\nu(x))=\Gamma\text{-}\lim_{n\to\infty}E_{n,\eta,\delta,\varepsilon}(u,Q_\rho^\nu(x)).$$

We obtain that

$$\begin{aligned}\varphi'(x,\nu)&=\lim_{\rho\to0}\frac{1}{\rho^{d-1}}E(u,Q_\rho^\nu(x))=\lim_{\rho\to0}\frac{1}{\rho^{d-1}}\lim_{n\to\infty}E_{n,\eta,\delta,\varepsilon}(u_{n,\rho},Q_\rho^\nu(x))\\ &=\lim_{\rho\to0}\frac{1}{\rho^{d-1}}\lim_{n\to\infty}\frac{1}{4}\sum_{\xi\in V}\sum_{i\in\left(Q_\rho^\nu(x)\right)_{\frac{1}{n}}}\frac{1}{n^{d-1}}c_{i,\xi}^{n,\eta,\delta,\varepsilon}((u_{n,\rho})_{\frac{i}{n}}-(u_{n,\rho})_{\frac{i+\xi}{n}})^2.\end{aligned}\tag{38}$$

There are five cases to investigate

i) $x \in (A_{k,\delta})^\circ$, $x \notin A_{j,\delta}$ for all $j \neq k$,for some $k = 1, \cdots, N$

ii) $x \in A_{k,\delta} \cap A_{j,\delta}$ for some $j, k \in \{1, \cdots, N\}$ such that $j \neq k$

iii) $x \in (A_{k,\delta+\eta} \setminus A_{k,\delta})^\circ$, $x \notin A_{j,\delta}$ for all $j \neq k$,for some $k = 1, \cdots, N$

iv) $x \in \partial A_{k,\delta}$ or $x \in \partial A_{k,\delta+\eta}$ for some $k = 1, \cdots, N$

v) otherwise

We only consder (i) and (iv). The cases (i)-(iii),(v) are treated analogously.

We first consider (i); i.e., let $x \in (A_{k,\delta})^\circ$ for some $k = 1, \cdots, N$ and $x \notin A_{j,\delta}$ for all $j \neq k, j = 1, \cdots, N$. By definition (37) we have that for ρ small enough

$$E_{n,\eta,\delta,\varepsilon}(u_{n,\rho}, Q_\rho^\nu(x)) = E_n^k(u_{n,\rho}, Q_\rho^\nu(x)),$$

and hence, by (38) and (36), we have that

$$\begin{aligned}\varphi'(x,\nu) &= \lim_{\rho\to 0} \frac{1}{\rho^{d-1}} \lim_{n\to\infty} E_{n,\eta,\delta,\varepsilon}(u_{n,\rho}, Q_\rho^\nu(x)) = \lim_{\rho\to 0} \frac{1}{\rho^{d-1}} \lim_{n\to\infty} E_n^k(u_{n,\rho}, Q_\rho^\nu(x)) \\ &= \lim_{\rho\to 0} \frac{1}{\rho^{d-1}} E(u, Q_\rho^\nu(x)) = f_{\eta,\delta}\Big(\frac{x}{\varepsilon}, \nu\Big).\end{aligned}$$

Now we treat the case (iv); i.e., either $x \in \partial A_{k,\delta}$ for some $k = 1, \cdots, N$ or $x \in \partial A_{k,\delta+\eta}$ for some $k = 1, \cdots, N$. In the first case we have that for ρ small enough $Q_\rho^\nu(x) \subset (A_{k,\delta+\eta})^\circ \setminus \bigcup_{j\neq k} A_{k,\delta+\eta}$ and hence

$$\begin{aligned}\varphi'(x,\nu) &= \lim_{\rho\to 0} \frac{1}{\rho^{d-1}} \liminf_{n\to\infty} E_{n,\eta,\delta,\varepsilon}(u_{n,\rho}, Q_\rho^\nu(x)) \\ &\geq \lim_{\rho\to 0} \frac{1}{\rho^{d-1}} \liminf_{n\to\infty} E_n^k(u_{n,\rho}, Q_\rho^\nu(x)) \\ &= \sum_{\xi\in V} c_\xi^k |\langle \nu, \xi\rangle| = f_{\eta,\delta}\Big(\frac{x}{\varepsilon}, \nu\Big).\end{aligned}$$

We now show that

$$\varphi'(x, \nu^k) \leq f_{\eta,\delta}\Big(\frac{x}{\varepsilon}, \nu^k\Big) \text{ for } \mathcal{H}^{d-1} \text{ almost every } x \in \partial A_{k,\delta}. \tag{39}$$

It is sufficient to prove that the Γ-limit agrees with $E_{\eta,\delta,\varepsilon}$, since $\nu_u = \nu^k$ $\mathcal{H}^{d-1}$-almost everywhere on $\partial A_{k,\delta}$. Let

$$\tilde{d} := \min_{\substack{1\leq j\leq N \\ j\neq k}} \operatorname{dist}_\infty(x, A_{j,\delta}).$$

Fix $r < \frac{1}{2}\min\{\tilde{d}, \delta\}$ small enough, assume that ν^k is the unit outer normal of the set $A_{k,\delta}$ and let $u \in BV_{\mathrm{loc}}(\mathbb{R}^d; \{\pm 1\})$ be defined by

$$u(z) = \chi_{Q^{\nu^k}_{\frac{r}{2}}(x)}\Big(z + \nu^k \frac{r}{2}\Big) - 2. \tag{40}$$

Let $x_n \to x$ be such that $\{x_n\}_n \subset (A_{k,\delta})^\circ$ and

$$\min_{\substack{1 \le j \le N \\ j \ne k}} \mathrm{dist}_\infty(x_n, \partial A_{j,\delta}) \ge \frac{3}{4}\tilde{d}.$$

Define $u_n \in BV_{\mathrm{loc}}(\mathbb{R}^d; \{\pm 1\})$ by

$$u_n(x) = \chi_{Q^{\nu^k}_{\frac{r}{2}}(x_n)}\Big(z + \nu^k \frac{r}{2}\Big) - 2. \tag{41}$$

We have

$$E(u) = \int_{\partial^*\{u=1\} \cap \partial A_{k,\delta}} \varphi'(y, \nu_u) \mathrm{d}\mathcal{H}^{d-1} + \int_{\partial^*\{u=1\} \setminus \partial A_{k,\delta}} \varphi'(y, \nu_u) \mathrm{d}\mathcal{H}^{d-1} \tag{42}$$

and for all $n \in \mathbb{N}$

$$E(u_n) = \int_{\partial^*\{u=1\} \cap \{\nu_{u_n} = \nu_k\}} \varphi'(y, \nu_u) \mathrm{d}\mathcal{H}^{d-1} + \int_{\partial^*\{u=1\} \setminus \{\nu_{u_n} = \nu_k\}} \varphi'(y, \nu_u) \mathrm{d}\mathcal{H}^{d-1} \tag{43}$$

Note that both in (42) and (43) $\nu_u = \nu^k$ in the first terms and that the second terms of (42) and (43) agree, because of (i). We have $u_n \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d; \{\pm 1\})$ and therefore by the lower semicontinuity of E we have

$$\liminf_{n \to +\infty} E(u_n) \ge E(u),$$

so that

$$r^{d-1} f_{\eta,\delta}\Big(\frac{x}{\varepsilon}, \nu^k\Big) = r^{d-1} \liminf_{n \to \infty} f_{\eta,\delta}\Big(\frac{x_n}{\varepsilon}, \nu^k\Big) \ge \int_{Q^{\nu^k}_{\frac{r}{2}}(x) \cap \partial A_{k,\delta}} \varphi'(y, \nu^k) \mathrm{d}\mathcal{H}^{d-1}.$$

Dividing by r^{d-1} and letting $r \to 0$, using the fact that x is a Lebesgue point with respect to $\mathcal{H}^{d-1}\lfloor_{\partial A_{k,\delta}}$, the claim follows. The other case of (iv) can be done analogously and this therefore yields Step 5.

Step 6. By the metrizability properties of Γ-convergence (see [[22],Theorem 10.22]), Steps 1-5 together with a diagonal argument, noting that η_k, N, ε_k can be chosen such that for all $k \in \mathbb{N}$ we have that $\eta_k \frac{N}{\varepsilon_k^d} \leq \sqrt{\eta_k}$, yields that there exists a sequence of coefficients $c_{i,\xi}^{n_k} = c_{i,\xi}^{n_k,\eta_k,\delta_k,\varepsilon_k}$ n_k-periodic such that

$$\theta_\xi(\{c_{i,\xi}^{n_k}\}) \to \theta_\xi \text{ as } k \to \infty \text{ for all } \xi \in V$$

and $E_k : BV(\Omega, \{\pm 1\}) \to [0, +\infty]$ defined by

$$E_k(u) = \frac{1}{4} \sum_{\xi \in V} \sum_{i \in \Omega_{1/n}} \frac{1}{n_k^{d-1}} c_{i,\xi}^{n_k} (u_{\frac{i}{n}} - u_{\frac{i+\xi}{n}})^2, \text{ for all } u \in PC_{\frac{1}{n_k}}(\Omega, \{\pm 1\})$$

Γ-converges as $k \to +\infty$ to the functional $E_\varphi : BV(\Omega, \{\pm 1\}) \to [0, \infty)$ defined by

$$E_\varphi(u) = \int_{\partial^*\{u=1\} \cap \Omega} \varphi(\nu_u(x)) \mathrm{d}\mathcal{H}^{d-1}.$$

This concludes the proof for $\varepsilon = \frac{1}{n_k}$. This sequence can be extended to all $\varepsilon > 0$. □

3.2.3 Proof of the optimality of the construction

In this section we prove that, if for $\varepsilon > 0$ fixed we define φ^ε as the homogenized energy densities of the $\{c_{i,\xi}^\varepsilon\}$ constructed in Theorem 11, then φ_ε converge to φ as $\varepsilon \to 0$. This implies that the bounds are optimal. The main difficulty in proving this result is that each homogenized energy can be described only by an asymptotic formula, so that a passage to the limit as $\varepsilon \to 0$ within this formula would involve an inversion of the order of two limits. Following an argument in [19], we extend our discrete energies to convex energies in a $W^{1,1}$ setting, where an equivalent single cell-problem formula describes the homogenized energy density. For this formula it is easier then to prove the desired convergence.

Theorem 12. *Let $\varphi : \mathbb{R}^d \to [0, +\infty)$ be convex, even, positively 1-homogeneous and such that*

$$\sum_{\xi \in V} \alpha_\xi |\langle \nu, \xi \rangle| \leq \varphi(\nu) \leq \sum_{\xi \in V} (\theta_\xi \beta_\xi + (1 - \theta_\xi)\alpha_\xi) |\langle \nu, \xi \rangle| \tag{44}$$

with $\theta_\xi \in [0, 1]$ for all $\xi \in V$. If $\theta \in [0, 1]$ satisfies (18) then $\varphi \in \boldsymbol{H}_{\alpha,\beta,V}(\theta)$.

To prove Theorem 12 we introduce the localization on regular open sets of $E_\varepsilon : BV(\Omega) \times \mathcal{A}^{reg}(\Omega) \to [0, +\infty]$ of the form

$$E_\varepsilon(u, A) = \begin{cases} \frac{1}{4} \sum_{\xi \in V} \sum_{i, i+\xi \in A_\varepsilon} \varepsilon^{d-1} c_{i,\xi}^\varepsilon (u_{\varepsilon i} - u_{\varepsilon(i+\xi)})^2 & \text{if } u \in \mathcal{PC}_\varepsilon(\Omega, \{\pm 1\}) \\ +\infty & \text{otherwise,} \end{cases} \tag{45}$$

and the localization on regular open sets of an auxiliary functional $F_\varepsilon : BV(\Omega)\times\mathcal{A}^{reg}(\Omega)\to[0,+\infty]$ defined by

$$F_\varepsilon(u,A)=\begin{cases}\frac{1}{2}\sum_{\xi\in V}\sum_{i,i+\xi\in A_\varepsilon}\varepsilon^{d-1}c^\varepsilon_{i,\xi}|u_{\varepsilon i}-u_{\varepsilon(i+\xi)}| & \text{if } u\in\mathcal{PC}_\varepsilon(\Omega)\\ +\infty & \text{otherwise.}\end{cases}\tag{46}$$

Note that F_ε is the positively 1-homogeneous extension of E_ε to $\mathcal{PC}_\varepsilon(\Omega)$, that is to say that for $u\in\mathcal{PC}_\varepsilon(\Omega,\{\pm1\})$ we have $\lambda E_\varepsilon(u,A)=\lambda F_\varepsilon(u,A)=F_\varepsilon(\lambda u,A)$.

Remark 13. By [3], up to subsequences it holds that

$$\Gamma\text{-}\lim_{\varepsilon\to0}E_\varepsilon(u,A)=\int_{\partial^*\{u=1\}\cap A}\varphi(x,\nu)\mathrm{d}\mathcal{H}^{d-1}=:E(u,A)$$

for all $(u,A)\in BV(\Omega,\{\pm1\})\times\mathcal{A}(\Omega)$ and for some $\varphi:\Omega\times\mathbb{R}^d\to[0,+\infty)$ one homogeneous. Analogously as in [3], using [[19], Remark 2.2 and Lemma 4.2] it can be shown that up to the same subsequence

$$\Gamma\text{-}\lim_{\varepsilon\to0}F_\varepsilon(u,A)=\frac{1}{2}\int_A\varphi\,(x,Du)=:F(u,A)$$

for all $(u,A)\in BV(\Omega)\times\mathcal{A}(\Omega)$, where the energy densities of E and F agree. Here we have used the shorthand of

$$\int_A\varphi(x,Du)=\int_A\varphi\Big(x,\frac{dDu}{d|Du|}\Big)\mathrm{d}|Du|.$$

Note that the functionals F_ε satisfy suitable growth conditions; i.e.,

$$\frac{1}{C}|Du|(A)\le F_\varepsilon(u,A)\le C|Du|(A)$$

for all $(u,A)\in PC_\varepsilon(A)\times\mathcal{A}^{reg}(\Omega)$.

The next proposition establishes a cell formula for the Γ-limit of the auxiliary functional to recover the energy density, provided it is homogeneous in the spatial variable. Proposition 15 and Proposition 16 show the convergence of the cell formulas of the approximating (in the sense of Γ-convergence) energies to the cell formula of the limiting energy. Those three propositions will then be used in the proof of Theorem 12.

Proposition 14. *Let $E : BV(\Omega)\times\mathcal{A}(\Omega)\to[0,+\infty]$ be defined by*

$$F(u,A)=\int_A\varphi(Du)\tag{47}$$

for some $\varphi : \mathbb{R}^d \to [0, \infty)$, convex, positively 1-homogeneous and such that

$$\frac{1}{C}|\nu| \leq \varphi(\nu) \leq C|\nu|$$

for some $C > 1$. Assume that $F(\cdot, A)$ is $L^1(A)$-lower semicontinuous for all $A \in \mathcal{A}(\Omega)$, then

$$\varphi(\nu) = \inf\Big\{ \int_{[0,1)^d} \varphi(Du) : u \in BV_{\rm loc}(\mathbb{R}^d) : u - \nu x \text{ 1-periodic} \Big\}. \tag{48}$$

Proof. We prove that

$$\inf\Big\{ \int_{[0,1)^d} \varphi(Du) : u \in BV_{\rm loc}(\mathbb{R}^d) : u - \nu x \text{ 1-periodic} \Big\} \tag{49}$$

$$= \inf\Big\{ \int_{Q} \varphi(Du) : u \in BV_{\rm loc}(\mathbb{R}^d) : u - \nu x \text{ 1-periodic}, |Du|(\partial Q) = 0 \Big\} \tag{50}$$

$$= \inf\Big\{ \int_{Q} \varphi(Du) : u \in W^{1,1}_{\rm loc}(\mathbb{R}^d) : u - \nu x \text{ 1-periodic} \Big\} = \varphi(\nu). \tag{51}$$

which yields the claim.

Note that

$$\begin{aligned}
&\inf\Big\{ \int_{[0,1)^d} \varphi(Du) : u \in BV_{\rm loc}(\mathbb{R}^d) : u - \nu x \text{ 1-periodic} \Big\} \\
\leq &\inf\Big\{ \int_{Q} \varphi(Du) : u \in BV_{\rm loc}(\mathbb{R}^d) : u - \nu x \text{ 1-periodic}, |Du|(\partial Q) = 0 \Big\} \\
\leq &\inf\Big\{ \int_{Q} \varphi(Du) : u \in W^{1,1}_{\rm loc}(\mathbb{R}^d) : u - \nu x \text{ 1-periodic} \Big\} \leq \varphi(\nu),
\end{aligned}$$

since we only decrease the set of admissible test functions in the minimum problems and in the last infimum $u(x) = \nu x$ is admissible.

We now prove that

$$\begin{aligned}
&\inf\Big\{ \int_{[0,1)^d} \varphi(Du) : u \in BV_{\rm loc}(\mathbb{R}^d) : u - \nu x \text{ 1-periodic} \Big\} \\
\geq &\inf\Big\{ \int_{Q} \varphi(Du) : u \in BV_{\rm loc}(\mathbb{R}^d) : u - \nu x \text{ 1-periodic}, |Du|(\partial Q) = 0 \Big\}.
\end{aligned} \tag{52}$$

To this end, let $u \in BV_{\rm loc}(\mathbb{R}^d)$ be such that $u - \nu x$ is 1-periodic. Let $\tau \in \mathbb{R}^d$ be such that $u_\tau \in BV_{\rm loc}(\mathbb{R}^d)$ defined by

$$u_\tau(x) = u(x + \tau)$$

satisfies $|Du_\tau|(\partial Q) = 0$ and

$$\int_Q \varphi(Du_\tau) = \int_{[0,1)^d} \varphi(Du_\tau) = \int_{[0,1)^d} \varphi(Du).$$

Then this yields (52).

Next, we prove that

$$\begin{aligned} &\inf\Big\{ \int_Q \varphi(Du) : u \in BV_{\mathrm{loc}}(\mathbb{R}^d) : u - \nu x \, 1\text{-periodic}, |Du|(\partial Q) = 0 \Big\} \\ \geq &\inf\Big\{ \int_Q \varphi(Du) : u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d) : u - \nu x \, 1\text{-periodic} \Big\}. \end{aligned} \tag{53}$$

Indeed, let $u \in BV_{\mathrm{loc}}(\mathbb{R}^d)$ be such that $u - \nu x$ is 1-periodic, $|Du|(\partial Q) = 0$. Set $u_\varepsilon = u * \rho_\varepsilon$, where $\{\rho_\varepsilon\}_\varepsilon$ is a family of positive symmetric mollifiers. We have that $u_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $|Du_\varepsilon|(Q) \to |Du|(Q)$ as $\varepsilon \to 0$. By the Reshetnyak Continuity Theorem [[7],Theorem 2.39] we have that

$$\lim_{\varepsilon \to 0} \int_Q \varphi(Du_\varepsilon) = \int_Q \varphi(Du)$$

and (53) follows.

We finally prove that

$$\inf\Big\{ \int_Q \varphi(Du) : u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d) : u - \nu x \, 1\text{-periodic} \Big\} \geq \varphi(\nu). \tag{54}$$

Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ be such that $u - \nu x$ is 1-periodic. For $n \in \mathbb{N}$, let $u_n \in W^{1,1}(Q)$ be defined by

$$u_n(x) = \frac{1}{n} u(nx).$$

We then have that $u_n \to \nu x$ in $L^1(Q)$ and

$$\int_Q \varphi(Du_n) \mathrm{d}x = \int_Q \varphi(Du) \mathrm{d}x. \tag{55}$$

By the lower semicontinuity of $F(\cdot, Q)$ we obtain

$$\varphi(\nu) = F(\nu x, Q) \leq \liminf_{n \to +\infty} F(u_n, Q) = \int_Q \varphi(Du) \mathrm{d}x \tag{56}$$

and the claim follows. $\square$

Proposition 15. *Let $F_\varepsilon : BV(\Omega)\times\mathcal{A}(\Omega)\to[0,\infty]$ be defined as in* (46) *and let F_ε Γ-converge with respect to the strong $L^1(\Omega)$-topology to the functional $F : BV(\Omega)\times\mathcal{A}(\Omega)\to[0,\infty]$ given by Remark* 13*. For $\nu\in S^{d-1}$ let*

$$\begin{aligned} m_\varepsilon(\nu) &:= \inf\Big\{\frac{1}{2}\sum_{\xi\in V}\sum_{i\in\left([-\frac{1}{2},\frac{1}{2})^d\right)_\varepsilon}\varepsilon^{d-1}c^\varepsilon_{i,\xi}|u_{\varepsilon i}-u_{\varepsilon(i+\xi)}| \\ &\qquad : u\in\mathcal{PC}_\varepsilon(\mathbb{R}^d), u-\nu x \text{ 1-periodic}\}, \\ m(\nu) &:= \inf\big\{F(u,[0,1)^d) : u\in BV_{\rm loc}(\mathbb{R}^d)\,, u-\nu x \text{ 1-periodic}\big\}. \end{aligned}$$

Then $\limsup_{\varepsilon\to0} m_\varepsilon(\nu)\le m(\nu)$.

Proof. Fix $\eta>0$ and let $M:=2\sup_{\xi\in V}||\xi||_\infty$. Let $u^\eta_\varepsilon\to\nu x$ be such that

$$\limsup_{\varepsilon\to0}F_\varepsilon(u^\eta_\varepsilon,(1+\eta)Q)\le F(\nu x,(1+\eta Q)\le(1+\eta)^d\varphi(\nu). \tag{57}$$

We now modify u^η_ε so that it can be used as a test function for $m_\varepsilon(\nu)$. Set

$$\delta_\varepsilon=\int_Q|u^\eta_\varepsilon-\nu x|\mathrm{d}x$$

and let $\varepsilon>0$ be small enough, let $k_\varepsilon\in\mathbb{N}$ be such that

$$\frac{\delta_\varepsilon}{\varepsilon}<<k_\varepsilon<<\frac{\eta}{\varepsilon}$$

and set $Q^i_{\eta,\varepsilon}=Q_{1-\eta-i\varepsilon M}$. Then, we get

$$\delta_\varepsilon\ge\int_{Q\setminus Q_{(1-2\eta)}}|u^\eta_\varepsilon-\nu x|\mathrm{d}x\ge\sum_{i=0}^{k_\varepsilon-1}\int_{Q^i_{\eta,\varepsilon}\setminus Q^{i+1}_{\eta,\varepsilon}}|u^\eta_\varepsilon-\nu x|\mathrm{d}x,$$

so that there exists $i_\varepsilon\in\{0,\cdots,k_\varepsilon\}$ such that

$$\sum_{i\in\left(Q^{i_\varepsilon}_{\eta,\varepsilon}\setminus Q^{i_\varepsilon+1}_{\eta,\varepsilon}\right)_\varepsilon}\varepsilon^d|(u^\eta_\varepsilon)_{\varepsilon i}-(\nu x)_{\varepsilon i}|\le C\int_{Q^{i_\varepsilon}_{\eta,\varepsilon}\setminus Q^{i_\varepsilon+1}_{\eta,\varepsilon}}|u^\eta_\varepsilon-\nu x|\mathrm{d}x<<\frac{\delta_\varepsilon}{k_\varepsilon}<<\varepsilon. \tag{58}$$

Now define $v^\eta_\varepsilon\in\mathcal{PC}_\varepsilon((1+\eta)Q)$ by

$$v^\eta_\varepsilon(i)=\begin{cases}u^\eta_\varepsilon(i) & i\in Q^{i_\varepsilon}_{\eta,\varepsilon}\cap\varepsilon\mathbb{Z}^d\\ \nu i & \text{otherwise on } \varepsilon\mathbb{Z}^d.\end{cases}$$

Note that, since $v_\varepsilon^\eta = \nu i$ on $(1+\eta)Q \setminus (1-\eta)Q$, it can be extended to the whole of $\mathbb{R}^d$ in such a way that $v_\varepsilon^\eta - \nu x$ is 1-periodic. Thus, we have

$$
\begin{aligned}
m_\varepsilon(\nu) &\leq \sum_{\xi \in V} \sum_{i \in \left([-\frac{1}{2},\frac{1}{2})^d\right)_\varepsilon} \varepsilon^{d-1} c_{i,\xi}^\varepsilon |(v_\varepsilon^\eta)_{\varepsilon i} - (v_\varepsilon^\eta)_{\varepsilon(i+\xi)}| \leq F_\varepsilon(v_\varepsilon^\eta, (1+\eta)Q) \\
&\leq F_\varepsilon(u_\varepsilon^\eta, (1+\eta)Q) + F_\varepsilon(\nu x, (1+\eta)Q \setminus (1-2\eta)Q) \\
&+ \sum_{\xi \in V} \sum_{i \in \left(Q_{\eta,\varepsilon}^{i_\varepsilon} \setminus Q_{\eta,\varepsilon}^{i_\varepsilon+1}\right)_\varepsilon} \varepsilon^{d-1} c_{i,\xi}^\varepsilon \left(|(u)_{\varepsilon i} - (\nu x)_{\varepsilon(i+\xi)}| + |(u)_{\varepsilon(i+\xi)} - (\nu x)_{\varepsilon i}| \right).
\end{aligned}
$$

Noting that

$$
|u_{\varepsilon(i+\xi)} - (v)_{\varepsilon i}| \leq |(u)_{\varepsilon(i+\xi)} - (v)_{\varepsilon(i+\xi)}| + |(v)_{\varepsilon(i+\xi)} - (v)_{\varepsilon i}|,
$$

using (58), and the growth conditions in Remark 13 we obtain

$$
m_\varepsilon(\nu) \leq F_\varepsilon(u_\varepsilon^\eta, (1+\eta)Q) + C\eta^d + o(1).
$$

Therefore by (57) we obtain

$$
\limsup_{\varepsilon \to 0} m_\varepsilon(\nu) \leq (1+\eta)^d \varphi(\nu) + C\eta^d.
$$

The claim follows by letting $\eta \to 0$. $\square$

Proposition 16. *Let $F_\varepsilon : BV(\Omega) \times \mathcal{A}(\Omega) \to [0,\infty]$ be defined as in (46) with $c_{i,\xi}^\varepsilon$ $\frac{1}{\varepsilon}$-periodic and let F_ε Γ-converge with respect to the strong $L^1(\Omega)$-topology to the functional $F : BV(\Omega) \times \mathcal{A}(\Omega) \to [0,\infty]$ given by Remark* 13. *Then* $\liminf_{\varepsilon \to 0} m_\varepsilon(\nu) \geq m(\nu)$.

Proof. Let $u^\varepsilon \in \mathcal{PC}_\varepsilon(\mathbb{R}^d)$ be such that $u_\varepsilon - \nu x$ is 1-periodic, $\int_Q u_\varepsilon = 0$ and

$$
\sum_{\xi \in V} \sum_{i \in \left([-\frac{1}{2},\frac{1}{2})^d\right)_\varepsilon} \varepsilon^{d-1} c_{i,\xi}^\varepsilon |(u^\varepsilon)_{\varepsilon i} - (u^\varepsilon)_{\varepsilon(i+\xi)}| \leq m_\varepsilon(\nu) + \varepsilon. \tag{59}
$$

Since $u^\varepsilon - \nu x$ is 1-periodic, $c_{i,\xi}^\varepsilon$ are $\frac{1}{\varepsilon}$-periodic, and by the growth condition of F_ε we have that

$$
\sup_{\varepsilon > 0} |Du^\varepsilon|(B_R) \leq C_R < +\infty.
$$

Hence, by the Poincaré inequality we have that

$$
\sup_{\varepsilon > 0} ||u^\varepsilon||_{BV(B_R)} < +\infty,
$$

and, therefore, up to subsequences we have that $u^\varepsilon \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $u \in BV_{\mathrm{loc}}(\mathbb{R}^d)$ and $u - \nu x$ is 1-periodic. In order to use u as a test function for m, which can be compared to m_ε, it is necessary to translate it so that it does not concentrate energy on the boundary. Choose $x_0 \in \mathbb{R}^d$, such that $|Du|(\partial Q(x_0)) = 0$. By the 1-periodicity of $u_\varepsilon - \nu x$ and $u - \nu x$, and the $\frac{1}{\varepsilon}$-periodicity of $c^\varepsilon_{i,\xi}$we have that

$$\sum_{\xi\in V}\sum_{i\in\left([-\frac{1}{2},\frac{1}{2})^d\right)_\varepsilon} c^\varepsilon_{i,\xi}|(u^\varepsilon)_{\varepsilon i} - (u^\varepsilon)_{\varepsilon(i+\xi)}| = \sum_{\xi\in V}\sum_{i\in\left(x_0+[-\frac{1}{2},\frac{1}{2})^d\right)_\varepsilon} \varepsilon^{d-1}c^\varepsilon_{i,\xi}|(u^\varepsilon)_{\varepsilon i} - (u^\varepsilon)_{\varepsilon(i+\xi)}|.$$

Furthermore, we have that

$$\int_{[-\frac{1}{2},\frac{1}{2})^d}\varphi(Du) = \int_{x_0+[-\frac{1}{2},\frac{1}{2})^d}\varphi(Du) = \int_{Q(x_0)}\varphi(Du).$$

Using (59) and using that F_ε Γ-converges to F we have that

$$\begin{aligned}
\liminf_{\varepsilon\to 0} m_\varepsilon(\nu) &\geq \liminf_{\varepsilon\to 0}\frac{1}{2}\sum_{\xi\in V}\sum_{i\in\left([-\frac{1}{2},\frac{1}{2})^d\right)_\varepsilon}\varepsilon^{d-1}c^\varepsilon_{i,\xi}|(u^\varepsilon)_{\varepsilon i} - (u^\varepsilon)_{\varepsilon(i+\xi)}| \\
&= \liminf_{\varepsilon\to 0}\frac{1}{2}\sum_{\xi\in V}\sum_{i\in\left(x_0+[-\frac{1}{2},\frac{1}{2})^d\right)_\varepsilon}\varepsilon^{d-1}c^\varepsilon_{i,\xi}|(u^\varepsilon)_{\varepsilon i} - (u^\varepsilon)_{\varepsilon(i+\xi)}| \\
&\geq \liminf_{\varepsilon\to 0} F_\varepsilon(u^\varepsilon, Q(x_0)) \\
&\geq F(u, Q(x_0)) = \int_{[-\frac{1}{2},\frac{1}{2})^d}\varphi(Du) \geq m(\nu),
\end{aligned}$$

and the claim follows. □

Proof of Theorem 12. Let $\theta \in [0,1]$, and let φ be given as in Theorem 12. By Theorem 11, we know that there exists $\{c^\varepsilon_{ij}\}_\varepsilon$,which we may assume $\frac{1}{\varepsilon}$-periodic, with $\theta(\{c^\varepsilon_{i,\xi}\}) \to \theta$ and such that the sequence $E_{1,\varepsilon}$ Γ-converges with respect to the strong $L^1(\Omega)$-topology to E, where $E_{\eta,\varepsilon} : P_{\eta\varepsilon}(\Omega, \{\pm1\}) \times \mathcal{A}(\Omega) \to [0, +\infty]$ is defined by

$$E_{\eta,\varepsilon}(u, A) = \frac{1}{4}\sum_{\xi\in V}\sum_{i,i+\xi\in A_{\eta\varepsilon}}(\eta\varepsilon)^{d-1}c^\varepsilon_{i,\xi}(u_{\eta\varepsilon i} - u_{\eta\varepsilon(i+\xi)})^2$$

and $E : BV(\Omega, \{\pm1\}) \to [0, +\infty]$ is defined by

$$E(u, A) = \int_{\partial^*\{u=1\}\cap A}\varphi(\nu_u(x))\mathrm{d}\mathcal{H}^{d-1}.$$

Introducing the auxiliary functionals $F_{\eta,\varepsilon} : P_{\eta\varepsilon}(\Omega)\times\mathcal{A}(\Omega)\to[0,+\infty)$ as in (46) by [[3],Theorem 4.7] up to subsequences it holds that

$$\Gamma\text{-}\lim_{\eta\to 0} E_{\eta,\varepsilon}(u,A)=\int_{\partial^*\{u=1\}\cap A}\varphi_\varepsilon(\nu_u(x))\mathrm{d}\mathcal{H}^{d-1}=: E_\varepsilon(u,A). \tag{60}$$

By Remark 13

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_{1,\varepsilon}(u,A)=\frac{1}{2}\int_A\varphi(Du)=: F(u,A), \tag{61}$$

$$\Gamma\text{-}\lim_{\eta\to 0} F_{\eta,\varepsilon}(u,A)=\frac{1}{2}\int_A\varphi_\varepsilon(Du)=: F_\varepsilon(u,A). \tag{62}$$

The normalization factor $\frac{1}{2}$ appears in order that the functionals E and F agree on functions in $BV(\Omega,\{\pm1\})$ (as mentioned in Remark 13). By (60), noting that the period of the coefficients is fixed with fixed ε, we have that $\varphi_\varepsilon\in\mathbf{H}_{\alpha,\beta,V}(\theta)$.

We then have that

$$\begin{aligned}&\inf\Big\{\sum_{\xi\in V}\sum_{i\in\;[-\frac{1}{2\varepsilon\eta},\frac{1}{2\varepsilon\eta})^d\cap\mathbb{Z}^d}(\varepsilon\eta)^{d-1}c^\varepsilon_{i,\xi}|u_{\varepsilon\eta i}-u_{\varepsilon\eta(i+\xi)}|: u\in\mathcal{PC}_{\varepsilon\eta}(\mathbb{R}^d), u-\nu x 1\text{-periodic}\Big\}\\ =\;&\inf\Big\{\sum_{\xi\in V}\sum_{i\in\left([-\frac12,\frac12)^d\right)_\varepsilon}\varepsilon^{d-1}c^\varepsilon_{i,\xi}|u_{\varepsilon i}-u_{\varepsilon(i+\xi)}|: u\in\mathcal{PC}_{\varepsilon}(\mathbb{R}^d), u-\nu x 1\text{-periodic}\Big\}.\end{aligned} \tag{63}$$

In fact, for every $u\in\mathcal{PC}_{\varepsilon\eta}(\mathbb{R}^d)$ with $u-\nu x$ 1-periodic we can define $\tilde u\in\mathcal{PC}_\varepsilon(\mathbb{R}^d)$, $\tilde u-\nu x$ 1-periodic by setting

$$\tilde u(z)=\eta^{d-1}\sum_{i\in\left([-\frac12,\frac12)^d\right)_\eta}u(\eta(z+i)).$$

By convexity, positive 1-homogeneity and the periodicity of $\{c^\varepsilon_{i,\xi}\}$ the inequality

$$\sum_{\xi\in V}\sum_{i\in\left([-\frac12,\frac12)^d\right)_{\varepsilon\eta}}(\varepsilon\eta)^{d-1}c^\varepsilon_{i,\xi}|u_{\varepsilon\eta i}-u_{\varepsilon\eta(i+\xi)}|\geq\sum_{\xi\in V}\sum_{i\in\left([-\frac12,\frac12)^d\right)_\varepsilon}\varepsilon^{d-1}c^\varepsilon_{i,\xi}|\tilde u_{\varepsilon i}-\tilde u_{\varepsilon(i+\xi)}|$$

holds. On the other hand, for every $u\in\mathcal{PC}_\varepsilon(\mathbb{R}^d)$ with $u-\nu x$ 1-periodic, one defines $\tilde u\in\mathcal{PC}_{\varepsilon\eta}(\mathbb{R}^d)$, $\tilde u-\nu x$ 1-periodic by

$$\tilde u(z)=\eta\, u\Big(\frac{z}{\eta}\Big).$$

By convexity, positive 1-homogeneity and the periodicity of the $c^\varepsilon_{i,\xi}$

$$\sum_{\xi\in V}\sum_{i\in\left([-\frac12,\frac12)^d\right)_{\varepsilon\eta}}(\varepsilon\eta)^{d-1}c^\varepsilon_{i,\xi}|\tilde u_{\varepsilon\eta i}-\tilde u_{\varepsilon\eta(i+\xi)}|\leq\sum_{\xi\in V}\sum_{i\in\left([-\frac12,\frac12)^d\right)_\varepsilon}\varepsilon^{d-1}c^\varepsilon_{i,\xi}|u_{\varepsilon i}-u_{\varepsilon(i+\xi)}|$$

holds. By (15), (16), (63) and (61) for all $\nu \in S^{d-1}$ we have

$$\lim_{\varepsilon\to 0} \frac{1}{2}\varphi_\varepsilon(\nu) = \lim_{\varepsilon\to 0} m_\varepsilon(\nu) = m(\nu) = \frac{1}{2}\varphi(\nu)$$

and therefore $\varphi \in \mathbf{H}_{\alpha,\beta,V}(\theta)$. This proves the claim. $\square$

4 A localization principle

The goal of this section is the computation of the *G-closure* of mixtures subject to the design constraint; i.e., all possible limits of mixtures, where the interaction coefficients $\{c^\varepsilon_{i,\xi}\}$ need not be periodic anymore. We show a *localization principle*, which states that this computation can be reduced to the optimal bounds of periodic mixtures. We state this in the two theorems below. Before stating them we preliminarily remark that among the integrands giving the same energy we may consider that characterized by a derivation formula, so that the integrand is determined as a limit of minimum problems involving the energy. Many proofs in the following will be then reduced to proving properties of those minima.

Remark 17. We deal with surface energies $E : BV(\Omega; \{\pm 1\}) \times \mathcal{A}(\Omega) \to [0, \infty)$ of the form

$$E(u, A) = \int_{\partial^*\{u=1\}\cap A} g(x, \nu_u(x))\mathrm{d}\mathcal{H}^{d-1}, \tag{64}$$

where $g : \Omega \times \mathbb{R}^d \to [0, \infty)$ satisfies

$$\frac{1}{C}|\nu| \leq g(x, \nu) \leq C|\nu| \tag{65}$$

for all $(x, \nu) \in \Omega \times \mathbb{R}^d$ and $E(\cdot, A)$ is $L^1(A)$-lower semicontinuous.

Let $\varphi : \Omega \times \mathbb{R}^d \to [0, +\infty)$ defined by

$$\varphi(x, \nu) = \limsup_{\rho\to 0} \frac{m(x, \nu, \rho)}{w_{d-1}r^{d-1}}, \tag{66}$$

where $m(x, \nu, \rho) : \Omega \times \mathbb{R}^d \times (0, \mathrm{dist}(x, \partial\Omega)) \to [0, \infty)$ is the one homogeneous extension in the second variable of

$$\begin{aligned} m(x, \nu, \rho) = \inf\{E(v, B_\rho(x)) : v \in BV(\Omega; \{\pm 1\}) \\ v = u_{x,\nu} \text{ in a neighborhood of } \partial B_\rho(x)\}. \end{aligned} \tag{67}$$

By [18] we have that

$$E(u, A) = \int_{\partial^*\{u=1\}\cap A} \varphi(x, \nu_u(x))\mathrm{d}\mathcal{H}^{d-1} \tag{68}$$

for all $(u, A) \in BV(\Omega; \{\pm 1\}) \times \mathcal{A}(\Omega)$, so that φ is equivalent to g in the sense that they define the same functional in $BV(\Omega; \{\pm 1\}) \times \mathcal{A}(\Omega)$.

In this section we write $c^\varepsilon_{i,\xi} \in \{\alpha_\xi, \beta_\xi\}^{\Omega_\varepsilon}$ as a shorthand for a system of bonds $\{c^\varepsilon_{i,\xi} : i \in \Omega_\varepsilon, \xi \in V\}$ satisfying the design constraint $c^\varepsilon_{i,\xi} \in \{\alpha_\xi, \beta_\xi\}$ for all i and ξ. For such a system we define (with abuse of notation) the *local volume fraction of β-bonds* by

$$\theta_\xi(\{c^\varepsilon_{i,\xi}\}) = \sum_{\substack{i\in\Omega_\varepsilon \\ c^\varepsilon_{i,\xi}=\beta_\xi}} \varepsilon^d \delta_{\varepsilon i}, \quad \theta(\{c^\varepsilon_{i,\xi}\}) = \frac{1}{\#V} \sum_{\xi\in V} \sum_{\substack{i\in\Omega_\varepsilon \\ c^\varepsilon_{i,\xi}=\beta_\xi}} \varepsilon^d \delta_{\varepsilon i}. \tag{69}$$

We can now state the two results that characterize all the limits of such systems in terms of the *limit local volume fraction of β-bonds* θ and the bonds describing $\mathbf{H}_{\alpha,\beta,V}(\theta(x))$ at almost every x. Namely, Theorem 18 establishes the fact that at almost every $x \in \Omega$ we can reduce to the periodic setting and Theorem 19 establishes the optimality of this condition; i.e., every surface energy whose energy density satisfies for almost every $x \in \Omega$ that $\varphi(x, \cdot) \in \mathbf{H}_{\alpha,\beta,V}(\theta(x))$ for some measurable function $0 \le \theta \le 1$ can be recovered as the Γ-limit of some discrete energies of the form (2), whose local volume fractions $\theta(\{c^\varepsilon_{i,\xi}\})$ of β-bonds converge (weakly*) to the limiting volume fraction θ. Note that the assumptions of Theorem 18 are always satisfied, up to a subsequence.

Theorem 18. *Let $\{c^\varepsilon_{i,\xi}\}_\varepsilon \in \{\alpha_\xi, \beta_\xi\}^{\Omega_\varepsilon}$ and let*

$$E_\varepsilon(u) = \frac{1}{4} \sum_{\xi\in V} \sum_{i,i+\xi\in\Omega_\varepsilon} \varepsilon^{d-1} c^\varepsilon_{i,\xi} (u_{\varepsilon i} - u_{\varepsilon(i+\xi)})^2. \tag{70}$$

Assume that $\theta(\{c^\varepsilon_{i,\xi}\}) \overset{*}{\rightharpoonup} \theta$ and E_ε Γ-converges to $E : BV(\Omega; \{\pm 1\}) \to [0, \infty)$ given by

$$E(u) = \int_{\partial^*\{u=1\}\cap\Omega} \varphi(x, \nu_u(x)) \mathrm{d}\mathcal{H}^{d-1}$$

with φ satisfying (66). *Then $\varphi(x, \cdot) \in \boldsymbol{H}_{\alpha,\beta,V}(\theta(x))$ for almost every $x \in \Omega$.*

Theorem 19. *Let $\theta : \Omega \to [0, 1]$ be measurable and $\varphi : \Omega \times \mathbb{R}^d \to [0, +\infty)$ be positively* 1*-homogeneous and even in the second variable such that the trivial bounds* (19) *are satisfied and $\varphi(x, \cdot) \in \boldsymbol{H}_{\alpha,\beta,V}(\theta(x))$ for almost every $x \in \Omega$. Then there exist $\{c^\varepsilon_{i,\xi}\}_\varepsilon \in \{\alpha_\xi, \beta_\xi\}^{\Omega_\varepsilon}$ such that $E_\varepsilon : L^1(\Omega) \to [0, +\infty]$ Γ-converges as $\varepsilon \to 0$ with respect to the strong $L^1(\Omega)$-topology to $E : L^1(\Omega) \to [0, +\infty]$, where*

$$E_\varepsilon(u) = \begin{cases} \frac{1}{4} \sum_{\xi\in V} \sum_{i,i+\xi\in\Omega_\varepsilon} \varepsilon^{d-1} c^\varepsilon_{i,\xi} (u_{\varepsilon i} - u_{\varepsilon(i+\xi)})^2 & u \in \mathcal{PC}_\varepsilon(\Omega; \{\pm 1\}) \\ +\infty & \textit{otherwise}, \end{cases}$$

$$E(u) = \begin{cases} \int_{\partial^*\{u=1\}\cap\Omega} \varphi(x, \nu_u(x)) \mathrm{d}\mathcal{H}^{d-1}, & u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \textit{otherwise}, \end{cases}$$

and $\theta(\{c^\varepsilon_{i,\xi}\}) \overset{*}{\rightharpoonup} \theta$ as $\varepsilon \to 0$.

4.1 Proof of Theorem 18

The proof of Theorem 18 consists in proving two estimates. The lower bound follows by direct comparison with the energy with interaction coefficients $c^{\varepsilon}_{i,\xi} = \alpha_\xi$ for all $\xi \in V, i \in \mathbb{Z}^d$, while the upper bound is obtained by a blow-up argument. To obtain the correct estimate from above we select suitable plane-like competitors, similar as in the proof of Proposition 8, to get the upper bound in terms of local volume averages of the interaction coefficients.

Proof of Theorem 18. By Remark 13 and [[18] Remark 3.8] we have that $\varphi(x,\cdot)$ is convex, even and positively 1-homogeneous. Thus it suffices to show that

$$\sum_{\xi\in V} \alpha_\xi |\langle \nu,\xi\rangle| \le \varphi(x,\nu) \le \sum_{\xi\in V} (\theta_\xi(x)\beta_\xi + (1-\theta_\xi(x))\alpha_\xi)|\langle \nu,\xi\rangle| \tag{71}$$

for almost every $x\in\Omega$ and with the weak*-limit θ of $\theta(\{c^{\varepsilon}_{i,\xi}\})$ satisfying (18). Note that the lower bound in (71) is trivial since $c^{\varepsilon}_{i,\xi} \ge \alpha_\xi$.

We have that $\theta(\{c^{\varepsilon}_{i,\xi}\}) \overset{*}{\rightharpoonup} \theta_\xi$, $\theta_\xi \in [0,1]$ and $\frac{1}{\#V}\sum_{\xi\in V}\theta_\xi = \theta$. We prove the estimate for all points in E where

$$E := \{x\in\Omega : \varphi(x,\cdot) \text{ is convex and } x \text{ is a Lebesgue point for } \theta_\xi \text{ for all } \xi\in V\}.$$

Since $\varphi(x,\cdot)$ is convex, it suffices to prove that for all $\nu = \frac{v}{||v||}$ with $v\in V$ we have

$$\varphi\left(x,\nu\right) \le \sum_{\xi\in V} (\theta_\xi(x)\beta_\xi + (1-\theta_\xi(x))\alpha_\xi)|\langle \nu,\xi\rangle| =: \psi\left(x,\nu\right), \tag{72}$$

since $\psi(x,\nu) = \sup\{g : g\in D_1(\mathbb{R}^d), g(v)\le \psi\left(x,v\right) \text{ for all } v\in V\}$.

We know that for all $x\in E$ and for all $\delta>0$, $\xi, v\in V$ we have

$$\lim_{\rho\to 0}\frac{1}{|B^v_{\rho,\delta}(x)|}\int_{B^v_{\rho,\delta}(x)} |\theta_\xi(y)-\theta_\xi(x)|\mathrm{d}y = 0,$$

where $B^v_{\rho,\delta}(x) = B_\rho(x)\cap\{y\in\mathbb{R}^d : |\langle y-x, \frac{v}{||v||}\rangle| \le \rho\delta\}$.

We now prove (72) for $v\in V$. To this end we construct a suitable competitor in the minimum problem for $m_\varepsilon(x,\frac{v}{||v||},\rho)$. Note that by [[3],Theorem 4.9] for suitable $\rho\to 0$ we have $m_\varepsilon(x,\frac{v}{||v||},\rho) \to m(x,\frac{v}{||v||},\rho)$ as $\varepsilon\to 0$. Let $k\in (B^v_{\rho,\delta}(x))_\varepsilon$ and let $u^{k,\rho}_\varepsilon \in \mathcal{PC}_\varepsilon(B_\rho,\{\pm1\})$ be defined by

$$u^{k,\rho}_\varepsilon(z) = \begin{cases} u_{\varepsilon k,\frac{v}{||v||}}(z) & \mathrm{dist}(z,\partial B_\rho(x)) > \rho\delta, \\ u_{x,\frac{v}{||v||}}(z) & \text{otherwise.}\end{cases}$$

We then have

$$E_\varepsilon(u_\varepsilon^{k,\rho}, B_\rho(x)) \le \sum_{\xi\in V} \sum_{i\in\Pi^\xi_{\frac{v}{||v||}}(k)\cap(B_\rho(x))_\varepsilon} \varepsilon^{d-1} c^\varepsilon_{i,\xi} + O(\delta\rho^{d-1}),$$

where $O(\delta\rho^{d-1})$ takes into account the contribution due to the interactions close to the boundary of $B_\rho(x)$. We choose $k_0 = k_0^{\rho,\delta} \in (B^v_{\rho,\delta}(x))_\varepsilon$ such that

$$E_\varepsilon(u_\varepsilon^{k_0,\rho}, B_\rho(x)) \le \frac{1}{\# B^v_{\rho,\delta}(x))_\varepsilon} \sum_{k\in B^v_{\rho,\delta}(x))_\varepsilon} E_\varepsilon(u_\varepsilon^{k,\rho}, B_\rho(x)),$$

so that

$$\begin{aligned}
m_\varepsilon\Big(x, \frac{v}{||v||}, \rho\Big) &\le E_\varepsilon(u_\varepsilon^{k_0,\rho}, B_\rho(x)) \\
&\le \frac{1}{\#(B^v_{\rho,\delta}(x))_\varepsilon} \sum_{\xi\in V} \sum_{k\in(B^v_{\rho,\delta}(x))_\varepsilon} \sum_{i\in\Pi^\xi_{\frac{v}{||v||}}(k)\cap(B_\rho)_\varepsilon} \varepsilon^{d-1} c^\varepsilon_{i,\xi} + O(\delta\rho^{d-1}) \\
&\le \frac{\varepsilon^d}{|B^v_{\rho,\delta}(x)|} \sum_{\xi\in V} \sum_{i\in(B_\rho)_\varepsilon} \varepsilon^{d-1} c^\varepsilon_{i,\xi} \#\{k\in(B^v_{\rho,\delta}(x))_\varepsilon : i\in\Pi^\xi_{\frac{v}{||v||}}(k)\} + O(\delta\rho^{d-1}) \\
&\le \frac{\varepsilon^d}{|B^v_{\rho,\delta}(x)|} \sum_{\xi\in V} \sum_{i\in(B^v_{\rho,\delta}(x))_\varepsilon} \varepsilon^{d-1} c^\varepsilon_{i,\xi} \Big|\Big\langle \frac{v}{||v||}, \xi\Big\rangle\Big| w_{d-1} \frac{\rho^{d-1}}{\varepsilon^{d-1}} + O(\delta\rho^{d-1}) \\
&= \frac{w_{d-1}\rho^{d-1}}{|B^v_{\rho,\delta}(x)|} \sum_{\xi\in V} \Big(\theta^\varepsilon_\xi(B^v_{\rho,\delta}(x))\beta_\xi \\
&\qquad + (|B^v_{\rho,\delta}(x)| - \theta^\varepsilon_\xi(B^v_{\rho,\delta}(x))\alpha_\xi\Big) \Big|\Big\langle \frac{v}{||v||}, \xi\Big\rangle\Big| + O(\delta\rho^{d-1}).
\end{aligned}$$

Dividing by $w_{d-1}\rho^{d-1}$, taking the limit as $\varepsilon \to 0$, the lim sup as $\rho \to 0$ and using the weak* convergence of measures, together with the fact that $\theta_\xi(\partial B^v_{\rho,\delta}(x)) = 0$, we obtain that

$$\begin{aligned}
\varphi\Big(x, \frac{v}{||v||}\Big) &\le \sum_{\xi\in V} \limsup_{\rho\to 0} \Big(\frac{1}{|B^v_{\rho,\delta}(x)|} \int_{B^v_{\rho,\delta}(x)} \theta_\xi(y)\mathrm{d}y \beta_\xi \\
&\qquad + \Big(1 - \frac{1}{|B^v_{\rho,\delta}(x)|} \int_{B^v_{\rho,\delta}(x)} \theta_\xi(y)\mathrm{d}y\Big)\alpha_\xi\Big) |\langle \frac{v}{||v||}, \xi\rangle| + O(\delta) \\
&= \sum_{\xi\in V} (\theta_\xi(x)\beta_\xi + (1-\theta_\xi(x))\alpha_\xi) \Big|\Big\langle \frac{v}{||v||}, \xi\Big\rangle\Big| + O(\delta).
\end{aligned}$$

The claim follows by letting $\delta \to 0$. □

4.2 Proof of Theorem 19

The proof of Theorem 19 is performed by a diagonalization argument. Theorem 23 and Lemma 24 below are used in order to be able to reduce to surface energy densities $\varphi : \Omega \times \mathbb{R}^d \to [0, +\infty)$ that are lower-semicontinuous in the first variable. In order to construct the converging sequence $\{\varphi_n\}_n$ we use a Lusin-type argument in order to obtain φ_n that agree with φ every outside a set of arbitrarily small Lebesgue measure and such that the $\varphi_n(\cdot, \nu)$ are lower-semicontinuous. Furthermore we construct φ_n in such a way that the minimum problems $m_{\varphi_n}(x, \nu, \rho)$ defined similarly as in (67) converge to the corresponding minimum problems $m_\varphi(x, \nu, \rho)$ on a dense subset of $\Omega \times S^{d-1} \times \mathrm{dist}(x, \partial\Omega)$. This together with Theorem 23 ensures the Γ-convergence of the energies.

By approximation this allows us to reduce further to energies whose integrands φ are continuous in the first variable. For these energies we use the results obtained in Section 3 to construct coefficients $c^\varepsilon_{i,\xi}$ whose energies Γ-converge to the continuum energy that φ defines. Furthermore the local volume fraction of β-bonds also converges. This construction is possible due to the continuity of $\varphi(\cdot, \nu)$, since this ensures that the choice of coefficients only varies on large scales.

We need first to establish some properties of the function m defined in (67).

Proposition 20. *The following statements hold true:*

i) For all $x \in \Omega$ and $\nu \in S^{d-1}$ there exists a countable set $E(x, \nu) \subset (0, \mathrm{dist}(x, \partial\Omega))$ such that $\rho \mapsto m(x, \nu, \rho)$ is continuous on $(0, \mathrm{dist}(x, \partial\Omega)) \setminus E(x, \nu)$.

ii) there exists a modulus of continuity $w : [0, \infty) \to [0, \infty)$ such that

$$|m(x, \nu_1, \rho) - m(x, \nu_2, \rho)| \leq \rho^{d-1} w(|\nu_1 - \nu_2|)$$

for all $x \in \Omega$, $\nu_1, \nu_2 \in S^{d-1}$ and $\rho \in (0, \mathrm{dist}(x, \partial\Omega))$.

iii) Let $x_0 \in \Omega, \rho_0 \in (0, \mathrm{dist}(x_0, \partial\Omega)), \nu \in S^{d-1}$ and assume that $\rho \mapsto m(x_0, \nu, \rho)$ is continuous at ρ_0, then $x \mapsto m(x, \nu, \rho_0)$ is continuous at x_0.

Proof. (i) With fixed $x \in \Omega$ and $\nu \in S^{d-1}$, set $r := \mathrm{dist}(x, \partial\Omega)$ and define $m : (0, d) \to [0, +\infty)$ by

$$m(\rho) = m(x, \nu, \rho) + \int_{\Pi_\nu(x) \cap (B_r(x) \setminus B_\rho(x))} g(y, \nu(y)) \mathrm{d}\mathcal{H}^{d-1}.$$

Claim: For $0 < \rho_1 < \rho_2 < r$ it holds that $m(\rho_2) \leq m(\rho_1)$.

Proof of the claim. Let $\varepsilon > 0$, $0 < \rho_1 < \rho_2 < r$ and let $u \in BV(\Omega; \{\pm 1\})$ be such that $u = u_{x,\nu}$ in a neighbourhood of $\partial B_{\rho_1}(x)$ and

$$E(u, B_{\rho_1}(x)) \leq m(x, \nu, \rho) + \varepsilon.$$

Define $\tilde{u} \in BV(\Omega; \{\pm 1\})$ by

$$\tilde{u}(z) = \begin{cases} u(z) & \text{if } z \in B_{\rho_1}(x), \\ u_{x,\nu}(z) & \text{otherwise.} \end{cases}$$

Note that $\tilde{u} = u_{x,\nu}$ in a neighbourhood of $\partial B_{\rho_2}(x)$. Hence we obtain

$$\begin{aligned} m(\rho_2) &\leq E(\tilde{u}, B_{\rho_2}(x)) + \int_{\Pi_\nu(x)\cap\left(B_r(x)\setminus B_{\rho_2}(x)\right)} g(y,\nu(y))\mathrm{d}\mathcal{H}^{d-1} \\ &\leq E(u, B_{\rho_1}(x)) + E(\tilde{u}, B_{\rho_2}(x) \setminus B_{\rho_1}(x)) + \int_{\Pi_\nu(x)\cap\left(B_r(x)\setminus B_{\rho_2}(x)\right)} \varphi(y,\nu(y))\mathrm{d}\mathcal{H}^{d-1} \\ &\leq E(u, B_{\rho_1}(x)) + \int_{\Pi_\nu(x)\cap\left(B_r(x)\setminus B_{\rho_1}(x)\right)} g(y,\nu(y))\mathrm{d}\mathcal{H}^{d-1} \\ &\leq m(x,\nu,\rho_1) + \int_{\Pi_\nu(x)\cap\left(B_r(x)\setminus B_{\rho_1}(x)\right)} g(y,\nu(y))\mathrm{d}\mathcal{H}^{d-1} + \varepsilon, \end{aligned}$$

where we have used the fact that $\mathcal{H}^{d-1}(\partial^*\{\tilde{u}=1\} \cap \partial B_{\rho_1}) = 0$ in the last inequality. The claim follows letting $\varepsilon \to 0$. Since $\rho \mapsto m(\rho)$ is a monotone function, it has countably many discontinuity points $E = E(x,\nu)$. Moreover, since $\mathcal{H}^{d-1}(\partial^*\{u_{x,\nu}=1\} \cap \partial B_\rho) = 0$ for all $0 < \rho < r$ we have that

$$\rho \mapsto \int_{\Pi_\nu(x)\cap(B_r(x)\setminus B_\rho(x))} g(y,\nu)\mathrm{d}\mathcal{H}^{d-1}$$

is a continuous function. Hence, we obtain that

$$\rho \mapsto m(x,\nu,\rho) = m(\rho) - \int_{\Pi_\nu(x)\cap(B_r(x)\setminus B_\rho(x))} g(y,\nu)\mathrm{d}\mathcal{H}^{d-1}$$

is continuous for all but countably many $\rho \in (0,r)$.

(ii) By [[18],Lemma 3.1] it holds that

$$|m(x,\nu_1,\rho) - m(x,\nu_2,\rho)| \leq C \int_{\partial B_\rho(x)} |\mathrm{tr}(u_{x,\nu_1} - u_{x,\nu_2})|\mathrm{d}\mathcal{H}^{d-1}. \tag{73}$$

Since

$$\int_{\partial B_\rho(x)} |\mathrm{tr}(u_{x,\nu_1} - u_{x,\nu_2})|\mathrm{d}\mathcal{H}^{d-1} \leq C\rho^{d-1}\arccos(\langle\nu_1,\nu_2\rangle) \leq \rho^{d-1} w(|\nu_1 - \nu_2|)$$

holds with w a modulus of continuity and the claim follows.

(iii) Claim: For all $0 < \rho_1 < \rho_2 < r$, $x_1, x_2 \in \Omega$ such that $|x_1 - x_2| < \min\{\rho_1, |\rho_1 - \rho_2|\}$, we have

$$m(x_2, \nu, \rho_2) \le m(x_1, \nu, \rho_1) + C\rho_1^{d-2}|x_1 - x_2| + C\Big(\rho_2^{d-1} - \Big(\sqrt{\rho_1^2 - |x_1 - x_2|^2}\Big)^{d-1}\Big) \quad (74)$$

Proof of the claim. Let $\varepsilon > 0$, and let $u \in BV(\Omega, \{\pm 1\})$ be such that $u = u_{x_1,\nu}$ in a neighborhood of $\partial B_{\rho_1}(x_1)$ and

$$E(u, B_{\rho_1}(x_1)) \le m(x_1, \nu, \rho_1) + \varepsilon.$$

Let $\tilde{u} \in BV(\Omega, \{\pm 1\})$ be defined by

$$\tilde{u}(z) = \begin{cases} u(z) & \text{if } z \in B_{\rho_1}(x_1), \\ u_{x_2,\nu}(z) & \text{otherwise.} \end{cases}$$

We then have $\tilde{u} = u_{x_2,\nu}$ in a neighborhood of $\partial B_{\rho_2}(x_2)$ and

$$\begin{aligned} m(x_2, \nu, \rho_2) &\le E(\tilde{u}, B_{\rho_2}(x_2)) \le E(\tilde{u}, B_{\rho_1}(x_1)) + E(\tilde{u}, B_{\rho_2}(x_2) \setminus B_{\rho_1}(x_1)) \\ &\le E(u, B_{\rho_1}(x_1)) + C\mathcal{H}^{d-1}(\partial^*\{\tilde{u} = 1\} \cap \partial B_{\rho_1}(x_1)) \\ &\qquad + C\mathcal{H}^{d-1}(\partial^*\{u_{x_2,\nu} = 1\} \cap \big(B_{\rho_2}(x_2) \setminus \overline{B}_{\rho_1}(x_1)\big)) \\ &\le E(u, B_{\rho_1}(x_1)) + C\rho_1^{d-2}|x_1 - x_2| + C\Big(\rho_2^{d-1} - \Big(\sqrt{\rho_1^2 - |x_1 - x_2|^2}\Big)^{d-1}\Big) \\ &\le m(x_1, \nu, \rho_1) + C\rho_1^{d-2}|x_1 - x_2| + C\Big(\rho_2^{d-1} - \Big(\sqrt{\rho_1^2 - |x_1 - x_2|^2}\Big)^{d-1}\Big) + \varepsilon. \end{aligned}$$

The claim follows by letting $\varepsilon \to 0$.

Claim: Let $x_0 \in \Omega$, $\rho_0 \in (0, \mathrm{dist}(\mathrm{x}_0, \partial\Omega))$ and $\nu \in S^{d-1}$, and assume that $\rho \mapsto m(x_0, \rho, \nu)$ is continuous at ρ_0, then $x \mapsto m(x, \rho_0, \nu)$ is continuous at x_0.

Proof of the claim. Fix $\varepsilon > 0$. First we prove that

$$m(x, \nu, r_0) \ge m(x_0, \nu, r_0) - \varepsilon \text{ for all } |x - x_0| < \delta = \delta(\varepsilon, r_0)$$

To this end, let $r_\varepsilon > r_0$ and $0 < \delta < \min\{r_0, |r_\varepsilon - r_0|\}$ be such that

$$|m(x_0, \nu, r) - m(x_0, \nu, r_0)| < \frac{\varepsilon}{2}, \quad r_\varepsilon^{d-1} - \Big(\sqrt{r_0^2 - \delta^2}\Big)^{d-1} \le \frac{\varepsilon}{4C}, \quad \delta r_\varepsilon^{d-2} \le \frac{\varepsilon}{4Cr_0}. \quad (75)$$

By (74) and (75) we then have

$$\begin{aligned} m(x_0, \nu, r_0) &\le m(x_0, \nu, r_\varepsilon) + \frac{\varepsilon}{2} \le m(x, \nu, r_0) + Cr_0^{d-2}\delta + C\Big(r_\varepsilon^{d-1} - \Big(\sqrt{r_0^2 - \delta^2}\Big)^{d-1}\Big) \\ &\le m(x, \nu, r_0) + \varepsilon \end{aligned}$$

for all $|x-x_0|<\delta$. On the other hand by (74) and (75) we have

$$\begin{aligned} m(x,\nu,r_0) &\le m(x_0,\nu,r_\varepsilon) + Cr_0^{d-2}\delta + C\Big(r_\varepsilon^{d-1} - \Big(\sqrt{r_0^2-\delta^2}\Big)^{d-1}\Big) \\ &\le m(x_0,\nu,r_0) + Cr_0^{d-2}\delta + C\Big(r_\varepsilon^{d-1} - \Big(\sqrt{r_0^2-\delta^2}\Big)^{d-1}\Big) + \frac{\varepsilon}{2} \\ &\le m(x_0,\nu,r_0) + \varepsilon \text{ for all } |x-x_0|<\delta. \end{aligned}$$

This yields the claim. □

Remark 21. Note that if there exist $m_1, m_2 : \Omega\times S^{d-1}\times(0,\mathrm{dist}(x,\partial\Omega)) \to [0,\infty)$ satisfying (i)-(iii) of Proposition 20 and there exists a countable and dense set $\mathcal{D}_1\times\mathcal{D}_2 \subset \Omega\times S^{d-1}$ such that for all $(x,\nu)\in\mathcal{D}_1\times\mathcal{D}_2$ there exists a countable and dense set $\mathcal{D}_3(x,\nu)\subset(0,\mathrm{dist}(x,\partial\Omega))$ such that

$$m_1(x,\nu,\rho) = m_2(x,\nu,\rho) \tag{76}$$

for all $(x,\nu,\rho)\in\mathcal{D}_1\times\mathcal{D}_2\times\mathcal{D}_3(x,\nu)$, then for all $(x,\nu)\in\Omega\times S^{d-1}$ we have that

$$m_1(x,\nu,\rho) = m_2(x,\nu,\rho)$$

for all $\rho\in(0,\mathrm{dist}(x,\partial\Omega))\setminus E(x)$, where $E(x)$ is countable.

Indeed, by (ii) it suffices to prove equality (76) on $\Omega\times\mathcal{D}_2\times(0,\mathrm{dist}(x,\partial\Omega))$, with $E(x)$ countable. We set

$$E(x) = \Big(\bigcup_{\nu\in\mathcal{D}_2} E(x,\nu)\Big) \cup \Big(\bigcup_{y\in\mathcal{D}_1,\nu\in\mathcal{D}_2} E(y,\nu)\Big),$$

where $E(z,\nu)$ is the countable set of discontinuity points of $m_1(z,\nu,\cdot)$ and $m_2(z,\nu,\cdot)$ given by Proposition 20. If $x\in\mathcal{D}_1$ and $\nu\in S^{d-1}$, then we have that $m_1(x,\nu,\rho) = m_2(x,\nu,\rho)$ for all ρ in a countable dense set $\mathcal{D}_3$ and both $m_1(x,\nu,\cdot)$ and $m_2(x,\nu,\cdot)$ are continuous on $(0,\mathrm{dist}(x,\partial\Omega))\setminus E(x)$. Therefore, for every $\rho\in(0,\mathrm{dist}(x,\partial\Omega))\setminus E(x)$ we can find $\rho_k\to\rho$, $\{\rho_k\}_k\subset\mathcal{D}_3$ such that

$$m_1(x,\nu,\rho) = \lim_{k\to+\infty} m_1(x,\nu,\rho_k) = \lim_{k\to+\infty} m_2(x,\nu,\rho_k) = m_2(x,\nu,\rho).$$

Let now $(x_0,\nu,\rho_0)\in\Omega\times S^{d-1}\times((0,\mathrm{dist}(x_0,\partial\Omega))\setminus E(x_0)$. By the definition of $E(x_0)$ we have that $\rho\mapsto m_1(x_0,\nu,\rho), \rho\mapsto m_2(x_0,\nu,\rho)$ are both continuous at ρ_0, by (iii) of Proposition 20 there holds that $x\mapsto m_1(x,\nu,\rho_0), \rho\mapsto m_2(x,\nu,\rho_0)$ are continuous at x_0. Choose $x_k\to x_0$, $\{x_k\}_k\subset\mathcal{D}_1$. By the definition of $E(x_0)$ we have that

$$m_1(x_0,\nu,\rho_0) = \lim_{k\to+\infty} m_1(x_k,\nu,\rho_0) = \lim_{k\to+\infty} m_2(x_k,\nu,\rho_0) = m_2(x_0,\nu,\rho_0)$$

and the remark holds true.

The next goal is to prove Theorem 23 below, which relates Γ-convergence with the convergence of the corresponding minimum problems (67) and we then use it in the proof of Proposition 25. In order to prove Theorem 23 we apply Lemma 22 below, which shows, that every lower-semicontinuous surface energy functional is characterized by its infimum problems on balls.

Lemma 22. *Let $\varphi_i : \Omega \times \mathbb{R}^d \to [0, +\infty)$, $i = 1, 2$, be such that the trivial bounds* (19) *are satisfied and let $E_i : BV(\Omega, \{\pm 1\}) \times \mathcal{A}(\Omega) \to [0, +\infty)$ be defined by*

$$E_i(u, A) = \int_{\partial^*\{u=1\}\cap A} \varphi_i(x, \nu_u(x)) \mathrm{d}\mathcal{H}^{d-1}$$

for all $(u, A) \in BV(\Omega, \{\pm 1\}) \times \mathcal{A}(\Omega)$. For all $A \in \mathcal{A}(\Omega)$ let $E_i(\cdot, A)$ be $L^1(A)$-lower-semicontinuous and assume that

$$m_1(x, \rho, \nu) = m_2(x, \rho, \nu) \text{ for all } x \in \Omega, \nu \in S^{d-1}, \rho \in (0, \mathrm{dist}(x, \partial\Omega)) \setminus E(x), \tag{77}$$

where $E(x)$ is a countable set, then $E_1(u, A) = E_2(u, A)$ for all $u \in BV(\Omega, \{\pm 1\})$ and $A \in \mathcal{A}(\Omega)$.

Proof. Let $u \in BV(\Omega, \{\pm 1\})$. For fixed $\delta > 0$, let $x_i \in \partial^*\{u = 1\}$, $\rho_i > 0$, with $i \in \mathbb{N}$ be such that if we set $B_i^\delta = B_{\rho_i}(x_i)$ then the following properties hold

$$\begin{aligned} &B_i^\delta \subset A, \rho_i < \delta, B_i^\delta \cap B_j^\delta = \emptyset \text{ if } i \neq j \\ &(77) \text{ is satisfied}, \int_{\partial B_i^\delta} |\mathrm{tr}(u - u_{x_i,\nu_i})| \mathrm{d}\mathcal{H}^{d-1} < \rho_i^{d-1}\delta, \\ &\mathcal{H}^{d-1}\Big(\partial^*\{u = 1\}) \setminus \bigcup_{i=1}^{\infty} B_i^\delta\Big) = 0, \quad \mathcal{H}^{d-1}\big(\partial^*\{u = 1\}) \cap B_i^\delta\big) \geq \frac{1}{2} w_{d-1} \rho_i^{d-1}. \end{aligned} \tag{78}$$

Such a countable cover of $\partial^*\{u = 1\}$ exists by the Besicovitch Covering Theorem.

Let $m_i : BV(\Omega; \{\pm 1\}) \times \mathcal{A}(\Omega) \to [0, +\infty)$ be defined by

$$m_i(u, A) = \inf \Big\{ E_i(v, A) : v \in BV(\Omega, \{\pm 1\}),\ u = v \text{ on a neighborhood of } A \Big\}$$

for $i = 1, 2$. Note that by [[18],Lemma 3.1]

$$|m_i(u, A) - m_i(v, A)| \leq C \int_{\partial A} |\mathrm{tr}(u - v)| \mathrm{d}\mathcal{H}^{d-1} \tag{79}$$

holds for $i = 1, 2$. Therefore by (78) and (79) we have

$$
\begin{aligned}
E_1(u, A) &\geq \sum_{i=1}^{\infty} E_1(u, B_i^\delta) \geq \sum_{i=1}^{\infty} m_1(u, B_i^\delta) \\
&\geq \sum_{i=1}^{\infty} m_1(u_{x_i,\nu_i}, B_i^\delta) - C \sum_{i=1}^{\infty} \int_{\partial B_i^\delta} |\mathrm{tr}(u - u_{x_i,\nu_i})| \mathrm{d}\mathcal{H}^{d-1} \\
&\geq \sum_{i=1}^{\infty} m_2(u_{x_i,\nu_i}, B_i^\delta) - C\delta \sum_{i=1}^{\infty} \rho_i^{d-1} \\
&\geq \sum_{i=1}^{\infty} m_2(u, B_i^\delta) - C\delta \sum_{i=1}^{\infty} \rho_i^{d-1} \\
&\geq \sum_{i=1}^{\infty} m_2(u, B_i^\delta) - C\delta \mathcal{H}^{d-1}(\partial^*\{u = 1\}) \cap A).
\end{aligned}
$$

Now, choose $u_\delta^i \in BV(B_i^\delta, \{\pm 1\})$ such that $u_i^\delta = u$ in a neighbourhood of ∂B_i^δ and

$$
E_2(u_\delta^i, B_i^\delta) \leq \frac{1}{2^i}\delta + m_2(u, B_i^\delta),
$$

set $N^\delta = \Omega \setminus \bigcup_{i=1}^{\infty} B_i^\delta$ and define

$$
u^\delta(x) = \begin{cases} u_\delta^i & x \in B_i^\delta \\ u(x) & x \in N^\delta. \end{cases}
$$

By the coercivity assumption on φ_2 we have that $u^\delta \in BV(\Omega, \{\pm 1\})$ and

$$
\begin{aligned}
E_2(u^\delta, A) &\leq \sum_{i=1}^{\infty} E_2(u^\delta, B_i^\delta) + E_2(u^\delta, N^\delta) \leq \sum_{i=1}^{\infty} m_2(u_i^\delta, B_i^\delta) + \delta \\
&\leq E_1(u, A) + C(u)\delta.
\end{aligned}
$$

We claim that $u^\delta \to u$ in $L^1(A)$ as $\delta \to 0$. In fact,

$$
\begin{aligned}
||u^\delta - u||_{L^1(A)} = \sum_{i=1}^{\infty} ||u^\delta - u||_{L^1(B_i^\delta)} &\leq C\delta \sum_{i=1}^{\infty} \mathcal{H}^{d-1}(\partial^*\{u = 1\} \cap B_i^\delta) \\
&= C\delta \mathcal{H}^{d-1}(\partial^*\{u = 1\}).
\end{aligned}
$$

Therefore, by the lower semicontinuity of E_2 we obtain

$$
E_2(u, A) \leq \liminf_{\delta \to 0} E_2(u^\delta, A) \leq E_1(u, A).
$$

By exchanging the roles of E_1 and E_2 we obtain the statement. $\square$

Theorem 23. *Let $\varphi_n, \varphi : \Omega \times \mathbb{R}^d \to [0, +\infty]$ be Borel functions such that $\varphi_n(x, \cdot), \varphi(x, \cdot)$ satisfy the trivial bounds* (19) *for all $x \in \Omega$ and for all $n \in \mathbb{N}$. Let $E_{\varphi_n}, E_\varphi : BV(\Omega, \{\pm 1\}) \times \mathcal{A}(\Omega) \to [0, +\infty)$ be defined by*

$$\begin{aligned} E_{\varphi_n}(u, A) &= \int_{\partial^*\{u=1\}\cap A} \varphi_n(x, \nu_u(x))\mathcal{H}^{d-1}, \\ E_\varphi(u, A) &= \int_{\partial^*\{u=1\}\cap A} \varphi(x, \nu_u(x))\mathcal{H}^{d-1}, \end{aligned}$$

respectively, for all $(u, A) \in BV(\Omega, \{\pm 1\}) \times \mathcal{A}(\Omega)$. Then the following are equivalent:

i) $E_{\varphi_n}(\cdot, A)$ *Γ-converges with respect to the $L^1(A)$-topology to $E_\varphi(\cdot, A)$ for all $A \in \mathcal{A}(\Omega)$.*

ii) If we define

$$\begin{aligned} m_n(x, \nu, \rho) &= \inf\{E_{\varphi_n}(u, B_\rho(x)) : u = u_{x,\nu} \text{ in a neighborhood of } \partial B_\rho(x)\} \\ m(x, \nu, \rho) &= \inf\{E_\varphi(u, B_\rho(x)) : u = u_{x,\nu} \text{ in a neighborhood of } \partial B_\rho(x)\}. \end{aligned}$$

then $m_n(x, \nu, \rho) \to m(x, \nu, \rho)$ for all $(x, \nu, \rho) \in \Omega \times S^{d-1} \times (0, \mathrm{dist}(x, \partial\Omega)) \setminus E(x)$, where $E(x)$ is countable.

Proof. We first show that (i) implies (ii).

Step 1. We show that

$$\limsup_{n\to+\infty} m_n(x, \nu, \rho) \leq m(x, \nu, \rho) \text{ for all } (x, \nu, \rho) \in \Omega \times S^{d-1} \times (0, \mathrm{dist}(x, \partial\Omega))$$

To this end let $\varepsilon > 0$ and $u \in BV(\Omega, \{\pm 1\})$ be such that $u = u_{x,\nu}$ in a neighborhood of $\partial B_\rho(x)$ and

$$E_\varphi(u, B_\rho(x)) \leq m(x, \nu, \rho) + \varepsilon.$$

Since E_n Γ-converges with respect to the strong $L^1(A)$-topology to E there exists $u_n \in BV(\Omega, \{\pm 1\})$ such that

$$\limsup_{n\to+\infty} E_{\varphi_n}(u_n, B_\rho(x)) \leq E_\varphi(u, B_\rho(x)).$$

By a cut-off argument (see e.g. [[**?**] Lemma 4.4]) we construct $\tilde{u}_n \in BV(\Omega, \{\pm 1\})$ such that $\tilde{u}_n = u_{x,\nu}$ in a neighborhood of $\partial B_\rho(x)$ and

$$\limsup_{n\to+\infty} E_{\varphi_n}(\tilde{u}_n, B_\rho(x)) \leq \limsup_{n\to+\infty} E_{\varphi_n}(u_n, B_\rho(x)).$$

Hence we have

$$\limsup_{n\to+\infty} m_n(x,\nu,\rho) \le \limsup_{n\to+\infty} E_{\varphi_n}(\tilde{u}_n, B_\rho(x)) \le \limsup_{n\to+\infty} E_{\varphi_n}(u_n, B_\rho(x))$$
$$\le E_\varphi(u, B_\rho(x)) \le m(x,\nu,\rho) + \varepsilon.$$

The claim follows as $\varepsilon \to 0$. By Proposition 20 (ii) we have that that

$$\limsup_{\rho'\to\rho} m(x,\nu,\rho') = m(x,\nu,\rho) \tag{80}$$

for all but countably many $\rho \in (0, \mathrm{dist}(x,\partial\Omega))$, where ρ' is converging decreasingly to ρ.

Step 2. We show that

$$\liminf_{n\to+\infty} m_n(x,\nu,\rho) \ge \limsup_{\rho'\to+\rho} m(x,\nu,\rho')$$

with ρ' converging decreasingly to ρ. To prove this choose for all $n \in \mathbb{N}$, $u_n \in BV(\Omega, \{\pm 1\})$ such that $u_n = u_{x,\nu}$ in a neighborhood of $\partial B_\rho(x)$ and

$$E_{\varphi_n}(u_n, B_\rho(x)) \le m_n(x,\nu,\rho) + \frac{1}{n}.$$

Let $\rho' > \rho$ and define

$$\tilde{u}_n(z) = \begin{cases} u_n(z) & z \in B_\rho(x) \\ u_{x,\nu}(z) & \text{otherwise.} \end{cases}$$

We have

$$m_n(x,\nu,\rho) \ge E_{\varphi_n}(u_n, B_\rho(x)) \ge E_{\varphi_n}(\tilde{u}, B_{\rho'}(x)) - \frac{1}{n} - C|\rho - \rho'|.$$

By the coercivity assumption we know that $\tilde{u}_n \to v$ up to subsequences, and therefore that $v = u_{x,\nu}$ in a neighborhood of $\partial B_{\rho'}(x)$ we obtain

$$\liminf_{n\to+\infty} m_n(x,\nu,\rho) \ge \liminf_{n\to+\infty} E_{\varphi_n}(u_n, B_\rho(x)) \ge E_\varphi(v, B_{\rho'}(x)) - C|\rho - \rho'|$$

By (80) the claim follows for all such $\rho \in (0,1) \setminus E(x)$ as ρ' converges decreasingly to ρ.

Now we prove that (ii) implies (i). Take a subsequence (not relabeled) $\{E_{\varphi_n}\}_n$. By [[6],Theorem 3.2], up to subsequences $E_{\varphi_n}(\cdot, A)\,\Gamma$-converges to some $\tilde{E} : BV(\Omega, \{\pm 1\}) \times \mathcal{A}(\Omega) \to [0, +\infty)$ of the form

$$\tilde{E}(u, A) = \int_{\partial^*\{u=1\}\cap A} \tilde{\varphi}(x, \nu_u(x))\mathrm{d}\mathcal{H}^{d-1} \text{ for all } (u, A) \in BV(\Omega, \{\pm 1\}) \times A(\Omega)$$

and therefore if we denote by $\tilde{m}$ the associated minimum problems of the energy $\tilde{E}$ by (i) implies (ii) and by our assumption we have that

$$\tilde{m}(x,\rho,\nu) = \lim_{n\to+\infty} m_n(x,\rho,\nu) = m(x,\rho,\nu)$$

for all $x,\nu,\rho \in \Omega \times S^{d-1} \times (0, \mathrm{dist}(x,\partial\Omega)) \setminus E(x)$ where $E(x)$ is a countable set. By Lemma 22 we have that $\tilde{E} = E_\varphi$. Therefore every subsequence contains a further subsequences which Γ-converges to E_φ. By the Urysohn-property of Γ-convergence we have that E_{φ_n} Γ-converges to E_φ. □

Since the energies that are involved are equi-coercive we may use a metrizability argument for Γ-convergence [[22],Theorem 10.22]. We therefore may argue by a diagonalization procedure. To this end we need to define a *minimal volume fraction* θ_φ.

For $\varphi : \mathbb{R}^d \to [0,+\infty)$, even, convex, positively 1-homogeneous, satisfying (19), or equivalently $\varphi \in \mathbf{H}_{\alpha,\beta,V}(1)$, we define $\theta_\varphi \in [0,1]$ by

$$\theta_\varphi = \min\{s \in [0,1] : \varphi \in \mathbf{H}_{\alpha,\beta,V}(s)\} \tag{81}$$

Note that the minimum in (81) is attained by the definition of $\mathbf{H}_{\alpha,\beta,V}(\theta)$. If $\varphi : \Omega \times \mathbb{R}^d \to [0,+\infty)$ is such that $\varphi(x,\cdot) \in \mathbf{H}_{\alpha,\beta,V}(1)$ for all $x \in \Omega$, we define $\theta_\varphi : \Omega \to [0,1]$ by

$$\theta_\varphi(x) = \theta_{\varphi(x,\cdot)}.$$

Some properties of the minimal volume fraction θ thus defined are contained in the following lemma.

Lemma 24. *The following properties hold true:*

i) Let $0 \le \theta_1 < \theta_2 \le 1$ and $\varphi \in \boldsymbol{H}_{\alpha,\beta,V}(\theta_1)$ then there exists $r > 0$ such that $B_r(\varphi) \subset \boldsymbol{H}_{\alpha,\beta,V}(\theta_2)$.

ii) Let $\varphi, \varphi_n \in \boldsymbol{H}_{\alpha,\beta,V}(1)$ and $\mathrm{d}(\varphi_n,\varphi) \to 0$ as $n \to \infty$, then $\theta_{\varphi_n} \to \theta_\varphi$.

iii) Let $\varphi(\cdot,\nu)$ be continuous for all $\nu \in S^{d-1}$, then θ_φ is continuous.

iv) For every surface energy density φ the function θ_φ is measurable.

Proof. (i) Let $0 \le \theta_1 < \theta_2 \le 1$ and $\varphi \in \mathbf{H}_{\alpha,\beta,V}(\theta_1)$. For all $\varphi' \in B_r(\varphi)$ we have

$$\varphi'(\nu) \le \varphi(\nu) + r \le \sum_{\xi\in V} c_\xi |\langle \nu,\xi\rangle| + r \le \sum_{\xi\in V} (c_\xi + r)|\langle \nu,\xi\rangle| =: \sum_{\xi\in V} \tilde{c}_\xi |\langle \nu,\xi\rangle|,$$

where $\alpha_\xi \leq c_\xi \leq (\theta_\xi \beta_\xi + (1-\theta_\xi)\alpha_\xi)$ for some $\{\theta_\xi\}_{\xi\in V} \subset [0,1]$ satisfying (18) with θ_1, and

$$\begin{aligned}\tilde{c}_\xi &\leq (\theta_\xi \beta_\xi + (1-\theta_\xi)\alpha_\xi) + r = \Big(\Big(\theta_\xi + \frac{r}{\beta_\xi - \alpha_\xi}\Big)\beta_\xi + \Big(1 - (\theta_\xi + \frac{r}{\beta_\xi - \alpha_\xi})\Big)\alpha_\xi\Big) \\ &\leq (\tilde{\theta}_\xi \beta_\xi + (1-\tilde{\theta}_\xi)\alpha_\xi),\end{aligned}$$

if $\tilde{\theta}_\xi - \theta_\xi \geq \frac{r}{\beta_\xi - \alpha_\xi}$ and $\{\tilde{\theta}_\xi\}_\xi \subset [0,1]$ satisfy (18) with θ_2. Set

$$r := \min_{\xi\in V} \frac{(\tilde{\theta}_\xi - \theta_\xi)(\beta_\xi - \alpha_\xi)}{2}.$$

We have that $B_r(\varphi) \subset \mathbf{H}_{\alpha,\beta,V}(\theta_2)$.

(ii) Let $\mathrm{d}(\varphi_n, \varphi) \to 0$ as $n \to +\infty$. Up to subsequences we have that $\theta_{\varphi_n} \to \tilde{\theta}$. By the definition of θ_φ we have that $\theta_\varphi \leq \tilde{\theta}$, since $\varphi \in \mathbf{H}_{\alpha,\beta,V}(\tilde{\theta})$. Assume that $\theta_\varphi < \theta < \tilde{\theta}$ for some $\theta \in (0,1)$. By (i) there exists a neighbourhood $B_r(\varphi)$ of φ such that $B_r(\varphi) \subset \mathbf{H}_{\alpha,\beta,V}(\theta)$. Therefore for n large enough we have that $\varphi_n \in \mathbf{H}_{\alpha,\beta,V}(\theta)$ so that $\theta_{\varphi_n} \leq \theta$, which contradicts $\theta_{\varphi_n} \to \tilde{\theta}$.

(iii) is a direct consequences of (ii). As for (iv), it suffices to notice that if we define $\varphi_n = \rho_n * \varphi$, where ρ_n is a sequence of convolution kernels, we have that φ_n is a sequence of continuous functions and φ_n converges almost everywhere to φ. In view of (ii) and (iii), θ_φ is an almost-everywhere limit of a sequence of continuous functions, hence it is measurable. □

Proposition 25. *Let $\theta : \Omega \to [0,1]$ be measurable and $\varphi : \Omega \times \mathbb{R}^d \to [0,+\infty]$ be such that $\varphi(x,\cdot) \in \boldsymbol{H}_{\alpha,\beta,V}(\theta(x))$ for almost every $x \in \Omega$. Then there exist sequences $\{\theta_n\}_n, \{c_\xi^n\}_{\xi,n} \subset C(\Omega), 0 \leq \theta_n \leq 1$ such that $\varphi_n : \Omega \times \mathbb{R}^d \to [0,+\infty]$ defined by*

$$\varphi_n(x,\nu) = \sum_{\xi\in V} c_\xi^n(x)|\langle \nu, \xi\rangle| \tag{82}$$

satisfy $\varphi_n(x,\cdot) \in \boldsymbol{H}_{\alpha,\beta,V}(\theta_n(x))$ for all $x \in \Omega$. Furthermore E_{φ_n} Γ-converges to E_φ and $\theta_n \overset{*}{\rightharpoonup} \theta$.

Proof. Since the energies involved are all equicoercive by the metrizability-properties of Γ-convergence (see [[22],Theorem 10.22]) we can use a diagonal argument. It suffices to construct lower-semicontinuous densities of the form (82) such that the associated energies Γ-converge to E_φ. Every such function can be approximated from below by functions of the form (82) with continuous coefficients. Therefore the associated energies Γ-converge to the energy associated to the limit density and by Lemma 24 (ii) the associated local volume fractions converge weakly*.

We now prove that there exists $\{\varphi_n\}_n$ of the form (82) with $c^n_\xi : \Omega \to [0,\infty)$ lower semicontinuous and E_{φ_n} Γ-converges to E_φ. By Theorem 23 and Remark 21 it suffices to find φ_n such that

$$\lim_{n\to+\infty} m_n(x_i,\nu_i,\rho_i) = m(x_i,\nu_i,\rho_i),$$

where $\{x_i,\rho_i,\nu_i\}_{i\in\mathbb{N}} \subset \Omega \times (0,\operatorname{dist}(x,\partial\Omega)) \times S^{d-1}$ is a dense set. Moreover we assume, by the Besicovitch Covering Theorem and Remark 21, that $|\Omega \setminus \bigcup_{i=1}^\infty B_{\rho_i}(x_i)| = 0$.

Let $u^n_i \in BV(\Omega,\{\pm 1\})$ be such that $u^n_i = u_{x_i,\nu_i}$ in a neighborhood of $\partial B_{\rho_i}(x_i)$ and

$$E_\varphi(u^n_i, B_{\rho_i}(x_i)) \leq m(x_i,\rho_i,\nu_i) + \frac{1}{n}.$$

By Lusin's theorem and the rectifiability of $\partial^*\{u^n_i = 1\}$ there exists $K_{i,n}, C_{i,n} \subset \mathbb{R}^d$ compact and $a_\xi : K_{i,n} \to [0,\infty)$ such that

i) $K_{i,n} \subset \partial^*\{u^n_i = 1\} \cap B_{\rho_i}(x_i)$, $x \mapsto \nu_{u^n_i}(x)$, $x \mapsto \varphi(x,\nu_{u^n_i}(x))$ are continuous on $K_{i,n}$ and $\mathcal{H}^{d-1}(\partial^*\{u^n_i = 1\} \setminus K_{i,n}) < \frac{1}{n}$.

ii) there exist $C_{i,n} \subset B_{\rho_i}(x_i) \setminus K_{i,n}$ such that $|B_{\rho_i} \setminus C_{i,n}| = |(B_{\rho_i} \setminus K_{i,n}) \setminus C_{i,n}| < \frac{1}{n}2^{-i}$ and

$$\varphi(x,\nu) \leq \sum_{\xi\in V} a_\xi(x)|\langle \nu,\xi\rangle|$$

for all $x \in C_{i,n}, \nu \in S^{d-1}$. Moreover for all $\xi_0 \in V$ there holds

$$\varphi\Big(x, \frac{\xi_0}{||\xi_0||}\Big) = \sum_{\xi\in V} a_\xi(x)\Big|\langle \frac{\xi_0}{||\xi_0||},\xi\rangle\Big|.$$

Define

$$\varphi_{n,i}(x,\nu) = \begin{cases} \displaystyle\sum_{\xi\in V} c_\xi(x,\nu_{u^n_i}(x))|\langle\nu,\xi\rangle| & x \in K_{i,n}, \\ \displaystyle\sum_{\xi\in V} a_\xi(x)|\langle\nu,\xi\rangle| & x \in C_{i,n}, \\ \displaystyle\sum_{\xi\in V} \beta_\xi|\langle\nu,\xi\rangle| & \text{otherwise}, \end{cases}$$

where $c_\xi(x,\nu_{u^n_i}(x)) \in C(\Omega)$ are chosen such that

$$\sum_{\xi\in V} c_\xi(x,\nu_{u^n_i}(x))|\langle\nu_{u^n_i}(x),\xi\rangle| = \varphi(x,\nu_{u^n_i}(x)).$$

Set

$$\varphi_n(x,\xi) := \min_{1\leq i\leq n} \varphi_{n,i}(x,\xi).$$

$\varphi_n(\cdot,\xi)$ is lower semicontinuous and for all $1\le i\le n$ we have

$$\begin{aligned} m_n(x_i,\nu_i,\rho_i) &\le E_{\varphi_n}(u_i^n,B_{\rho_i}(x_i)) \\ &\le \int_{K_{i,n}} \varphi_n(x,\nu_{u_i^n}(x))\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial^*\{u=1\}\setminus K_{i,n}} \varphi_n(x,\nu_{u_i^n}(x))\mathrm{d}\mathcal{H}^{d-1} \\ &\le \int_{\partial^*\{u=1\}} \varphi(x,\nu_{u_i^n}(x))\mathrm{d}\mathcal{H}^{d-1} + C\mathcal{H}^{d-1}(\partial^*\{u_i^n=1\}\setminus K_{i,n}) \\ &\le m(x_i,\nu_i,\rho_i)+\frac{C}{n}. \end{aligned}$$

Now let $\varepsilon>0$ and let $u\in BV(\Omega,\{\pm1\})$ be such that $u=u_{x_i,\nu_i}$ in a neighborhood of $\partial B_{\rho_i}(x_i)$ and

$$E_{\varphi_n}(u,B_{\rho_i}(x_i))\le m_n(x_i,\nu_i,\rho_i)+\varepsilon.$$

Noting that $\nu_u=\nu_{u_i^n}$ $\mathcal{H}^{d-1}$-almost everywhere on $\partial^*\{u=1\}$, we have

$$\begin{aligned} &m_n(x_i,\rho_i,\nu_i)+\varepsilon \ge E_{\varphi_n}(u,B_{\rho_i}(x_i)) \\ &= \int_{\partial^*\{u=1\}\cap B_{\rho_i}(x_i)} \varphi_n(x,\nu_u(x))\mathrm{d}\mathcal{H}^{d-1} \\ &= \sum_{j=1}^n \int_{I_j^n\cap B_{\rho_i}(x_i)} \varphi_n(x,\nu_u(x))\mathrm{d}\mathcal{H}^{d-1} + \int_{(\partial^*\{u=1\}\setminus\bigcup_{j=1}^n I_j^n)\cap B_{\rho_i}(x_i)} \varphi_n(x,\nu_u(x))\mathrm{d}\mathcal{H}^{d-1} \\ &\ge \sum_{j=1}^n \int_{I_j^n\cap B_{\rho_i}(x_i)} \varphi(x,\nu_u(x))\mathrm{d}\mathcal{H}^{d-1} + \int_{(\partial^*\{u=1\}\setminus\bigcup_{j=1}^n I_j^n)\cap B_{\rho_i}(x_i)} \varphi(x,\nu_u(x))\mathrm{d}\mathcal{H}^{d-1} \\ &= E_\varphi(u,B_{\rho_i}(x_i))\ge m(x_i,\nu_i,\rho_i), \end{aligned}$$

where $I_j^n=K_{j,n}\setminus\bigcup_{j'<j}K_{j',n}$. Letting $\varepsilon\to0$, we obtain

$$|m_n(x_i,\nu_i,\rho_i)-m(x_i,\nu_i,\rho_i)|\le\frac{C}{n}\text{ for all }1\le i\le n.$$

Hence, $m_n(x_i,\nu_i,\rho_i)\to m(x_i,\nu_i,\rho_i)$ for all $i\in\mathbb{N}$, so that by Theorem 23 we have that E_{φ_n} Γ-converges to E_φ.

It remains to show that $\theta_n\overset{*}{\rightharpoonup}\theta$. By the definition of φ_n we have that

$$\theta_{\varphi_n}(x)=\begin{cases}\theta(x) & x\in\bigcup_{i=1}^n C_{i,n}\\ 1 & x\notin\bigcup_{i=1}^n C_{i,n}\end{cases}$$

where we remark that $|K_{i,n}|=0$. Now let $\delta>0$, and let $n_\delta\in\mathbb{N}$ be such that $|\Omega\setminus\bigcup_{i=1}^{n_\delta}B_{\rho_i}(x_i)|<\delta$. For $f\in L^1(\Omega)$ and for n large enough we have

$$\begin{aligned} \int_\Omega f(\theta_n-\theta)\mathrm{d}x &= \int_{\Omega\setminus\bigcup_{i=1}^n C_{i,n}} f(\theta_n-\theta)\mathrm{d}x + \int_{\bigcup_{i=1}^n C_{i,n}} f(\theta_n-\theta)\mathrm{d}x \\ &= \int_{\Omega\setminus\bigcup_{i=1}^n C_{i,n}} f(1-\theta)\mathrm{d}x. \end{aligned}$$

Note that $|f(1-\theta)| \leq 2f$ and

$$\Big|\Omega \setminus \bigcup_{i=1}^{n_\delta} C_{i,n}\Big| \leq \Big|\Omega \setminus \bigcup_{i=1}^{n_\delta} B_{\rho_i}(x_i)\Big| + \Big|\bigcup_{i=1}^{n_\delta} B_{\rho_i}(x_i) \setminus C_{i,n}\Big| \leq \delta + \frac{1}{n}.$$

Hence by the Dominated Convergence Theorem we have that

$$\int_{\Omega \setminus \bigcup_{i=1}^{n} C_{i,n}} f(1-\theta)\mathrm{d}x \to 0$$

as $n \to +\infty, \delta \to 0$. Therefore we have that $\theta_n \overset{*}{\rightharpoonup} \theta$ as $n \to +\infty$. □

Proof of Theorem 19. By the metrizability-properties of Γ-convergence and of the weak*-convergence on bounded sets we proceed by successive approximations and conclude then by a diagonal argument.

Step 1. By Proposition 25 we can assume that

$$\varphi(x,\nu) = \sum_{\xi \in V} c_\xi(x)|\langle \nu, \xi\rangle| \tag{83}$$

for all $(x,\nu) \in \Omega \times \mathbb{R}^d$ with $c_\xi \in C(\Omega, [\alpha_\xi, \beta_\xi])$ for all $\xi \in V$. Note that by (iii) of Lemma 24 we have that θ_φ is continuous.

Step 2. For every φ of the form (83) let $c_\xi^k : \Omega \to [\alpha_\xi, \beta_\xi]$ be defined by

$$c_\xi^k(x) = \begin{cases} \inf\limits_{z \in Q_{2^{-k}}(x_n)} c_\xi(z) & x \in Q_{2^{-k}}(x_n), x_n \in \mathcal{Z}_k, \\ \inf\limits_{z \in Q_{2^{-k}}(x_n) \cup Q_{2^{-k}}(x_{n'})} c_\xi(z) & x \in \overline{Q}_{2^{-k}}(x_n) \cap \overline{Q}_{2^{-k}}(x_{n'}), x_n, x_{n'} \in \mathcal{Z}_k, \end{cases}$$

where $\mathcal{Z}_k := 2^{-k}\mathbb{Z}^d + \sum_{i=1}^d e_i 2^{-k-1}$. We define $\varphi_k : \Omega \times \mathbb{R}^d \to [0,+\infty)$ by

$$\varphi_k(x,\nu) = \sum_{\xi \in V} c_\xi^k(x)|\langle \nu, \xi\rangle|. \tag{84}$$

Furthermore, we set $E_k : BV(\Omega, \{\pm 1\}) \to [0, +\infty)$

$$E_k(u) = \int_{\partial^*\{u=1\}\cap\Omega} \varphi_k(x, \nu_u(x))\mathrm{d}\mathcal{H}^{d-1}.$$

We have that $\Gamma\text{-}\lim\limits_{k\to+\infty} E_k(u) = E(u)$. This follows since c_ξ^k converges increasingly to c_ξ. Moreover, by Lemma 24 we have that $\theta_{\varphi_k} \overset{*}{\rightharpoonup} \theta_\varphi$ since $\varphi_k(x,\cdot) \to \varphi(x,\cdot)$ for all $x \in \Omega$.

Step 3. Every $E_k : BV(\Omega,\{\pm 1\}) \to [0,+\infty)$ can be approximated by $E_k^\delta : BV(\Omega,\{\pm 1\}) \to [0,+\infty)$ of the form

$$E_k^\delta(u) = \int_{\partial^*\{u=1\}\cap\Omega} \varphi_k^\delta(x,\nu_u(x))\mathrm{d}\mathcal{H}^{d-1}, \tag{85}$$

where $\varphi_k^\delta : \Omega\times\mathbb{R}^d \to [0,+\infty]$ is defined by

$$\varphi_k^\delta(x,\nu) = \sum_{\xi\in V} c_\xi^{k,\delta}(x)|\langle \nu,\xi\rangle|$$

with $c_\xi^{k,\delta} : \Omega \to [\alpha_\xi,\beta_\xi]$ defined by

$$c_\xi^{k,\delta}(x) = \begin{cases} c_\xi^k(x) & x\in Q_{2^{-k}(1-\delta)}(x_n), x_n\in\mathcal{Z}_k,\\ \beta_\xi & \text{otherwise.}\end{cases}$$

To prove Step 3 note first that since $c_\xi^{k,\delta} \geq c_\xi^k$ for all $\delta>0$, we have that

$$\Gamma\text{-}\liminf_{\delta\to 0} E_k^\delta(u) \geq E_k(u).$$

By [[6],Theorem 3.2] we have that

$$\Gamma\text{-}\limsup_{\delta\to 0} E_k^\delta(u) = \int_{\partial^*\{u=1\}\cap\Omega} \tilde\varphi_k(x,\nu_u(x))\mathrm{d}\mathcal{H}^{d-1}$$

for some $\tilde\varphi : \Omega\times\mathbb{R}^d \to [0,+\infty)$, where

$$\tilde\varphi_k(x,\nu) = \limsup_{\rho\to 0}\frac{m_k(x,\nu,\rho)}{w_{d-1}\rho^{d-1}}.$$

Using [[18],Lemma 4.3.5] we have that

$$\tilde\varphi_k(x,\nu) = \limsup_{\rho\to 0}\frac{m_k(x,\nu,\rho)}{w_{d-1}\rho^{d-1}} = \limsup_{\rho\to 0}\liminf_{\delta\to 0}\frac{m_k^\delta(x,\nu,\rho)}{w_{d-1}\rho^{d-1}}.$$

Hence it follows that $\tilde\varphi(x,\nu) = \varphi(x,\nu)$ for all $x\in Q_{2^{-k}}(x_n), x_n\in\mathcal{Z}_k$. From this fact and by the lower semicontinuity of the energy functional, as in the proof of (39), we conclude that $\tilde\varphi(x,\nu)\leq\varphi(x,\nu)$ for all $x\in\partial Q_{2^{-k}}(x_n)$, $x_n\in\mathcal{Z}_k$ with ν being the normal vector of $\partial Q_{2^{-k}}(x_k)$. For every $\delta>0$ small enough we have that $c_\xi^{k,\delta} = c_\xi^k$ for all $x\in Q_{2^{-k}(1-\delta)}(x_n), x_n\in\mathcal{Z}_k$ with $\lim_{\delta\to 0}|\Omega\setminus\bigcup_{x_n\in\mathcal{Z}_k} Q_{2^{-k}(1-\delta)}(x_n)| = 0$. Therefore, it follows that $\theta_{\varphi_k^\delta} \overset{*}{\rightharpoonup} \theta_{\varphi_k}$.

Step 4. For $E_k^\delta : BV(\Omega,\{\pm1\}) \to [0,+\infty)$ of the form (85) we can find energies $E_\varepsilon^{k,\delta} : BV(\Omega,\{\pm1\}) \to [0,+\infty)$ of the form

$$E_\varepsilon^{k,\delta}(u) = \frac{1}{4}\sum_{\xi\in V}\sum_{i,i+\xi\in\Omega_\varepsilon} \varepsilon^{d-1} c_{i,\xi}^{k,\delta,\varepsilon}(u_{\varepsilon i} - u_{\varepsilon(i+\xi)})^2 \tag{86}$$

for some $c_{i,\xi}^{k,\delta,\varepsilon} \in \{\alpha_\xi,\beta_\xi\}^{\mathbb{Z}^d}$ such that $\theta(\{c_{i,\xi}^{k,\delta,\varepsilon}\}) \overset{*}{\rightharpoonup} \theta_{\varphi_k^\delta}$.

Since $\varphi_k^\delta(x,\cdot) \in \mathbf{H}_{\alpha,\beta,V}(\theta_{\varphi_k^\delta}(x_n))$ if $x \in Q_{2^{-k}(1-\delta)}(x_n), x_n \in \mathcal{Z}_k$ there exists a sequence of homogenized densities $\{\varphi_N^{k,\delta,n}\}$ such that $\varphi_N^{k,n} \to \varphi(x_n,\cdot)$ as $N \to +\infty$. Furthermore, for every $N \in \mathbb{N}$ there exist $\{c_{i,\xi}^{k,n,N}\}$ T-periodic for some $T \in \mathbb{N}$ such that $\theta(\{c_{i,\xi}^{k,n,N}\}) \to \theta(x_n)$, where $\theta(\{c_{i,\xi}^{k,n,N}\})$ is given by (15) and

$$\Gamma\text{-}\lim_{\varepsilon\to0} E_\varepsilon^{k,n,N}(u) = \int_{\partial^*\{u=1\}\cap\Omega} \varphi_N^{k,n}(\nu_u(x))\mathrm{d}\mathcal{H}^{d-1}.$$

We define $E_{k,N,\delta} : BV(\Omega,\{\pm1\}) \to [0,+\infty)$ by

$$E_{k,N,\delta}(u) = \int_{\partial^*\{u=1\}\cap\Omega} \varphi_N^{k,\delta}(x,\nu_u(x))\mathrm{d}\mathcal{H}^{d-1}, \tag{87}$$

where

$$\varphi_N^{k,\delta}(x,\nu) = \begin{cases} \varphi_N^{k,n}(\nu) & x \in Q_{2^{-k}(1-\delta)}(x_n), x_n \in \mathcal{Z}_k, \\ \displaystyle\sum_{\xi\in V} \beta_\xi|\langle\nu,\xi\rangle| & \text{otherwise.} \end{cases}$$

We have that

$$\Gamma\text{-}\lim_{N\to+\infty} E_{k,N,\delta}(u) = E_k^\delta(u).$$

This follows from the convergence $\varphi_N^{k,n} \to \varphi(x_n,\cdot)$ and

$$\tilde{\varphi}_k^\delta(x,\nu) = \limsup_{\rho\to0} \frac{m_k^\delta(x,\nu,\rho)}{w_{d-1}\rho^{d-1}} = \limsup_{\rho\to0}\liminf_{N\to\infty} \frac{m_{k,N}^\delta(x,\nu,\rho)}{w_{d-1}\rho^{d-1}}.$$

Note that if we define $E_\varepsilon^{k,N,\delta} : BV(\Omega,\{\pm1\}) \to [0,+\infty)$ by (86) with

$$c_{i,\xi}^{k,\delta,N,\varepsilon} = \begin{cases} c_{i,\xi}^{k,n,N} & i \in (Q_{2^{-k}(1-\delta)}(x_n))_\varepsilon, x_n \in \mathcal{Z}_k, \\ \beta_\xi & \text{otherwise,} \end{cases}$$

then we have

$$\Gamma\text{-}\lim_{\varepsilon\to0} E_\varepsilon^{k,\delta,N}(u) = \Gamma\text{-}\lim_{\varepsilon\to0} \frac{1}{4}\sum_{\xi\in V}\sum_{i,i+\xi\in\Omega_\varepsilon} \varepsilon^{d-1} c_{i,\xi}^{k,\delta,N,\varepsilon}(u_{\varepsilon i} - u_{\varepsilon(i+\xi)})^2 = E_{k,N,\delta}(u)$$

with $E_{k,N,\delta}(u)$ of the form (87). The claim follows by a diagonal argument for the convergence of the energies and the local volume fractions. □

References

[1] E. Acerbi and G. Buttazzo. On the limits of periodic Riemannian metrics. *J. Anal. Math.* **43** (1983), 183–201.

[2] R. Alicandro, A. Braides, M. Cicalese. Phase and anti-phase boundaries in binary discrete systems: a variational viewpoint. *Netw. Heterog. Media* **1** (2006), 85–107

[3] R. Alicandro, M.S. Gelli. Local and non local continuum limits of Ising type energies for spin systems, *SIAM J. Math Anal.* **48** (2016), 895–931.

[4] G. Allaire. *Shape Optimization by the Homogenization Method.* Springer-Verlag, Berlin, 2002.

[5] L. Ambrosio and A. Braides. Functionals defined on partitions of sets of finite perimeter, I: integral representation and Γ-convergence. *J. Math. Pures. Appl.* **69** (1990), 285-305.

[6] L. Ambrosio and A. Braides. Functionals defined on partitions of sets of finite perimeter, II: semicontinuity, relaxation and homogenization. *J. Math. Pures. Appl.* **69** (1990), 307–333.

[7] L. Ambrosio, N. Fusco and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Vol. 254. Oxford: Clarendon Press, 2000.

[8] N. Ansini and O. Iosifescu. Approximation of anisotropic perimeter functionals by homogenization. *Boll. Un. Mat. Ital.* **3** (2010), 149–168.

[9] X. Blanc, C. Le Bris and P.L. Lions. From molecular models to continuum models. *C.R. Acad. Sci., Paris, Ser. I* **332** (2001), 949–956.

[10] A. Braides. Γ-*convergence for Beginners.* Oxford University Press, Oxford, 2002.

[11] A. Braides. A handbook of Γ-convergence. In *Handbook of Differential Equations. Stationary Partial Differential Equations, Volume* 3 (M. Chipot and P. Quittner, eds.), Elsevier, 2006, p. 101–213.

[12] A. Braides. An example of non-existence of plane-like minimizers for an almost-periodic Ising system. *J. Stat. Phys.* **157** (2015), 295–302.

[13] A. Braides, G. Buttazzo, and I. Fragalà. Riemannian approximation of Finsler metrics. *Asympt. Anal.* **31** (2002), 177–187

[14] A. Braides, and V. Chiadò Piat. A derivation formula for convex integral functionals defined on BV (Ω), *J. Convex Anal* **2** (1995), 69-85.

[15] A. Braides and A. Defranceschi. *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998.

[16] A. Braides and G. Francfort, Bounds on the effective behavior of a square conducting lattice. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **460** (2004), 1755–1769

[17] A. Braides and L. Kreutz. Optimal bounds for periodic mixtures of nearest-neighbour ferromagnetic interactions. *Rend. Lincei Mat. Appl.* **28** (2017), 103–117

[18] G. Bouchitté, I. Fonseca, L. Mascarenhas. A global method for relaxation, *Arch. Rational Mech. Analysis* **145** (1998) 51-98.

[19] A. Braides and A. Piatnitski. Homogenization of surface and length energies for spin systems. *J. Funct. Anal.* **264** (2013), 1296–1328

[20] A. Braides and M. Solci. Interfacial energies on Penrose lattices. *M3AS* **21** (2011), 1193–1210

[21] L. Caffarelli and R. de la Llave. Interfaces of ground states in Ising models with periodic coefficients. *J. Stat. Phys.* **118** (2005), 687–719.

[22] G. Dal Maso. *An Introduction to Gamma-convergence*. Birkhhauser, Boston, 1992.

[23] A. Davini and M. Ponsiglione. Homogenization of two-phase metrics and applications. *J. Analyse Math.* **103** (2007), 157–196.

[24] G.W. Milton. *The Theory of Composites*. Cambridge University Press, Cambridge, 2002.

[25] U. Raitums. On the local representation of G-closure. *Arch. Rational Mech. Anal.* **158** (2001), 213–234.