A quantum distance between von Neumann algebras and applications to quantum field theory

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A notion of distance between von Neumann algebras appers to be a useful tool in order to study the dependence of the algebras of local observables of QFT from the parameters of the model. We report here on work in which such a notion is defined by dualizing Rieffel's quantum Gromov–Hausdorff distance between compact quantum metric spaces. A simple application to the mass dependence of the algebras generated by a free quantum field is also presented.

Keywords: Noncommutative geometry; quantum Gromov–Hausdorff distance; algebraic QFT.

1. Motivations

In the algebraic approach to quantum field theory[1], the relevant object of study are the von Neumann algebras $\mathcal{A}(O)$ generated by (bounded) observables which are measurable in bounded regions O of (Minkowski) spacetime. These algebras depend in general from the parameters of the QFT model under consideration, such as masses, spins and coupling constants. It is therefore rather natural to ask what are the properties of such a dependence, and in particular if it is continuous in some suitable sense.

For instance, it is well known that the Klein–Gordon field ϕ_m depends continuously on the mass $m \geq 0$, *e.g.*, in the sense that for the n-point vacuum expectation values there holds

$$\lim_{m' \to m} \langle \Omega, \phi_{m'}(x_1) \dots \phi_{m'}(x_n)\Omega \rangle = \langle \Omega, \phi_m(x_1) \dots \phi_m(x_n)\Omega \rangle.$$

It is then tempting to try to find out a sense in which this property can be lifted to the associated local von Neumann algebras $\mathcal{A}_m(O)$.

A more interesting example is obtained by considering the algebraic formulation of the renormalization group proposed in Ref. 2. There, a model independent procedure is given which associates to a given system of local algebras $\mathcal{A}(O)$, their *scaling limit* algebras $\mathcal{A}_0(O)$, describing the short distance behavior of the original theory. The scaling limit algebras are obtained by applying a reconstruction theorem to the limit, as the scaling parameter $\lambda \to 0$, of the vacuum expectation values of suitable functions $\lambda \mapsto A_\lambda \in \mathcal{A}(\lambda O)$. A very natural question is therefore whether one can regard $\mathcal{A}_0(O)$ as the limit, as $\lambda \to 0$, of $\mathcal{A}(\lambda O)$ in some metric space of von Neumann algebras.

The above mentioned examples, among others, point to the necessity of a suitable notion of distance between von Neumann algebras. Such a distance has been defined in Ref. 3, by dualizing Rieffel's construction[4] of the quantum Gromov–Hausdorff distance between compact quantum metric spaces, which is in turn a noncommutative version of the Gromov–Hausdorff distance between ordinary compact metric spaces. As an application, it also shown that the local von Neumann algebras $\mathcal{A}_m(O)$ associated to the mass $m \geq 0$ Klein–Gordon field depend continuously on m.

In the present contribution we review the above mentioned results. In particular, in Sec. 2 we will briefly recall the definition and main properties of the Gromov–Hausdorff distance, and in Sec. 3 we will describe its quantum counterpart. Section 4 will be devoted to the discussion of the results of Ref. 3 on the dual quantum Gromov–Hausdorff distance between von Neumann algebras, and finally in Sec. 5 we will explain the application to the free scalar field.

2. The Gromov–Hausdorff distance between compact metric spaces

Let (Z, d) be a metric space. The classical *Hausdorff distance* between compact subsets $X, Y \subset Z$ is defined by

$$\mathrm{dist}_H(X, Y) := \inf\{r > 0 \,:\, X \subset \mathcal{N}_r(Y),\, Y \subset \mathcal{N}_r(X)\},$$

where $\mathcal{N}_r(X) := \bigcup_{x \in X}\{y \in Z \,:\, d(x, y) < r\}$ is the set of all points of Z which are within a distance r from some point of X. In other words, $\mathrm{dist}_H(X, Y) < r$ if and only if for each $x \in X$ there is a $y \in Y$ such that $d(x, y) < r$, and the same holds interchanging the roles of x and y.

It is then well known (see, *e.g.*, Ref. 5) that dist_H is a metric on the the space of all compact subsets of Z, and that such space is complete (resp. compact) if and only if Z is.

This distance was extended by Gromov[6] to a distance between abstract compact metric spaces X and Y, by considering the set $\mathcal{D}(X, Y)$ of all the metrics d on the (disjoint) union $X \sqcup Y$ such that the canonical embeddings $X, Y \hookrightarrow X \sqcup Y$ are isometries, then computing the Hausdorff distance $\mathrm{dist}^d_H(X, Y)$ between X and Y in $(X \sqcup Y, d)$, and finally taking the infimum of this quantity over $\mathcal{D}(X, Y)$. Namely, the *Gromov–Hausdorff distance* of X, Y is

$$\mathrm{dist}_{GH}(X, Y) := \inf_{d \in \mathcal{D}(X,Y)} \mathrm{dist}^d_H(X, Y).$$

Again there holds that dist_{GH} is a metric on the space of isometry classes of compact metric spaces (*i.e.*, if $\mathrm{dist}_{GH}(X, Y) = 0$ then X and Y are isometric), and this space is complete with this metric.

3. Quantum compact metric spaces and quantum Gromov–Hausdorff distance

The central idea of *noncommutative geometry*[7] is that, as ordinary spaces can be equivalently described by the (commutative) algebras of functions of their coordinates, noncommutative (or quantum) spaces are defined by suitable noncommutative algebras (or more general algebraic structures), whose elements are interpreted as functions of the corresponding "noncommutative coordinates". In this setting, ordinary points are then replaced by positive functionals (*states*) on the algebra.

Focusing then on compact metric spaces (X, d), one sees that the metric d can be recovered from the *Lipschitz seminorm* L on the space $C(X)$ of (complex) continuous functions over X:

$$L(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Indeed, there holds $d(x, y) = \sup_{L(f) \leq 1} |f(x) - f(y)|$.

This observation led Rieffel[8] to define a *compact quantum metric space* as a order–unit space[a] X together with a *Lip-seminorm* $L : X \to [0, +\infty)$ which vanishes only on the multiples of the unit, and such that the distance between elements φ, ψ of the state space $\mathcal{S}(X)$ of X, defined by analogy with ordinary metric spaces,

$$d_L(\varphi, \psi) := \sup_{x \in X,\, L(x) \leq 1} |\varphi(x) - \psi(x)|,$$

induces the natural w^*-topology on the states. Two compact quantum metric spaces X, Y with respective Lip-seminorms L_X, L_Y, will be said to be *Lip-isomorphic* if there is an isomorphism $\phi : X \to Y$ such that $L_Y \circ \phi = L_X$.

Then the *quantum Gromov–Hausdorff distance* $\mathrm{dist}_{qGH}(X, Y)$ of X and Y is defined by considering the set $\mathcal{L}(X, Y)$ of Lip-seminorms L on the direct sum $X \oplus Y$ (which is the noncommutative analog of the disjoint union) such that $L|_{X \oplus \{0\}} = L_X$, $L|_{\{0\} \oplus Y} = L_Y$, and by setting

$$\mathrm{dist}_{qGH}(X, Y) := \inf_{L \in \mathcal{L}(X,Y)} \mathrm{dist}_H^L(\mathcal{S}(X), \mathcal{S}(Y)),$$

where $\mathrm{dist}_H^L(\mathcal{S}(X), \mathcal{S}(Y))$ is the Hausdorff distance, in the metric space $(\mathcal{S}(X \oplus Y), d_L)$, of the obvious embeddings of $\mathcal{S}(X)$, $\mathcal{S}(Y)$. The main result of Ref. 4 is that this is a metric on the space of Lip-isomorphism classes of quantum compact metric spaces, which is then a complete (ordinary) metric space.

As the space of continuous function over a compact space is not just an order–unit space, but actually a (commutative) C*-algebra, and in view of applications to the (noncommutative) C*-algebras of quantum field theory, it seems desirable to measure the distance of two C*-algebras A, B by applying the above defined metric to their selfadjoint parts A_{sa}, B_{sa}, which are order–unit spaces. But one has to face immediately the problem that isomorphism of A_{sa}, B_{sa} as order–unit spaces does not entail isomorphism of A, B as C*-algebras.

[a]An order–unit space is a real vector space in which a partial order and a distinguished element (the unit), satisfying certain natural assumptions, are defined, and which can always be thought of as a subspace of the selfadjoint operators on a Hilbert space.

This was remedied by Kerr[9], who defined a *matricial quantum Gromov–Hausdorff distance* between two C*-algebras A, B endowed with densely defined seminorms L_A, L_B which are Lip-seminorms on A_{sa}, B_{sa}, as

$$\mathrm{dist}^2_{qGH}(A, B) := \inf_{L \in \mathcal{L}(A,B)} \mathrm{dist}^L_H(\mathcal{S}_2(A), \mathcal{S}_2(B)),$$

where $\mathcal{S}_2(A)$ is a space of matrix–valued states (more precisely: unital, completely positive maps) $\varphi : A \to M_2(\mathbb{C})$, and the distance of two such states φ, ψ is $d_L(\varphi, \psi) := \sup_{L(a) \leq 1} \|\varphi(a) - \psi(a)\|$, in analogy with Rieffel's definition. In this way, one gets a distance on the space of Lip-isomorphism classes of C*-algebras, which is however not complete[10].

4. Lip-von Neumann algebras and dual quantum Gromov-Hausdorff distance

If one would like to apply the above discussed notion of distance to the local von Neumann algebras $\mathcal{A}(O)$ of quantum field theory, the first question one has to answer is the choice of sensible Lip-seminorms on these algebras.

Candidates which are rather natural, especially having in mind the problem of convergence to the scaling limit algebras, are obtained by setting $L(A) := \|\Theta(A)\|$, with $\Theta : \mathcal{A}(O) \to X$, X a Banach space, one of the many compact (or nuclear) maps considered in the literature in order to characterize the phase–space properties of specific models (see, *e.g.*, Ref. 11 for a discussion of some of these maps). A moment's reflection shows however that the compactness of Θ is equivalent to the fact that L so defined induces the σ-weak topology on the (norm) unit ball $\mathcal{A}(O)_1$, which is the same as the w^*-topology of $\mathcal{A}(O)$ as a dual Banach space. In turn, this is also equivalent to the fact that actually the norm L_* defined on the predual $\mathcal{A}(O)_*$ (the space of σ-weakly continuous functionals on $\mathcal{A}(O)$) as

$$L_*(\varphi) := \sup_{A \in \mathcal{A}(O)} \frac{|\varphi(A)|}{L(A)}$$

is a Lip-norm.

Thus the situation suggested by algebraic quantum field theory is in a sense dual to Rieffel's one, in that one has a Lip-norm not on the algebra but on its predual, and the corresponding dual norm induces the w^*-topology on the algebra. This motivates the following definition.

Definition 4.1 (Def. 1.1 of Ref. 3). *A* Lip-von Neumann algebra *is a von Neumann algebra M which is equipped, besides the operator norm, with a* dual-Lip-norm *L, which is a norm that induces the σ-weak topology on bounded subsets of M.*

It is easy to see that this is equivalent to require that the predual M_* has a densely defined norm L_* such that the set $\{\varphi \in M : L_*(\varphi) \leq 1\}$ is compact with respect to the natural norm of M_*.

In this setting, the definition of distance is completely parallel to that of Rieffel and Kerr. In particular, in order to obtain a metric and not only a pseudo-metric, one has to consider matrices, and therefore we extend a dual-Lip-norm L on M to the algebra $\mathcal{M}_2(M)$ of 2×2 matrices with entries in M by $L((x_{ij})) := \max_{ij} L(x_{ij})$.

Definition 4.2 (Def. 2.3 of Ref. 3). *Given Lip-von Neumann algebras M, N with respective dual-Lip-norms L_M, L_N, denote by $\mathcal{L}(M,N)$ the set of seminorms L on $M \oplus N$ such that $L|_{M\oplus\{0\}} = L_M$, $L|_{\{0\}\oplus N} = L_N$. The* dual quantum Gromov–Hausdorff distance *between M and N is then defined as*

$$\mathrm{dist}_{qGH^*}(M,N) := \inf_{L\in\mathcal{L}(M,N)} \mathrm{dist}_H^L(\mathcal{M}_2(M)_{1,+}, \mathcal{M}_2(N)_{1,+}),$$

where $\mathcal{M}_2(M)_{1,+}$ is the positive part of the unit ball of $\mathcal{M}_2(M)$ (identified with its canonical embedding into $\mathcal{M}_2(M \oplus N)$).

Again, the Lip-von Neumann algebras M, N are called *Lip-isomorphic* if there is a von Neumman algebra isomorphism $\phi : M \to N$ such that $L_N(\phi(x)) = L_M(x)$ for all $x \in M$. The main result of Ref. 3 is the following one.

Theorem 4.1 (Thm. 2.13 of Ref. 3). dist_{qGH^*} *is a metric on the space of Lip-isomorphism classes of Lip-von Neumann algebras.*

Contrary to what usually happens, but similarly to the case of dist_{qGH}, the difficult part of the proof here is the zero–distance property (*i.e.*, the fact that $\mathrm{dist}_{qGH^*}(M,N) = 0$ implies that M, N are Lip-isomorhic), which is handled with techniques inspired by Rieffel's ones.

Continuing the parallelism with dist_{qGH}, the space of Lip-von Neumann algebras equipped with dist_{qGH^*} appears to be not complete, essentially because there is no way of controlling $L(xy)$ through $L(x)$, $L(y)$. It is expected however that one can obtain a complete space by extending dist_{qGH^*} to the so-called dual operator systems, and considering $n \times n$ matrices for all $n \in \mathbb{N}$.

5. An application to free quantum fields

Let $\phi_m(x)$ be the mass $m \geq 0$ Klein–Gordon field over 4-dimensional Minkowski spacetime. Given a bounded open spacetime region O, the corresponding von Neumann algebra $\mathcal{A}_m(O)$ is defined as the von Neumann algebra generated by the unitary operators $e^{i\phi_m(f)}$ acting on Fock space $\mathcal{H}$, where $\phi_m(f) = \int dx f(x)\phi_m(x)$ denotes the smearing of ϕ_m with a smooth, real valued test function f with support in O.

A classical result[12] asserts that there is a von Neumann algebra isomophism $\tau_m : \mathcal{A}_m(O) \to \mathcal{A}_0(O)$ such that $\tau_m(e^{i\phi_m(f)}) = e^{i\phi_0(f)}$. Moreover, denoting by H_m the mass m hamiltonian operator, and by Ω the vacuum vector, the map $\Theta_m :$ $\mathcal{A}_m(O) \to \mathcal{H}$ defined by $\Theta_m(A) := e^{-\beta H_m} A\Omega$, $\beta > 0$, is compact for all $m \geq 0$ (Ref. 11).

Theorem 5.1 (Prop. 3.1 and Thm. 3.3 of Ref. 3).
The map $L_m(A) := \|e^{-\beta H_m}\tau_m(A)\Omega\|$ *is a dual-Lip-norm on the von Neumann algebra* $\mathcal{A}_m(O)$, $m \geq 0$, *and, with respect to such dual-Lip-norms, there holds*

$$\lim_{m' \to m} \mathrm{dist}_{qGH^*}(\mathcal{A}_{m'}(O), \mathcal{A}_m(O)) = 0,$$

i.e., *the family of Lip-von Neumann algebras* $\mathcal{A}_m(O)$ *depends continuously on* $m \geq 0$.

Thus we see that the first problem raised in Sec. 1 has a positive solution. The proof of this theorem is obtained by observing that the dual-Lip-norms L_m, $L_{m'}$ can be transported to the same algebra $\mathcal{A}_0(O)$ through τ_m, $\tau_{m'}$, and using the following simple estimate, valid for a von Neumann algebra M with two different dual-Lip-norms L_1, L_2:

$$\mathrm{dist}_{qGH^*}((M, L_1), (M, L_2)) \leq \sup_{x \in M_1} L(x, -x),$$

where L is any seminorm on $M \oplus M$ restricting to L_1, L_2 on the two summands. This reduces the proof to showing that $\sup_{A \in \mathcal{A}_0(O)_1} \|(e^{-\beta H_{m'}} - e^{-\beta H_m})A\Omega\| \to 0$ as $m' \to m$.

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