Odd length for even hyperoctahedral groups and signed generating functions [1]

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Abstract

We define a new statistic on the even hyperoctahedral groups which is a natural analogue of the odd length statistic recently defined and studied on Coxeter groups of types A and B. We compute the signed (by length) generating function of this statistic over the whole group and over its maximal and some other quotients and show that it always factors nicely. We also present some conjectures.

1 Introduction

The signed (by length) enumeration of the symmetric group, and other finite Coxeter groups by various statistics is an active area of research (see, e.g., [1, 2, 4, 6, 7, 10, 11, 12, 13, 14, 19]). For example, the signed enumeration of classical Weyl groups by major index was carried out by Gessel-Simion in [19] (type A), by Adin-Gessel-Roichman in [1] (type B) and by Biagioli in [2] (type D), that by descent by Desarmenian-Foata in [7] (type A) and by Reiner in [13] (types B and D), while that by excedance by Mantaci in [11] and independently by Sivasubramanian in [14] (type A) and by Mongelli in [12] (other types).

In [9], [17] and [18] two statistics were introduced on the symmetric and hyperoctahedral groups, in connection with the enumeration of partial flags in a quadratic space and the study of local factors of representation zeta functions of certain groups, respectively (see [9] and [18], for details). These statistics combine combinatorial and parity conditions and have been called the "odd length" of the respective groups. In [9] and [18] it was conjectured that the signed (by length) generating functions of these statistics over all the quotients of the corresponding groups always factor in a very nice way, and this was proved in [4] (see also [5]) for types A and B and independently, and in a different way, in [10] for type B.

In this paper we define a natural analogue of these statistics for the even hyperoctahedral group and study the corresponding signed generating functions. More precisely, we show that certain general properties that these signed generating functions have in types A and B (namely "shifting" and "compressing") continue to hold in type D. We then show that these generating functions factor nicely for the whole group (i.e., for the trivial quotient) and for the maximal quotients. As a consequence of our results we show that the signed generating function over the whole even hyperoctahedral group is the square of the one for the symmetric group.

The organization of the paper is as follows. In the next section we recall some definitions, notation, and results that are used in the sequel. In §3 we define a new statistic on the even hyperoctahedral group which is a natural analogue of the odd length statistics that have already been defined in types A and B in [9] and [18], and study some general properties of the corresponding signed generating functions. These include a complementation property, the identification of subsets of the quotients over which the corresponding signed generating function always vanishes, and operations on a quotient that leave the corresponding signed generating function unchanged. In §4 we show that the signed generating function over the whole even hyperoctahedral group factors nicely. As a consequence of this result we obtain that this signed generating function is the square of the corresponding one for type A. In §5 we compute the signed generating functions of the maximal, and some other, quotients and show that these also always factor nicely. Finally, in §6, we present some conjectures naturally arising from the present work, and the evidence that we have in their favor.

2   Preliminaries

In this section we recall some notation, definitions, and results that are used in the sequel.

We let $\mathbb{P} := \{1, 2, \ldots\}$ be the set of positive integers and $\mathbb{N} := \mathbb{P} \cup \{0\}$. For all $m, n \in \mathbb{Z}$,

$m \leq n$ we let $[m, n] := \{m, m+1, \ldots, n\}$, $[n] := [1, n]$, and $[n]_\pm := [n] \cup [-n, -1]$. Given a set I we denote by $|I|$ its cardinality. For a real number x we denote by $\lfloor x \rfloor$ the greatest integer less than or equal to x and by $\lceil x \rceil$ the smallest integer greater than or equal to x. Given $J \subseteq [0, n-1]$ there are unique integers $a_1 < \cdots < a_s$ and $b_1 < \cdots < b_s$ such that $J = [a_1, b_1] \cup \cdots \cup [a_s, b_s]$ and $a_{i+1} - b_i > 1$ for $i = 1, \ldots, s-1$. We call the intervals $[a_1, b_1], \ldots, [a_s, b_s]$ the *connected components* of J.

For $n_1, \ldots, n_k \in \mathbb{N}$ and $n := \sum_{i=1}^{k} n_i$, we let $\begin{bmatrix} n \\ n_1, \ldots, n_k \end{bmatrix}_q$ denote the *q-multinomial coefficient*

$$\begin{bmatrix} n \\ n_1, \ldots, n_k \end{bmatrix}_q := \frac{[n]_q!}{[n_1]_q! \cdot \ldots \cdot [n_k]_q!},$$

where

$$[n]_q := \frac{1 - q^n}{1 - q}, \qquad [n]_q! := \prod_{i=1}^{n} [i]_q \qquad \text{and, in particular,} \qquad [0]_q! := 1.$$

The symmetric group S_n is the group of permutations of the set $[n]$. For $\sigma \in S_n$ we use both the one-line notation $\sigma = [\sigma(1), \ldots, \sigma(n)]$ and the disjoint cycle notation. We let $s_1, \ldots, s_{n-1}$ denote the standard generators of S_n, $s_i = (i, i+1)$.

The hyperoctahedral group B_n is the group of signed permutations, or permutations σ of the set $[-n, n]$ such that $\sigma(j) = -\sigma(-j)$. For a signed permutation σ we use the window notation $\sigma = [\sigma(1), \ldots, \sigma(n)]$ and the disjoint cycle notation. The standard generating set of B_n is $S = \{s_0^B, s_1, \ldots, s_{n-1}\}$, where $s_0^B = [-1, 2, 3, \ldots, n]$ and $s_1, \ldots, s_{n-1}$ are as above. By convention, we multiply (signed) permutations from the right. Thus, for $w \in B_n$ and $i \in [n-1]$, ws_i is obtained from w exchanging the values in position i and $i+1$, while ws_0^B is obtained from w by changing the sign of the value in the first position.

We follow [3] for notation and terminology about Coxeter groups. In particular, for a Coxeter system (W, S) we let ℓ be the Coxeter length and for $I \subseteq S$ we define the quotients:

$$W^I := \{w \in W \ : \ D(w) \subseteq S \setminus I\},$$

and

$$^I W := \{w \in W \ : \ D_L(w) \subseteq S \setminus I\},$$

where $D(w) = \{s \in S \ : \ \ell(ws) < \ell(w)\}$, and $D_L(w) = \{s \in S \ : \ \ell(sw) < \ell(w)\}$. The parabolic subgroup W_I is the subgroup generated by I. The following result is well known (see, e.g., [3, Proposition 2.4.4]).

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Proposition 2.1. *Let (W, S) be a Coxeter system, $J \subseteq S$, and $w \in W$. Then there exist unique elements $w^J \in W^J$ and $w_J \in W_J$ (resp., $^J w \in {}^J W$ and $_J w \in W_J$) such that $w = w^J w_J$ (resp., $_J w^J w$). Furthermore $\ell(w) = \ell(w^J) + \ell(w_J)$ (resp., $\ell(_J w) + \ell(^J w)$).*

It is well known that S_n and B_n, with respect to the above generating sets, are Coxeter systems and that the following results hold (see, e.g., [3, Propositions 1.5.2, 1.5.3, and §8.1]).

Proposition 2.2. *Let $\sigma \in S_n$. Then $\ell_A(\sigma) = |\{(i, j) \in [n]^2 : i < j, \sigma(i) > \sigma(j)\}|$ and $D(\sigma) = \{s_i : \sigma(i) > \sigma(i+1)\}$.*

For $\sigma \in B_n$ let

$$\mathrm{inv}(\sigma) := |\{(i, j) \in [n]^2 : i < j, \sigma(i) > \sigma(j)\}|,$$

$$\mathrm{neg}(\sigma) := |\{i \in [n] : \sigma(i) < 0\}|,$$

$$\mathrm{nsp}(\sigma) := |\{(i, j) \in [n]^2 : i < j, \sigma(i) + \sigma(j) < 0\}|.$$

Proposition 2.3. *Let $\sigma \in B_n$. Then*

$$\ell_B(\sigma) = \frac{1}{2} |\{(i, j) \in [-n, n]^2 : i < j, \sigma(i) > \sigma(j)\}| = \mathrm{inv}(\sigma) + \mathrm{neg}(\sigma) + \mathrm{nsp}(\sigma)$$

and $D(\sigma) = \{s_i : i \in [0, n-1], \sigma(i) > \sigma(i+1)\}$.

The group D_n of even-signed permutations is the subgroup of B_n of elements with an even number of negative entries in the window notation:

$$D_n = \{\sigma \in B_n : \mathrm{neg}(\sigma) \equiv 0 \pmod 2\}.$$

This is a Coxeter group of type D_n, with set of generators $S = \{s_0^D, s_1^D, \ldots, s_{n-1}^D\}$, where $s_0^D := [-2, -1, 3, \ldots n]$ and $s_i^D := s_i$ for $i \in [n-1]$. Moreover, the following holds (see, e.g., [3, Propositions 8.2.1 and 8.2.3]).

Proposition 2.4. *Let $\sigma \in D_n$. Then*

$$\ell_D(\sigma) = \frac{1}{2} |\{(i, j) \in [n]_\pm^2 : i < j, \sigma(i) > \sigma(j)\}| = \mathrm{inv}(\sigma) + \mathrm{nsp}(\sigma)$$

and $D(\sigma) = \{s_i^D : i \in [0, n-1], \sigma(i) > \sigma(i+1)\}$, where $\sigma(0) := \sigma(-2)$.

Thus, for a subset of the generators $I \subseteq S$, that we identify with the corresponding subset $I \subseteq [0, n-1]$, we have the following description of the quotient

$$D_n^I = \{\sigma \in D_n : \sigma(i) < \sigma(i+1) \text{ for all } i \in I\}$$

where $\sigma(0) := -\sigma(2)$.

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Note that the length ℓ_D is well defined also on $B_n \setminus D_n$. In the sequel we will sometimes evaluate it also on elements in this set.

The following statistic was first defined in [9]. Our definition is not the original one, but is equivalent to it (see [9, Definition 5.1 and Lemma 5.2]) and is the one that is best suited for our purposes.

Definition 2.5. Let $n \in \mathbb{P}$. The statistic $L_A : S_n \to \mathbb{N}$ is defined as follows. For $\sigma \in S_n$

$$L_A(\sigma) := |\{(i,j) \in [n]^2 : i < j, \sigma(i) > \sigma(j), i \not\equiv j \pmod{2}\}|.$$

The following statistic was introduced in [17] and [18], and is a natural analogue of the statistic L_A introduced above, for Coxeter groups of type B.

Definition 2.6. Let $n \in \mathbb{P}$. The statistic $L_B : B_n \to \mathbb{N}$ is defined as follows. For $\sigma \in B_n$

$$L_B(\sigma) := \frac{1}{2}|\{(i,j) \in [-n,\,n]^2 : i < j, \sigma(i) > \sigma(j), i \not\equiv j \pmod{2}\}|.$$

For example, if $n = 4$ and $\tau = [-2, 4, 3, -1]$ then $L_B(\tau) = \frac{1}{2}|\{(-4,-3), (-4,1), (-3,-2),$ $(-1,0), (-1,4), (0,1), (2,3), (3,4)\}| = 4$.

We call these statistics L_A and L_B the *odd length* of the symmetric and hyperoctahedral groups, respectively. Note that if $\sigma \in S_n \subset B_n$ then $L_B(\sigma) = L_A(\sigma)$.

The odd length of an element $\sigma \in B_n$ also has a description in terms of statistics of the window notation of σ. Given $\sigma \in B_n$ we let

$$\mathrm{oinv}(\sigma) := |\{(i,j) \in [n]^2 : i < j, \sigma(i) > \sigma(j), i \not\equiv j \pmod{2}\}|,$$
$$\mathrm{oneg}(\sigma) := |\{i \in [n] : \sigma(i) < 0, i \not\equiv 0 \pmod{2}\}|,$$
$$\mathrm{onsp}(\sigma) := |\{(i,j) \in [n]^2 : i < j, \sigma(i) + \sigma(j) < 0, i \not\equiv j \pmod{2}\}|.$$

The following result appears in [4, Proposition 5.1].

Proposition 2.7. *Let* $\sigma \in B_n$. *Then* $L_B(\sigma) = \mathrm{oinv}(\sigma) + \mathrm{oneg}(\sigma) + \mathrm{onsp}(\sigma)$.

The signed generating function of the odd length factors very nicely both on quotients of S_n and of B_n. The following result was conjectured in [9, Conjecture C] and proved in [4].

Theorem 2.8. *Let* $n \in \mathbb{P}$, $I \subseteq [n-1]$, *and* $I_1, \ldots, I_s$ *be the connected components of* I. *Then*

$$\sum_{\sigma \in S_n^I} (-1)^{\ell_A(\sigma)} x^{L_A(\sigma)} = \left[\begin{matrix} m \\ \left\lfloor \frac{|I_1|+1}{2} \right\rfloor, \ldots, \left\lfloor \frac{|I_s|+1}{2} \right\rfloor \end{matrix} \right]_{x^2} \prod_{k=2m+2}^{n} \left(1 + (-1)^{k-1} x^{\left\lfloor \frac{k}{2} \right\rfloor} \right) \qquad (1)$$

where $m := \sum_{k=1}^{s} \left\lfloor \frac{|I_k|+1}{2} \right\rfloor$.

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In particular, for the whole group we have the following.

Corollary 2.9. *Let $n \in \mathbb{P}$, $n \geq 2$. Then*

$$\sum_{\sigma \in S_n} (-1)^{\ell_A(\sigma)} x^{L_A(\sigma)} = \prod_{i=2}^{n} \left(1 + (-1)^{i-1} x^{\lfloor \frac{i}{2} \rfloor} \right).$$

For $J \subseteq [0, n-1]$ we define $J_0 \subseteq J$ to be the connected component of J which contains 0, if $0 \in J$, or $J_0 := \emptyset$ otherwise. Let $J_1, \ldots, J_s$ be the remaining ordered connected components. The following result was conjectured in [18, Conjecture 1.6] and proved in [4] and independently in [10].

Theorem 2.10. *Let $n \in \mathbb{P}$, $J \subseteq [0, n-1]$, and $J_0, \ldots, J_s$ be the connected components of J indexed as just described. Then*

$$\sum_{\sigma \in B_n^J} (-1)^{\ell_B(\sigma)} x^{L_B(\sigma)} = \frac{\prod\limits_{j=a+1}^{n} (1 - x^j)}{\prod\limits_{i=1}^{m} (1 - x^{2i})} \left[\begin{array}{c} m \\ \left\lfloor \frac{|J_1|+1}{2} \right\rfloor, \ldots, \left\lfloor \frac{|J_s|+1}{2} \right\rfloor \end{array} \right]_{x^2}$$

where $m := \sum_{i=1}^{s} \left\lfloor \frac{|J_i|+1}{2} \right\rfloor$ and $a := \min\{[0, n] \setminus J\}$.

3 Definition and general properties

In this section we define a new statistic, on the even hyperoctahedral group D_n, which is a natural analogue of the odd length statistics that have already been defined and studied in types A and B, and study some of its basic properties.

Given the descriptions of L_A and L_B in terms of odd inversions, odd negatives and odd negative sum pairs, and the relation between the Coxeter lengths of the Weyl groups of types B and D (see, e.g., [3, Propositions 8.1.1 and 8.2.1]), the following definition is natural.

Definition 3.1. Let $\sigma \in D_n$. We let

$$L_D(\sigma) := L_B(\sigma) - \mathrm{oneg}(\sigma) = \mathrm{oinv}(\sigma) + \mathrm{onsp}(\sigma).$$

Equivalently, in analogy with the formula for ℓ_D in Proposition 2.4,

$$L_D(\sigma) = \frac{1}{2} |\{(i, j) \in [n]_{\pm}^2 : i < j, \sigma(i) > \sigma(j), i \not\equiv j \pmod{2} \}|.$$

For example let $n = 5$, $\sigma = [2, -1, 5, -4, 3]$. Then $L_D(\sigma) = 5$. We call L_D the *odd length* of type D. Note that the statistic L_D is well defined also on S_n (where it coincides with L_A) and on B_n. In fact, the signed distribution of L_D over any quotient of D_n and

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over its "complement" in B_n, is exactly the same, as we now show. For $I \subseteq [0, n-1]$ let $(B_n \setminus D_n)^I := \{\sigma \in B_n \setminus D_n : \sigma(i) < \sigma(i+1) \text{ for all } i \in I\}$ where $\sigma(0) := -\sigma(2)$. Note that $(B_n \setminus D_n)^I = B_n^I \setminus D_n^I$ if $I \subseteq [n-1]$.

Lemma 3.2. *Let $n \in \mathbb{P}$ and $I \subseteq [0, n-1]$. Then*

$$\sum_{\sigma \in D_n^I} y^{\ell_D(\sigma)} x^{L_D(\sigma)} = \sum_{\sigma \in (B_n \setminus D_n)^I} y^{\ell_D(\sigma)} x^{L_D(\sigma)}.$$

In particular, $\sum_{\sigma \in D_n^I} (-1)^{\ell_D(\sigma)} x^{L_D(\sigma)} = \sum_{\sigma \in (B_n \setminus D_n)^I} (-1)^{\ell_D(\sigma)} x^{L_D(\sigma)}$.

Proof. Left multiplication by s_0^B (that is, changing the sign of 1 in the window notation) is a bijection between D_n^I and $(B_n \setminus D_n)^I$. Moreover, (odd) inversions and (odd) negative sum pairs are preserved by this operation so $L_D(s_0^B \sigma) = L_D(\sigma)$, and $\ell_D(s_0^B \sigma) = \ell_D(\sigma)$, for all $\sigma \in D_n$ and the result follows. $\qquad\square$

In what follows, since we are mainly concerned with distributions in type D, we omit the subscript and write just ℓ and L for the length and odd length, respectively, on D_n. We now show that the generating function of $(-1)^{\ell(\cdot)} x^{L(\cdot)}$ over any quotient D_n^I such that $0 \notin I$ can be reduced to elements for which n (or $-n$) is in certain positions. More precisely, we prove that, for a given quotient, our generating function is zero over all elements for which n (or $-n$) is sufficiently far from I.

Lemma 3.3. *Let $n \in \mathbb{P}$, $n \geq 3$, $I \subseteq [0, n-1]$ and $a \in [2, n-1]$ such that $a+1 \notin I$. Suppose that the following hold: if $a = 3$ then $0, 1 \notin I$; if $a \geq 4$ then $a - 2 \notin I$. Then*

$$\sum_{\substack{\{\sigma \in D_n^I : \\ \sigma(a) = n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\substack{\{\sigma \in D_n^I : \\ \sigma(a) = -n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = 0.$$

Proof. Under our hypotheses, if $\sigma \in D_n^I$ and $\sigma(a) = n$ then $\sigma^a := \sigma(-a-1, -a+1)(a-1, a+1)$ also has these properties. Clearly $(\sigma^a)^a = \sigma$ and $|\ell(\sigma) - \ell(\sigma^a)| = 1$, while, since $\sigma(a) = n$, $L(\sigma^a) = L(\sigma)$. Therefore we have that

$$\sum_{\substack{\{\sigma \in D_n^I : \\ \sigma(a) = n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\substack{\{\sigma \in D_n^I : \sigma(a) = n, \\ \sigma(a-1) < \sigma(a+1)\}}} \left((-1)^{\ell(\sigma)} x^{L(\sigma)} + (-1)^{\ell(\sigma^a)} x^{L(\sigma^a)} \right) = 0.$$

The proof of the second equality is exactly analogous and is therefore omitted. $\qquad\square$

Although we do not know of any definition of our (or of any other) odd length statistics in Coxeter theoretic language, it is natural to expect that the only automorphism of the Dynkin diagram of D_n preserves the corresponding signed generating function. This is indeed the case, as we now show.

Proposition 3.4. *Let $n \in \mathbb{P}$, $n \geq 2$, and $I \subseteq [2, n-1]$. Then*

$$\sum_{\sigma \in D_n^{I \cup \{0\}}} y^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in D_n^{I \cup \{1\}}} y^{\ell(\sigma)} x^{L(\sigma)}.$$

In particular, $\sum_{\sigma \in D_n^{I \cup \{0\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in D_n^{I \cup \{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}$.

Proof. Right multiplication by s_0^B (i.e., changing the sign of the leftmost element in the window notation) is a bijection between $D_n^{I \cup \{0\}}$ and $(B_n \setminus D_n)^{I \cup \{1\}}$. Furthermore, if $\sigma \in D_n$, then

$$
\begin{aligned}
\mathrm{oinv}(\sigma s_0^B) &= \mathrm{oinv}(\sigma) - |\{i \in [2, n] : i \equiv 0 \pmod 2, \ \sigma(1) > \sigma(i)\}| \\
&\quad + |\{i \in [2, n] : i \equiv 0 \pmod 2, \ -\sigma(1) > \sigma(i)\}|, \\
\mathrm{onsp}(\sigma s_0^B) &= \mathrm{onsp}(\sigma) - |\{i \in [2, n] : i \equiv 0 \pmod 2, \ \sigma(1) + \sigma(i) < 0\}| \\
&\quad + |\{i \in [2, n] : i \equiv 0 \pmod 2, \ -\sigma(1) + \sigma(i) < 0\}|, \\
\mathrm{inv}(\sigma s_0^B) &= \mathrm{inv}(\sigma) - |\{i \in [2, n] : \ \sigma(1) > \sigma(i)\}| + |\{i \in [2, n] : \ -\sigma(1) > \sigma(i)\}|,
\end{aligned}
$$

and

$$\mathrm{nsp}(\sigma s_0^B) = \mathrm{nsp}(\sigma) - |\{i \in [2, n] : \ \sigma(1) + \sigma(i) < 0\}| + |\{i \in [2, n] : \ -\sigma(1) + \sigma(i) < 0\}|.$$

Therefore $L(\sigma s_0^B) = L(\sigma)$ and $\ell(\sigma s_0^B) = \ell(\sigma)$. Hence

$$\sum_{\sigma \in (B_n \setminus D_n)^{I \cup \{1\}}} y^{\ell(\sigma)} \, x^{L(\sigma)} = \sum_{\sigma \in D_n^{I \cup \{0\}}} y^{\ell(\sigma s_0^B)} \, x^{L(\sigma s_0^B)} = \sum_{\sigma \in D_n^{I \cup \{0\}}} y^{\ell(\sigma)} \, x^{L(\sigma)},$$

and the result follows from Lemma 3.2. $\qquad\square$

Remark 3.5. For $n = 4$ there are more automorphisms of the Dynkin diagram than the one considered in Proposition 3.4. Indeed, it can be verified that the signed generating function over quotients of D_4 reflects these symmetries. For example,

$$\sum_{\sigma \in D_4^{\{0,2\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in D_4^{\{1,2\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in D_4^{\{2,3\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (1 - x^2)^3$$

and

$$\sum_{\sigma \in D_4^{\{0,1,2\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in D_4^{\{0,2,3\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in D_4^{\{1,2,3\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (1 - x^2)^2;$$

they are anyhow explained by the property of shifting, cf. Propositions 3.6 and 3.7. We also record here the formula for $n = 4$, $I = \{0, 1, 3\}$; cf. Conjecture 6.2 and 6.4 for $n \geq 5$:

$$\sum_{\sigma \in D_4^{\{0,1,3\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (1 + 3x^2)(1 - x^2).$$

We conclude this section by showing that when I does not contain 0, each connected component can be shifted to the left or to the right, as long as it remains a connected component, without changing the generating function over the corresponding quotient. The proof is analogous to that of [4, Proposition 3.3]. However, for completeness, and for the reader's convenience, we include it here.

Proposition 3.6. *Let* $I \subseteq [n-1]$, $i \in \mathbb{P}$, $k \in \mathbb{N}$ *be such that* $[i, i+2k]$ *is a connected component of* I *and* $i + 2k + 2 \notin I$. *Then*

$$\sum_{\sigma \in D_n^I} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in D_n^{I \cup \tilde{I}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in D_n^{\tilde{I}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \tag{2}$$

where $\tilde{I} := (I \setminus \{i\}) \cup \{i + 2k + 1\}$.

Proof. We have

$$\sum_{\sigma \in D_n^I} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\substack{\{\sigma \in D_n^I : \sigma(i) > \\ \sigma(i+2k+2)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + \sum_{\substack{\{\sigma \in D_n^I : \sigma(i+2k+1) < \\ \sigma(i+2k+2)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

$$+ \sum_{j=1}^{2k+1} \sum_{\substack{\{\sigma \in D_n^I : \sigma(i+j-1) < \\ \sigma(i+2k+2) < \sigma(i+j)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}.$$

Let $j \in [k]$. Then

$$\sum_{\substack{\{\sigma \in D_n^I : \sigma(i+2j-1) < \\ \sigma(i+2k+2) < \sigma(i+2j)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + \sum_{\substack{\{\sigma \in D_n^I : \sigma(i+2j) < \\ \sigma(i+2k+2) < \sigma(i+2j+1)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

$$= \sum_{\substack{\{\sigma \in D_n^I : \sigma(i+2j-1) < \\ \sigma(i+2k+2) < \sigma(i+2j)\}}} \left((-1)^{\ell(\sigma)} x^{L(\sigma)} + (-1)^{\ell(\tilde{\sigma})} x^{L(\tilde{\sigma})} \right)$$

where $\tilde{\sigma} := \sigma(i+2j, i+2k+2)(-i-2j, -i-2k-2)$. But $\ell(\tilde{\sigma}) = \ell(\sigma) - 1$ and $L(\tilde{\sigma}) = L(\sigma)$, so the above sum is equal to zero. Similarly,

$$\sum_{\substack{\{\sigma \in D_n^I : \\ \sigma(i+2k+2) < \sigma(i)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + \sum_{\substack{\{\sigma \in D_n^I : \sigma(i) < \\ \sigma(i+2k+2) < \sigma(i+1)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = 0.$$

Hence

$$\sum_{\sigma \in D_n^I} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\substack{\{\sigma \in D_n^I : \sigma(i+2k+1) < \\ \sigma(i+2k+2)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)},$$

This proves the left equality in (2). The proof of the right equality is exactly analogous and is therefore omitted. $\square$

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Shifting is also allowed when I contains 0, but only for connected components which are sufficiently far from 0, as stated in the next result.

Proposition 3.7. *Let $I \subseteq [0, n-1]$, $i \in \mathbb{P}$, $i > 2$ and $k \in \mathbb{N}$ such that $[i, i+2k]$ is a connected component of I and $i + 2k + 2 \notin I$. Then*

$$\sum_{\sigma \in D_n^I} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in D_n^{I \cup \tilde{I}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in D_n^{\tilde{I}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

where $\tilde{I} := (I \setminus \{i\}) \cup \{i + 2k + 1\}$.

Proof. The proof is analogous to that of Proposition 3.6 noting that, since $i > 2$, $\sigma \in D_n^I$ if and only if $\sigma(i+2j, i+2k+2)(-i-2j, -i-2k-2) \in D_n^I$. $\qquad \square$

4 Trivial quotient

In this section, using the results in the previous one, we compute the generating function of $(-1)^{\ell(\cdot)} x^{L(\cdot)}$ over the whole even hyperoctahedral group D_n. In particular, we obtain that this generating function is the square of the corresponding one for type A (i.e., for the symmetric group).

Theorem 4.1. *Let $n \in \mathbb{P}$, $n \geq 2$. Then*

$$\sum_{\sigma \in D_n} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \prod_{j=2}^{n} (1 + (-1)^{j-1} x^{\lfloor \frac{j}{2} \rfloor})^2.$$

Proof. We proceed by induction on n. By direct computation, the result holds for $n = 2$: $\sum_{\sigma \in D_2} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (1-x)^2$.

By Lemma 3.3, the sum over all elements for which n or $-n$ appears in positions different from 1 and n is zero. So the generating function over D_n reduces to

$$
\begin{aligned}
\sum_{\sigma \in D_n} (-1)^{\ell(\sigma)} x^{L(\sigma)} = & \sum_{\substack{\{\sigma \in D_n: \\ \sigma(1) = n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + \sum_{\substack{\{\sigma \in D_n: \\ \sigma(n) = n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \\
& + \sum_{\substack{\{\sigma \in D_n: \\ \sigma(1) = -n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + \sum_{\substack{\{\sigma \in D_n: \\ \sigma(n) = -n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \\
= & \sum_{\sigma \in D_{n-1}} (-1)^{\ell(\tilde{\sigma})} x^{L(\tilde{\sigma})} + \sum_{\sigma \in D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \\
& + \sum_{\sigma \in B_{n-1} \setminus D_{n-1}} (-1)^{\ell(\hat{\sigma})} x^{L(\hat{\sigma})} + \sum_{\sigma \in B_{n-1} \setminus D_{n-1}} (-1)^{\ell(\check{\sigma})} x^{L(\check{\sigma})},
\end{aligned}
$$

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where $\tilde{\sigma} := [n, \sigma(1), \ldots, \sigma(n-1)]$, $\hat{\sigma} := [-n, \sigma(1), \ldots, \sigma(n-1)]$, and $\check{\sigma} := [\sigma(1), \ldots, \sigma(n-1), -n]$. But, by our definition and Proposition 8.2.1 of [3], we have that

$$L(\tilde{\sigma}) = L(\sigma) + m, \qquad \ell(\tilde{\sigma}) = \ell(\sigma) + n - 1 \tag{3}$$

$$L(\hat{\sigma}) = L(\sigma) + m, \qquad \ell(\hat{\sigma}) = \ell(\sigma) + n - 1 \tag{4}$$

$$L(\check{\sigma}) = L(\sigma) + 2m, \qquad \ell(\check{\sigma}) = \ell(\sigma) + 2(n-1), \tag{5}$$

where $m := \lfloor \frac{n}{2} \rfloor$. Therefore

$$\sum_{\sigma \in D_{n-1}} (-1)^{\ell(\tilde{\sigma})} x^{L(\tilde{\sigma})} = (-1)^{n-1} x^m \sum_{\sigma \in D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

and, similarly,

$$\sum_{\sigma \in B_{n-1} \backslash D_{n-1}} (-1)^{\ell(\hat{\sigma})} x^{L(\hat{\sigma})} = (-1)^{n-1} x^m \sum_{\sigma \in B_{n-1} \backslash D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

$$\sum_{\sigma \in B_{n-1} \backslash D_{n-1}} (-1)^{\ell(\check{\sigma})} x^{L(\check{\sigma})} = x^{2m} \sum_{\sigma \in B_{n-1} \backslash D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)}.$$

So by Lemma 3.2 and our induction hypothesis we obtain that

$$\begin{aligned} \sum_{\sigma \in D_n} (-1)^{\ell(\sigma)} x^{L(\sigma)} &= (1 + (-1)^{n-1} x^m) \sum_{\sigma \in D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + \\ &+ ((-1)^{n-1} x^m + x^{2m}) \sum_{\sigma \in B_{n-1} \backslash D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \\ &= (1 + 2(-1)^{n-1} x^m + x^{2m}) \sum_{\sigma \in D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \\ &= \left(1 + (-1)^{n-1} x^{\lfloor \frac{n}{2} \rfloor}\right)^2 \sum_{\sigma \in D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)}, \end{aligned}$$

and the result follows by induction. $\square$

As an immediate consequence of Theorem 4.1 and of Corollary 2.9 we obtain the following result.

Corollary 4.2. *Let $n \in \mathbb{P}$, $n \geq 2$. Then*

$$\sum_{\sigma \in D_n} (-1)^{\ell(\sigma)} x^L(\sigma) = \left(\sum_{\sigma \in S_n} (-1)^{\ell_A(\sigma)} x^{L_A(\sigma)} \right)^2 . \square$$

It would be interesting to have a direct proof of Corollary 4.2.

5 Maximal and other quotients

In this section we compute, using the results in §3, the signed generating function of the odd length over the maximal, and some other, quotients of D_n. In particular, we obtain that these generating functions always factor nicely.

Theorem 5.1. *Let $n \in \mathbb{P}$, $n \geq 2$ and $i \in [0, n-1]$. Then*

$$\sum_{\sigma \in D_n^{\{i\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = 1 - x$$

if $n = 2$, and

$$\sum_{\sigma \in D_n^{\{i\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (1 - x^2) \prod_{j=4}^{n} (1 + (-1)^{j-1} x^{\lfloor \frac{j}{2} \rfloor})^2,$$

if $n \geq 3$.

Proof. By Propositions 3.4 (with $I = \emptyset$) and 3.6, we may assume $i = 1$.

We proceed by induction on $n > 3$, the result being true for $n = 3$ (and $n = 2$) by direct verification: $\sum_{\sigma \in D_3^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = 1 - x^2$.

By Lemma 3.3 we have that the sum over $\sigma \in D_n^{\{1\}}$ such that n or $-n$ appear in the window in any position but $1, 3$, or n is zero. Furthermore, if $\sigma \in D_n^{\{1\}}$ then $\sigma^{-1}(n) \neq 1$. Thus we have:

$$\sum_{\sigma \in D_n^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\substack{\{\sigma \in D_n^{\{1\}}: \\ \sigma(3)=n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + \sum_{\substack{\{\sigma \in D_n^{\{1\}}: \\ \sigma(n)=n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} +$$

$$+ \sum_{\substack{\{\sigma \in D_n^{\{1\}}: \\ \sigma(1)=-n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + \sum_{\substack{\{\sigma \in D_n^{\{1\}}: \\ \sigma(3)=-n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + \sum_{\substack{\{\sigma \in D_n^{\{1\}}: \\ \sigma(n)=-n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

$$= \sum_{\substack{\{\sigma \in D_n^{\{1\}}: \\ \sigma(3)=n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + \sum_{\sigma \in D_{n-1}^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + \sum_{\sigma \in B_{n-1} \setminus D_{n-1}} (-1)^{\ell(\hat{\sigma})} x^{L(\hat{\sigma})} +$$

$$+ \sum_{\substack{\{\sigma \in D_n^{\{1\}}: \\ \sigma(3)=-n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + \sum_{\sigma \in (B_{n-1} \setminus D_{n-1})^{\{1\}}} (-1)^{\ell(\check{\sigma})} x^{L(\check{\sigma})}$$

where $\hat{\sigma} := [-n, \sigma(1), \ldots, \sigma(n-1)]$ and $\check{\sigma} := [\sigma(1), \ldots, \sigma(n-1), -n]$. Now

$$\sum_{\substack{\{\sigma \in D_n^{\{1\}}: \\ \sigma(3)=n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\substack{\{\sigma \in D_{n-1}: \\ \sigma(1)>\sigma(2)\}}} (-1)^{\ell(\bar{\sigma})} x^{L(\bar{\sigma})}$$

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where $\bar{\sigma} := [\sigma(2), \sigma(1), n, \sigma(3), \ldots, \sigma(n-1)]$. But $\ell(\bar{\sigma}) = \text{inv}(\sigma) + n - 4 + \text{nsp}(\sigma) = \ell(\sigma) + n - 4$, and $L(\bar{\sigma}) = \text{oinv}(\sigma) - 1 + \lceil \frac{n-3}{2} \rceil + \text{onsp}(\sigma) = L(\sigma) + \lceil \frac{n-5}{2} \rceil = L(\sigma) + m - 2$, where $m := \lceil \frac{n-1}{2} \rceil = \lfloor \frac{n}{2} \rfloor$, so

$$\sum_{\substack{\{\sigma \in D_n^{\{1\}}: \\ \sigma(3)=n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (-1)^n x^{m-2} \sum_{\substack{\{\sigma \in D_{n-1}: \\ \sigma(1) > \sigma(2)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

$$= (-1)^n x^{m-2} \left(\sum_{\sigma \in D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)} - \sum_{\sigma \in D_{n-1}^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right).$$

Similarly,

$$\sum_{\substack{\{\sigma \in D_n^{\{1\}}: \\ \sigma(3)=-n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\substack{\{\sigma \in B_{n-1} \setminus D_{n-1}: \\ \sigma(1) > \sigma(2)\}}} (-1)^{\ell(\bar{\sigma})} x^{L(\bar{\sigma})}$$

where $\bar{\bar{\sigma}} := [\sigma(2), \sigma(1), -n, \sigma(3), \ldots, \sigma(n-1)]$ and $\ell(\bar{\bar{\sigma}}) = \text{inv}(\sigma) + 1 + \text{nsp}(\sigma) + n - 1 = \ell(\sigma) + n$, $L(\bar{\bar{\sigma}}) = \text{oinv}(\sigma) + \text{onsp}(\sigma) + 1 + \lceil \frac{n-3}{2} \rceil = L(\sigma) + \lceil \frac{n-1}{2} \rceil = L(\sigma) + m$. So, by Lemma 3.2,

$$\sum_{\substack{\{\sigma \in D_n^{\{1\}}: \\ \sigma(3)=-n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (-1)^n x^m \sum_{\substack{\{\sigma \in B_{n-1} \setminus D_{n-1}: \\ \sigma(1) > \sigma(2)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

$$= (-1)^n x^m \left(\sum_{\sigma \in B_{n-1} \setminus D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)} - \sum_{(B_{n-1} \setminus D_{n-1})^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right)$$

$$= (-1)^n x^m \left(\sum_{\sigma \in D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)} - \sum_{\sigma \in D_{n-1}^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right).$$

Moreover, by (4) and (5) we have

$$\sum_{\sigma \in B_{n-1} \setminus D_{n-1}} (-1)^{\ell(\hat{\sigma})} x^{L(\hat{\sigma})} = (-1)^{n-1} x^m \sum_{\sigma \in B_{n-1} \setminus D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

and

$$\sum_{\sigma \in (B_{n-1} \setminus D_{n-1})^{\{1\}}} (-1)^{\ell(\check{\sigma})} x^{L(\check{\sigma})} = x^{2m} \sum_{\sigma \in (B_{n-1} \setminus D_{n-1})^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}.$$

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Thus we get, again by Lemma 3.2,

$$
\begin{aligned}
\sum_{\sigma \in D_n^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \;=\;& (-1)^n x^{m-2} \left(\sum_{\sigma \in D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)} - \sum_{\sigma \in D_{n-1}^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right) \\
&+ \sum_{\sigma \in D_{n-1}^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + (-1)^{n-1} x^m \sum_{\sigma \in D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \\
&+ (-1)^n x^m \left(\sum_{\sigma \in D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)} - \sum_{\sigma \in D_{n-1}^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right) \\
&+ x^{2m} \sum_{\sigma \in D_{n-1}^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \\
=\;& (-1)^n x^{m-2} \sum_{\sigma \in D_{n-1}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \\
&+ \left(1 + (-1)^{n-1} x^{m-2} + (-1)^{n-1} x^m + x^{2m} \right) \sum_{\sigma \in D_{n-1}^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}
\end{aligned}
$$

and the result follows by Theorem 4.1 and our induction hypothesis. $\qquad\square$

We note the following consequence of Theorems 4.1 and 5.1.

Corollary 5.2. *Let $n \in \mathbb{P}$, $n \geq 3$, and $i \in [0, n-1]$. Then*

$$
\sum_{\sigma \in D_n} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (1 - x^2) \sum_{\sigma \in D_n^{\{i\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}.
$$

Proof. This follows immediately from Theorems 4.1 and 5.1. $\qquad\square$

The results obtained up to now compute $\sum_{\sigma \in D_n^I} (-1)^{\ell(\sigma)} x^{L(\sigma)}$ when $|I| \leq 1$. A natural next step is to try to compute these generating functions if $|I \setminus \{0\}| \leq 1$. We are able to do this for $I = \{0, 1\}$, and $I = \{0, 2\}$. The computation for $I = \{0, 2\}$ follows easily from results that we have already obtained.

Corollary 5.3. *Let $n \in \mathbb{P}$, $n \geq 3$. Then*

$$
\sum_{\sigma \in D_n^{\{0,2\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (1 - x^2) \prod_{j=4}^{n} (1 + (-1)^{j-1} x^{\lfloor \frac{j}{2} \rfloor})^2.
$$

Proof. By Proposition 3.4 and Proposition 3.6 we have

$$
\sum_{\sigma \in D_n^{\{0,2\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in D_n^{\{1,2\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in D_n^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)},
$$

and the result follows by Theorem 5.1. $\qquad\square$

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We conclude this section by computing $\sum_{\sigma \in D_n^I}(-1)^{\ell(\sigma)}x^{L(\sigma)}$ when $I = \{0,1\}$. When $n = 2$, the quotient $D_2^{\{0,1\}}$ only consists of the identity, thus the corresponding generating function is 1. For $n \geq 3$ we have the following.

Theorem 5.4. *Let $n \in \mathbb{P}$, $n \geq 3$. Then*

$$\sum_{\sigma \in D_n^{\{0,1\}}}(-1)^{\ell(\sigma)}x^{L(\sigma)} = (1+x^2)\prod_{j=4}^{n}(1+(-1)^{j-1}x^{\lfloor \frac{j}{2} \rfloor})^2.$$

Proof. It is easy to check that $\sum_{\sigma \in D_3^{\{0,1\}}}(-1)^{\ell(\sigma)}x^{L(\sigma)} = 1+x^2$. We proceed by induction.

By Lemma 3.3 we have that the sum over $\sigma \in D_n^{\{0,1\}}$ such that n or $-n$ appear in the window in any position but $1, 3$, or n is zero; moreover for $\sigma \in D_n^{\{0,1\}}$ we always have $\sigma^{-1}(\pm n) \neq 1$. Thus

$$\sum_{\sigma \in D_n^{\{0,1\}}}(-1)^{\ell(\sigma)}x^{L(\sigma)} = \sum_{\substack{\{\sigma \in D_n^{\{0,1\}}:\\ |\sigma(3)|=n\}}}(-1)^{\ell(\sigma)}x^{L(\sigma)} + \sum_{\substack{\{\sigma \in D_n^{\{0,1\}}:\\ |\sigma(n)|=n\}}}(-1)^{\ell(\sigma)}x^{L(\sigma)}.$$

By (5) and Lemma 3.2 we have

$$\sum_{\substack{\{\sigma \in D_n^{\{0,1\}}:\\ |\sigma(n)|=n\}}}(-1)^{\ell(\sigma)}x^{L(\sigma)} = \left(1+x^{2\lfloor \frac{n}{2} \rfloor}\right)\sum_{\sigma \in D_{n-1}^{\{0,1\}}}(-1)^{\ell(\sigma)}x^{L(\sigma)}$$

$$= \left(1+x^{2\lfloor \frac{n}{2} \rfloor}\right)(1+x^2)\prod_{j=4}^{n-1}(1+(-1)^{j-1}x^{\lfloor \frac{j}{2} \rfloor})^2$$

by our induction hypothesis. Moreover,

$$\sum_{\substack{\{\sigma \in D_n^{\{0,1\}}:\\ \sigma(3)=n\}}}(-1)^{\ell(\sigma)}x^{L(\sigma)} = \sum_{\sigma \in D_{n-1}^{\{1\}}\setminus D_{n-1}^{\{0,1\}}}(-1)^{\ell(\bar\sigma)}x^{L(\bar\sigma)}$$

where $\bar\sigma := [-\sigma(2), -\sigma(1), n, \sigma(3), \ldots, \sigma(n-1)]$. But

$$\begin{aligned}
\mathrm{inv}(\bar\sigma) &= \mathrm{inv}([n, \sigma(3), \ldots, \sigma(n-1)]) + |\{j \in [3, n-1] : -\sigma(1) > \sigma(j)\}| \\
&\quad + |\{j \in [3, n-1] : -\sigma(2) > \sigma(j)\}|, \\
\mathrm{nsp}(\bar\sigma) &= \mathrm{nsp}([n, \sigma(3), \ldots, \sigma(n-1)]) + |\{j \in [3, n-1] : -\sigma(1) + \sigma(j) < 0\}| \\
&\quad + |\{j \in [3, n-1] : -\sigma(2) + \sigma(j) < 0\}|, \\
\mathrm{oinv}(\bar\sigma) &= \mathrm{oinv}([n, \sigma(3), \ldots, \sigma(n-1)]) + |\{j \in [3, n-1] : j \equiv 0 \pmod 2, \sigma(1) + \sigma(j) < 0\}| \\
&\quad + |\{j \in [3, n-1] : j \equiv 1 \pmod 2, \sigma(2) + \sigma(j) < 0\}|,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{onsp}(\bar\sigma) &= \mathrm{onsp}([n, \sigma(3), \ldots, \sigma(n-1)]) + |\{j \in [3, n-1] : j \equiv 0 \pmod 2, \sigma(1) > \sigma(j)\}| \\
&\quad + |\{j \in [3, n-1] : j \equiv 1 \pmod 2, \sigma(2) > \sigma(j)\}|.
\end{aligned}$$

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Therefore, $\ell(\bar{\sigma}) = \ell(\sigma) + n - 4$, and $L(\bar{\sigma}) = \text{oinv}(\sigma) - 1 + \lceil \frac{n-3}{2} \rceil + \text{onsp}(\sigma) = L(\sigma) + m - 2$, where $m := \lfloor \frac{n}{2} \rfloor$. Similarly,

$$\sum_{\substack{\{\sigma \in D_n^{\{0,1\}}: \\ \sigma(3) = -n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\substack{\{\sigma \in B_{n-1} \setminus D_{n-1}: \\ \sigma(1) < \sigma(2) < -\sigma(1)\}}} (-1)^{\ell(\bar{\bar{\sigma}})} x^{L(\bar{\bar{\sigma}})}$$

where $\bar{\bar{\sigma}} := [-\sigma(2), -\sigma(1), -n, \sigma(3), \ldots, \sigma(n-1)]$ and $\ell(\bar{\bar{\sigma}}) = \ell(\sigma) + n$, $L(\bar{\bar{\sigma}}) = \text{oinv}(\sigma) + \text{onsp}(\sigma) + 1 + \lceil \frac{n-3}{2} \rceil = L(\sigma) + m$. But $\{\sigma \in B_{n-1} \setminus D_{n-1} : \sigma(1) < \sigma(2) < -\sigma(1)\} = (B_{n-1} \setminus D_{n-1})^{\{1\}} \setminus (B_{n-1} \setminus D_{n-1})^{\{0,1\}}$, so by Lemma 3.2, Theorem 5.1 and our induction hypothesis

$$\sum_{\substack{\{\sigma \in D_n^{\{0,1\}}: \\ |\sigma(3)| = n\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (-1)^n x^{m-2}(1 + x^2) \left(\sum_{\sigma \in D_{n-1}^{\{1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} - \sum_{\sigma \in D_{n-1}^{\{0,1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right)$$

$$= 2(-1)^{n-1} x^m (1 + x^2) \prod_{j=4}^{n-1} (1 + (-1)^{j-1} x^{\lfloor \frac{j}{2} \rfloor})^2$$

Thus

$$\sum_{\sigma \in D_n^{\{0,1\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (1 + (-1)^{n-1} x^m)^2 (1 + x^2) \prod_{j=4}^{n-1} (1 + (-1)^{j-1} x^{\lfloor \frac{j}{2} \rfloor})^2$$

and the result follows. □

6 Open problems

In this section we present some conjectures naturally arising from the present work and the evidence that we have in their favor.

In this paper we have given closed product formulas for $\sum_{\sigma \in D_n^I} (-1)^{\ell(\sigma)} x^{L(\sigma)}$ when $|I| \leq 1$, $I = \{0,1\}$ and $I = \{0,2\}$.

We feel that such formulas always exist. In particular, if $|I \setminus \{0,1\}| \leq 1$, we feel that the following holds. For $n \in \mathbb{P}$ and $I \subseteq [0, n-1]$ let, for brevity, $D_n^I(x) := \sum_{\sigma \in D_n^I} (-1)^{\ell(\sigma)} x^{L(\sigma)}$.

Conjecture 6.1. *Let $n \in \mathbb{P}$, $n \geq 5$, and $i \in [3, n-1]$. Then*

$$D_n^{\{0,i\}}(x) = \prod_{j=4}^{n} (1 + (-1)^{j-1} x^{\lfloor \frac{j}{2} \rfloor})^2.$$

Conjecture 6.2. *Let $n \in \mathbb{P}$, $n \geq 5$, and $i \in [3, n-1]$. Then*

$$D_n^{\{0,1,i\}}(x) = (1 - x^4) \prod_{j=5}^{n} (1 + (-1)^{j-1} x^{\lfloor \frac{j}{2} \rfloor})^2.$$

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We have verified these conjectures for $n \leq 8$. Note that, by Proposition 3.6, it is enough to prove Conjectures 6.1 and 6.2 for $i = 3$.

Note that, by Theorem 2.8, Conjectures 6.1 and 6.2 may be formulated in the following equivalent way. For $n \in \mathbb{P}$ and $J \subseteq [n-1]$ let $S_n^J(x) := \sum_{\sigma \in S_n^J} (-1)^{\ell_A(\sigma)} x^{L_A(\sigma)}$.

Conjecture 6.3. *Let $n \in \mathbb{P}$, $n \geq 5$, and $i \in [3, n-1]$. Then*

$$D_n^{\{0,i\}}(x) = (S_n^{\{i\}}(x))^2.$$

Conjecture 6.4. *Let $n \in \mathbb{P}$, $n \geq 5$, and $i \in [3, n-1]$. Then*

$$D_n^{\{0,1,i\}}(x) = (1 - x^4)\,(S_n^{\{1,i\}}(x))^2.$$

We feel that the presence of the factor $\prod_{j=6}^n (1 + (-1)^{j-1} x^{\lfloor \frac{j}{2} \rfloor})^2$ in Conjectures 6.1 and 6.2 is not a coincidence. More generally, we feel that the following holds.

Conjecture 6.5. *Let $n \in \mathbb{P}$, $n \geq 3$, and $J \subseteq [0, n-1]$. Let $J_0, J_1, \ldots, J_s$ be the connected components of J indexed as described before Theorem 2.10. Then there exists a polynomial $M_J(x) \in \mathbb{Z}[x]$ such that*

$$D_n^J(x) = M_J(x) \prod_{j=2m+2}^n (1 + (-1)^{j-1} x^{\lfloor \frac{j}{2} \rfloor})^2,$$

where $m := \sum_{i=0}^s \left\lfloor \frac{|J_i|+1}{2} \right\rfloor$. Furthermore, $M_J(x)$ only depends on $(|J_0|, |J_1|, \ldots, |J_s|)$ and is a symmetric function of $|J_1|, \ldots, |J_s|$.

This conjecture has been verified for $n \leq 8$.

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