Information Spreading in Stationary Markovian Evolving Graphs *

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Abstract

Markovian evolving graphs [2] are dynamic-graph models where the links among a fixed set of nodes change during time according to an arbitrary Markovian rule. They are extremely general and they can well describe important dynamic-network scenarios.
We study the speed of information spreading in the *stationary phase* by analyzing the completion time of the *flooding mechanism*. We prove a general theorem that establishes an upper bound on flooding time in *any* stationary Markovian evolving graph in terms of its node-expansion properties.
We apply our theorem in two natural and relevant cases of such dynamic graphs: *edge-Markovian evolving graphs* [24, 7] where the probability of existence of any edge at time t depends on the existence (or not) of the same edge at time $t-1$; *geometric Markovian evolving graphs* [4, 10, 9] where the Markovian behaviour is yielded by n mobile radio stations, with fixed transmission radius, that perform n independent random walks over a square region of the plane. In both cases, the obtained upper bounds are shown to be nearly tight and, in fact, they turn out to be tight for a large range of the values of the input parameters.

1 Introduction

Markovian evolving graphs and Flooding. Graphs that evolve over time are currently a very hot topic in computer science. They arise from several areas such as mobile networks, networks of users exchanging e-mail or instant messages, citation networks and hyperlinks networks, peer-to-peer networks, social networks (who-trust-whom, who-talks-to-whom, etc.), and many other more [2, 11, 8, 21, 16, 22, 23].

Markovian evolving graphs are a natural and very general class of models for evolving graphs introduced in [2]. In these models, the set of nodes is fixed and the edge set at time t stochastically depends on the edge set at time $t-1$: so, we have an infinite sequence of graphs that is a Markov chain. It is important to observe that, on one hand, these

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models make the underlying mechanism of how the graph evolves explicit; on the other hand, they are very general since, by a suitable choice of the *matrix transition probability* yielding the graph Markovian process, it is possible to model several important network scenarios such as *faulty-networks* and *geometric-mobile* networks (such scenarios will be described later).

In [2], the hitting time and cover time of random walks in some specific cases of Markovian evolving graphs have been analytically studied. We instead investigate the speed of *information spreading* on general Markovian evolving graphs. Reaching all nodes from a given source/initiator node is typically required to disseminate or retrieve information: this task is performed via suitable protocols that aim to achieve low delay and message overhead. However, when the network topology is highly dynamic and unknown, (e.g. unstructured peer-to-peer networks, faulty/mobile networks, etc), it is very hard to design efficient protocols for that task and, as a result, the *flooding mechanism* is often adopted [6, 12, 13, 19]. In the flooding mechanism, any *informed* node (i.e. any node that has the source message) always sends the source message to all its neighbors. So, the source is informed since the beginning and, clearly, any other node gets informed at time step t iff any of its neighbors (w.r.t. the edge set at time t) is informed at time step $t-1$.
The completion time of the flooding mechanism (shortly *flooding time*) is the first time step in which all nodes of the network are informed.

It is important to observe that flooding time in dynamic networks plays the same role of *diameter* in static networks. Indeed, flooding time and diameter represent "natural" lower bounds for broadcast protocols in dynamic networks and static ones, respectively. For this reason, flooding is often used in order to evaluate the relative efficiency of alternative protocols, especially in networks with unknown dynamic topology [6, 12, 21].

Our results. We study flooding time in *stationary* Markovian evolving graphs, i.e., when the *initial* graph is random with a *stationary distribution* of the underlying Markov chain [1]. In network mobility simulation, this corresponds to the important concept of *perfect simulation* (see [18, 5]).

We prove an upper bound on flooding time in *any* stationary Markovian evolving graph. This upper bound is expressed in terms of the parameterized node-expansion properties satisfied by the stationary graphs. As far as we know, this is the first analytical result on the speed of information spreading in so general dynamic models.
We then show the tightness (so the "goodness") of this bound in two relevant and natural dynamic scenarios: *edge-Markovian evolving graphs* (in short, *edge-MEG*) and *geometric Markovian evolving graphs* (in short, *geometric-MEG*).

Edge-Markovian evolving graphs. In several networks scenarios, there is a strong dependence between the existence (or the absence) of a link between two nodes at a given time step and the existence (or the absence) of the same link at the previous time step. Important examples of this behavior arise in faulty communication networks, peer-to-peer networks[1], and social networks.

We thus consider *edge-MEG*, special Markovian evolving graphs, recently studied in [7], which are a time-discrete version of the *reciprocity graph model* introduced in the context of evolving social networks [24]. At every time step, every edge changes its state (existing or not) according to a two-state Markovian process with probabilities $p(n)$ and

[1]Notice that, in some of these settings, there is an underlying physical network that supports the abstraction of point-to-point communication.

$q(n)$ where n is the number of nodes. If an edge exists at time t then at time $t+1$ it dies with probability $q(n)$ (i.e. *death-rate*). If instead the edge does not exist at time t, then it will come into existence at time $t+1$ with probability $p(n)$ (i.e.*birth-rate*). For brevity's sake, functions $p(n)$ and $q(n)$ will be simply denoted as p and q, respectively[2]. Observe that setting $q = 1 - p$ yields the (time-independent) *dynamic random graph* model [8] where links, at every time, are chosen independently at random. So, edge-MEG are (in turn) a wide generalization of dynamic random graphs. Observe that when $0 < p, q < 1$, the stationary distribution is unique.

We first prove that stationary edge-MEG, yielding *connected* graphs, satisfy certain parameterized node-expansion properties. Thanks to these properties, we can apply our general result and achieve an upper bound on flooding time. The obtained bound is shown to be tight whenever flooding time is $\Omega(\log\log n)$: this includes, for instance, the relevant case where the expected node-degree is $O(\text{polylog } n)$. In general, our upper bound for edge-MEG is thus at most an $O(\log\log n)$ factor larger than the optimum.

In [7], it has been studied the *maximal* flooding time in edge-MEG with respect to *any* initial probability distribution. In that paper, in fact, almost tight bounds for the *worst-case* flooding time have been derived. However, those results do not say whether flooding can be (significantly) faster in *stationary* edge-MEG. Interestingly enough, our stationary bound implies that, whenever the birth-rate p is $O(1/n^{1+\epsilon})$ and the death-rate q is $O(np/\log n)$, there is an *exponential* gap between the stationary case and the worst-case. An exponential gap also holds whenever $p = O(\log n/n)$ and $q = O(p\sqrt{n})$ (for instance, set $q = \text{polylog } n/n$).

Geometric Markovian evolving graphs. We consider a model of evolving graphs that is based on *node mobility*. It is the discrete version of the well-known *random-walk* model [4, 9, 14]. In this model, denoted here as *geometric-MEG*, nodes (i.e. radio stations) move on a region of the plane (typically a square region) and each node performs, independently from the others, a sort of Brownian motion. At any time there is an edge (i.e. a bidirectional connection link) between two nodes if they are at distance at most R (typically R represents the transmission range). We make time discrete and consider a square grid as a node support-space (see Section 4 for details). This model can also be viewed as the *walkers* model [9] on the square grid.

Differently from edge-MEG, geometric-MEG yield *spatial* stochastic dependency among the dynamic edges, i.e., the probability of an edge depends on the existence of other edges.

Similarly to the edge-MEG case, we first prove that stationary geometric MEG, yielding *connected* graphs, satisfy certain parameterized node-expansion properties. We then apply our general result and achieve an upper bound on flooding time. The obtained bound is shown to be tight whenever flooding time is $\Omega(\log\log n)$. Informally speaking, this happens whenever *(i)* the transmission radius is *not* "almost" equal to the diameter of the square region and *(ii)* the maximal *node-velocity* is less than the *message-transmission* speed. Both assumptions are satisfied by most of real mobile networks. In general, our upper bound is thus at most an $O(\log\log n)$ factor larger than the optimum.

Further mobility models. The node-expansion properties of geometric-MEG are mainly due to the fact that the stationary distribution of node positions is *almost uniform.* In this paper, we provide formal results and proofs only for flooding in geometric-MEG.

[2]Hence, any inequality $p \leqslant (\geqslant) b(n)$ means that $p(n)$ is eventually not larger (not smaller) than $b(n)$. The same holds for $q = q(n)$.

However, our *expansion* technique can be applied to *any* mobility model yielding a uniform or almost uniform stationary distribution of node positions. Several variants of the *random waypoint* model, one of the most commonly used mobility models [17, 4], enjoy this uniformity property. Among the others, we mention the random-direction model *with reflection* (also called the *billiard* model) [3, 18, 20], the random waypoint *on a torus* [14, 15, 18, 20] and the random waypoint *on a sphere* [18]. Furthermore, the uniformity property is also satisfied by the *walkers* model on a toroidal grid [10].

To the best of our knowledge, our results are the the first *analytical* (tight) bounds on flooding time for natural and relevant models of *mobile* networks.

We finally remark that our flooding analysis does not take care about the *interference* problem in message transmissions: this is typically managed at the MAC layer of a wireless network architecture [3, 8]. The impact of message interferences in geometric-MEG is a further interesting issue which is out of the scope of our work focussing, instead, on *dynamic-connectivity* properties of MEG.

Organization of the paper. In Section 2, we prove our upper bound for flooding time in general Markovian evolving graphs. The results for edge-MEG and geometric-MEG are described in Sections 3 and 4, respectively. Finally, further generalizations of our general theorem and some potential applications to other important classes of evolving graphs are discussed in Section 5.

2 Markovian evolving graphs: the general theorem

Through this paper, the set $[n] = \{1, \ldots, n\}$ will represent the set of n nodes. Let $G = ([n], E)$ be a graph and $I \subseteq [n]$ be a subset of nodes. We denote by $N(I)$ the out-neighborhood of I, i.e.

$$N(I) = \{v \in [n] \setminus I \, : \, \{u, v\} \in E, \text{ for some } u \in I\}$$

Given a source node $s \in [n]$, the *flooding* process can be represented by the sequence $\{I_t \subseteq [n] \, : \, t \in \mathbb{N}\}$ where I_t is the subset of informed nodes defined recursively as follows

$$\begin{cases} I_0 & = & \{s\} \\ I_{t+1} & = & I_t \cup N(I_t) \end{cases}$$

Notice that the subset $N(I_t)$ refers to the graph at time step t. Let $T(s)$ be the first time step such that all nodes are informed. The *flooding time* is the maximum $T(s)$ over all possible choices of source s.

Definition 2.1 (Markovian evolving graph) *Let* $\mathbf{G}$ *be a family of graphs with the same node set* $[n]$*. A* Markovian evolving graph $\mathcal{M} = \{G_t \, : \, t \in \mathbb{N}\}$ *is a Markov chain with state space* $\mathbf{G}$*.*
A stationary *Markovian evolving graph is a Markovian evolving graph* $\mathcal{M} = \{G_t \, : \, t \in \mathbb{N}\}$ *such that* G_0 *is random with a stationary distribution of* $\mathcal{M}$*.*

The following definition concerns a sort of parameterized node-expansion. This is a key-ingredient, in our analysis of flooding in Markovian evolving graphs, to cope with the difficulties due to the stochastic dependence.

Definition 2.2 (Expander) *A graph* $G = ([n], E)$ *is a* (h, k)-expander *if, for every set of nodes* $I \subseteq [n]$ *with* $|I| \leqslant h$*, it holds that* $|N(I)| \geqslant k|I|$*.*

The above definition naturally extends to random variables and their probability distributions.

Definition 2.3 (Expander II) *Let X be a random variable with values in a family of graphs with the same node set $[n]$. Then X is a (h,k)-*expander *with probability p if*

$$\mathbf{P}\left(X \textit{ is a } (h,k)\textit{-expander}\right) \geqslant p$$

In this case, we also say that the probability distribution of X yields *an (h,k)-expander with probability p.*

We are now able to provide our main result for general stationary Markovian evolving graphs.

Theorem 2.4 *Let $\mathcal{M} = \{G_t : t \in \mathbb{N}\}$ be a stationary Markovian evolving graph. Assume an increasing sequence $1 = h_0 \leqslant h_1 < \cdots < h_s = n/2$ and a decreasing sequence $k_1 \geqslant \cdots \geqslant k_s$ of positive real numbers exist such that, for every $i = 1, \ldots, s$, the stationary distribution of $\mathcal{M}$ yields an (h_i, k_i)-expander with probability $1 - \frac{1}{n^4}$. Then flooding time in $\mathcal{M}$ is w.h.p.*

$$\mathcal{O}\left(\sum_{i=1}^{s} \frac{\log(h_i/h_{i-1})}{\log(1+k_i)}\right)$$

Proof. We first give an idea of the proof. Since by Theorem's hypothesis, the distribution of G_0 is the stationary distribution of the Markov chain $\mathcal{M}$, then for every $t \in \mathbb{N}$, r.v. G_t has the same distribution of G_0.
Let us call $m_t = |I_t|$ the number of informed nodes at time step t; at the beginning we have $m_0 = 1$. Since every r.v. G_t is an (h_1, k_1)-expander w.h.p. then, as long as $m_t \leqslant h_1$, the recurrence

$$m_{t+1} \geqslant (1+k_1)m_t \tag{1}$$

holds w.h.p. Indeed, whatever the stochastic dependence were till time step t, the node-expansion property guarantees that $|N(I_t)|$ is at least $k_1 m_t$. The closed form of the above recurrence is $m_t \geqslant (1+k_1)^t m_0$, hence

$$\mathcal{O}\left(\frac{\log(h_1/m_0)}{\log(1+k_1)}\right)$$

time steps are enough to get $m_t \geqslant h_1$ w.h.p. However, the latter bound might be $o(1)$ and this requires some technical care to be treated. Now, let t_1 be the smallest time step such that $m_{t_1} \geqslant h_1$. From that time step on, we cannot use Recurrence (1) anymore, but as G_t is also an (h_2, k_2)-expander w.h.p., as long as $m_t \leqslant h_2$, the new recurrence $m_{t+1+t_1} \geqslant (1+k_2)m_{t+t_1}$ holds w.h.p. By solving the recurrence, we obtain $m_{t+t_1} \geqslant (1+k_2)^t m_{t_1} \geqslant (1+k_2)^t h_1$. So the number of time steps required to reach h_2 informed nodes is w.h.p.

$$\mathcal{O}\left(\frac{\log(h_2/h_1)}{\log(1+k_2)}\right) + \mathcal{O}\left(\frac{\log h_1}{\log(1+k_1)}\right)$$

We apply this way of reasoning over all sequence of expansion parameters and we thus get that at least $n/2$ nodes will be informed within a number of time steps that is w.h.p.

within the bound of the theorem.
Once there are $n/2$ informed nodes, a symmetric argument shows that the number of non-informed nodes decreases at the same rate.
We now provide the formal proof. Let us call $\bar{t} = \sum_{i=1}^{s} t_i$ where

$$t_i = \frac{\log(h_i/h_{i-1})}{\log(1+k_i)}$$

The proof is splitted in two parts: in the first part we prove that after $\mathcal{O}(\bar{t})$ time steps, there are at least $n/2$ informed nodes w.h.p.; in the second part we assume to have at least $n/2$ informed nodes and then prove that, after further $\mathcal{O}(\bar{t})$ time steps, all nodes will be informed w.h.p.
First part (From 1 to $n/2$). For $i = 0, 1, \ldots, s$, let T_i be the random variable defined by

$$T_i = \min\{t \in \mathbb{N} \,:\, m_t \geqslant h_i\}$$

where m_t is the r.v. counting the number of informed nodes at time step t. We now show that $T_s \in \mathcal{O}(\bar{t})$ w.h.p.

Claim 1 *If $t_i \geqslant 1$ then*

$$\mathbf{P}\left(T_i - T_{i-1} > 2t_i\right) \leqslant \frac{1}{n^2}$$

Proof. (of Claim.) For $t \in \mathbb{N}$ define the events

$$\begin{aligned}
\mathcal{E}_t^i &= \left\{m_{t+T_{i-1}} \geqslant (1+k_i)m_{t-1+T_{i-1}}\right\} \\
\mathcal{Z}_t^i &= \left\{m_{t-1+T_{i-1}} \leqslant h_i\right\} \\
\mathcal{F}_t^i &= \left\{m_{t+T_{i-1}} < (1+k_i)^t m_{T_{i-1}}\right\}
\end{aligned}$$

Firstly, it holds that [3]

$$\mathcal{Z}_{t+1} \subseteq \mathcal{Z}_t \tag{2}$$

Indeed, if the number of informed nodes were less than h_i in time step $t + T_{i-1}$, then they would be less than that in the previous time step. Moreover, we have that

$$\bigcap_{j=1}^{t} \mathcal{E}_j \subseteq \overline{\mathcal{F}_t}$$

Indeed, if $m_{j+T_{i-1}} \geqslant (1+k_i)m_{j-1+T_{i-1}}$ for every $j = 1, \ldots, t$, then it holds that $m_{t+T_{i-1}} \geqslant (1+k_i)^t m_{T_{i-1}}$. By considering the complementary sets, we get

$$\mathcal{F}_t \subseteq \bigcup_{j=1}^{t} \overline{\mathcal{E}_j} \tag{3}$$

Finally, define $\hat{t}_i = \lceil t_i \rceil = \left\lceil \frac{\log(h_i/h_{i-1})}{\log(1+k_i)} \right\rceil$ and notice that since $t_i \geqslant 1$ then $t_i \leqslant \hat{t}_i \leqslant 2t_i$. Since $m_{T_{i-1}} > h_{i-1}$, it holds that

$$\{m_{\hat{t}_i+T_{i-1}} < h_i\} \subseteq \left\{m_{\hat{t}_i+T_{i-1}} < \frac{h_i}{h_{i-1}} m_{T_{i-1}}\right\} \subseteq \mathcal{F}_{\hat{t}_i} \tag{4}$$

[3] In what follows, we omit superscript i to simplify the notation

We also easily have that $\{m_{\hat{t}_i+T_{i-1}} < h_i\} \subseteq \mathcal{Z}_{\hat{t}_i}$. Now we can evaluate

$$
\begin{aligned}
\mathbf{P}\left(T_i - T_{i-1} > 2t_i\right) \leqslant \mathbf{P}\left(T_i - T_{i-1} > \hat{t}_i\right) &= \mathbf{P}\left(m_{\hat{t}_i+T_{i-1}} < h_i\right) \\
&= \mathbf{P}\left(\{m_{\hat{t}_i+T_{i-1}} < h_i\} \cap \mathcal{Z}_{\hat{t}_i}\right) \leqslant \mathbf{P}\left(\mathcal{F}_{\hat{t}_i} \cap \mathcal{Z}_{\hat{t}_i}\right) \\
&\leqslant \mathbf{P}\left(\left(\bigcup_{j=1}^{\hat{t}_i} \mathcal{E}_j\right) \cap \mathcal{Z}_{\hat{t}_i}\right) = \mathbf{P}\left(\bigcup_{j=1}^{\hat{t}_i} \left(\mathcal{E}_j \cap \mathcal{Z}_{\hat{t}_i}\right)\right) \\
&\leqslant \mathbf{P}\left(\bigcup_{j=1}^{\hat{t}_i} \left(\mathcal{E}_j \cap \mathcal{Z}_j\right)\right) \leqslant \sum_{j=1}^{\hat{t}_i} \mathbf{P}\left(\mathcal{E}_j \mid \mathcal{Z}_j\right) \mathbf{P}\left(Z_j\right) \\
&\leqslant \sum_{j=1}^{\hat{t}_i} \mathbf{P}\left(\mathcal{E}_j \mid \mathcal{Z}_j\right)
\end{aligned}
$$

In the second line we used (4), in the third (3), and in the fourth line we used (2). For $j = 1, \ldots, \hat{t}_i$, r.v. G_j is a (h_i, k_i)-expander with probability at least $1 - 1/n^4$; indeed G_j has the same distribution of G_0 and this one is a (h_i, k_i)-expander w.h.p. by hypothesis. Hence $\mathbf{P}\left(\mathcal{E}_j \mid \mathcal{Z}_j\right) \leqslant 1/n^4$. We can bound $\hat{t}_i \leqslant n \log n$: indeed, we surely have $h_i \leqslant n$ and $k_i \geqslant 1/n$. This gives the bound

$$
\mathbf{P}\left(T_i - T_{i-1} > 2t_i\right) \leqslant \frac{\hat{t}_i}{n^4} \leqslant \frac{1}{n^2}
$$

$\square$

(Theorem's Proof follows). Let $\{\alpha(j)\}_j$ be a subsequence of $\{0, 1, \ldots, s\}$ defined recursively as follows: $\alpha(0) = 0$,

$$
\alpha(j) = \min\left\{\alpha \in [s] \, : \, \sum_{i=\alpha(j-1)+1}^{\alpha} t_i \geqslant 1\right\}
$$

if the set is not empty, and

$$
\alpha(j) = s \quad \text{otherwise}
$$

Let us call $\bar{j}$ the last index of sequence $\{\alpha(j)\}$ so that $\alpha(\bar{j}) = s$. For every $j = 1, \ldots, \bar{j}$, define

$$
l(j) = \sum_{i=\alpha(j-1)+1}^{\alpha(j)} t_i
$$

By using union bound, we get

$$
\mathbf{P}\left(T_s > 2\bar{t}\right) \leqslant \sum_{j=1}^{\bar{j}} \mathbf{P}\left(T_{\alpha(j)} - T_{\alpha(j-1)} > 2l(j)\right)
$$

In what follows, we show that, for every $j = 1, \ldots, \bar{j}$, it holds that

$$
\mathbf{P}\left(T_{\alpha(j)} - T_{\alpha(j-1)} > 2l(j)\right) \leqslant \frac{1}{n^2} \tag{5}
$$

By definition of $\alpha(j)$, If $\alpha(j) = \alpha(j-1)+1$ then $t_{\alpha(j)} \geqslant 1$; hence

$$\mathbf{P}\left(T_{\alpha(j)} - T_{\alpha(j-1)} > 2l(j)\right) = \mathbf{P}\left(T_{\alpha(j)} - T_{\alpha(j)-1} > 2t_{\alpha(j)}\right)$$

Thus, from Claim 1 we get (5).
Assume now that $\alpha(j) > \alpha(j-1)+1$ and observe that, by union bound, it holds that

$$\mathbf{P}\left(T_{\alpha(j)} - T_{\alpha(j-1)} > 2l(j)\right) \leqslant \mathbf{P}\left(T_{\alpha(j)-1} - T_{\alpha(j-1)} > l(j)\right)$$
$$+\mathbf{P}\left(T_{\alpha(j)} - T_{\alpha(j)-1} > l(j)\right)$$

If $m_{T_{\alpha(j-1)}} > h_s = n/2$ then the first part is proved. Otherwise, let

$$\gamma = \min\{i \in [s] \;:\; m_{T_{\alpha(j-1)}} \leqslant h_i\}$$

The graph is w.h.p. a (h_γ, k_γ)-expander and it holds that $m_{T_{\alpha(j-1)}} \leqslant h_\gamma$; so, we have w.h.p. that

$$m_{T_{\alpha(j-1)}+1} \geqslant (1+k_\gamma) m_{T_{\alpha(j-1)}}$$

Since γ is the minimum index such that $m_{T_{\alpha(j-1)}} \leqslant h_\gamma$, we also have that $m_{T_{\alpha(j-1)}} > h_{\gamma-1}$ and so w.h.p.

$$m_{T_{\alpha(j-1)}+1} > (1+k_\gamma) h_{\gamma-1}$$

If $(1+k_\gamma)h_{\gamma-1} > h_s = n/2$, then, in this step, we achieve $n/2$ informed nodes w.h.p. Otherwise, define

$$\Gamma = \min\{i \in [s] \;:\; h_i \geqslant (1+k_\gamma)h_{\gamma-1}\}$$

Claim 2 $\Gamma \geqslant \alpha(j)$.

Proof.

$$\sum_{i=\gamma}^{\Gamma} t_i = \sum_{i=\gamma}^{\Gamma} \frac{\log(h_i/h_{i-1})}{\log(1+k_i)}$$
$$\geqslant \frac{1}{\log(1+k_\gamma)} \sum_{i=\gamma}^{\Gamma} \log(h_i/h_{i-1}) = \frac{\log(h_\Gamma/h_{\gamma-1})}{\log(1+k_\gamma)} \geqslant 1$$

where we used the fact that sequence $\{k_i\}$ is decreasing and the definition of Γ. Since $\gamma \geqslant \alpha(j-1)+1$ then

$$1 \leqslant \sum_{i=\gamma}^{\Gamma} t_i \leqslant \sum_{i=\alpha(j-1)+1}^{\Gamma} t_i$$

But, by definition, $\alpha(j)$ is the smallest index that satisfies

$$\sum_{i=\alpha(j-1)+1}^{\alpha(j)} t_i \geqslant 1$$

$\square$

(Theorem's Proof follows). Thanks to the above claim and the (h_γ, k_γ)-expansion property, we get

$$m_{T_{\alpha(j-1)}+1} \geqslant (1+k_\gamma)h_{\gamma-1} > h_{\Gamma-1} \geqslant h_{\alpha(j)-1}$$

Since

$$\{m_{T_{\alpha(j-1)}+1} > h_{\alpha(j)-1}\} = \{T_{\alpha(j)-1} \leqslant T_{\alpha(j-1)} + 1\}$$

then we have that

$$\mathbf{P}\left(T_{\alpha(j)-1} - T_{\alpha(j-1)} > l(j)\right) \leqslant \mathbf{P}\left(T_{\alpha(j)-1} - T_{\alpha(j-1)} > 1\right)$$

$$\leqslant \mathbf{P}\left(G_0 \text{ is not an } (h_\gamma, k_\gamma)\text{-expander}\right) \leqslant \frac{1}{n^4}$$

Finally consider $\mathbf{P}\left(T_{\alpha(j)} - T_{\alpha(j)-1} > l(j)\right)$. If $t_{\alpha(j)} \geqslant 1$ then we can apply Claim 1 and get

$$\mathbf{P}\left(T_{\alpha(j)} - T_{\alpha(j)-1} > l(j)\right) \leqslant \mathbf{P}\left(T_{\alpha(j)} - T_{\alpha(j)-1} > t_{\alpha(j)}\right) \leqslant \frac{1}{n^2}$$

Otherwise, if

$$t_{\alpha(j)} = \frac{\log(h_{\alpha(j)}/h_{\alpha(j)-1})}{\log(1 + k_{\alpha(j)})} < 1$$

then $h_{\alpha(j)} \leqslant (1 + k_{\alpha(j)})h_{\alpha(j)-1}$ and since, by the expansion property, it holds w.h.p.

$$m_{T_{\alpha(j)}} \geqslant (1 + k_{\alpha(j)})m_{T_{\alpha(j)-1}}$$

we have that

$$\mathbf{P}\left(T_{\alpha(j)} - T_{\alpha(j)-1} > l(j)\right) \leqslant \mathbf{P}\left(T_{\alpha(j)} - T_{\alpha(j)-1} > 1\right)$$

$$\leqslant \mathbf{P}\left(m_{T_{\alpha(j)}} < (1 + k_{\alpha(j)})m_{T_{\alpha(j)-1}}\right) \leqslant \frac{1}{n^4}$$

Thus, we can conclude that

$$\mathbf{P}\left(T_{\alpha(j)} - T_{\alpha(j-1)} > 2l(j)\right) \leqslant \mathbf{P}\left(T_{\alpha(j)} - T_{\alpha(j)-1} > l(j)\right) +$$

$$+ \mathbf{P}\left(T_{\alpha(j)-1} - T_{\alpha(j-1)} > l(j)\right) \leqslant \frac{2}{n^2},$$

$$and \mathbf{P}\left(T_s > 2\bar{t}\right) \leqslant \sum_{j=1}^{\bar{j}} \mathbf{P}\left(T_{\alpha(j)} - T_{\alpha(j-1)} > 2l(j)\right) \leqslant \frac{2s}{n^2} \leqslant \frac{1}{n}$$

Second part (From $n/2$ to n).
In the first part we showed that after $\mathcal{O}(\bar{t})$ time steps there are at least $n/2$ informed nodes w.h.p. In this second part, define for $t \in \mathbb{N}$ the random variable $\bar{m}_t$ that counts the number of *non-informed* nodes at time steps t, i.e. $\bar{m}_t = n - m_t$. At the beginning of the second part, we have $\bar{m}_{T_s} \leqslant n/2$ w.h.p.

Given a graph $G = ([n], E)$ and a subset of nodes $J \subseteq [n]$, let us define

$$W(J) = \{u \in J \ : \ \{u, v\} \in E, \ \text{ for some } v \in [n] \setminus J\}$$

Observe that if J is the set of non-informed nodes in a fixed time step, then $W(J)$ is the set of nodes that will be informed in the next time step.

Claim 3 *Let G be a (h, k)-expander, then for every set $J \subseteq [n]$ with $|J| \leqslant h$ it holds that*

$$|W(J)| \geqslant \frac{k}{k+1}|J|$$

Proof. For every set $J \subseteq V$ it holds that $N(J \setminus W(J)) \subseteq W(J)$. So, if $|J| \leqslant h$ we have that $|J \setminus W(J)| \leqslant h$ and, from the definition of (h,k)-expander, it holds that

$$|W(J)| \geqslant |N(J \setminus W(J))| \geqslant k|J \setminus W(J)| = k(|J| - |W(J)|)$$

$\square$

Since r.v. G_t is w.h.p. a (h_i, k_i)-expander, the previous claim implies that if $\bar{m}_{t-1} \leqslant h_i$ then the recurrence

$$\bar{m}_t \leqslant \bar{m}_{t-1} - \frac{k_i}{1+k_i}\bar{m}_{t-1}$$

holds w.h.p. If t_0 is the first time step such that $\bar{m}_{t_0} \leqslant h_i$ then the closed form of the recurrence is $\bar{m}_{t+t_0} \leqslant \left(\frac{1}{1+k_i}\right)^t \bar{m}_{t_0}$ and, in order to have $\bar{m}_{t+t_0} \leqslant h_{i-1}$, it suffices to have

$$\left(\frac{1}{1+k_i}\right)^t \leqslant h_{i-1}/\bar{m}_{t_0} \tag{6}$$

If $t \geqslant \frac{\log(h_i/h_{i-1})}{\log(1+k_i)}$ then (6) is satisfied. Thus, for each stage $i = 1, \ldots, s$, we have the same expression of the first part, and the proof can proceed in the same way.
Formally, for each $i = 0, 1, \ldots, s$ define the r.v. $\bar{T}_i = \min\{t \in \mathbb{N} \,:\, \bar{m}_t \leqslant h_i\}$. Symmetrically to the first part, it comes out that $\bar{T}_{\alpha(j-1)} - \bar{T}_{\alpha(j)} \in \mathcal{O}\left(\sum_{i=\alpha(j-1)+1}^{\alpha(j)} t_i\right)$ w.h.p.
$\square$

An easy consequence of Theorem 2.4 is the following

Corollary 2.5 *Let $\mathcal{M} = \{G_t \,:\, t \in \mathbb{N}\}$ be a stationary Markovian evolving graph. Assume a decreasing sequence $k_1 \geqslant \cdots \geqslant k_{n/2}$ of positive real numbers exists such that, for every $i = 1, \ldots, n/2$, the stationary distribution of $\mathcal{M}$ yields an (i, k_i)-expander with probability $1 - \frac{1}{n^4}$. Then flooding time in $\mathcal{M}$ is w.h.p.*

$$\mathcal{O}\left(\sum_{i=1}^{n/2} \frac{1}{i\,\log(1+k_i)}\right)$$

Definition 2.1 naturally extends in the following way. We will need this generalization to include geometric-MEG (see Section 4).

Definition 2.6 (Markovian Evolving Graph II) *Let $\mathbf{G}$ be a family of graphs with the same node set $[n]$. A* Markovian evolving graph *$\mathcal{G} = \{G_t \,:\, t \in \mathbb{N}\}$ is a sequence of random variables with state space $\mathbf{G}$ and such that there exist both a Markov chain $\mathcal{X} = \{X_t \,:\, t \in \mathbb{N}\}$ and a function f so that $G_t = f(X_t)$.*
A stationary *Markovian evolving graph is a Markovian evolving graph $\mathcal{G} = \{G_t \,:\, t \in \mathbb{N}\}$ such that G_0 is random with a stationary distribution of $\mathcal{X}$ translated by f.*

It is not hard to show that Theorem 2.4 easily extends to the above generalized definition of Markovian evolving graphs.

3 Edge-Markovian evolving graphs

We recall the model introduced in [7, 24]. An edge-MEG $\mathcal{M}(n,p,q) = \{G_t \,:\, t \in \mathbb{N}\}$ is a Markov chain such that $G_t = ([n], E_t)$ with

$$E_t = \left\{ e \in \binom{[n]}{2} \,:\, X_t(e) = 1 \right\}$$

where $\{X_t(e) \,:\, e \in \binom{[n]}{2}\}$ are independent Markov chains with transition matrix

$$M = \left(\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-p & p \\ 1 & q & 1-q \end{array} \right)$$

Remind that p is the birth-rate and q is the death-rate and notice that an edge-MEG is a Markovian evolving graph according to Definition 2.1. Observe that if $0 < p, q, < 1$ the Markov chains $\{X_t(e) \,:\, t \in \mathbb{N}\}$ are irreducible and aperiodic; so there is a unique stationary distribution

$$\pi_e = \left(\frac{q}{p+q}, \frac{p}{p+q} \right)$$

Hence, the stationary distribution of $\mathcal{M}(n,p,q)$ is $G_{n,\hat{p}}$ (i.e. Erdös-Rényi distribution in which each possible edge occurs independently with probability $\hat{p}$) where here and in the sequel

$$\hat{p} = \frac{p}{p+q}$$

Stationary edge-MEG enjoy the following node-expansion properties.

Theorem 3.1 *Let $\mathcal{M}(n,p,q)$ be an edge-MEG such that $\hat{p} \geqslant c\frac{\log n}{n}$ for a sufficiently large constant c. Then, the stationary distribution of $\mathcal{M}(n,p,q)$ yields, with probability at least $1 - \frac{1}{n^4}$, a $\left(h, \frac{n\hat{p}}{c}\right)$-expander for $1 \leqslant h \leqslant \frac{1}{\hat{p}}$ and a $\left(h, \frac{n}{ch}\right)$-expander for $\frac{1}{\hat{p}} \leqslant h \leqslant \frac{n}{2}$.*

The proof of the theorem is a simple consequence of the following lemma.

Lemma 3.2 *Let $\hat{p} \geqslant c\frac{\log n}{n}$ for a sufficiently large constant c. With probability $1 - \frac{1}{n^4}$ for $G_{n,\hat{p}}$ it holds that for any $I \subseteq [n]$ with $|I| \leqslant \frac{n}{2}$,*

$$|N(I)| \geqslant \min \left\{ \frac{|I| n\hat{p}}{c}, \frac{n}{c} \right\}$$

Proof. Fix the constant $c = 28$. We first consider the case when $|I| \leqslant \frac{1}{\hat{p}}$ and prove that, with probability at least $1 - \frac{1}{n^4}$, it holds $|N(I)| \geqslant \frac{|I|n\hat{p}}{c}$. Then we consider the case $\frac{1}{\hat{p}} \leqslant |I| \leqslant \frac{n}{2}$ and prove that, with probability at least $1 - \frac{1}{n^4}$, it holds $|N(I)| \geqslant \frac{n}{c}$.
Let $m = |I| \leqslant \frac{1}{\hat{p}}$. For any $u \in [n] \setminus I$ consider the random variable X_u so that $X_u = 1$ if $u \in N(I)$ and $X_u = 0$ otherwise. Since $\mathbf{P}\left(X_v = 1\right) \geqslant m\hat{p}$ we have

$$\begin{aligned} \mathbf{E}\left[|N(I)|\right] &= \sum_{u \in [n] \setminus I} \mathbf{E}\left[X_u\right] = (n-m)m\hat{p} \\ &\geqslant \left(n - \frac{1}{\hat{p}}\right) m\hat{p} \geqslant \frac{1}{2} nm\hat{p} \end{aligned}$$

From Chernoff's bound we get

$$\mathbf{P}\left(|N(I)| \leqslant \frac{1}{c} nm\hat{p}\right) \leqslant e^{-\frac{1}{4}nm\hat{p}\left(\frac{c-2}{c}\right)^2}$$

$$\leqslant e^{-\frac{1}{4}m\log n\frac{(c-2)^2}{c}} \leqslant n^{-\frac{c-4}{4}m}$$

Therefore

$$\mathbf{P}\left(\exists I \subseteq [n],\, 1 \leqslant |I| \leqslant 1/\hat{p} \;:\; |N(I)| \leqslant \frac{1}{c} nm\hat{p}\right)$$

$$\leqslant \sum_{\substack{I \subseteq [n] \\ 1 \leqslant |I| \leqslant 1/\hat{p}}} \mathbf{P}\left(|N(I)| \leqslant \frac{1}{c} nm\hat{p}\right)$$

$$\leqslant \sum_{m=1}^{\lfloor 1/\hat{p} \rfloor} \binom{n}{m} n^{-\frac{c-4}{4}m} \quad \leqslant \quad \sum_{m=1}^{\lfloor 1/\hat{p} \rfloor} n^m n^{-\frac{c-4}{4}m}$$

$$\leqslant \quad \frac{1}{\hat{p}} n^{-\frac{c-8}{4}} \quad \leqslant n^{-\frac{c-12}{4}} \;\leqslant\; n^{-4}$$

Now consider the case where $\frac{1}{\hat{p}} \leqslant |I| = m \leqslant \frac{n}{2}$. Notice that $|N(I)| \leqslant \frac{n}{c}$ if and only if there exists a set $A \subseteq [n] \setminus (I \cup N(I))$ such that $|A| \geqslant n - m - \frac{n}{c}$. Hence

$$\mathbf{P}\left(\exists I \subseteq [n], |I| = m : |N(I)| \leqslant \frac{n}{c}\right) = \binom{n}{m}\binom{n-m}{|A|}(1-\hat{p})^{m|A|}$$

From the following inequalities

- $\binom{n}{m} \leqslant \left(\frac{en}{m}\right)^m \leqslant e^{m\log(en)} = e^{mn\hat{p}\frac{\log(en)}{n\hat{p}}} \leqslant e^{mn\hat{p}\frac{\log(en)}{c\log n}} = e^{mn\hat{p}\left(\frac{1}{c\log n}+\frac{1}{c}\right)}$
- $\binom{n-m}{|A|} = \binom{n-m}{n-m-|A|} = \binom{n-m}{\frac{n}{c}} \leqslant \binom{n}{\frac{n}{c}} \leqslant (ec)^{\frac{n}{c}} = e^{\frac{n}{c}\log(ec)} \leqslant e^{mn\hat{p}\left(\frac{1}{c}+\frac{\log c}{c}\right)}$
- $(1-\hat{p})^{m|A|} \leqslant e^{-m\hat{p}|A|} \leqslant e^{-m\hat{p}\left(n-\frac{n}{2}-\frac{n}{c}\right)} = e^{-mn\hat{p}\left(\frac{1}{2}-\frac{1}{c}\right)}$

we get

$$\mathbf{P}\left(\exists I \subseteq [n],\; |I| = m : |N(I)| \leqslant \frac{n}{c}\right) \leqslant e^{-mn\hat{p}\left(\frac{1}{2}-\frac{3}{c}-\frac{\log c}{c}-\frac{1}{c\log n}\right)}$$

$$\leqslant e^{-\frac{n}{5}}$$

Hence

$$\mathbf{P}\left(\exists I \subseteq [n], \frac{1}{\hat{p}} \leqslant |I| \leqslant \frac{n}{2} : |N(I)| \leqslant \frac{n}{c}\right)$$

$$\leqslant \sum_{I \subseteq [n] 1/\hat{p} \leqslant m \leqslant n/2} \mathbf{P}\left(\exists I \subseteq [n], |I| = m : |N(I)| \leqslant \frac{n}{c}\right)$$

$$\leqslant \sum_{m=\lfloor 1/\hat{p} \rfloor}^{\lceil n/2 \rceil} e^{-\frac{n}{5}} \leqslant ne^{-\frac{n}{5}} \leqslant n^{-4}$$

where the last inequality holds for sufficiently large n. $\square$

The expansion properties of stationary edge-MEG, stated in Theorem 3.1, allow us to apply Corollary 2.5 and, thus, getting the following

Theorem 3.3 *Let $\mathcal{M}(n,p,q)$ be a stationary edge-MEG such that $\hat{p} \geqslant c\frac{\log n}{n}$ for a sufficiently large constant c. Then flooding time in $\mathcal{M}(n,p,q)$ is w.h.p.*

$$\mathcal{O}\left(\frac{\log n}{\log(n\hat{p})} + \log\log(n\hat{p})\right)$$

Proof. Thanks to Theorem 3.1, we can apply Corollary 2.5 with sequence

$$k_i = \begin{cases} \frac{n\hat{p}}{c} & \text{for } 1 \leqslant i \leqslant \left\lfloor \frac{1}{\hat{p}} \right\rfloor \\ \frac{n}{ci} & \text{for } \left\lfloor \frac{1}{\hat{p}} \right\rfloor < i \leqslant \frac{n}{2} \end{cases}$$

Thus we obtain that the order of flooding time is w.h.p. bounded by

$$\sum_{i=1}^{\lfloor 1/\hat{p} \rfloor} \frac{1}{i \log(1 + \frac{n\hat{p}}{c})} + \sum_{i=\lfloor 1/\hat{p} \rfloor + 1}^{\lceil n/c \rceil - 1} \frac{1}{i \log(1 + \frac{n}{ci})}$$

$$+ \sum_{i=\lceil n/c \rceil}^{n/2} \frac{1}{i \log(1 + \frac{n}{ci})}$$

We now evaluate the above sums separately. For the first sum, by using $\sum_{i=1}^{m} \frac{1}{i} \leqslant \log m + 1$, we have

$$\sum_{i=1}^{\lfloor 1/\hat{p} \rfloor} \frac{1}{i \log(1 + \frac{n\hat{p}}{c})} = \frac{\log \frac{1}{\hat{p}} + 1}{\log(1 + \frac{n\hat{p}}{c})} = \mathcal{O}\left(\frac{\log n}{\log(n\hat{p})}\right)$$

For the second sum, by using $\log(1+x) \geqslant \log x$ for $x > 1$, we have

$$\sum_{i=\lfloor 1/\hat{p} \rfloor + 1}^{\lceil n/c \rceil - 1} \frac{1}{i \log(1 + \frac{n}{ci})} \leqslant \sum_{i=\lfloor 1/\hat{p} \rfloor + 1}^{\lceil n/c \rceil - 1} \frac{1}{i \log \frac{n}{ci}}$$

$$\leqslant \int_{\lfloor 1/\hat{p} \rfloor}^{\lceil n/c \rceil - 1} \frac{1}{x \log \frac{n}{cx}} dx = \left[-\log\log \frac{n}{cx} \right]_{\lfloor 1/\hat{p} \rfloor}^{\lceil n/c \rceil - 1} = \mathcal{O}(\log\log(n\hat{p}))$$

For the third sum, we apply $\log(1+x) \geqslant x/(1+x)$ for $x < 1$ and get

$$\sum_{i=\lceil n/c \rceil}^{n/2} \frac{1}{i \log(1 + \frac{n}{ci})} \leqslant \sum_{i=\lceil n/c \rceil}^{n/2} \frac{(1 + \frac{n}{ci})}{i \frac{n}{ci}}$$

$$= \sum_{i=\lceil n/c \rceil}^{n/2} \left(\frac{c}{n} + \frac{1}{i}\right) \leqslant \sum_{i=\lceil n/c \rceil}^{n/2} \left(\frac{c}{n} + \frac{c}{n}\right) = \mathcal{O}(1)$$

□

Next theorem gives a lower bound on flooding time in stationary edge-MEG.

Theorem 3.4 *Let $\mathcal{M}(n,p,q)$ be a stationary edge-MEG such that $\hat{p} \geqslant c\frac{\log n}{n}$ for a sufficiently large constant c. Then flooding time in $\mathcal{M}(n,p,q)$ is w.h.p.*

$$\Omega\left(\frac{\log n}{\log(n\hat{p})}\right)$$

Proof of Theorem 3.4. Consider the sequence $\mathcal{M}(n,p,q) = \{G_t \ : \ t \in \mathbb{N}\}$, where each graph G_t, is random with distribution $G_{n.\hat{p}}$. For a sufficiently large c (say $c = 4$), it holds w.h.p. that, for every $t = O(n)$, the maximal node degree of G_t is $O(n\hat{p})$. Then, by evaluating the maximum number of new informed nodes at every time step, we get the thesis. □

By comparing the upper bound of Theorem 3.3 and the lower bound of Theorem 3.4 we obtain the following

Corollary 3.5 *Let $\mathcal{M}(n,p,q)$ be a stationary edge-MEG such that $c\frac{\log n}{n} \leqslant \hat{p} \leqslant \frac{n^{\frac{1}{\log\log n}}}{n}$, for a sufficiently large constant c. Then flooding time in $\mathcal{M}(n,p,q)$ is w.h.p.*

$$\Theta\left(\frac{\log n}{\log(n\hat{p})}\right)$$

4 Geometric Markovian evolving graph

We introduce a model of dynamic graphs that is a discrete version of the *random walk mobility* model for radio networks [4]. In the latter model, nodes (i.e. radio stations) move on a bounded region of the plane (typically a square region) and each node performs, independently from the others, a sort of Brownian motion. At any time there is an edge (i.e. a bidirectional connection link) between two nodes if they are at distance at most R (typically R represents the transmission range). In our model we discretize time and space. We choose to keep constant the density (i.e. the ratio between the number of nodes and the area) as the number n of nodes grows. The node region is a square of side $\sqrt{n}$ and the density equals to 1. This choice is only for the sake of simplicity and all the results can be scaled to any density $\delta(n)$. The nodes can assume positions whose coordinates are integer multiple of a resolution coefficient $\epsilon > 0$. Formally, nodes move on the following set of points

$$L_{n,\epsilon} = \{(i\epsilon, j\epsilon) \mid i,j \in \mathbb{N} \wedge i,j \leqslant \frac{\sqrt{n}}{\epsilon}\}$$

At any time step, a node can move to one of the positions of $L_{n,\epsilon}$ within distance r from the previous position. The positive real number r is a fixed parameter that we call *move radius*. It can be interpreted as the maximum velocity of a node[4]. Formally, we introduce the *move graph* $M_{n,r,\epsilon} = (L_{n,\epsilon}, E_{n,r,\epsilon})$, where

$$E_{n,r,\epsilon} = \{(\mathbf{x},\mathbf{y}) \mid \mathbf{x},\mathbf{y} \in L_{n,\epsilon} \ \ d(\mathbf{x},\mathbf{y}) \leqslant r\}$$

and $d(\cdot,\cdot)$ is the Euclidean distance. A node in position $\mathbf{x}$, in one time step, can move in any position in $\Gamma(\mathbf{x})$, where $\Gamma(\mathbf{x}) = \{\mathbf{y} \mid (\mathbf{x},\mathbf{y}) \in E_{n,r,\epsilon}\}$. The nodes are identified by the first n positive integers $[n]$. The time-evolution of the movement of a single node i is represented by a Markov chain $\{P_{i,t} \ ; \ t \in \mathbb{N}\}$ where $P_{i,t}$ are random variables whose state-space is $L_{n,\epsilon}$ and

$$\mathbf{P}\left(P_{i,t+1} = \mathbf{x}\right) \quad = \quad \begin{cases} \frac{1}{|\Gamma(P_{i,t})|} & \text{if } \mathbf{x} \in \Gamma(P_{i,t}) \\ 0 & \text{otherwise} \end{cases}$$

[4]Indeed, a node can run through a distance of at most r in a unit of time.

In other words, $P_{i,t}$ is the position of node i at time t. Thus, the time-evolution of the movements of all the nodes is represented by a Markov chain $\mathcal{P}(n,r,\epsilon) = \{P_t \,:\, t \in \mathbb{N}\}$ whose state-space is $L_{n,\epsilon} \times L_{n,\epsilon} \times \cdots \times L_{n,\epsilon}$ (n times) and

$$P_t = (P_{1,t}, P_{2,t}, \ldots, P_{n,t})$$

Let us fix a *transmission radius* $R > 0$. A *geometric-MEG* is a sequence of random variables $\mathcal{G}(n,r,R,\epsilon) = \{G_t \,:\, t \in \mathbb{N}\}$ such that $G_t = ([n], E_t)$ with

$$E_t = \{(i,j) \mid d(P_{i,t}, P_{j,t}) \leqslant R\}$$

Clearly, a geometric-MEG is a Markovian evolving graph according to Definition 2.6.
As for the stationary case, we observe that the stationary distribution π_i of Markov chain $\{P_{i,t} \,;\, t \in \mathbb{N}\}$ is (see [1])

$$\pi_i(\mathbf{x}) \quad = \quad \frac{|\Gamma(\mathbf{x})|}{\sum_{\mathbf{y} \in L_{n,\epsilon}} |\Gamma(\mathbf{y})|}$$

Notice that π_i is almost uniform since, for any two positions $\mathbf{x}$ and $\mathbf{y}$, the values $\pi_i(\mathbf{x})$ and $\pi_i(\mathbf{y})$ can differ by at most a constant factor. Moreover, the stationary distribution of $\mathcal{P}(n,r,\epsilon)$ is the product of the independent distributions π_i for all $i \in [n]$. We say that a geometric-MEG $\mathcal{G}(n,r,R,\epsilon) = \{G_t \,:\, t \in \mathbb{N}\}$ is a *stationary* geometric-MEG if the underlying P_0 is random with the stationary distribution of the Markov chain $\mathcal{P}(n,r,\epsilon) = \{P_t \,:\, t \in \mathbb{N}\}$. Notice that if $\mathcal{G}(n,r,R,\epsilon) = \{G_t \,:\, t \in \mathbb{N}\}$ is a stationary geometric-MEG then all random variables G_t are random with the same probability distribution that we call *stationary distribution* of $\mathcal{G}(n,r,R,\epsilon)$.
Stationary geometric-MEG enjoy of the following expansion properties.

Theorem 4.1 *If $\epsilon \leqslant 1$ and $c\sqrt{\log n} \leqslant R \leqslant \sqrt{n}$ for a sufficiently large constant c, then constants $\alpha, \beta > 0$ exist such that, with probability $1 - \frac{1}{n^4}$, the stationary distribution of $\mathcal{G}(n,r,R,\epsilon)$ yields:*

- *A $(h, \alpha \frac{R^2}{h})$-expander for $1 \leqslant h \leqslant \alpha R^2$;*
- *A $(h, \beta \frac{R}{\sqrt{h}})$-expander for $\alpha R^2 \leqslant h \leqslant n/2$.*

Proof. Let $m = \lceil \sqrt{5n}/R \rceil$. Consider the partition of the square $\sqrt{n} \times \sqrt{n}$ into $m \times m$ congruent sub-squares, called cells. Every cell can be identified by the pair of indices (i,j), for $1 \leqslant i,j \leqslant m$, such that i is the index of row and j is the index of column of the cell. Let $c_{i,j}$ be the subset of the points of $L_{n,\epsilon}$ that fall into the cell (i,j). Notice that the side length ℓ of a cell satisfies $R/(\sqrt{5}+1) \leqslant \ell \leqslant R/\sqrt{5}$. Thus, any point of a cell is at distance less than R from any point of a side-by-side adjacent cell.
Through the following, we assume that the positions of the nodes are random with the stationary distribution of the Markov chain $\mathcal{P}(n,r,\epsilon)$. Moreover, we say that a node belongs to a cell whenever its position belongs to the cell. Let $N_{i,j}$ be the random variable counting the number of nodes in cell $c_{i,j}$. Now, we prove a simple but crucial claim.

Claim 4 *If $\epsilon \leqslant 1$ and $R \geqslant c\sqrt{\log n}$ for a sufficiently large constant c, then a constant $\lambda \geqslant 1$ exists such that, with probability $1 - \frac{1}{n^4}$, it holds that, for every $1 \leqslant i,j \leqslant m$,*

$$\frac{R^2}{\lambda} \quad \leqslant \quad N_{i,j} \quad \leqslant \quad \lambda R^2$$

Proof. Firstly, consider a fixed cell (i,j). For every $u \in [n]$, let X_u be the $\{0,1\}$ random variable that is 1 iff node u is in the cell $c_{i,j}$. Clearly, these are independent random variables and it holds that $N_{i,j} = \sum_{u \in [n]} X_u$. As for the probability distribution of X_u, we have that

$$\mathbf{P}\left(X_u = 1\right) \quad = \quad \sum_{\mathbf{x} \in c_{i,j}} \pi_u(\mathbf{x})$$

Since $\pi_u(\mathbf{x}) = \frac{|\Gamma(\mathbf{x})|}{\sum_{y \in L_{n,\epsilon}} |\Gamma(\mathbf{y})|}$, it is easy to see that, if ϵ is sufficiently small (say $\epsilon \leqslant 1$) then there is a constant $\gamma \geqslant 1$ such that, for every $\mathbf{x} \in L_{n,\epsilon}$, it holds that

$$\frac{1}{\gamma |L_{n,\epsilon}|} \quad \leqslant \quad \pi_u(\mathbf{x}) \quad \leqslant \quad \frac{\gamma}{|L_{n,\epsilon}|}$$

This implies that

$$\frac{|c_{i,j}|}{\gamma |L_{n,\epsilon}|} \quad \leqslant \quad \mathbf{P}\left(X_u = 1\right) \quad \leqslant \quad \frac{\gamma |c_{ij}|}{|L_{n,\epsilon}|}$$

By taking into account the side length of the cells, it is easy to verify that

$$\frac{R^2}{10n} \quad \leqslant \quad \frac{|c_{i,j}|}{|L_{n,\epsilon}|} \quad \leqslant \quad \frac{2R^2}{5n}$$

It follows that

$$\frac{R^2}{10\gamma n} \leqslant \mathbf{P}\left(X_u = 1\right) \leqslant \frac{2\gamma R^2}{5n} \quad \text{and} \quad \frac{R^2}{10\gamma} \leqslant \mathbf{E}\left[N_{i,j}\right] \leqslant \frac{2\gamma R^2}{5}$$

Now, in virtue of the Chernoff's bound, if $R \geqslant c\sqrt{\log n}$, for a sufficiently large constant c, then a constant $\lambda \geqslant 1$ exists such that

$$\frac{R^2}{\lambda} \quad \leqslant \quad N_{i,j} \quad \leqslant \quad \lambda R^2$$

with probability at least $1 - \frac{1}{n^5}$. Since the number of cells is less than n, a simple application of the union bound proves the thesis of the claim. □

Let $\mathcal{B}$ be the event that occurs when, for every $1 \leqslant i,j \leqslant m$,

$$\frac{R^2}{\lambda} \quad \leqslant \quad N_{i,j} \quad \leqslant \quad \lambda R^2$$

where λ is the constant of Claim 4. We now prove event $\mathcal{B}$ implies the expansion properties stated in the thesis of the theorem.

Claim 5 *If event $\mathcal{B}$ holds then the graph induced by R and by the positions of the nodes is a $(h, \alpha\frac{R^2}{h})$-expander for $1 \leqslant h \leqslant \alpha R^2$, where $\alpha = 1/(2\lambda)$.*

Proof. Let $I \subseteq [n]$ be such that $|I| \leqslant \alpha R^2$. Consider a node u in I and let $c_{i,j}$ be the cell that contains u. Since $\mathcal{B}$ holds, $N_{i,j} \geqslant \frac{R^2}{\lambda}$. All the nodes in $c_{i,j}$ are adjacent to u. Thus, there are at least $N_{i,j} - |I|$ nodes that are adjacent to u and that are not in I. It follows that

$$|N(I)| \;\geqslant\; N_{i,j} - |I| \;\geqslant\; \frac{R^2}{\lambda} - \alpha R^2 \;\geqslant\; \frac{R^2}{2\lambda} \;=\; \alpha R^2$$

In other terms, $|N(I)| \geqslant \alpha \frac{R^2}{|I|} |I|$. □

Claim 6 *If event $\mathcal{B}$ holds then the graph induced by $R \leqslant \sqrt{n}$ and by the positions of the nodes is a $(h, \beta\frac{R}{\sqrt{h}})$-expander for $\alpha R^2 \leqslant h \leqslant n/2$, where $\beta = \frac{1}{8\lambda^2}$.*

Proof. Let $I \subseteq [n]$ be any subset of nodes with $|I| \leqslant n/2$. We say that a cell is *black* if it contains at least a node in I. We say that a cell is *white* if it does not contain any node in I. Let B be the random variable that is the set of *black* cells. Let J be the random variable defined as follows

$$J = \{u \in [n] \mid u \notin I \ \wedge \ \exists c \in B \ : \ \text{node } u \text{ belongs to } c\}$$

Now, two cases are possibile: either $|J| \geqslant \beta R\sqrt{|I|}$ or not. Firstly, suppose that $|J| \geqslant \beta R\sqrt{|I|}$. Since every node in J is in a *black* cell, it holds that $J \subseteq N(I)$, and thus

$$|N(I)| \ \geqslant \ |J| \ \geqslant \ \beta R\sqrt{|I|}$$

In other terms, $N(I) \geqslant \beta \frac{R}{\sqrt{|I|}}|I|$ and the expansion property is proved.

Consider now the case $|J| < \beta R\sqrt{|I|}$. We say that a row (column) of cells is *black* if all the cells of the row (column) are *black*. Similarly, we say that a row (column) is *white* if all the cells of the row (column) are *white*. A row (column) that is neither *black* nor *white* is said to be *gray*. Notice that a *gray* row (column) contains at least two adjacent cells such that one is *white* and the other is *black*. Let B_r and B_c be, respectively, the number of *black* rows and the number of *black* columns. Three cases may arise.

$[B_r \geqslant 1]$: Observe that in this case all the columns are either *black* or *gray*. Let Y be the number of *gray* columns. It holds that

$$Y \ \geqslant \ m - B_c \ \geqslant \ m - \frac{|B|}{m}$$

Since event $\mathcal{B}$ holds, the number of nodes in non-*black* cells is bounded by $\lambda R^2(m^2 - |B|)$ and thus

$$\lambda R^2(m^2 - |B|) \quad \geqslant \quad n - |I| - |J| \quad \geqslant \quad n - |I| - \beta R\sqrt{|I|}$$

It follows that

$$|B| \quad \leqslant \quad m^2 - \frac{n - |I| - \beta R\sqrt{|I|}}{\lambda R^2}$$

By combining this bound with the previous bound on Y we obtain

$$Y \ \geqslant \ \frac{n - |I| - \beta R\sqrt{|I|}}{\lambda R^2 m} \ \geqslant \ \frac{n - |I| - \beta R\sqrt{|I|}}{\lambda 2\sqrt{5n}R}$$

where the last inequality follows from $m = \lceil \sqrt{5n}/R \rceil$ and $R \leqslant \sqrt{n}$.

Observe that every *gray* column contains at least a *white* cell that is adjacent to a *black* cell. So, all the nodes belonging to those *white* cells are included in $N(I)$. Since event $\mathcal{B}$ holds, it follows that

$$|N(I)| \ \geqslant \ Y\frac{R^2}{\lambda} \ \geqslant \ R\left(\frac{n - |I| - \beta R\sqrt{|I|}}{\lambda^2 2\sqrt{5n}}\right)$$

Now, recalling that $\beta = \frac{1}{8\lambda^2}$, $|I| \leqslant n/2$, and $R \leqslant \sqrt{n}$, it is easy to verify that

$$\frac{n - |I| - \beta R\sqrt{|I|}}{\lambda^2 2\sqrt{5n}} \geqslant \beta\sqrt{|I|}$$

It follows that $|N(I)| \geqslant \beta R\sqrt{|I|}$ and the expansion property holds.

$[B_c \geqslant 1$ (and $B_r = 0)]$: This case is symmetric to the previous one.

[$B_r = 0$ and $B_c = 0$]: In this case, all the rows and columns are either *gray* or *white*. Let Y_r and Y_c be the number of *gray* rows and the number of *gray* columns, respectively. Since there are neither *black* rows nor *black* columns, it must be the case that every *black* cell belongs to both a *gray* row and a *gray* column. As a consequence it holds that $Y_r \cdot Y_c \geqslant |B|$. Without loss of generality, assume that $Y_r \geqslant Y_c$. Then $Y_r^2 \geqslant |B|$ and thus $Y_r \geqslant \sqrt{|B|}$. Since event $\mathcal{B}$ holds and every *gray* row contains a *white* cell adjacent to a *black* one, it holds that

$$|N(I)| \geqslant Y_r \frac{R^2}{\lambda} \geqslant \sqrt{|B|}\frac{R^2}{\lambda}$$

By using again the fact (implied by event $\mathcal{B}$) that every cell contains at most λR^2 nodes, we have that $|B|\lambda R^2 \geqslant |I|$ and thus $\sqrt{|B|} \geqslant \frac{\sqrt{|I|}}{\sqrt{\lambda}R}$. It follows that

$$|N(I)| \geqslant \frac{R\sqrt{|I|}}{\lambda\sqrt{\lambda}} \geqslant \beta R\sqrt{|I|}$$

and the expansion property holds. □

Since, by Claim 4, event $\mathcal{B}$ occurs with probability at least $1 - \frac{1}{n^4}$, Claims 5 and 6 imply that also the expansion properties will hold with probability $1 - \frac{1}{n^4}$. □

Thanks to the general bound given by Corollary 2.5, the above expansion properties can be exploited in order to bound the flooding time in stationary geometric-MEG.

Theorem 4.2 *Let $\mathcal{G}(n, r, R, \epsilon)$ be a stationary geometric-MEG. If $\epsilon \leqslant 1$ and $c\sqrt{\log n} \leqslant R \leqslant \sqrt{n}$ for a sufficiently large constant c, then flooding time in $\mathcal{G}(n, r, R, \epsilon)$ is w.h.p.*

$$\mathcal{O}\left(\frac{\sqrt{n}}{R} + \log\log R\right)$$

Proof. From Theorem 4.1, the stationary geometric-MEG $\mathcal{G}(n, r, R, \epsilon)$ enjoys, with probability $1 - \frac{1}{n^4}$, of the following expansion properties:

- $(h, \alpha\frac{R^2}{h})$-expander for $1 \leqslant h \leqslant \alpha R^2$
- $(h, \beta\frac{R}{\sqrt{h}})$-expander for $\alpha R^2 \leqslant h \leqslant n/2$.

Thus, by applying Theorem 2.4, we obtain that flooding time is w.h.p.

$$\mathcal{O}\left(\sum_{h=1}^{\alpha R^2} \frac{1}{h\log(1+\alpha\frac{R^2}{h})} + \sum_{h=\alpha R^2}^{n/2} \frac{1}{h\log(1+\beta\frac{R}{\sqrt{h}})}\right)$$

We now evaluate the above two sums separately. For the sake of convenience, set $T = \alpha R^2$. It holds that

$$\sum_{h=1}^{T} \frac{1}{h\log(1+\frac{T}{h})} \leqslant 2\sum_{h=1}^{T} \frac{1}{h\log(1+\frac{T}{h})}\frac{T}{(T+h)}$$

This holds because $\frac{T}{(T+h)} \geqslant 1/2$ for $h \leqslant T$. Moreover,

$$\begin{array}{rcl}
\sum_{h=1}^{T} \frac{1}{h\log(1+\frac{T}{h})}\frac{T}{(T+h)} & = & \frac{T}{(T+1)\log(T+1)} + \\
+ \sum_{h=2}^{T} \frac{1}{h\log(1+\frac{T}{h})}\frac{T}{(T+h)} & \leqslant & 1 + \int_1^T \frac{T}{x(T+x)\log(1+\frac{T}{x})}dx \\
 & = & 1 + [-\log\log(1+\frac{T}{x})]_1^T \\
 & = & \log\log(T) + c
\end{array}$$

where c is a constant. Therefore we have shown that

$$\sum_{h=1}^{\alpha R^2} \frac{1}{h \log(1+\alpha \frac{R^2}{h})} = \mathcal{O}(\log\log R)$$

Now consider the second sum. By using the inequality $\log(1+x) \geqslant \frac{x}{1+x}$ we have that

$$\begin{aligned} \sum_{h=\alpha R^2}^{n/2} \frac{1}{h \log(1+\beta \frac{R}{\sqrt{h}})} &\leqslant \sum_{h=\alpha R^2}^{n/2} \frac{\sqrt{h}+\beta R}{h \beta R} \\ &\leqslant \frac{1+\frac{\beta}{\sqrt{\alpha}}}{\beta R} \sum_{h=\alpha R^2}^{n/2} \frac{1}{\sqrt{h}}. \end{aligned}$$

where the last inequality comes from inequality

$$\sqrt{h}+\beta R \leqslant (1+\frac{\beta}{\sqrt{\alpha}})\sqrt{h}$$

for $h \geqslant \alpha R^2$. Moreover, it holds that

$$\sum_{h=\alpha R^2}^{n/2} \frac{1}{\sqrt{h}} \leqslant \int_{\alpha R^2 - 1}^{n/2} \frac{dx}{\sqrt{x}} \leqslant 2\sqrt{n}$$

By combining the above inequalities we obtain

$$\sum_{h=\alpha R^2}^{n/2} \frac{1}{h \log(1+\beta \frac{R}{\sqrt{h}})} \leqslant 2\frac{1+\frac{\beta}{\sqrt{\alpha}}}{\beta R}\sqrt{n}$$

that is,

$$\sum_{h=\alpha R^2}^{n/2} \frac{1}{h \log(1+\beta \frac{R}{\sqrt{h}})} = \mathcal{O}\left(\frac{\sqrt{n}}{R}\right)$$

□

We remark that the proof of the expansion properties of Theorem 4.1 only relies on the fact that the stationary distribution of node positions is almost uniform. In fact we can get the same expansion properties for any mobility model yielding a stationary distribution of node position that is uniform or almost uniform. As mentioned in the Introduction, several relevant mobility models enjoy of this *uniformity* property. So, thanks to our Theorem 2.4, we can get an upper bound on flooding time similar to that of Theorem 4.2.
Next theorem shows a lower bound on flooding time in stationary geometric-MEG.

Theorem 4.3 *Let $\mathcal{G}(n, r, R, \epsilon)$ be a stationary geometric-MEG. If $\epsilon \leqslant 1$, then flooding time in $\mathcal{G}(n, r, R, \epsilon)$ is w.h.p.*

$$\Omega\left(\frac{\sqrt{n}}{R+r}\right)$$

Proof. Since the geometric-MEG is stationary, it is not hard to see that, w.h.p., at time 0 there exist at least two nodes u and v that are at distance greater than $\sqrt{n}/2$. Consider the flooding process with source node v. Let $\mathbf{x}_0$ be the position of v at time 0. For any t, let d_t be the minimum distance from $\mathbf{x}_0$ that node u has ever reached during

the first t time steps. It is immediate to see that $d_{t+1} \geqslant d_t - r$. Since $d_0 \geqslant \sqrt{n}/2$, it holds that $d_t \geqslant \sqrt{n}/2 - r \cdot t$.
Let D_t be the maximal distance from $\mathbf{x}_0$ that any informed node has ever reached during the first t time steps. It is easy to see that $D_{t+1} \leqslant D_t + R + r$. Since $D_0 = 0$, it holds that $D_t \leqslant (R + r)t$.
Let τ be the time step in which node u gets informed. It must be the case that $D_\tau \geqslant d_\tau$. It follows that

$$(R + r)\tau \;\geqslant\; D_\tau \;\geqslant\; d_\tau \;\geqslant\; \sqrt{n}/2 - r \cdot \tau.$$

It follows that $\tau \geqslant \sqrt{n}/(2(R + 2r))$. Therefore, the flooding cannot be completed in less than $\Omega\left(\frac{\sqrt{n}}{R+r}\right)$ time steps. □
By comparing Theorem 4.2 and Theorem 4.3 we obtain the following

Corollary 4.4 *Let $\mathcal{G}(n, r, R, \epsilon)$ be a stationary geometric-MEG. If $\epsilon \leqslant 1$, $r = \mathcal{O}(R)$, and $c\sqrt{\log n} \leqslant R \leqslant \frac{\sqrt{n}}{\log \log n}$ for a sufficiently large constant c, then flooding time in $\mathcal{G}(n, r, R, \epsilon)$ is w.h.p.*

$$\Theta\left(\frac{\sqrt{n}}{R}\right)$$

Under the very reasonable conditions of the above corollary, the general bound on flooding time in Markovian evolving graphs thus turns out to be asymptotically tight for stationary geometric-MEG.

5 Conclusions

As a general remark, we can say that our results provide an *analytical evidence* of the phenomenon that certain node mobility *do not slow down* information spreading. In [7], some cases have been shown where certain dynamicity even *significantly speeds up* information spreading. Our general upper bound for Markovian evolving graphs has been used by considering node-expansion properties of *connected* (random) graphs. However, since such graphs are dynamic, we strongly believe that our approach may work *below* the *connectivity threshold* of such random graphs as well. To this aim, it might be useful to consider a *dynamic* version of the parameterized node-expansion properties similar to that used in [7]. This would allow us to provide new strong upper bounds on flooding time in stationary Markovian evolving graphs that are *almost never* connected. Such results would be an analytical evidence of the phenomenon that node mobility can even speed up information spreading.

Our method provides “good” upper bounds in Markovian evolving graphs having an *almost homogeneous* topology. Another important issue is to investigate evolving graphs that are somewhat *non homogeneous*. For instance, we can consider mobility models yielded by node random walks over highly-irregular support graphs. A further instance is that yielded by the random-waypoint model over a non-convex, irregular region.

References

[1] D. Aldous and J. Fill. *Reversible Markov Chains and Random Walks on Graphs.* http://stat-www.berkeley.edu/users/aldous/RWG/book.html, 2002.

[2] C. Avin, M. Koucky, and Z. Lotker. How to explore a fast-changing world. In *Proc. of 35th International Colloquium on Automata, Languages and Programming (ICALP'08)*, volume 5125 of *LNCS*, pages 121–132. Springer, 2008.

[3] N. Bansal and Z. Liu. Capacity, delay, and mobility in wireless ad-hoc networks. In *Proc. of 22nd IEEE INFOCOM*, 2003.

[4] T. Camp, J. Boleng, and V. Davies. A survey of mobility models for ad hoc network research. *Wireless Communication and Mobile Computing*, 2(5):483–502, 2002.

[5] T. Camp, W. Navidi, and N. Bauer. Improving the accuracy of random waypoint simulations through steady-state initialization. In *Proc. of 15th Int. Conf. on Modelling and Simulation*, pages 319–326, 2004.

[6] N.B. Chang and M. Liu. Optimal contolled flooding search in a large wireless network. In *Proc. of 3rd International Symposium on Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WIOPT'05)*, pages 229–237, 2005.

[7] A. Clementi, C. Macci, A. Monti, F. Pasquale, and R. Silvestri. Flooding time in edge-markovian dynamic graphs. In *Proc. of 27th Annual ACM SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC'08)*, pages 213–222. ACM Press, 2008.

[8] A. Clementi, A. Monti, F. Pasquale, and R. Silvestri. Communication in dynamic radio networks. In *Proc. of 26th Annual ACM SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC'07)*, pages 205–214. ACM Press, 2007.

[9] J. Diaz, D. Mitsche, and X. Perez-Gimenez. On the connectivity of dynamic random geometric graphs. In *Proc. of 19th annual ACM-SIAM symposium on Discrete algorithms (SODA'08)*, pages 601–610, 2008.

[10] J. Diaz, X. Perez, M.J. Serna, and N.C. Wormald. Walkers on the cycle and the grid. *SIAM J. Discrete Math.*, 22(2):747–775, 2008.

[11] S.N. Dorogovtsev and J.F.F. Mendes. *Evolution of Networks.* Oxford University Press, 2003.

[12] M.M.C. Gkantsidis and A. Saberi. Hybrid search schemes for unstructured peer-to-peer networks. In *Proc. of 24th INFOCOM*, pages 1526–1537. IEEE Computer Society, 2005.

[13] Gnutella. Gnutella rfc. `http://rfc-gnutella.sourceforge.net`, 2002.

[14] R.A. Guerin. Channel occupancy time distribution in a cellular radio system. *IEEE Trans. on Veichular Technology*, 36(3):89–99, 1987.

[15] Z.J. Haas. The routing algorithm for the reconfigurable wireless networks. In *Proc. of the ICUPC*, 1997.

[16] S.M. Hedetniemi, S.T. Hedetniemi, and A.L. Liestman. A survey of gossiping and broadcasting in communication networks. *Networks*, 18(4):319–349, 1988.

[17] D.B. Johnson and D.A. Maltz. Dynamic source routing in ad-hoc wireless networks. *Mobile Computing*, pages 153–181, 1996.

[18] J.-Y. Leboudec and M. Vojnovic. Perfect simulation and the stationarity of a class of mobility models. In *Proc. of 24th IEEE INFOCOM*, pages 2743–2754, 2005.

[19] Q. Lv, P. Cao, E. Cohen, K. Li, and S. Shenker. Search and replication in unstructured peer-to-peer networks. In *Proc. of 16th International Conference on Supercomputing (ICS'02)*, pages 84–95, 2002.

[20] P. Naine, D.F. Towsley, B. Liu, and Z. Liu. Properties of random direction models. In *Proc. of 24th IEEE INFOCOM*, pages 1896–1907, 2005.

[21] K. Oikonomou and I. Stavrakakis. Performance analysis of probabilistic flooding using random graphs. In *Proc. of 1st IEEE WoWMoM Workshop on Autonomic and Opportunistic Communications (AOC)*, pages 1–6. IEEE Computer Society, 2007.

[22] B. Pittel. On spreading a rumor. *SIAM Journal on Applied Mathematics*, 47(1):213–223, 1987.

[23] C. Scheideler. Models and techniques for communication in dynamic networks. In *Proc. of 19th Symposium on Theoretical Aspects of Computer Science (STACS'02)*, volume 2285 of *LNCS*, pages 27–49. Springer, 2002.

[24] S. Wasserman. Analyzing social networks as stochastic processes. *Journal of the American Statistical Association*, 75:280–294, 1980.