Asymptotic results for finite superpositions of Ornstein-Uhlenbeck processes*

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Abstract

A model of intermittency based on superposition of Lévy driven Ornstein-Uhlenbeck processes is studied in [6]. In particular, as shown in Theorem 5.1 in that paper, finite superpositions obey a (sample path) central limit theorem under suitable hypotheses. In this paper we prove large (and moderate) deviation results associated with this central limit theorem.

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1 Introduction

Lévy driven (or non-Gaussian) Ornstein-Uhlenbeck processes are widely studied in the literature; they play an important role in theory (see e.g. Section 17 in [8] for the connections with the selfdecomposable distributions) and applications (see e.g. [1] for their uses in finance).

Some recent results for superpositions of Lévy driven Ornstein-Uhlenbeck (OU for short) processes are presented in [6]. In particular it was shown that, while the partial sums of finite superpositions obey the (sample-path) central limit theorem, partial sums of infinite long-range dependent superpositions provide examples of intermittent processes. The phenomenon of intermittency has been widely discussed in physics literature (see the references cited in [6]); for a formal definition of intermittency appearing in the theory of stochastic partial differential equations see [2] and Chapter 7 in [7]. The term intermittency is used in the literature to describe models exhibiting high degree of variability and enormous fluctuations which escape from the scope of the usual limit theory; moreover other terms as multifractality, separation of scales and dynamo effect are often used interchangeably with intermittency.

The aim of this paper is to present large (and moderate) deviation results associated to Theorem 5.1 in [6], namely the (sample-path) central limit theorem for partial sums of finite superpositions; see the next Theorem 1.1 with the correct expression of c_K (in [6] there is a typo). Throughout the paper we use the notation $[x] := \max\{k \in \mathbb{Z} : k \leq x\}$. The theory of large deviations is a collection of techniques which gives an asymptotic estimate of small probabilities on an exponential scale; see e.g. [4] as a reference on this topic.

Theorem 1.1. *Let $\{X^{(k)} : k \in \{1, \dots, K\}\}$ be independent one-dimensional stationary OU processes, namely*

$$dX^{(k)}(t) = -\lambda_k X^{(k)}(t) + dZ^{(k)}(\lambda_k t), \tag{1}$$

*where $\lambda_k > 0$ and $\{Z^{(k)}(t) : k \geq 1\}$ are independent Lévy processes which satisfy suitable conditions (see Assumption **A** (with equation (5)) in [6]; in particular, for all $k \geq 1$, they assume that the*

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self-decomposable distribution of the random variables $\{X^{(k)}(t) : t \geq 0\}$ *has finite moments of order* $p \geq 2$*). Moreover we consider the partial sums*

$$S_K(t) := \sum_{i=1}^{[t]} X_K(i), \text{ where } X_K(t) := \sum_{k=1}^{K} X^{(k)}(t).$$

Then

$$\left\{ \frac{1}{c_K N^{1/2}} (S_K([Nt]) - \mathbb{E}[S_K([Nt])]) \right\}_{t \in [0,1]} \text{ converges to } \{B(t)\}_{t \in [0,1]} \text{ as } N \to \infty$$

(we mean the convergence in the sense of weak convergence in Skorokhod space $D[0,1]$*), where* $\{B(t)\}_{t \in [0,1]}$ *is a standard Brownian motion on* $[0,1]$*,*

$$c_K := \left(\sum_{k=1}^{K} \mathrm{Var}[X^{(k)}] \frac{1 + e^{-\lambda_k}}{1 - e^{-\lambda_k}} \right)^{1/2}, \tag{2}$$

and $\mathrm{Var}[X^{(k)}]$ *is the (common) variance of the random variables* $\{X^{(k)}(t) : t \geq 0\}$*.*

We remark that we are implicitly assuming that $c_K > 0$, namely we are neglecting the case where $Z^{(1)}, \ldots, Z^{(K)}$ are deterministic Lévy processes. Moreover, by (1), we have

$$X^{(k)}(t) = e^{-\lambda_k t} \left(X^{(k)}(0) + \int_0^t e^{\lambda_k s} dZ^{(k)}(\lambda_k s) \right). \tag{3}$$

In this paper we present two kind of results:

1. large deviation principle of $\left\{ \frac{S_K([N\cdot])}{N} : n \geq 1 \right\}$ with speed N, for Gaussian OU processes;
2. for every sequence of positive numbers $\{\varepsilon_N : N \geq 1\}$ such that $\varepsilon_N \to 0$ and $N\varepsilon_N \to \infty$ as $N \to \infty$, a class of large deviation principles of $\left\{ \frac{S_K([N\cdot]) - \mathbb{E}[S_K([N\cdot])]}{\sqrt{N/\varepsilon_N}} : n \geq 1 \right\}$ with speed $1/\varepsilon_N$.

The second one concerns moderate deviations, which fill the gap between the two following regimes:

- the asymptotic Normality result in Theorem 5.1 in [6] (case $\varepsilon_N = 1$ and therefore only the condition $N\varepsilon_N \to \infty$ holds);
- the convergence of the centered sequence $\left\{ \frac{S_K([N\cdot]) - \mathbb{E}[S_K([N\cdot])]}{N} : n \geq 1 \right\}$ to the null function (case $\varepsilon_N = 1/N$ and therefore only the condition $\varepsilon_N \to 0$ holds).

We conclude with the outline of the paper. We start with some preliminaries in Section 2. In Section 3 we study some useful functions which play a crucial role in the proofs of the results. Large and moderate deviation results are presented in Sections 4 and 5, respectively.

2 Preliminaries

In this section we present some preliminaries on large deviations and on Lévy processes.

2.1 Preliminaries on large deviations

We start by recalling some basic definitions on large deviations (see e.g. [4]). Let $\mathcal{Y}$ be a Hausdorff topological space with Borel σ-algebra $\mathcal{B}(\mathcal{Y})$. Let $\{v_N : N \geq 1\}$ be a sequence such that $v_N \to \infty$ as $N \to \infty$, called speed. Let $I : \mathcal{Y} \to [0, \infty]$ be a lower semi-continuous function, called rate function. Then a sequence of $\mathcal{Y}$-valued random variables $\{Y_N : N \geq 1\}$ satisfies the *large deviation principle* (LDP for short) with rate function I and speed v_N if

$$\limsup_{N\to\infty} \frac{1}{v_N} \log P(Y_N \in C) \leq - \inf_{y\in C} I(y) \quad \text{for all closed sets } C \subset \mathcal{Y}$$

and

$$\liminf_{N\to\infty} \frac{1}{v_N} \log P(Y_N \in G) \geq - \inf_{y\in G} I(y) \quad \text{for all open sets } G \subset \mathcal{Y}.$$

A rate function I is said to be good if all its level sets $\{\{y \in \mathcal{Y} : I(y) \leq \eta\} : \eta \geq 0\}$ are compact. We also recall that a sequence $\{Y_N : N \geq 1\}$ is said to be exponentially tight (with respect to the speed v_N) if, for all $\eta > 0$, there exists a compact set K_η, $K_\eta \subset \mathcal{Y}$, such that

$$\limsup_{N\to\infty} \frac{1}{v_N} \log P(Y_N \notin K_\eta) \leq -\eta.$$

We conclude by recalling some large deviation theorems used in this paper. We apply Gärtner Ellis Theorem (see e.g. Theorem 2.3.6 in [4]) to prove the results with finite dimensional distributions, namely Proposition 4.1 for large deviations and Proposition 5.1 for moderate deviations. Moreover we apply Dawson Gärtner Theorem (see e.g. Theorem 4.6.1 in [4]) to obtain sample path large deviation results with respect to the topology of pointwise convergence; typically this is the first step, and later we obtain sample path large deviation results with respect to the topology of uniform convergence (namely Proposition 4.2 for large deviations and Proposition 5.2 for moderate deviations) by considering suitable polygonal approximations of the processes.

2.2 Preliminaries on Lévy processes and OU processes

Here we recall some known results for Lévy processes and for Lévy driven stationary OU processes. In view of what follows we refer to $\{Z^{(k)}(t) : t \geq 0\}$ and $\{X^{(k)}(t) : t \geq 0\}$ in (1), or in (3). Throughout the paper we use the symbol "i" for the imaginary unit.

Firstly we can say that

$$\mathbb{E}[e^{\mathrm{i}\theta Z^{(k)}(t)}] = \exp(t\tilde{\Psi}_{Z^{(k)}}(\theta)),$$

where $\tilde{\Psi}_{Z^{(k)}}(\cdot)$ is the logarithm of the characteristic function of the random variable $Z^{(k)}(1)$; moreover, for $a_k \in \mathbb{R}$, $b_k \geq 0$ and a measure ν_k (called Lévy measure) with suitable properties, we have

$$\tilde{\Psi}_{Z^{(k)}}(\theta) = \mathrm{i}a_k\theta - \frac{b_k\theta^2}{2} + \int_{\mathbb{R}} (e^{\mathrm{i}x\theta} - 1 - \mathrm{i}\theta x 1_{\{|x|\leq 1\}})\nu_k(dx). \tag{4}$$

Moreover

$$\mathbb{E}\left[e^{\mathrm{i}\int_{t_1}^{t_2} f(s)dZ^{(k)}(t)}\right] = \exp\left(\int_{t_1}^{t_2} \tilde{\Psi}_{Z^{(k)}}(f(s))ds\right) \text{ (for all } t_2 \geq t_1 \geq 0) \tag{5}$$

for all continuous and locally bounded functions f (see e.g. Lemma 15.1 in [3]). Furthermore, as far as the logarithm of the (common) characteristic function of the random variables $\{X^{(k)}(t) : t \geq 0\}$ is concerned, i.e.

$$\tilde{\Psi}_{X^{(k)}}(\theta) := \log \mathbb{E}\left[e^{\mathrm{i}\theta X^{(k)}(t)}\right]$$

which does not depend on t, we have

$$\tilde{\Psi}_{X^{(k)}}(\theta) = \int_0^\infty \lambda_k \tilde{\Psi}_{Z^{(k)}}(\theta e^{-\lambda_k s})ds \tag{6}$$

(see e.g. Proposition 15.4 in [3]; it is a slight modification of equation (15.22) in that reference).

Remark 2.1. *Throughout this paper we often deal with logarithm of moment generating functions, usually denoted by* Ψ*, instead of logarithm of characteristic functions. So, when* $\Psi(\theta)$ *is finite (we usually assume that the logarithm of moment generating functions are finite in the neighborhood of the origin), we have* $\Psi(\theta)=\tilde{\Psi}(-\mathrm{i}\theta)$*. Thus, when* $\Psi_{Z^{(k)}}(\theta):=\log\mathbb{E}[e^{\theta Z^{(k)}(1)}]$ *is finite, we have*

$$\Psi_{Z^{(k)}}(\theta)=a_k\theta+\frac{b_k\theta^2}{2}+\int_{\mathbb{R}}(e^{x\theta}-1-\theta x1_{\{|x|\leq 1\}})\nu_k(dx)$$

by (4)*; moreover, when* $\Psi_{X^{(k)}}(\theta):=\log\mathbb{E}[e^{\theta X^{(k)}(t)}]$ *(which does not depend on* t*) is finite,*

$$\Psi_{X^{(k)}}(\theta)=\int_0^\infty\lambda_k\Psi_{Z^{(k)}}(\theta e^{-\lambda_k s})ds$$

by (6)*.*

We recall that (6) yields the following formulas for the (common) expected value $\mathbb{E}[X^{(k)}]$ and variance $\mathrm{Var}[X^{(k)}]$ of the random variables $\{X^{(k)}(t):t\geq 0\}$, i.e.

$$\mathbb{E}[X^{(k)}]=\mathbb{E}[Z^{(k)}(1)]$$

and

$$\mathrm{Var}[X^{(k)}]=\frac{1}{2}\mathrm{Var}[Z^{(k)}(1)]=\frac{1}{2}\left(b_k+\int_{\mathbb{R}}x^2\nu_k(dx)\right). \tag{7}$$

We also recall some details on the class of infinitely divisible distributions called self-decomposable distributions. This class coincides with the class of stationary distributions of Lévy driven OU processes (see e.g. Proposition 15.4 in [3]). A random variable Y has self-decomposable distribution if, for all $b>1$, there exists a random variable $X^{(b)}$, independent of X, such that X is distributed as $\frac{X}{b}+X^{(b)}$. Self-decomposable distributions can be characterized in terms of the Lévy measures which appears in the representation of their characteristic function (see e.g. Proposition 15.3 in [3]).

3 Some useful functions

In this section we consider some useful functions where N is fixed. These functions will appear in the proofs of the asymptotic results as $N\to\infty$ (both large and moderate deviations).

3.1 Some functions for the results on finite-dimensional distributions

Here we consider the functions $\Lambda_N(\cdot,\underline{t}_m)$ and $\{\Psi_{N,k}(\cdot,\underline{t}_m):k\in\{1,\ldots,K\}\}$ defined below, which play a crucial role in the proofs of the results for finite dimensional distributions. More precisely we mean the finite dimensional distributions of the processes with m time instants $\underline{t}_m=(t_1,\ldots,t_m)$ such that

$$0=t_0<t_1<\cdots<t_m.$$

So we often consider column vectors $\underline{v}_m=(v_1,\ldots,v_m)^{\mathrm{tr}}\in\mathbb{R}^m$, where $(\cdot)^{\mathrm{tr}}$ is the transpose operator for vectors or matrices; we use the notation $\langle\cdot,\cdot\rangle$ for the inner product in $\mathbb{R}^m$ and $\underline{0}_m$ is the null vector in $\mathbb{R}^m$.

In view of what follows, we need some other notation. We start with the matrix $T=(t_{h\ell})_{h,\ell\in\{1,\ldots,m\}}$ defined by

$$t_{h\ell}:=\begin{cases}1 & \text{for } \ell\geq h\\ 0 & \text{for } \ell\leq h-1;\end{cases}$$

we denote by $T_{(v)}$ the v-th row of the matrix T and, for $\underline{\theta}_m=(\theta_1,\ldots,\theta_m)^{\mathrm{tr}}\in\mathbb{R}^m$, we have the vector

$$T\underline{\theta}_m=((T\underline{\theta}_m)_1,\ldots,(T\underline{\theta}_m)_m)^{\mathrm{tr}}\in\mathbb{R}^m,\ \text{where } (T\underline{\theta}_m)_h=T_{(h)}\underline{\theta}_m=\sum_{\ell=h}^m\theta_\ell\ \text{(for } h\in\{1,\ldots,m\}).$$

Moreover let $\underline{L}_m^{(v)}(N) = (L_1^{(v)}(N), \ldots, L_m^{(v)}(N))^{\mathrm{tr}}$ be the vector defined by

$$L_h^{(v)}(N) := \begin{cases} 0 & \text{for } h \leq v-1 \\ \hat{c}_{v,v}(N) & \text{for } h = v \\ c_{h,v}(N)(1 - e^{-\lambda_k([Nt_v]-[Nt_{v-1}])}) & \text{for } h \geq v+1, \end{cases} \tag{8}$$

where

$$c_{h,v}(N) := \frac{e^{-\lambda_k([Nt_{h-1}]-[Nt_v]+1)}(1 - e^{-\lambda_k([Nt_h]-[Nt_{h-1}])})}{1 - e^{-\lambda_k}} \text{ (for } 0 \leq v < h \leq m) \tag{9}$$

and

$$\hat{c}_{v,v}(N) := [Nt_v] - [Nt_{v-1}] - \frac{e^{-\lambda_k}(1 - e^{-\lambda_k([Nt_v]-[Nt_{v-1}])})}{1 - e^{-\lambda_k}} \text{ (for } v \in \{1, \ldots, m\}); \tag{10}$$

let $Q^{(v)}(N) = (Q_{h\ell}^{(v)}(N))_{h,\ell \in \{1,\ldots,m\}}$ be the symmetric $m \times m$ matrix defined by (we consider the case $h \geq \ell$)

$$Q_{h\ell}^{(v)}(N) := \begin{cases} 0 & \text{for } \ell \leq h \leq v-1 \\ \frac{1+e^{-\lambda_k}}{1-e^{-\lambda_k}} \Big([Nt_v] - [Nt_{v-1}] \\ \quad - \frac{2e^{-\lambda_k}(1-e^{-\lambda_k([Nt_v]-[Nt_{v-1}])})}{1-e^{-\lambda_k}} + \frac{e^{-2\lambda_k}(1-e^{-2\lambda_k([Nt_v]-[Nt_{v-1}])})}{1-e^{-2\lambda_k}} \Big) & \text{for } h = \ell = v \\ (1+e^{-\lambda_k})c_{h,v}(N) \left(\frac{1-e^{-\lambda_k([Nt_v]-[Nt_{v-1}])}}{1-e^{-\lambda_k}} - \frac{e^{-\lambda_k}(1-e^{-2\lambda_k([Nt_v]-[Nt_{v-1}])})}{1-e^{-2\lambda_k}} \right) & \text{for } h > \ell = v \\ c_{h,v}(N)c_{\ell,v}(N)(1 - e^{-2\lambda_k([Nt_v]-[Nt_{v-1}])}) & \text{for } h \geq \ell \geq v+1; \end{cases} \tag{11}$$

let $\underline{D}_m^{(v,q)}(N) = (D_1^{(v,q)}(N), \ldots, D_m^{(v,q)}(N))^{\mathrm{tr}}$ be the vector defined by

$$D_h^{(v,q)}(N) := \begin{cases} 0 & \text{for } h \leq v-1 \\ \frac{e^{-\lambda_k}(1-e^{-\lambda_k q})}{1-e^{-\lambda_k}} & \text{for } h = v \\ c_{h,v}(N)e^{-\lambda_k q} & \text{for } h \geq v+1. \end{cases} \tag{12}$$

Now we are ready to present the main functions studied in this section. We start with

$$\Lambda_N(\underline{\theta}_m, \underline{t}_m) := \log \mathbb{E}\left[\exp\left(\sum_{\ell=1}^m \theta_\ell S_K([Nt_\ell])\right)\right]$$
$$= \log \mathbb{E}\left[\exp\left(\sum_{\ell=1}^m \theta_\ell \sum_{i=1}^{[Nt_\ell]} X_K(i)\right)\right] = \log \mathbb{E}\left[\exp\left(\sum_{\ell=1}^m \theta_\ell \sum_{i=1}^{[Nt_\ell]} \sum_{k=1}^K X^{(k)}(i)\right)\right],$$

where $N \geq 1$, $m \geq 1$ and $\underline{\theta}_m = (\theta_1, \ldots, \theta_m)^{\mathrm{tr}} \in \mathbb{R}^m$. We remark that, by the independence of the OU processes $\{X^{(k)} : k \geq 1\}$, we have

$$\Lambda_N(\underline{\theta}_m, \underline{t}_m) = \sum_{k=1}^K \Psi_{N,k}(\underline{\theta}_m, \underline{t}_m), \text{ where } \Psi_{N,k}(\underline{\theta}_m, \underline{t}_m) := \log \mathbb{E}\left[\exp\left(\sum_{\ell=1}^m \theta_\ell \sum_{i=1}^{[Nt_\ell]} X^{(k)}(i)\right)\right]. \tag{13}$$

In the next Lemma 3.1 we give an explicit formula for $\Psi_{N,k}(\underline{\theta}_m, \underline{t}_m)$ (thus, by (13), we immediately get an explicit expression for $\Lambda_N(\underline{\theta}_m, \underline{t}_m)$).

Lemma 3.1. *Let us consider the functions*

$$\varphi_0^{(k)}(\theta) := \log \mathbb{E}\left[\exp\left(\theta X^{(k)}(0)\right)\right]$$

and

$$\varphi_{j-1,j}^{(k)}(\theta) := \log \mathbb{E}\left[\exp\left(\theta \int_{j-1}^j e^{\lambda_k s} dZ^{(k)}(\lambda_k s)\right)\right] \text{ (for } j \in \{0, \ldots, [Nt_m]\}).$$

Then, if $\Psi_{N,k}(\underline{\theta}_m, \underline{t}_m)$ in (13) is finite, we have

$$\Psi_{N,k}(\underline{\theta}_m, \underline{t}_m) = \varphi_0^{(k)}\left(\sum_{h=1}^{m}(T\underline{\theta}_m)_h c_{h,0}(N)\right)$$
$$+\sum_{v=1}^{m}\left\{\left(a_k - \int_{[-1,1]} x\nu_k(dx)\right)\langle \underline{L}_m^{(v)}(N), T\underline{\theta}_m\rangle + \frac{b_k}{4}\langle T\underline{\theta}_m, Q^{(v)}(N)T\underline{\theta}_m\rangle\right.$$
$$\left.+\lambda_k \sum_{q=1}^{[Nt_v]-[Nt_{v-1}]}\int_0^1 \left(\int_{\mathbb{R}}(\exp\left(x\langle \underline{D}_m^{(v,q)}(N), T\underline{\theta}_m\rangle e^{\lambda_k u}\right) - 1)\nu_k(dx)\right)du\right\}.$$

Actually the equality holds even if $\Psi_{N,k}(\underline{\theta}_m, \underline{t}_m)$ is equal to infinity (and the right hand side is not finite).

We remark that, by (4) and (6) (see also Remark 2.1), we have

$$\varphi_0^{(k)}(\theta) = \int_0^\infty \lambda_k \tilde{\Psi}_{Z^{(k)}}(-\mathrm{i}\theta e^{-\lambda_k s})ds$$
$$= \lambda_k \int_0^\infty \left(a_k\theta e^{-\lambda_k s} + \frac{b_k(\theta e^{-\lambda_k s})^2}{2} + \int_{\mathbb{R}}(e^{x\theta e^{-\lambda_k s}} - 1 - \theta e^{-\lambda_k s}x1_{\{|x|\le 1\}})\nu(dx)\right)ds$$
$$= \left(a_k - \int_{[-1,1]} x\nu_k(dx)\right)\theta + \frac{b_k\theta^2}{4} + \lambda_k\int_0^\infty \left(\int_{\mathbb{R}}(e^{x\theta e^{-\lambda_k s}} - 1)\nu_k(dx)\right)ds.$$

In our results $\Psi_{N,k}(\underline{\theta}_m, \underline{t}_m)$ is finite in a neighborhood of the origin $\underline{\theta}_m = \underline{0}_m$. This yields the finiteness of the moments of the involved random variables.

Proof. We remark that

$$\sum_{\ell=1}^{m}\theta_\ell \sum_{i=1}^{[Nt_\ell]} X^{(k)}(i) = \sum_{\ell=1}^{m}\theta_\ell \sum_{i=1}^{[Nt_\ell]} e^{-\lambda_k i}\left(X^{(k)}(0) + \int_0^i e^{\lambda_k s}dZ^{(k)}(\lambda_k s)\right)$$

by (3), and

$$\int_0^i e^{\lambda_k s}dZ^{(k)}(\lambda_k s) = \sum_{j=1}^{i}\int_{j-1}^{j} e^{\lambda_k s}dZ^{(k)}(\lambda_k s)$$

(note that the summands $\{\int_{j-1}^j e^{\lambda_k s}dZ^{(k)}(\lambda_k s) : j \in \{1, \ldots, [Nt_m]\}\}$ are independent, and independent of $X^{(k)}(0)$). Thus

$$\sum_{\ell=1}^{m}\theta_\ell \sum_{i=1}^{[Nt_\ell]} X^{(k)}(i) = \sum_{\ell=1}^{m}\theta_\ell \sum_{h=1}^{\ell}\sum_{i=[Nt_{h-1}]+1}^{[Nt_h]} e^{-\lambda_k i}\left(X^{(k)}(0) + \sum_{j=1}^{i}\int_{j-1}^{j} e^{\lambda_k s}dZ^{(k)}(\lambda_k s)\right)$$
$$= \sum_{\ell=1}^{m}\theta_\ell \sum_{h=1}^{\ell}\sum_{i=[Nt_{h-1}]+1}^{[Nt_h]} e^{-\lambda_k i}X^{(k)}(0) + \sum_{\ell=1}^{m}\theta_\ell \sum_{h=1}^{\ell}\sum_{i=[Nt_{h-1}]+1}^{[Nt_h]} e^{-\lambda_k i}\sum_{j=1}^{i}\int_{j-1}^{j} e^{\lambda_k s}dZ^{(k)}(\lambda_k s).$$

Moreover, if we exchange the order of the indices h and ℓ in the first sum and the order of (j, h) and (i, ℓ) in the last sum, we get

$$\sum_{\ell=1}^{m}\theta_\ell \sum_{i=1}^{[Nt_\ell]} X^{(k)}(i)$$
$$= \sum_{h=1}^{m}(T\underline{\theta}_m)_h \sum_{i=[Nt_{h-1}]+1}^{[Nt_h]} e^{-\lambda_k i}X^{(k)}(0) + \sum_{j=1}^{[Nt_m]}\int_{j-1}^{j} e^{\lambda_k s}dZ^{(k)}(\lambda_k s)\sum_{h=1}^{m}(T\underline{\theta}_m)_h \sum_{i=([Nt_{h-1}]+1)\vee j}^{[Nt_h]} e^{-\lambda_k i}.$$

Then we have a linear combination of independent random variables, namely

$$\sum_{\ell=1}^{m}\theta_\ell\sum_{i=1}^{[Nt_\ell]}X^{(k)}(i)=\tilde{\theta}_0(N)X^{(k)}(0)+\sum_{j=1}^{[Nt_m]}\tilde{\theta}_j(N)\int_{j-1}^{j}e^{\lambda_k s}dZ^{(k)}(\lambda_k s),$$

where the coefficients $\{\tilde{\theta}_j(N):j\in\{0,1,\dots,[Nt_m]\}\}$ are defined by

$$\tilde{\theta}_j(N):=\sum_{h=1}^{m}(T\underline{\theta}_m)_h\sum_{i=([Nt_{h-1}]+1)\vee j}^{[Nt_h]}e^{-\lambda_k i};\tag{14}$$

thus, by the independence of the random variables cited above, we have

$$\Psi_{N,k}(\underline{\theta}_m,\underline{t}_m)=\varphi_0^{(k)}(\tilde{\theta}_0(N))+\sum_{j=1}^{[Nt_m]}\varphi_{j-1,j}^{(k)}(\tilde{\theta}_j(N)).$$

Now, in view of what follows, we consider a more explicit expression of $\tilde{\theta}_j(N)$ in (14) in different cases; namely for $j=0$ and, for every choice of $v\in\{1,\dots,m\}$, for $j\in\{[Nt_{v-1}]+1,\dots,[Nt_v]\}$. In order to do that we refer to $\{c_{h,v}(N):N\geq 1\}$ in (9), which are bounded coefficients with respect to N for every choice of v and h (and this will be important when we take the limit as N go to infinity in the next sections). We have

$$\tilde{\theta}_0(N):=\sum_{h=1}^{m}(T\underline{\theta}_m)_h\sum_{i=[Nt_{h-1}]+1}^{[Nt_h]}e^{-\lambda_k i}=\sum_{h=1}^{m}(T\underline{\theta}_m)_h c_{h,0}(N)$$

for $j=0$ and, for $v\in\{1,\dots,m\}$,

$$\tilde{\theta}_j(N)=(T\underline{\theta}_m)_v\frac{e^{-\lambda_k j}(1-e^{-\lambda_k([Nt_v]-j+1)})}{1-e^{-\lambda_k}}+e^{-\lambda_k[Nt_v]}\sum_{h=v+1}^{m}c_{h,v}(N)(T\underline{\theta}_m)_h\tag{15}$$

for $j\in\{[Nt_{v-1}]+1,\dots,[Nt_v]\}$ (note that we do not have any contribution to the sum in (14) when $h\in\{1,\dots,v-1\}$).

We also recall that, by (4) and (5) with $f(s)=-\mathrm{i}\theta e^{\lambda_k s}$ and $(t_1,t_2)=(j-1,j)$ (see also Remark 2.1), we have

$$\begin{aligned}\varphi_{j-1,j}^{(k)}(\theta)&=\int_{j-1}^{j}\lambda_k\tilde{\Psi}_{Z^{(k)}}(-\mathrm{i}\theta e^{\lambda_k s})ds\\&=\int_{j-1}^{j}\lambda_k\left(a_k\theta e^{\lambda_k s}+\frac{b_k(\theta e^{\lambda_k s})^2}{2}+\int_{\mathbb{R}}(e^{x\theta e^{\lambda_k s}}-1-\theta e^{\lambda_k s}x1_{\{|x|\leq 1\}})\nu_k(dx)\right)ds\\&=\left(a_k-\int_{[-1,1]}x\nu_k(dx)\right)(e^{\lambda_k j}-e^{\lambda_k(j-1)})\theta\\&\quad+\frac{b_k(e^{2\lambda_k j}-e^{2\lambda_k(j-1)})\theta^2}{4}+\lambda_k\int_{j-1}^{j}\left(\int_{\mathbb{R}}(e^{x\theta e^{\lambda_k s}}-1)\nu_k(dx)\right)ds;\end{aligned}$$

then we get (we consider the change of variable $u=s-(j+1)$ in each integral over $(j-1,j)$ for the last equality)

$$\Psi_{N,k}(\underline{\theta}_m,\underline{t}_m)=\varphi_0^{(k)}(\tilde{\theta}_0(N))+\sum_{v=1}^{m}\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}\varphi_{j-1,j}^{(k)}(\tilde{\theta}_j(N))$$

$$=\varphi_0^{(k)}(\tilde{\theta}_0(N))+\sum_{v=1}^{m}\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}\left\{\left(a_k-\int_{[-1,1]}x\nu_k(dx)\right)(e^{\lambda_k j}-e^{\lambda_k(j-1)})\tilde{\theta}_j(N)\right.$$

$$\left.+\frac{b_k(e^{2\lambda_k j}-e^{2\lambda_k(j-1)})(\tilde{\theta}_j(N))^2}{4}+\lambda_k\int_{j-1}^{j}\left(\int_{\mathbb{R}}(e^{x\tilde{\theta}_j(N)e^{\lambda_k s}}-1)\nu_k(dx)\right)ds\right\}$$

$$=\varphi_0^{(k)}(\tilde{\theta}_0(N))+\sum_{v=1}^{m}\left\{\left(a_k-\int_{[-1,1]}x\nu_k(dx)\right)\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}(e^{\lambda_k j}-e^{\lambda_k(j-1)})\tilde{\theta}_j(N)\right.$$

$$\left.+\frac{b_k}{4}\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}(e^{2\lambda_k j}-e^{2\lambda_k(j-1)})(\tilde{\theta}_j(N))^2+\lambda_k\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}\int_0^1\left(\int_{\mathbb{R}}(e^{x\tilde{\theta}_j(N)e^{\lambda_k(u+j-1)}}-1)\nu_k(dx)\right)du\right\}.$$

So we have to manipulate the following three sums:

1. $\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}(e^{\lambda_k j}-e^{\lambda_k(j-1)})\tilde{\theta}_j(N)$;
2. $\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}(e^{2\lambda_k j}-e^{2\lambda_k(j-1)})(\tilde{\theta}_j(N))^2$;
3. $\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}\int_0^1\left(\int_{\mathbb{R}}(e^{x\tilde{\theta}_j(N)e^{\lambda_k(u+j-1)}}-1)\nu_k(dx)\right)du$.

We start with the *sum 1*. We take into account (15) and, with some computations, we get

$$\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}(e^{\lambda_k j}-e^{\lambda_k(j-1)})\tilde{\theta}_j(N)=\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}e^{\lambda_k j}(1-e^{\lambda_k})\tilde{\theta}_j(N)$$

$$=(T\underline{\theta}_m)_v\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}(1-e^{-\lambda_k([Nt_v]-j+1)})+(1-e^{-\lambda_k})\sum_{h=v+1}^{m}c_{h,v}(N)(T\underline{\theta}_m)_h\frac{1-e^{-\lambda_k([Nt_v]-[Nt_{v-1}])}}{1-e^{-\lambda_k}}$$

$$=(T\underline{\theta}_m)_v\left([Nt_v]-[Nt_{v-1}]-\frac{e^{-\lambda_k}(1-e^{-\lambda_k([Nt_v]-[Nt_{v-1}])})}{1-e^{-\lambda_k}}\right)+\sum_{h=v+1}^{m}c_{h,v}(N)(T\underline{\theta}_m)_h(1-e^{-\lambda_k([Nt_v]-[Nt_{v-1}])}).$$

Thus

$$\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}(e^{\lambda_k j}-e^{\lambda_k(j-1)})\tilde{\theta}_j(N)=\langle\underline{L}_m^{(v)}(N),T\underline{\theta}_m\rangle.$$

Now the *sum 2*. We take into account (15) and, with some computations, we get

$$\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]} (e^{2\lambda_k j} - e^{2\lambda_k (j-1)})(\tilde{\theta}_j(N))^2 = \sum_{j=[Nt_{v-1}]+1}^{[Nt_v]} e^{2\lambda_k j}(1-e^{2\lambda_k})(\tilde{\theta}_j(N))^2$$

$$= (1-e^{-2\lambda_k})\left\{ \sum_{j=[Nt_{v-1}]+1}^{[Nt_v]} e^{2\lambda_k j}\left((T\underline{\theta}_m)_v \frac{e^{-\lambda_k j}(1-e^{-\lambda_k([Nt_v]-j+1)})}{1-e^{-\lambda_k}} \right)^2 \right.$$

$$+ 2\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]} e^{2\lambda_k j}\left((T\underline{\theta}_m)_v \frac{e^{-\lambda_k j}(1-e^{-\lambda_k([Nt_v]-j+1)})}{1-e^{-\lambda_k}} \right) e^{-\lambda_k [Nt_v]} \sum_{h=v+1}^{m} c_{h,v}(N)(T\underline{\theta}_m)_h$$

$$\left. + \sum_{j=[Nt_{v-1}]+1}^{[Nt_v]} e^{2\lambda_k j}\left(e^{-\lambda_k[Nt_v]} \sum_{h=v+1}^{m} c_{h,v}(N)(T\underline{\theta}_m)_h \right)^2 \right\}$$

$$= \frac{1-e^{-2\lambda_k}}{(1-e^{-\lambda_k})^2}((T\underline{\theta}_m)_v)^2 \sum_{j=[Nt_{v-1}]+1}^{[Nt_v]} (1-e^{-\lambda_k([Nt_v]-j+1)})^2$$

$$+ 2\frac{1-e^{-2\lambda_k}}{1-e^{-\lambda_k}}(T\underline{\theta}_m)_v \sum_{h=v+1}^{m} c_{h,v}(N)(T\underline{\theta}_m)_h \sum_{j=[Nt_{v-1}]+1}^{[Nt_v]} (1-e^{-\lambda_k([Nt_v]-j+1)})e^{-\lambda_k([Nt_v]-j)}$$

$$+ (1-e^{-2\lambda_k})\left(\sum_{h=v+1}^{m} c_{h,v}(N)(T\underline{\theta}_m)_h \right)^2 \sum_{j=[Nt_{v-1}]+1}^{[Nt_v]} e^{-2\lambda_k([Nt_v]-j)}$$

$$= \frac{1+e^{-\lambda_k}}{1-e^{-\lambda_k}}((T\underline{\theta}_m)_v)^2 \Big([Nt_v]-[Nt_{v-1}]$$

$$\left. - \frac{2e^{-\lambda_k}(1-e^{-\lambda_k([Nt_v]-[Nt_{v-1}])})}{1-e^{-\lambda_k}} + \frac{e^{-2\lambda_k}(1-e^{-2\lambda_k([Nt_v]-[Nt_{v-1}])})}{1-e^{-2\lambda_k}} \right)$$

$$+2(1+e^{-\lambda_k})(T\underline{\theta}_m)_v \sum_{h=v+1}^{m} c_{h,v}(N)(T\underline{\theta}_m)_h \left(\frac{1-e^{-\lambda_k([Nt_v]-[Nt_{v-1}])}}{1-e^{-\lambda_k}} - \frac{e^{-\lambda_k}(1-e^{-2\lambda_k([Nt_v]-[Nt_{v-1}])})}{1-e^{-2\lambda_k}} \right)$$

$$+ \left(\sum_{h=v+1}^{m} c_{h,v}(N)(T\underline{\theta}_m)_h \right)^2 (1-e^{-2\lambda_k([Nt_v]-[Nt_{v-1}])}).$$

Thus

$$\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]} (e^{2\lambda_k j} - e^{2\lambda_k (j-1)})(\tilde{\theta}_j(N))^2 = \langle T\underline{\theta}_m, Q^{(v)}(N)T\underline{\theta}_m \rangle.$$

Finally the *sum 3*. We take into account (15) and, if we set

$$q := [Nt_v]-j+1 \text{ (for } q \in \{1,\ldots,[Nt_v]-[Nt_{v-1}]\}),$$

with some computations we get

$$\tilde{\theta}_j(N)e^{\lambda_k(j-1)} = (T\underline{\theta}_m)_v \frac{e^{-\lambda_k}(1-e^{-\lambda_k([Nt_v]-j+1)})}{1-e^{-\lambda_k}} + e^{-\lambda_k([Nt_v]-j+1)} \sum_{h=v+1}^{m} c_{h,v}(N)(T\underline{\theta}_m)_h$$

$$= (T\underline{\theta}_m)_v \frac{e^{-\lambda_k}(1-e^{-\lambda_k q})}{1-e^{-\lambda_k}} + e^{-\lambda_k q} \sum_{h=v+1}^{m} c_{h,v}(N)(T\underline{\theta}_m)_h.$$

Thus

$$\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]} \int_0^1 \left(\int_{\mathbb{R}} (e^{x\tilde{\theta}_j(N) e^{\lambda_k(u+j-1)}} - 1)\nu_k(dx) \right) du$$
$$= \sum_{q=1}^{[Nt_v]-[Nt_{v-1}]} \int_0^1 \left(\int_{\mathbb{R}} (\exp\left(x \langle \underline{D}_m^{(v,q)}(N), T\underline{\theta}_m \rangle e^{\lambda_k u} \right) - 1)\nu_k(dx) \right) du.$$

□

3.2 Some auxiliary functions for polygonal approximations

Here we consider some auxiliary functions which play a crucial role when we check the exponential tightness. These functions are defined by considering suitable polygonal approximations of the processes.

More precisely, for $t_2 > t_1 > 0$, we consider the auxiliary function $\Psi_{N,k}^{(\mathrm{pol})}(\cdot, (t_1, t_2))$ defined by

$$\Psi_{N,k}^{(\mathrm{pol})}((\theta_1, \theta_2), (t_1, t_2)) := \log \mathbb{E} \left[\exp\left(\sum_{\ell=1}^{2} \theta_\ell \left\{ \sum_{i=1}^{[Nt_\ell]} X^{(k)}(i) + (Nt_\ell - [Nt_\ell]) X^{(k)}([Nt_\ell]+1) \right\} \right) \right], \tag{16}$$

where the pairs (θ_1, θ_2) are such that $(T\underline{\theta}_2)_1 = \theta_1 + \theta_2 = 0$. We remark that

$$\sum_{\ell=1}^{2} \theta_\ell \left\{ \sum_{i=1}^{[Nt_\ell]} X^{(k)}(i) + (Nt_\ell - [Nt_\ell]) X^{(k)}([Nt_\ell]+1) \right\}$$
$$= \sum_{\ell=1}^{2} \theta_\ell \sum_{i=1}^{[Nt_\ell]} X^{(k)}(i) + \sum_{\ell=1}^{2} \theta_\ell (Nt_\ell - [Nt_\ell]) X^{(k)}([Nt_\ell]+1)$$
$$= \sum_{\ell=1}^{2} \theta_\ell \sum_{i=1}^{[Nt_\ell]} X^{(k)}(i) + \sum_{\ell=1}^{2} \theta_\ell (Nt_\ell - [Nt_\ell]) e^{-\lambda_k([Nt_\ell]+1)} X^{(k)}(0)$$
$$+ \sum_{\ell=1}^{2} \theta_\ell (Nt_\ell - [Nt_\ell]) e^{-\lambda_k([Nt_\ell]+1)} \sum_{j=1}^{[Nt_\ell]+1} \int_{j-1}^{j} e^{\lambda_k s} dZ^{(k)}(\lambda_k s)$$

and, as far as the last sum is concerned, we have

$$\sum_{\ell=1}^{2} \theta_\ell (Nt_\ell - [Nt_\ell]) e^{-\lambda_k([Nt_\ell]+1)} \sum_{h=1}^{[Nt_\ell]+1} \int_{j-1}^{j} e^{\lambda_k s} dZ^{(k)}(\lambda_k s)$$
$$= \sum_{j=1}^{[Nt_2]+1} \int_{j-1}^{j} e^{\lambda_k s} dZ^{(k)}(\lambda_k s) \sum_{\ell:[Nt_\ell]+1 \geq j}^{m} \theta_\ell (Nt_\ell - [Nt_\ell]) e^{-\lambda_k([Nt_\ell]+1)};$$

thus, for $j \in \{[Nt_{v-1}]+1, \ldots, [Nt_v]\}$, the coefficient of $\int_{j-1}^{j} e^{\lambda_k s} dZ^{(k)}(\lambda_k s)$ is

$$\sum_{\ell:[Nt_\ell]+1 \geq j}^{2} \theta_\ell (Nt_\ell - [Nt_\ell]) e^{-\lambda_k([Nt_\ell]+1)}$$
$$= \sum_{\ell=v}^{2} \theta_\ell (Nt_\ell - [Nt_\ell]) e^{-\lambda_k([Nt_\ell]+1)} + 1_{\{j=[Nt_{v-1}]+1\}} \theta_{v-1} (Nt_{v-1} - [Nt_{v-1}]) e^{-\lambda_k([Nt_{v-1}]+1)},$$

with $\theta_0 = 0$ for the case $v = 1$.

So, if we put the pieces together, we get

$$\sum_{\ell=1}^{2} \theta_\ell \left\{ \sum_{i=1}^{[Nt_\ell]} X^{(k)}(i) + (Nt_\ell - [Nt_\ell]) X^{(k)}([Nt_\ell]+1) \right\}$$

$$= \tilde{\theta}_0^{(\mathrm{pol})}(N) X^{(k)}(0) + \sum_{v=1}^{2} \sum_{j=[Nt_{v-1}]+1}^{[Nt_v]} \tilde{\theta}_j^{(\mathrm{pol})}(N) \int_{j-1}^{j} e^{\lambda_k s} dZ^{(k)}(\lambda_k s)$$
$$+ \tilde{\theta}_{[Nt_2]+1}^{(\mathrm{pol})}(N) \int_{[Nt_2]}^{[Nt_2]+1} e^{\lambda_k s} dZ^{(k)}(\lambda_k s)$$

where, if we refer to the coefficients $\tilde{\theta}_j(N)$ in the proof of Lemma 3.1 (see (14)), we have

$$\tilde{\theta}_0^{(\mathrm{pol})}(N) := \tilde{\theta}_0(N) + \sum_{\ell=1}^{2} \theta_\ell (Nt_\ell - [Nt_\ell]) e^{-\lambda_k([Nt_\ell]+1)},$$

$$\tilde{\theta}_j^{(\mathrm{pol})}(N) := \tilde{\theta}_j(N) + \begin{cases} \sum_{\ell=1}^{2} \theta_\ell (Nt_\ell - [Nt_\ell]) e^{-\lambda_k([Nt_\ell]+1)} & \text{for } j \in \{1, \ldots, [Nt_1]\} \\ \theta_2 (Nt_2 - [Nt_2]) e^{-\lambda_k([Nt_2]+1)} & \\ +1_{\{j=[Nt_1]+1\}} \theta_1 (Nt_1 - [Nt_1]) e^{-\lambda_k([Nt_1]+1)} & \text{for } j \in \{[Nt_1]+1, \ldots, [Nt_2]\} \end{cases}$$

and

$$\tilde{\theta}_{[Nt_2]+1}^{(\mathrm{pol})}(N) := \theta_2 (Nt_2 - [Nt_2]) e^{-\lambda_k([Nt_2]+1)}.$$

Then, if we refer to some expressions of the functions $\{\varphi_{j-1,j}^{(k)} : j \in \{1, \ldots, [Nt_2]\}\}$ in the proof of Lemma 3.1 (here we also have to consider the case $j = [Nt_2] + 1$), we have

$$\Psi_{N,k}^{(\mathrm{pol})}((\theta_1, \theta_2), (t_1, t_2))$$
$$= \varphi_0^{(k)}(\tilde{\theta}_0^{(\mathrm{pol})}(N)) + \sum_{v=1}^{2} \sum_{j=[Nt_{v-1}]+1}^{[Nt_v]} \varphi_{j-1,j}^{(k)}(\tilde{\theta}_j^{(\mathrm{pol})}(N)) + \varphi_{[Nt_2],[Nt_2]+1}^{(k)}(\tilde{\theta}_{[Nt_2]+1}^{(\mathrm{pol})}(N)).$$

Moreover, by taking into account the structure of the coefficients $\tilde{\theta}_j(N)$ in the proof of Lemma 3.1, one can check that

$$\tilde{\theta}_0^{(\mathrm{pol})}(N) = \theta_2 \frac{e^{-\lambda_k([Nt_1]+1)}(1 - e^{-\lambda_k([Nt_2]-[Nt_1])})}{1 - e^{-\lambda_k}} + \sum_{\ell=1}^{2} \theta_\ell (Nt_\ell - [Nt_\ell]) e^{-\lambda_k([Nt_\ell]+1)},$$

and, for some bounded sequences $\{\{d_{ij}^{(\mathrm{pol})}(N) : N \geq 1\} : i, j \in \{1, 2\}\}$,

$$\tilde{\theta}_j^{(\mathrm{pol})}(N) = \begin{cases} (d_{11}^{(\mathrm{pol})}(N)\theta_1 + d_{12}^{(\mathrm{pol})}(N)\theta_2) e^{-\lambda[Nt_1]} & \text{for } j \in \{1, \ldots, [Nt_1]\} \\ \theta_2 \left(\frac{e^{-\lambda_k j}(1 - e^{-\lambda_k([Nt_2]-j+1)})}{1 - e^{-\lambda_k}} + d_{22}^{(\mathrm{pol})}(N) e^{-\lambda_k[Nt_2]} \right) & \\ \qquad +1_{\{j=[Nt_1]+1\}} \theta_1 d_{21}^{(\mathrm{pol})}(N) e^{-\lambda_k[Nt_1]} & \text{for } j \in \{[Nt_1]+1, \ldots, [Nt_2]\} \\ \theta_2 d_{22}^{(\mathrm{pol})}(N) e^{-\lambda_k[Nt_2]} & \text{for } j = [Nt_2]+1. \end{cases}$$

Decomposition of the function in (16). We consider the following decomposition

$$\Psi_{N,k}^{(\mathrm{pol})}((\theta_1, \theta_2), (t_1, t_2)) = \Psi_{N,k}^{(\mathrm{pol},1)}((\theta_1, \theta_2), (t_1, t_2)) + \Psi_{N,k}^{(\mathrm{pol},2)}((\theta_1, \theta_2), (t_1, t_2)),$$

where we define the first term only and, obviously, the second term is

$$\Psi_{N,k}^{(\mathrm{pol},2)}((\theta_1, \theta_2), (t_1, t_2)) := \Psi_{N,k}^{(\mathrm{pol})}((\theta_1, \theta_2), (t_1, t_2)) - \Psi_{N,k}^{(\mathrm{pol},1)}((\theta_1, \theta_2), (t_1, t_2)).$$

More precisely, if we consider the coefficients

$$\tilde{\theta}_j^{(\mathrm{pol},1)}(N) := \begin{cases} \tilde{\theta}_j^{(\mathrm{pol})}(N) & \text{for } j \in \{0,1,\dots,[Nt_1]\} \\ \theta_2 \left(\frac{e^{-\lambda_k j}(1-e^{-\lambda_k([Nt_2]-j+1)})}{1-e^{-\lambda_k}} + d_{22}^{(\mathrm{pol})}(N)e^{-\lambda_k[Nt_2]}\right) & \text{for } j \in \{[Nt_1]+1,\dots,[Nt_2]\} \\ 0 & \text{for } j = [Nt_2]+1, \end{cases}$$

we consider the function $\Psi_{N,k}^{(\mathrm{pol},1)}((\theta_1,\theta_2),(t_1,t_2))$ defined by

$$\begin{aligned}\Psi_{N,k}^{(\mathrm{pol},1)}&((\theta_1,\theta_2),(t_1,t_2)) \\ &:= \varphi_0^{(k)}(\tilde{\theta}_0^{(\mathrm{pol},1)}(N)) + \sum_{v=1}^{2}\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]} \varphi_{j-1,j}^{(k)}(\tilde{\theta}_j^{(\mathrm{pol},1)}(N)) + \varphi_{[Nt_2],[Nt_2]+1}^{(k)}(\tilde{\theta}_{[Nt_2]+1}^{(\mathrm{pol},1)}(N));\end{aligned}$$

(actually the last term can be neglected because $\varphi_{[Nt_2],[Nt_2]+1}^{(k)}(\tilde{\theta}_{[Nt_2]+1}^{(\mathrm{pol},1)}(N)) = 0$).

The following remark on the *first function* of the decomposition plays a crucial role in some proof in the next sections.

Remark 3.1. *The coefficients $\tilde{\theta}_j(N)$ in the proof of Lemma 3.1 can be expressed as the coefficients $\tilde{\theta}_j^{(\mathrm{pol},1)}(N)$ here, with different bounded sequences $\{\{d_{ij}(N) : N \geq 1\} : i,j \in \{1,2\}\}$; thus we can give a formula for $\Psi_{N,k}^{(\mathrm{pol},1)}((\theta_1,\theta_2),(t_1,t_2))$ as the one in Lemma 3.1 for $\Psi_{N,k}((\theta_1,\theta_2),(t_1,t_2))$. Obviously the sequences $\{\underline{L}_m^{(v)}(N) : N \geq 1\}$, $\{Q^{(v)}(N) : N \geq 1\}$ and $\{\underline{D}_m^{(v,q)}(N) : N \geq 1\}$ in that formula change accordingly but, for our aims, $\Psi_{N,k}^{(\mathrm{pol},1)}((\theta_1,\theta_2),(t_1,t_2))$ behaves like $\Psi_{N,k}((\theta_1,\theta_2),(t_1,t_2))$ (as $N \to \infty$).*

So, in what follows, we concentrate the attention on the *second function* of the decomposition. We have the following manipulations.

$$\begin{aligned}\Psi_{N,k}^{(\mathrm{pol},2)}&((\theta_1,\theta_2),(t_1,t_2)) \\ &= \sum_{v=1}^{2}\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]} [\varphi_{j-1,j}^{(k)}(\tilde{\theta}_j^{(\mathrm{pol})}(N)) - \varphi_{j-1,j}^{(k)}(\tilde{\theta}_j^{(\mathrm{pol};1)}(N))] + \varphi_{[Nt_2],[Nt_2]+1}^{(k)}(\tilde{\theta}_{[Nt_2]+1}^{(\mathrm{pol})}(N))\end{aligned}$$

and, by taking into account the expressions of the functions $\varphi_{j-1,j}^{(k)}$ which appear in the proof of

Lemma 3.1, we get

$$
\begin{aligned}
&\Psi_{N,k}^{(\mathrm{pol},2)}((\theta_1,\theta_2),(t_1,t_2))\\
&\quad=\sum_{v=1}^{2}\Bigg\{\left(a_k-\int_{[-1,1]}x\nu_k(dx)\right)\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}e^{\lambda_k j}(1-e^{-\lambda_k})(\tilde\theta_j^{(\mathrm{pol})}(N)-\tilde\theta_j^{(\mathrm{pol};1)}(N))\\
&\qquad\qquad+\frac{b_k}{4}\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}e^{2\lambda_k j}(1-e^{-2\lambda_k})[(\tilde\theta_j^{(\mathrm{pol})}(N))^2-(\tilde\theta_j^{(\mathrm{pol};1)}(N))^2]\\
&\qquad+\lambda_k\sum_{j=[Nt_{v-1}]+1}^{[Nt_v]}\int_0^1\left(\int_{\mathbb{R}}(e^{x\tilde\theta_j^{(\mathrm{pol})}(N)e^{\lambda_k(u+j-1)}}-e^{x\tilde\theta_j^{(\mathrm{pol};1)}(N)e^{\lambda_k(u+j-1)}})\nu_k(dx)\right)du\Bigg\}\\
&\qquad\qquad+\left(a_k-\int_{[-1,1]}x\nu_k(dx)\right)e^{\lambda_k([Nt_2]+1)}(1-e^{-\lambda_k})\theta_2 d_{22}^{(\mathrm{pol})}(N)e^{-\lambda_k[Nt_2]}\\
&\qquad\qquad\quad+\frac{b_k}{4}e^{2\lambda_k([Nt_2]+1)}(1-e^{-2\lambda_k})\theta_2^2(d_{22}^{(\mathrm{pol})}(N))^2e^{-2\lambda_k[Nt_2]}\\
&\qquad\qquad+\lambda_k\int_0^1\left(\int_{\mathbb{R}}(e^{x\theta_2 d_{22}^{(\mathrm{pol})}(N)e^{-\lambda_k[Nt_2]}e^{\lambda_k(u+[Nt_2]+1-1)}}-1)\nu_k(dx)\right)du\\
&\qquad\qquad=\left(a_k-\int_{[-1,1]}x\nu_k(dx)\right)e^{\lambda_k([Nt_1]+1)}(1-e^{-\lambda_k})\theta_1 d_{21}^{(\mathrm{pol})}(N)e^{-\lambda_k[Nt_1]}\\
&\qquad\qquad\quad+\frac{b_k}{4}e^{2\lambda_k([Nt_1]+1)}(1-e^{-2\lambda_k})\Bigg[\theta_1^2(d_{21}^{(\mathrm{pol})}(N))^2e^{-2\lambda_k[Nt_1]}\\
&\quad+2\theta_1\theta_2 d_{21}^{(\mathrm{pol})}(N)e^{-\lambda_k[Nt_1]}\left(\frac{e^{-\lambda_k([Nt_1]+1)}(1-e^{-\lambda_k([Nt_2]-[Nt_1])})}{1-e^{-\lambda_k}}+d_{22}^{(\mathrm{pol})}(N)e^{-\lambda_k[Nt_2]}\right)\Bigg]\\
&\quad+\lambda_k\int_0^1\Bigg(\int_{\mathbb{R}}\exp\left(x\theta_2(\frac{e^{-\lambda_k([Nt_1]+1)}(1-e^{-\lambda_k([Nt_2]-[Nt_1])})}{1-e^{-\lambda_k}}+d_{22}^{(\mathrm{pol})}(N)e^{-\lambda_k[Nt_2]})e^{\lambda_k(u+[Nt_1])}\right)\\
&\qquad\qquad\qquad\times(e^{x\theta_1 d_{21}^{(\mathrm{pol})}(N)e^{-\lambda_k[Nt_1]}e^{\lambda_k(u+[Nt_1])}}-1)\nu_k(dx)\Bigg)du\\
&\quad+\left(a_k-\int_{[-1,1]}x\nu_k(dx)\right)e^{\lambda_k}(1-e^{-\lambda_k})\theta_2 d_{22}^{(\mathrm{pol})}(N)+\frac{b_k}{4}e^{2\lambda_k}(1-e^{-2\lambda_k})\theta_2^2(d_{22}^{(\mathrm{pol})}(N))^2\\
&\qquad\qquad\qquad+\lambda_k\int_0^1\left(\int_{\mathbb{R}}(e^{x\theta_2 d_{22}^{(\mathrm{pol})}(N)e^{\lambda_k u}}-1)\nu_k(dx)\right)du\\
&\qquad=\left(a_k-\int_{[-1,1]}x\nu_k(dx)\right)e^{\lambda_k}(1-e^{-\lambda_k})\sum_{v=1}^{2}\theta_v d_{2v}^{(\mathrm{pol})}(N)+\frac{b_k}{4}e^{2\lambda_k}(1-e^{-2\lambda_k})\\
&\times\left[\sum_{v=1}^{2}\theta_v^2(d_{2v}^{(\mathrm{pol})}(N))^2+2\theta_1\theta_2 d_{21}^{(\mathrm{pol})}(N)\left(\frac{e^{-\lambda_k}(1-e^{-\lambda_k([Nt_2]-[Nt_1])})}{1-e^{-\lambda_k}}+d_{22}^{(\mathrm{pol})}(N)e^{-\lambda_k([Nt_2]-[Nt_1])}\right)\right]\\
&\quad+\lambda_k\int_0^1\Bigg(\int_{\mathbb{R}}\exp\left(x\theta_2(\frac{e^{-\lambda_k}(1-e^{-\lambda_k([Nt_2]-[Nt_1])})}{1-e^{-\lambda_k}}+d_{22}^{(\mathrm{pol})}(N)e^{-\lambda_k([Nt_2]-[Nt_1])})e^{\lambda_k u}\right)\\
&\qquad\qquad\times(e^{x\theta_1 d_{21}^{(\mathrm{pol})}(N)e^{\lambda_k u}}-1)\nu_k(dx)\Bigg)du+\lambda_k\int_0^1\left(\int_{\mathbb{R}}(e^{x\theta_2 d_{22}^{(\mathrm{pol})}(N)e^{\lambda_k u}}-1)\nu_k(dx)\right)du.
\end{aligned}
$$

4 Large deviations for the Gaussian case

In this section we always consider Gaussian OU processes, namely we assume that $\nu_1,\dots,\nu_K$ are null measures. Thus the moment generating functions $\{\varphi_0^{(k)}:k\in\{1,\dots,K\}\}$ are finite everywhere;

more precisely, for all $k \in \{1, \dots, K\}$, we have

$$\varphi_0^{(k)}(\theta) = a_k\theta + \frac{b_k\theta^2}{4} \text{ (for all } \theta \in \mathbb{R}).$$

We start with a result for finite dimensional distributions. Note that, since $\nu_1, \dots, \nu_K$ are null measures, σ_0^2 below coincides with c_K^2 by (2) and (7).

Proposition 4.1. *We assume that $\nu_1, \dots, \nu_K$ are null measures. Then the family of random variables*

$$\left\{ \left(\frac{S_K([Nt_1])}{N}, \dots, \frac{S_K([Nt_m])}{N} \right)^{\mathrm{tr}} : N \geq 1 \right\}$$

satisfies the LDP with speed N and good rate function $\bar{\Lambda}^*(\cdot, \underline{t}_m)$ defined by

$$\bar{\Lambda}^*(\underline{x}_m, \underline{t}_m) := \sum_{v=1}^{m} (t_v - t_{v-1}) \frac{\left(\frac{x_v - x_{v-1}}{t_v - t_{v-1}} - \mu \right)^2}{2\sigma_0^2},$$

where $x_0 = t_0 = 0$, $\mu := \sum_{k=1}^{K} a_k$ and $\sigma_0^2 := \sum_{k=1}^{K} \frac{b_k}{2} \frac{1+e^{-\lambda_k}}{1-e^{-\lambda_k}}$.

Proof. For $m \geq 1$ and $0 = t_0 < t_1 < \dots < t_m$ we set $\bar{\Lambda}(\underline{\theta}_m, \underline{t}_m) := \sum_{k=1}^{K} \bar{\Psi}_k(\underline{\theta}_m, \underline{t}_m)$, where

$$\bar{\Psi}_k(\underline{\theta}_m, \underline{t}_m) := \sum_{v=1}^{m} (t_v - t_{v-1}) \left\{ a_k (T\underline{\theta}_m)_v + \frac{b_k}{4} \frac{1 + e^{-\lambda_k}}{1 - e^{-\lambda_k}} ((T\underline{\theta}_m)_v)^2 \right\};$$

thus, by taking into account the values μ and σ_0^2, we have

$$\bar{\Lambda}(\underline{\theta}_m, \underline{t}_m) = \sum_{v=1}^{m} (t_v - t_{v-1}) \left\{ \mu (T\underline{\theta}_m)_v + \frac{\sigma_0^2}{2} ((T\underline{\theta}_m)_v)^2 \right\}.$$

We want to apply Gärtner Ellis Theorem. In order to do that, if we refer to the functions in (13), we have to prove that

$$\lim_{N \to \infty} \frac{\Lambda_N(\underline{\theta}_m, \underline{t}_m)}{N} = \bar{\Lambda}(\underline{\theta}_m, \underline{t}_m)$$

or, equivalently,

$$\lim_{N \to \infty} \frac{\Psi_{N,k}(\underline{\theta}_m, \underline{t}_m)}{N} = \bar{\Psi}_k(\underline{\theta}_m, \underline{t}_m), \tag{17}$$

for each fixed $k \in \{1, \dots, K\}$.

The limit (17) will be checked by considering the expression of $\Psi_{N,k}(\underline{\theta}_m, \underline{t}_m)$ in Lemma 3.1. Firstly, by (9), we remark that

$$c_{h,0}(N) = \frac{e^{-\lambda_k([Nt_{h-1}]+1)}(1 - e^{-\lambda_k([Nt_h]-[Nt_{h-1}])})}{1 - e^{-\lambda_k}} \to \begin{cases} \frac{e^{-\lambda_k}}{1-e^{-\lambda_k}} & \text{if } h = 1 \\ 0 & \text{if } h \in \{2, \dots, m\} \end{cases} \quad \text{(as } N \to \infty),$$

and therefore

$$\lim_{N \to \infty} \sum_{h=1}^{m} (T\underline{\theta}_m)_h c_{h,0}(N) = \frac{e^{-\lambda_k}}{1 - e^{-\lambda_k}} (T\underline{\theta}_m)_1;$$

thus

$$\lim_{N \to \infty} \frac{\varphi_0^{(k)} \left(\sum_{h=1}^{m} (T\underline{\theta}_m)_h c_{h,0}(N) \right)}{N} = 0.$$

Then we check (17) noting that

$$\lim_{N \to \infty} \frac{L_h^{(v)}(N)}{N} := \begin{cases} t_v - t_{v-1} & \text{for } h = v \\ 0 & \text{otherwise} \end{cases}$$

(because we always have bounded quantities divided by N except $[Nt_v]-[Nt_{v-1}]$ in $\hat{c}_{v,v}(N)$; see (8), (9) and (10)) and

$$\lim_{N\to\infty}\frac{Q_{h\ell}^{(v)}(N)}{N}:=\begin{cases}\frac{1+e^{-\lambda_k}}{1-e^{-\lambda_k}}(t_v-t_{v-1}) & \text{for } h=\ell=v\\ 0 & \text{otherwise}\end{cases}$$

(because we always have bounded quantities divided by N except $[Nt_v]-[Nt_{v-1}]$ in $Q_{vv}^{(v)}(N)$; see (11)).

Then Gärtner Ellis Theorem can be applied and the desired LDP is proved if we show that the function $\bar{\Lambda}^*(\cdot,\underline{t}_m)$ defined by

$$\begin{aligned}\bar{\Lambda}^*(\underline{x}_m,\underline{t}_m)&:=\sup_{\underline{\theta}_m\in\mathbb{R}^m}\left\{\sum_{v=1}^m\theta_v x_v-\bar{\Lambda}(\underline{\theta}_m,\underline{t}_m)\right\}\\ &=\sup_{\underline{\theta}_m\in\mathbb{R}^m}\left\{\sum_{v=1}^m\theta_v x_v-\sum_{v=1}^m(t_v-t_{v-1})\left\{\mu(T\underline{\theta}_m)_v+\frac{\sigma_0^2}{2}((T\underline{\theta}_m)_v)^2\right\}\right\}\end{aligned}$$

coincides with the one in the statement of the proposition. In order to do that we consider a change of variables in the supremum, namely the vector $\underline{\gamma}_m\in\mathbb{R}^m$ defined by

$$\gamma_v=(T\underline{\theta}_m)_v \ (\text{for } v\in\{1,\dots,m\}).$$

Then we have $\sum_{v=1}^m\theta_v x_v=\sum_{v=1}^m\gamma_v(x_v-x_{v-1})$ (where $x_0=0$), and we get

$$\begin{aligned}\bar{\Lambda}^*(\underline{x}_m,\underline{t}_m)&=\sup_{\underline{\gamma}_m\in\mathbb{R}^m}\left\{\sum_{v=1}^m\gamma_v(x_v-x_{v-1})-\sum_{v=1}^m(t_v-t_{v-1})\left\{\mu\gamma_v+\frac{\sigma_0^2}{2}\gamma_v^2\right\}\right\}\\ &\leq\sum_{v=1}^m(t_v-t_{v-1})\sup_{\gamma_v\in\mathbb{R}}\left\{\gamma_v\frac{x_v-x_{v-1}}{t_v-t_{v-1}}-\left\{\mu\gamma_v+\frac{\sigma_0^2}{2}\gamma_v^2\right\}\right\}=\sum_{v=1}^m(t_v-t_{v-1})\frac{\left(\frac{x_v-x_{v-1}}{t_v-t_{v-1}}-\mu\right)^2}{2\sigma_0^2};\end{aligned}$$

moreover, in order to get the inverse inequality, for all $v\in\{1,\dots,m\}$ we can take a sequence $\{\gamma_v^{(h)}:h\geq 1\}$ which attains the supremum with respect to $\gamma_v\in\mathbb{R}$, we have

$$\begin{aligned}\bar{\Lambda}^*(\underline{x}_m,\underline{t}_m)&\geq\sum_{v=1}^m\gamma_v^{(h)}(x_v-x_{v-1})-\sum_{v=1}^m(t_v-t_{v-1})\left\{\mu\gamma_v^{(h)}+\frac{\sigma_0^2}{2}(\gamma_v^{(h)})^2\right\}\\ &=\sum_{v=1}^m(t_v-t_{v-1})\left\{\gamma_v^{(h)}\frac{x_v-x_{v-1}}{t_v-t_{v-1}}-\left\{\mu\gamma_v^{(h)}+\frac{\sigma_0^2}{2}(\gamma_v^{(h)})^2\right\}\right\}\end{aligned}$$

and we obtain the inverse inequality by letting h go to infinity. □

Now we are ready to give the sample-path large deviation result.

Proposition 4.2. *We consider the same hypotheses and notation in Proposition 4.1. Moreover let* $\{W_N:N\geq 1\}$ *be the family of stochastic processes defined by*

$$W_N(t):=\frac{S_K([Nt])}{N}\ (\textit{for } t\in[0,1]).$$

Then $\{W_N:N\geq 1\}$ *satisfies the LDP with respect to the topology of uniform convergence, with speed* N *and good rate function* I_{LD} *defined by*

$$I_{\mathrm{LD}}(f):=\begin{cases}\int_0^1\bar{\Lambda}^*(\dot{f}(t),1)dt & \textit{if } f\in AC[0,1] \textit{ and } f(0)=0\\ \infty & \textit{otherwise,}\end{cases}$$

where $\dot{f}(t)$ *is the almost everywhere derivative of* $f(t)$ *and* $AC[0,1]$ *is the family of all absolutely continuous functions on* $[0,1]$.

Proof. The proof consists of several steps, and we refer to the polygonal approximations $\{W_N^{(\mathrm{pol})} : N \geq 1\}$ defined by

$$W_N^{(\mathrm{pol})}(t) := W_N(t) + \frac{Nt - [Nt]}{N} X_K([Nt]+1). \tag{18}$$

Firstly we check that $\{W_N : N \geq 1\}$ and $\{W_N^{(\mathrm{pol})} : N \geq 1\}$ are exponentially equivalent (see e.g. Definition 4.2.10 in [4]) with respect to the speed N, namely

$$\limsup_{N\to\infty} \frac{1}{N} \log P\left(\sup_{t\in[0,1]} \left|W_N^{(\mathrm{pol})}(t) - W_N(t)\right| > \delta\right) = -\infty \text{ for all } \delta > 0.$$

In fact we have

$$\left\{\sup_{t\in[0,1]} \left|W_N^{(\mathrm{pol})}(t) - W_N(t)\right| > \delta\right\} = \left\{\sup_{t\in[0,1]} \left|\left(t - \frac{[Nt]}{N}\right) X_K([Nt]+1)\right| > \delta\right\}$$
$$\subset \left\{\max_{i\in\{1,\dots,N+1\}} |X_K(i)| > N\delta\right\} = \bigcup_{i=1}^{N+1} \{|X_K(i)| > N\delta\} \subset \bigcup_{i=1}^{N+1} \left\{\sum_{k=1}^{K} \left|X^{(k)}(i)\right| > N\delta\right\}$$

and, by the union bound and the stationarity of the processes,

$$P\left(\sup_{t\in[0,1]} \left|W_N^{(\mathrm{pol})}(t) - W_N(t)\right| > \delta\right) \leq \sum_{i=1}^{N+1} P\left(\sum_{k=1}^{K} \left|X^{(k)}(0)\right| > N\delta\right);$$

thus, for all $\eta > 0$, we have

$$P\left(\sup_{t\in[0,1]} \left|W_N^{(\mathrm{pol})}(t) - W_N(t)\right| > \delta\right) \leq (N+1)\mathbb{E}\left[e^{\eta \sum_{k=1}^{K} |X^{(k)}(0)|}\right] e^{-\eta N\delta},$$

$$\frac{1}{N} \log P\left(\sup_{t\in[0,1]} \left|W_N^{(\mathrm{pol})}(t) - W_N(t)\right| > \delta\right) \leq \frac{1}{N}\log(N+1) + \frac{1}{N}\sum_{k=1}^{K} \log \mathbb{E}\left[e^{\eta |X^{(k)}(0)|}\right] - \eta\delta,$$

$$\limsup_{N\to\infty} \frac{1}{N} \log P\left(\sup_{t\in[0,1]} \left|W_N^{(\mathrm{pol})}(t) - W_N(t)\right| > \delta\right) = -\eta\delta$$

(because $\sum_{k=1}^{K} \log \mathbb{E}\left[e^{\eta |X^{(k)}(0)|}\right] < \infty$ for all $\eta > 0$ by the hypotheses), and we get the desired exponential equivalence condition by letting η go to infinity.

Then, by Theorem 4.2.13 in [4] and this exponential equivalence condition, we have the following statements.

1. It suffices to prove the LDP of $\{W_N^{(\mathrm{pol})} : N \geq 1\}$ to get the desired LDP of $\{W_N : N \geq 1\}$.

2. The LDP for finite dimensional distributions in Proposition 4.1 can also be stated for the finite dimensional distributions of $\{W_N^{(\mathrm{pol})} : N \geq 1\}$.

As a consequence of Statement 2 we have the LDP of $\{W_N^{(\mathrm{pol})} : N \geq 1\}$ with respect to the topology of pointwise convergence; more precisely, by Dawson Gärtner Theorem, $\{W_N^{(\mathrm{pol})} : N \geq 1\}$ satisfies the LDP with speed N and good rate function I_{LD} defined by

$$I_{\mathrm{LD}}(f) := \sup_{m\geq 1, \underline{t}_m} \bar{\Lambda}^*((f(t_1), \dots, f(t_m)), \underline{t}_m); \tag{19}$$

moreover we can check that the rate function I_{LD} here coincides with the one in the statement of the proposition (this can be checked with some standard computation; for instance see a part of Lemma 5.1.6 in [4]).

Then we complete the proof showing that $\{W_N^{(\mathrm{pol})}:N\geq 1\}$ is exponentially tight; in fact, by Corollary 4.2.6 in [4], the LDP of $\{W_N^{(\mathrm{pol})}:N\geq 1\}$ holds with respect to the topology of uniform convergence and this completes the proof by taking into account Statement 1 above.

So in what follows we check that $\{W_N^{(\mathrm{pol})}:N\geq 1\}$ is exponentially tight. In order to do that we refer to Theorem 3 in [9] which is proved as a consequence of Theorems 1 and 2 in the same reference; see also [5] (Theorem A.1 in [5] coincides with Theorem 1 in [9]) and [10], and we have to check two conditions. The *first condition* is

$$\lim_{R\to\infty}\limsup_{N\to\infty}\frac{1}{N}\log P(|W_N^{(\mathrm{pol})}(0)|>R)=-\infty. \tag{20}$$

Then, for all $\gamma>0$, we have (in particular we take into account (18) and $W_N(0)=\frac{S_K(0)}{N}$)

$$\begin{aligned}P(|W_N^{(\mathrm{pol})}(0)|>R)&=P(|S_K(0)|>NR)\\&\leq\mathbb{E}\left[\exp\left(\gamma|S_K(0)|\right)\right]e^{-\gamma NR}\leq\prod_{k=1}^K\mathbb{E}\left[\exp\left(\gamma|X^{(k)}(0)|\right)\right]e^{-\gamma NR}\end{aligned}$$

and

$$\limsup_{N\to\infty}\frac{1}{N}\log P(|S_K(0)|>NR)\leq-\gamma R+\limsup_{N\to\infty}\frac{1}{N}\sum_{k=1}^K\log\mathbb{E}\left[\exp\left(\gamma|X^{(k)}(0)|\right)\right]$$

thus (20) holds because $\sum_{k=1}^K\log\mathbb{E}\left[\exp\left(\gamma|X^{(k)}(0)|\right)\right]<\infty$ (by the hypotheses of Proposition 4.1). The *second condition* to check is the following: there exist $\alpha,\gamma>0$ and $C\in\mathbb{R}$ such that, for all $t_1,t_2\in[0,1]$ with $t_2>t_1$, we have

$$\frac{1}{N}\log\mathbb{E}\left[\exp\left(\frac{N\gamma}{(t_2-t_1)^\alpha}|W_N^{(\mathrm{pol})}(t_2)-W_N^{(\mathrm{pol})}(t_1)|\right)\right]\leq C,\ \text{eventually.}$$

Then it suffices to show that there exist $\gamma>0$ and $C\in\mathbb{R}$ such that, for all $t_1,t_2\in[0,1]$ with $t_2>t_1$, we have

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}\left[\exp\left(\frac{N\gamma}{\sqrt{t_2-t_1}}|W_N^{(\mathrm{pol})}(t_2)-W_N^{(\mathrm{pol})}(t_1)|\right)\right]\leq C \tag{21}$$

(thus we have $\alpha=1/2$). We remark that, for all $\gamma>0$, we have

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\frac{N\gamma}{\sqrt{t_2-t_1}}|W_N^{(\mathrm{pol})}(t_2)-W_N^{(\mathrm{pol})}(t_1)|\right)\right]\\&\leq\mathbb{E}\left[\exp\left(\frac{N\gamma}{\sqrt{t_2-t_1}}(W_N^{(\mathrm{pol})}(t_2)-W_N^{(\mathrm{pol})}(t_1))\right)\right]+\mathbb{E}\left[\exp\left(\frac{-N\gamma}{\sqrt{t_2-t_1}}(W_N^{(\mathrm{pol})}(t_2)-W_N^{(\mathrm{pol})}(t_1))\right)\right];\end{aligned}$$

moreover

$$\log\left[\exp\left(\frac{\pm N\gamma}{\sqrt{t_2-t_1}}(W_N^{(\mathrm{pol})}(t_2)-W_N^{(\mathrm{pol})}(t_1))\right)\right]=\sum_{k=1}^K\Psi_{N,k}^{(\mathrm{pol})}\left(\pm\left(-\frac{\gamma}{\sqrt{t_2-t_1}},\frac{\gamma}{\sqrt{t_2-t_1}}\right),(t_1,t_2)\right).$$

Then, by taking into account the decomposition of the function in (16) illustrated in Section 3 (we recall that here we are assuming that all the Lévy measures $\nu_1,\ldots,\nu_K$ are null measures), we have

$$\lim_{N\to\infty}\frac{\Psi_{N,k}^{(\mathrm{pol},1)}\left(\pm\left(-\frac{\gamma}{\sqrt{t_2-t_1}},\frac{\gamma}{\sqrt{t_2-t_1}}\right),(t_1,t_2)\right)}{N}=\bar{\Psi}_k\left(\pm\left(-\frac{\gamma}{\sqrt{t_2-t_1}},\frac{\gamma}{\sqrt{t_2-t_1}}\right),(t_1,t_2)\right)$$

by Remark 3.1, and

$$\lim_{N\to\infty}\frac{\Psi_{N,k}^{(\mathrm{pol},2)}\left(\pm\left(-\frac{\gamma}{\sqrt{t_2-t_1}},\frac{\gamma}{\sqrt{t_2-t_1}}\right),(t_1,t_2)\right)}{N}=0$$

because we take the limit of bounded terms divided by N; thus we get

$$\lim_{N\to\infty}\frac{\Psi_{N,k}^{(\mathrm{pol})}\left(\pm\left(-\frac{\gamma}{\sqrt{t_2-t_1}},\frac{\gamma}{\sqrt{t_2-t_1}}\right),(t_1,t_2)\right)}{N}=\bar{\Psi}_k\left(\pm\left(-\frac{\gamma}{\sqrt{t_2-t_1}},\frac{\gamma}{\sqrt{t_2-t_1}}\right),(t_1,t_2)\right)$$
$$=(t_2-t_1)\left\{\pm\frac{a_k\gamma}{\sqrt{t_2-t_1}}+\frac{b_k}{4}\frac{1+e^{-\lambda_k}}{1-e^{-\lambda_k}}\frac{\gamma^2}{(\sqrt{t_2-t_1})^2}\right\}.$$

Finally we have

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}\left[\exp\left(\frac{\gamma}{\sqrt{t_2-t_1}}|W_N^{(\mathrm{pol})}(t_2)-W_N^{(\mathrm{pol})}(t_1)|\right)\right]$$
$$\leq\max\left\{\sum_{k=1}^K\bar{\Psi}_k\left(\left(-\frac{\gamma}{\sqrt{t_2-t_1}},\frac{\gamma}{t_2-t_1}\right),(t_1,t_2)\right),\sum_{k=1}^K\bar{\Psi}_k\left(\left(\frac{\gamma}{\sqrt{t_2-t_1}},-\frac{\gamma}{\sqrt{t_2-t_1}}\right),(t_1,t_2)\right)\right\}$$

by Lemma 1.2.15 in [4], and the desired condition with inequality (21) can be easily checked. □

We conclude this section with a brief discussion on the difficulties in lifting the assumptions beyond Gaussian case. In the proof of Proposition 4.1 it is not clear how to prove an extended version of (17) with a more general limit function which depends on the Lévy measure ν_k (in place of $\bar{\Psi}_k(\underline{\theta}_m,\underline{t}_m)$); in fact, for each fixed $\underline{\theta}_m\in\mathbb{R}^m$, we need to take the limit as $N\to\infty$ through an integral of a function which depends on $\underline{\theta}_m$. Moreover the limit function could be equal to infinity for some $\underline{\theta}_m\in\mathbb{R}^m$; this could give rise to some difficulties for the application of Gärtner Ellis Theorem (the steepness condition for the limit function could fail) in the proof of Proposition 4.1, and some difficulties to check (20) in the proof of Proposition 4.2.

5 Moderate deviations

We start with a result for finite dimensional distributions, namely the analogue of Proposition 4.1. Here we do not restrict the attention on the case of Gaussian OU processes.

Proposition 5.1. *We assume that, at least for N large enough, the functions $\{\Psi_{N,k}(\cdot,\underline{t}_m):k\in\{1,\dots,K\}\}$ assume finite values in a fixed open neighborhood $\mathcal{I}$ of the origin $\underline{0}_m\in\mathbb{R}^m$. Then, for every sequence of positive numbers $\{\varepsilon_N:N\geq 1\}$ such that $\varepsilon_N\to 0$ and $N\varepsilon_N\to\infty$ as $N\to\infty$, the family of random variables*

$$\left\{\left(\frac{S_K([Nt_1])-\mathbb{E}[S_K([Nt_1])]}{\sqrt{N/\varepsilon_N}},\dots,\frac{S_K([Nt_m])-\mathbb{E}[S_K([Nt_m])]}{\sqrt{N/\varepsilon_N}}\right)^{\mathrm{tr}}:N\geq 1\right\}$$

satisfies the LDP with speed $1/\varepsilon_N$ and good rate function $\Upsilon^*(\cdot,\underline{t}_m)$ defined by

$$\Upsilon^*(\underline{x}_m,\underline{t}_m)=\sum_{v=1}^m(t_v-t_{v-1})\frac{(x_v-x_{v-1})^2}{2c_K^2(t_v-t_{v-1})^2}=\sum_{v=1}^m\frac{(x_v-x_{v-1})^2}{2c_K^2(t_v-t_{v-1})},$$

where $x_0=0$ and c_K is as in (2) (see also (7)).

The finiteness of the functions $\{\Psi_{N,k}(\cdot,\underline{t}_m):k\in\{1,\dots,K\}\}$ in a fixed open neighborhood $\mathcal{I}$ of the origin $\underline{0}_m\in\mathbb{R}^m$ yields the finiteness of the moments which can be obtained by considering taking the derivatives for $\underline{\theta}_m=\underline{0}_m$.

Proof. We start noting that, if we refer to the functions $\Lambda_N(\cdot,\underline{t}_m)$ and $\{\Psi_{N,k}(\cdot,\underline{t}_m):k\in\{1,\dots,K\}\}$ in (13), we have

$$\nabla\Lambda_N(\underline{0}_m,\underline{t}_m)=(\mathbb{E}[S_K([Nt_1])],\dots,\mathbb{E}[S_K([Nt_m])])^{\mathrm{tr}}$$

and

$$\nabla\Psi_{N,k}\left(\underline{0}_m,\underline{t}_m\right)=\left(\sum_{i=1}^{[Nt_1]}\mathbb{E}[X^{(k)}(i)],\ldots,\sum_{i=1}^{[Nt_m]}\mathbb{E}[X^{(k)}(i)]\right)^{\mathrm{tr}}.$$

In what follows we prove that

$$\lim_{N\to\infty}\varepsilon_N\left(\Psi_{N,k}\left(\frac{1}{\sqrt{N\varepsilon_N}}\cdot\underline{\theta}_m,\underline{t}_m\right)-\left\langle\frac{1}{\sqrt{N\varepsilon_N}}\cdot\underline{\theta}_m,\nabla\Psi_{N,k}(\underline{0}_m,\underline{t}_m)\right\rangle\right)=\frac{1}{2}\langle\underline{\theta}_m,C_k\underline{\theta}_m\rangle,\qquad(22)$$

for a suitable (covariance) matrix C_k; this will imply that

$$\lim_{N\to\infty}\varepsilon_N\left(\Lambda_N\left(\frac{1}{\sqrt{N\varepsilon_N}}\cdot\underline{\theta}_m,\underline{t}_m\right)-\left\langle\frac{1}{\sqrt{N\varepsilon_N}}\cdot\underline{\theta}_m,\nabla\Lambda_N(\underline{0}_m,\underline{t}_m)\right\rangle\right)=\Upsilon(\underline{\theta}_m,\underline{t}_m)$$

where $\Upsilon(\underline{\theta}_m,\underline{t}_m):=\frac{1}{2}\langle\underline{\theta}_m,C\underline{\theta}_m\rangle$ and $C:=\sum_{k=1}^K C_k$. Then Gärtner Ellis Theorem allows to say that the desired LDP holds with good rate function $\Upsilon^*(\cdot,\underline{t}_m)$ defined by

$$\Upsilon^*(\underline{x}_m,\underline{t}_m):=\sup_{\underline{\theta}_m\in\mathbb{R}^m}\left\{\sum_{v=1}^m\theta_v x_v-\Upsilon(\underline{\theta}_m,\underline{t}_m)\right\};\qquad(23)$$

moreover in the final part of the proof we show that the expression of $\Upsilon^*(\underline{x}_m,\underline{t}_m)$ coincides with the rate function expression in the statement of the proposition.

The limit (22) will be checked by considering the expression of the function $\Psi_{N,k}(\cdot,\underline{t}_m)$ in Lemma 3.1. We start noting that, for each fixed $k\in\{1,\ldots,K\}$, $\frac{1}{\sqrt{N\varepsilon_N}}\cdot\underline{\theta}_m\in\mathcal{I}$ for N large enough. Thus, eventually, we consider the Mac-Laurin formula arrested at order 2 and we have

$$\begin{aligned}&\varepsilon_N\left(\Psi_{N,k}\left(\frac{1}{\sqrt{N\varepsilon_N}}\cdot\underline{\theta}_m,\underline{t}_m\right)-\left\langle\frac{1}{\sqrt{N\varepsilon_N}}\cdot\underline{\theta}_m,\nabla\Psi_{N,k}(\underline{0}_m,\underline{t}_m)\right\rangle\right)\\&\qquad=\varepsilon_N\frac{1}{2}\left\langle\frac{1}{\sqrt{N\varepsilon_N}}\cdot\underline{\theta}_m,C_{N,k}\frac{1}{\sqrt{N\varepsilon_N}}\cdot\underline{\theta}_m\right\rangle+\varepsilon_N\cdot o\left(\frac{1}{N\varepsilon_N}\cdot\|\underline{\theta}_m\|^2\right),\end{aligned}$$

where $C_{N,k}$ is the Hessian matrix defined by

$$C_{N,k}:=\left(\frac{\partial^2\Psi_{N,k}(\underline{0}_m,\underline{t}_m)}{\partial\theta_i\theta_j}\right)_{i,j\in\{1,\ldots,m\}}$$

and $\lim_{N\to\infty}\frac{o\left(\frac{1}{N\varepsilon_N}\cdot\|\underline{\theta}_m\|^2\right)}{\frac{1}{N\varepsilon_N}}=0$. Thus

$$\begin{aligned}&\varepsilon_N\left(\Psi_{N,k}\left(\frac{1}{\sqrt{N\varepsilon_N}}\cdot\underline{\theta}_m,\underline{t}_m\right)-\left\langle\frac{1}{\sqrt{N\varepsilon_N}}\cdot\underline{\theta}_m,C_{n,k}\right\rangle\right)\\&\qquad=\frac{1}{2}\left\langle\underline{\theta}_m,\frac{1}{N}C_{N,k}\cdot\underline{\theta}_m\right\rangle+\varepsilon_N\cdot o\left(\frac{1}{N\varepsilon_N}\cdot\|\underline{\theta}_m\|^2\right),\end{aligned}$$

and we get (22) if we show that, for some limit matrix C_k (which will be specified below), we have

$$\lim_{N\to\infty}\frac{1}{N}C_{N,k}=C_k.\qquad(24)$$

In view of what follows we consider the notation $(\underline{c}_{m,0}(N))^{\mathrm{tr}}:=(c_{1,0}(N),\ldots,c_{m,0}(N))$ and we recall that $(\varphi_0^{(k)})''(0)$ coincide with the stationary variance $\mathrm{Var}[X^{(k)}]$ in (7), namely

$$(\varphi_0^{(k)})''(0)=\frac{1}{2}\left(b_k+\int_{\mathbb{R}}x^2\nu_k(dx)\right).$$

Then, by taking into account the expression of $\Psi_{N,k}(\underline{\theta}_m,\underline{t}_m)$ in Lemma 3.1, after some computations we get

$$\frac{1}{N}C_{N,k}=\frac{1}{N}\sum_{v=1}^{m}\Bigg\{(\varphi_0^{(k)})''(0)T^{\mathrm{tr}}\underline{c}_{m,0}(N)(\underline{c}_{m,0}(N))^{\mathrm{tr}}T+\frac{b_k}{2}T^{\mathrm{tr}}Q^{(v)}(N)T$$
$$+\lambda_k\sum_{q=1}^{[Nt_v]-[Nt_{v-1}]}\int_0^1\int_{\mathbb{R}}x^2e^{2\lambda_k u}\nu_k(dx)du\;T^{\mathrm{tr}}\underline{D}_m^{(v,q)}(N)(\underline{D}_m^{(v,q)}(N))^{\mathrm{tr}}T\Bigg\}$$
$$=\frac{1}{N}\sum_{v=1}^{m}\Bigg\{(\varphi_0^{(k)})''(0)T^{\mathrm{tr}}\underline{c}_{m,0}(N)(\underline{c}_{m,0}(N))^{\mathrm{tr}}T+\frac{b_k}{2}T^{\mathrm{tr}}Q^{(v)}(N)T$$
$$+\frac{1}{2}\int_{\mathbb{R}}x^2\nu_k(dx)(e^{2\lambda_k}-1)\sum_{q=1}^{[Nt_v]-[Nt_{v-1}]}T^{\mathrm{tr}}\underline{D}_m^{(v,q)}(N)(\underline{D}_m^{(v,q)}(N))^{\mathrm{tr}}T\Bigg\}.$$

Now, in order to check (24), we let N go to infinity. Obviously

$$\frac{1}{N}\sum_{v=1}^{m}(\varphi_0^{(k)})''(0)T^{\mathrm{tr}}\underline{c}_{m,0}(N)(\underline{c}_{m,0}(N))^{\mathrm{tr}}T$$

in negligible because it converges to the null matrix as $N\to\infty$ (because the numerator is bounded). Moreover

$$\lim_{N\to\infty}\frac{1}{N}\sum_{v=1}^{m}\frac{b_k}{2}T^{\mathrm{tr}}Q^{(v)}(N)T=\frac{b_k}{2}\frac{1+e^{-\lambda_k}}{1-e^{-\lambda_k}}(t_v-t_{v-1})T_{(v)}^{\mathrm{tr}}T_{(v)}$$

by (11). Finally we remark that, if we set

$$\underline{D}_m^{(v,q,1)}:=(D_1^{(v,q,1)},\ldots,D_m^{(v,q,1)})\text{ with }D_h^{(v,q,1)}:=\inf_{N\geq1}D_h^{(v,q)}(N)$$

and

$$\underline{D}_m^{(v,q,2)}:=(D_1^{(v,q,2)},\ldots,D_m^{(v,q,2)})\text{ with }D_h^{(v,q,2)}:=\sup_{N\geq1}D_h^{(v,q)}(N),$$

for all $v\in\{1,\ldots,m\}$ and $i\in\{1,2\}$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{q=1}^{[Nt_v]-[Nt_{v-1}]}T^{\mathrm{tr}}\underline{D}_m^{(v,q,i)}(\underline{D}_m^{(v,q,i)})^{\mathrm{tr}}T=\left(\frac{e^{-\lambda_k}}{1-e^{-\lambda_k}}\right)^2(t_v-t_{v-1})T_{(v)}^{\mathrm{tr}}T_{(v)}$$

by taking into account (12) and by applying Cesaro Theorem; then we can say that

$$\frac{1}{2}\int_{\mathbb{R}}x^2\nu_k(dx)(e^{2\lambda_k}-1)\lim_{N\to\infty}\frac{1}{N}\sum_{q=1}^{[Nt_v]-[Nt_{v-1}]}T^{\mathrm{tr}}\underline{D}_m^{(v,q)}(N)(\underline{D}_m^{(v,q)}(N))^{\mathrm{tr}}T$$
$$=\frac{1}{2}\int_{\mathbb{R}}x^2\nu_k(dx)\frac{1+e^{-\lambda_k}}{1-e^{-\lambda_k}}(t_v-t_{v-1})T_{(v)}^{\mathrm{tr}}T_{(v)}.$$

So, if we put the pieces together, we can say that (24) holds with

$$C_k:=\sum_{v=1}^{m}(t_v-t_{v-1})\frac{1}{2}\left\{b_k+\int_{\mathbb{R}}x^2\nu_k(dx)\right\}\frac{1+e^{-\lambda_k}}{1-e^{-\lambda_k}}T_{(v)}^{\mathrm{tr}}T_{(v)}.$$

In the final part of the proof we consider some standard manipulations of the Legendre transform in (23) with a change of variables in the supremum, namely the vector $\underline{\gamma}_m\in\mathbb{R}^m$ defined by

$$\gamma_v:=\langle T_{(v)}^{\mathrm{tr}},\underline{\theta}_m\rangle,\text{ or equivalently }\gamma_v:=(T\underline{\theta}_m)_v.$$

In fact we have $\sum_{v=1}^m \theta_v x_v = \sum_{v=1}^m \gamma_v(x_v - x_{v-1})$ where $x_0 = 0$ and, noting that

$$C = \sum_{k=1}^K C_k = \sum_{k=1}^K \sum_{v=1}^m (t_v - t_{v-1}) \frac{1}{2} \left\{ b_k + \int_{\mathbb{R}} x^2 \nu_k(dx) \right\} \frac{1 + e^{-\lambda_k}}{1 - e^{-\lambda_k}} T_{(v)}^{\mathrm{tr}} T_{(v)} = c_K^2 \sum_{v=1}^m (t_v - t_{v-1}) T_{(v)}^{\mathrm{tr}} T_{(v)}$$

(the last equality holds by combining (7) and (2)), we get

$$\Upsilon(\underline{\theta}_m, \underline{t}_m) = \frac{1}{2} \langle \underline{\theta}_m, C \underline{\theta}_m \rangle = \frac{c_K^2}{2} \sum_{v=1}^m (t_v - t_{v-1}) \langle \underline{\theta}_m, T_{(v)}^{\mathrm{tr}} T_{(v)} \underline{\theta}_m \rangle = \frac{c_K^2}{2} \sum_{v=1}^m (t_v - t_{v-1}) \gamma_v^2.$$

Then

$$\Upsilon^*(\underline{x}_m, \underline{t}_m) = \sum_{v=1}^m (t_v - t_{v-1}) \sup_{\gamma_v \in \mathbb{R}} \left\{ \gamma_v \frac{x_v - x_{v-1}}{t_v - t_{v-1}} - \frac{c_K^2}{2} \gamma_v^2 \right\}$$
$$= \sum_{v=1}^m (t_v - t_{v-1}) \frac{(x_v - x_{v-1})^2}{2 c_K^2 (t_v - t_{v-1})^2} = \sum_{v=1}^m \frac{(x_v - x_{v-1})^2}{2 c_K^2 (t_v - t_{v-1})}$$

and the proof is complete. □

Now we are ready to give the sample-path moderate deviation result.

Proposition 5.2. *We consider the same hypotheses of Proposition 5.1. Moreover, for a sequence of positive numbers $\{\varepsilon_N : N \geq 1\}$, let $\{W_N(\cdot; \varepsilon_N) : N \geq 1\}$ be the family of stochastic processes defined by*

$$W_N(t; \varepsilon_N) := \frac{S_K([Nt]) - \mathbb{E}[S_K([Nt])]}{\sqrt{N/\varepsilon_N}} \ (\textit{for } t \in [0, 1]).$$

Then, for every sequence of positive numbers $\{\varepsilon_N : N \geq 1\}$ such that $\varepsilon_N \to 0$ and $N\varepsilon_N \to \infty$ as $N \to \infty$, $\{W_N(\cdot; \varepsilon_N) : N \geq 1\}$ satisfies the LDP with respect to the topology of uniform convergence, with speed $1/\varepsilon_N$ and good rate function I_{MD} defined by

$$I_{\mathrm{MD}}(f) := \begin{cases} \int_0^1 \Upsilon^*(\dot{f}(t), 1) dt = \int_0^1 \frac{(\dot{f}(t))^2}{2 c_K^2} dt & \text{if } f \in AC[0, 1] \text{ and } f(0) = 0 \\ \infty & \text{otherwise,} \end{cases}$$

where $\dot{f}(t)$ is the almost everywhere derivative of $f(t)$ and $AC[0, 1]$ is the family of all absolutely continuous functions on $[0, 1]$.

Proof. The proof follows the same lines of the proof of Proposition 4.2. In this case, for every sequence of positive numbers $\{\varepsilon_N : N \geq 1\}$ such that $\varepsilon_N \to 0$ and $N\varepsilon_N \to \infty$ as $N \to \infty$, we have to consider the polygonal approximations $\{W_N^{(\mathrm{pol})}(\cdot; \varepsilon_N) : N \geq 1\}$ defined by

$$W_N^{(\mathrm{pol})}(t; \varepsilon_N) := W_N(t; \varepsilon_N) + \frac{Nt - [Nt]}{\sqrt{N/\varepsilon_N}} (X_K([Nt] + 1) - \mathbb{E}[X_K([Nt] + 1)]) \tag{25}$$

in place of $W_N^{(\mathrm{pol})}(\cdot)$ in (18). Here some parts are omitted (because we can simply adapt some parts of the proof of Proposition 4.2) and we only give some details on the main differences.

Firstly we check that $\{W_N(\cdot; \varepsilon_N) : N \geq 1\}$ and $\{W_N^{(\mathrm{pol})}(\cdot; \varepsilon_N) : N \geq 1\}$ are exponentially equivalent with respect to the speed $1/\varepsilon_N$. In fact we have

$$\left\{ \sup_{t \in [0,1]} \left| W_N^{(\mathrm{pol})}(t; \varepsilon_N) - W_N(t; \varepsilon_N) \right| > \delta \right\}$$
$$= \left\{ \sup_{t \in [0,1]} \left| \frac{Nt - [Nt]}{\sqrt{N/\varepsilon_N}} (X_K([Nt] + 1) - \mathbb{E}[X_K([Nt] + 1)]) \right| > \delta \right\}$$
$$\subset \bigcup_{i=1}^{N+1} \left\{ \sum_{k=1}^K \left| X^{(k)}(i) - \mathbb{E}[X^{(k)}(i)] \right| > \sqrt{\frac{N}{\varepsilon_N}} \delta \right\}$$

and, by the union bound and the stationarity of the processes,

$$P\left(\sup_{t\in[0,1]}\left|W_N^{(\mathrm{pol})}(t;\varepsilon_N)-W_N(t;\varepsilon_N)\right|>\delta\right)\leq\sum_{i=1}^{N+1}P\left(\sum_{k=1}^{K}\left|X^{(k)}(0)-\mathbb{E}[X^{(k)}(0)]\right|>\sqrt{\frac{N}{\varepsilon_N}}\delta\right);$$

thus

$$P\left(\sup_{t\in[0,1]}\left|W_N^{(\mathrm{pol})}(t;\varepsilon_N)-W_N(t;\varepsilon_N)\right|>\delta\right)\leq(N+1)\mathbb{E}\left[e^{\eta\sum_{k=1}^{K}|X^{(k)}(0)-\mathbb{E}[X^{(k)}(0)]|}\right]e^{-\eta\sqrt{N/\varepsilon_N}\delta},$$

$$\begin{aligned}\log P&\left(\sup_{t\in[0,1]}\left|W_N^{(\mathrm{pol})}(t;\varepsilon_N)-W_N(t;\varepsilon_N)\right|>\delta\right)\\&\leq\log(N+1)+\sum_{k=1}^{K}\log\mathbb{E}\left[e^{\eta|X^{(k)}(0)-\mathbb{E}[X^{(k)}(0)]|}\right]-\eta\sqrt{\frac{N}{\varepsilon_N}}\delta\\&=\sqrt{\frac{N}{\varepsilon_N}}\left(\frac{\log(N+1)+\sum_{k=1}^{K}\log\mathbb{E}\left[e^{\eta|X^{(k)}(0)-\mathbb{E}[X^{(k)}(0)]|}\right]}{\sqrt{N/\varepsilon_N}}-\eta\delta\right),\end{aligned}$$

and we get the desired exponential equivalence condition noting that, if we take $\eta>0$ small enough to have $\sum_{k=1}^{K}\log\mathbb{E}\left[e^{\eta|X^{(k)}(0)-\mathbb{E}[X^{(k)}(0)]|}\right]<\infty$ (this is possible by the hypotheses; in fact the functions $\{\varphi_0^{(k)}:k\in\{1,\dots,K\}\}$ are finite in a neighborhood of $0\in\mathbb{R}$), we have

$$\begin{aligned}\frac{1}{1/\varepsilon_N}\log P&\left(\sup_{t\in[0,1]}\left|W_N^{(\mathrm{pol})}(t;\varepsilon_N)-W_N(t;\varepsilon_N)\right|>\delta\right)\\&\leq\sqrt{N\varepsilon_N}\left(\frac{\log(N+1)+\sum_{k=1}^{K}\log\mathbb{E}\left[e^{\eta|X^{(k)}(0)-\mathbb{E}[X^{(k)}(0)]|}\right]}{\sqrt{N/\varepsilon_N}}-\eta\delta\right)\to-\infty\ (\text{as } N\to\infty).\end{aligned}$$

Now we can simply adapt some parts of the proof of Proposition 4.2: we have the same statements which follow from the exponential equivalence and Theorem 4.2.13 in [4] (obviously here we have to refer to Proposition 5.1 instead of Proposition 4.1); we can consider again the application of Dawson Gärtner Theorem with the rate function I_{MD} defined by

$$I_{\mathrm{MD}}(f):=\sup_{m\geq1,\underline{t}_m}\Upsilon^*((f(t_1),\dots,f(t_m)),\underline{t}_m)$$

instead of I_{LD} in (19), and the rate function I_{MD} here coincides with the one in the statement of the proposition; we conclude the proof showing that $\{W_N^{(\mathrm{pol})}(\cdot;\varepsilon_N):N\geq1\}$ is exponentially tight, and we refer again to Theorem 3 in [9].

We give some details on the two conditions for the applications of Theorem 3 in [9]. The *first condition*, namely the analogue of (20), is

$$\lim_{R\to\infty}\limsup_{N\to\infty}\frac{1}{1/\varepsilon_N}\log P(|W_N^{(\mathrm{pol})}(0;\varepsilon_N)|>R)=-\infty. \tag{26}$$

Then, for all $\gamma>0$, we have (in particular we take into account (25) and $W_N(0;\varepsilon_N)=\frac{S_K(0)-\mathbb{E}[S_K(0)]}{\sqrt{N/\varepsilon_N}}$)

$$\begin{aligned}P(|W_N^{(\mathrm{pol})}(0;\varepsilon_N)|>R)&=P\left(|S_K(0)-\mathbb{E}[S_K(0)]|>\sqrt{\frac{N}{\varepsilon_N}}R\right)\\&\leq\mathbb{E}\left[\exp\left(\gamma|S_K(0)-\mathbb{E}[S_K(0)]|\right)\right]e^{-\gamma\sqrt{N/\varepsilon_N}R}\\&\leq\mathbb{E}\left[\exp\left(\gamma\sum_{k=1}^{K}|X^{(k)}(0)-\mathbb{E}[X^{(k)}(0)]|\right)\right]e^{-\gamma\sqrt{N/\varepsilon_N}R};\end{aligned}$$

thus (26) holds because, if we take $\eta > 0$ small enough to have $\sum_{k=1}^{K} \log \mathbb{E}\left[e^{\eta |X^{(k)}(0) - \mathbb{E}[X^{(k)}(0)]|}\right] < \infty$ (this is possible as we said above), we have

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{1/\varepsilon_N} \log P(|W_N^{(\mathrm{pol})}(0;\varepsilon_N)| > R) \\ \leq \limsup_{N\to\infty} \varepsilon_N \left(\sum_{k=1}^{K} \log \mathbb{E}\left[\exp\left(\gamma |X^{(k)}(0) - \mathbb{E}[X^{(k)}(0)]|\right)\right] - \gamma\sqrt{N/\varepsilon_N} R \right) \\ = \limsup_{N\to\infty} \varepsilon_N \sum_{k=1}^{K} \log \mathbb{E}\left[\exp\left(\gamma |X^{(k)}(0) - \mathbb{E}[X^{(k)}(0)]|\right)\right] - \gamma\sqrt{N\varepsilon_N} R = -\infty\end{aligned}$$

for all $R > 0$. For the *second condition* we show that there exist $\gamma > 0$ and $C \in \mathbb{R}$ such that, for all $t_1, t_2 \in [0,1]$ with $t_2 > t_1$, we have

$$\limsup_{N\to\infty} \frac{1}{1/\varepsilon_N} \log \mathbb{E}\left[\exp\left(\frac{\gamma}{\varepsilon_N\sqrt{t_2 - t_1}} |W_N^{(\mathrm{pol})}(t_2;\varepsilon_N) - W_N^{(\mathrm{pol})}(t_1;\varepsilon_N)|\right)\right] \leq C \tag{27}$$

(thus we still have $\alpha = 1/2$; note that (27) is the analogue of (21) in the proof of Proposition 4.2). We remark that, for all $\gamma > 0$, we have

$$\begin{aligned}\mathbb{E}\left[\exp\left(\frac{\gamma}{\varepsilon_N\sqrt{t_2 - t_1}} |W_N^{(\mathrm{pol})}(t_2;\varepsilon_N) - W_N^{(\mathrm{pol})}(t_1;\varepsilon_N)|\right)\right] \\ \leq \mathbb{E}\left[\exp\left(\frac{\gamma}{\varepsilon_N\sqrt{t_2 - t_1}} (W_N^{(\mathrm{pol})}(t_2;\varepsilon_N) - W_N^{(\mathrm{pol})}(t_1;\varepsilon_N))\right)\right] \\ + \mathbb{E}\left[\exp\left(\frac{-\gamma}{\varepsilon_N\sqrt{t_2 - t_1}} (W_N^{(\mathrm{pol})}(t_2;\varepsilon_N) - W_N^{(\mathrm{pol})}(t_1;\varepsilon_N))\right)\right];\end{aligned}$$

moreover

$$\begin{aligned}\log\left[\exp\left(\frac{\pm\gamma}{\varepsilon_N\sqrt{t_2 - t_1}} (W_N^{(\mathrm{pol})}(t_2;\varepsilon_N) - W_N^{(\mathrm{pol})}(t_1;\varepsilon_N))\right)\right] \\ = \sum_{k=1}^{K} \left\{ \Psi_{N,k}^{(\mathrm{pol})}\left(\pm\left(\frac{-\gamma}{\sqrt{N\varepsilon_N(t_2 - t_1)}}, \frac{\gamma}{\sqrt{N\varepsilon_N(t_2 - t_1)}} \right), (t_1, t_2) \right) \right. \\ \left. - \left\langle \frac{\pm 1}{\sqrt{N\varepsilon_N(t_2 - t_1)}} (-\gamma, \gamma), \nabla\Psi_{N,k}^{(\mathrm{pol})}((0,0),(t_1,t_2)) \right\rangle \right\}.\end{aligned}$$

Then, by taking into account the decomposition of the function in (16) illustrated in Section 3, after some computations we can check that

$$\begin{aligned}\lim_{N\to\infty} \varepsilon_N \left\{ \Psi_{N,k}^{(\mathrm{pol},1)}\left(\pm\left(\frac{-\gamma}{\sqrt{N\varepsilon_N(t_2 - t_1)}}, \frac{\gamma}{\sqrt{N\varepsilon_N(t_2 - t_1)}} \right), (t_1, t_2) \right) \right. \\ \left. - \left\langle \frac{\pm 1}{\sqrt{N\varepsilon_N(t_2 - t_1)}} (-\gamma, \gamma), \nabla\Psi_{N,k}^{(\mathrm{pol},1)}((0,0),(t_1,t_2)) \right\rangle \right\} \\ = \frac{1}{2}(t_2 - t_1)\mathrm{Var}[X^{(k)}] \frac{1 + e^{-\lambda_k}}{1 - e^{-\lambda_k}} \frac{\gamma^2}{(\sqrt{t_2 - t_1})^2}\end{aligned}$$

and

$$\begin{aligned}\lim_{N\to\infty} \varepsilon_N \left\{ \Psi_{N,k}^{(\mathrm{pol},2)}\left(\pm\left(\frac{-\gamma}{\sqrt{N\varepsilon_N(t_2 - t_1)}}, \frac{\gamma}{\sqrt{N\varepsilon_N(t_2 - t_1)}} \right), (t_1, t_2) \right) \right. \\ \left. - \left\langle \frac{\pm 1}{\sqrt{N\varepsilon_N(t_2 - t_1)}} (-\gamma, \gamma), \nabla\Psi_{N,k}^{(\mathrm{pol},2)}((0,0),(t_1,t_2)) \right\rangle \right\} = 0\end{aligned}$$

(in particular we take into account Remark 3.1 for the first limit); thus we get

$$\lim_{N\to\infty} \varepsilon_N \left\{ \Psi_{N,k}^{(\mathrm{pol})}\left(\pm\left(\frac{-\gamma}{\sqrt{N\varepsilon_N(t_2-t_1)}}, \frac{\gamma}{\sqrt{N\varepsilon_N(t_2-t_1)}} \right), (t_1,t_2) \right) - \left\langle \frac{\pm 1}{\sqrt{N\varepsilon_N(t_2-t_1)}}(-\gamma,\gamma), \nabla\Psi_{N,k}^{(\mathrm{pol})}((0,0),(t_1,t_2)) \right\rangle \right\}$$
$$= \frac{1}{2}(t_2-t_1)\mathrm{Var}[X^{(k)}]\frac{1+e^{-\lambda_k}}{1-e^{-\lambda_k}}\frac{\gamma^2}{(\sqrt{t_2-t_1})^2}.$$

Finally we have

$$\limsup_{N\to\infty} \frac{1}{1/\varepsilon_N} \log \mathbb{E}\left[\exp\left(\frac{\gamma}{\varepsilon_N\sqrt{t_2-t_1}} |W_N^{(\mathrm{pol})}(t_2;\varepsilon_N) - W_N^{(\mathrm{pol})}(t_1;\varepsilon_N)| \right) \right]$$
$$\leq \frac{\gamma^2}{2}\sum_{k=1}^K \mathrm{Var}[X^{(k)}]\frac{1+e^{-\lambda_k}}{1-e^{-\lambda_k}}$$

by Lemma 1.2.15 in [4], and the desired condition is checked (in fact, for every choice of t_1 and t_2, (27) holds with $C = \frac{\gamma^2}{2}\sum_{k=1}^K \mathrm{Var}[X^{(k)}]\frac{1+e^{-\lambda_k}}{1-e^{-\lambda_k}}$). □

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