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A procedure for the direct determination of Bishop's χ parameter from changes in Pore Size Distribution

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Abstract

Most of the recent works about the concept of effective stress in unsaturated soils focus on the proposal by Bishop, and, more particularly, on the search for suitable relationships between Bishop's χ parameter and the main controlling variables. These relationships are generally formulated by theoretical derivations and back-analyses of the dependency of mechanical parameters on hydraulic variables like suction or saturation. In this note, a new procedure is proposed to evaluate directly, and without any *a priori* assumptions, values for Bishop's χ parameter. In the first part, a general derivation based on the definition of work conjugated variables allows defining χ parameter as the ratio of the change of water volume over the change in pore volume during a process at constant suction. This definition is further exploited to evaluate Bishop's parameter from the changes suffered by material pore size distribution during loading. Method is applied to data obtained by Mercury Intrusion Porosimetry (MIP) tests on low plasticity silt (Jossiny silt), low plasticity sandy clay (Lean clay) and highly plastic clay (FEBEX clay). Values obtained for these materials show that χ parameter is close to the effective degree of saturation rather than total degree of saturation.

Keywords:

Effective stress, Unsaturated Soils, Mercury Intrusion Porosimetry

Introduction

It is now well established that the behaviour of unsaturated soils should be described using two stress-like variables defined as a combination of total stress σ_{ij}, pore water pressure p_w and pore air pressure p_a (Fredlund & Morgenstern, 1977; Houlsby, 1997; Tarantino *et al.*, 2000; Pereira *et al.*, 2005; Coussy *et al.*, 2010). Among all the possible combinations, the pair (suction $s = p_a - p_w$, net stresses $\sigma"_{ij} = \sigma_{il} - \delta_{ij}p_a$ where δ_{ij} is Kronecker symbol) have played historically a central role because of their experimental convenience, while Bishop proposal $\sigma"_{ij} = \sigma_{il} - \delta_{ij}(\chi p_w - (1-\chi)p_a)$ took relevance in constitutive modelling, because it provides a natural and smooth extension of Terzaghi's effective stress. Value of χ parameter has been essentially explored by looking at relationships with volumetric variables measuring water amount in the soil: degree of saturation (Jommi, 2000; Loret & Khalili, 2000; Khalili *et al.*, 2004; Pereira *et al.*, 2005), Lagrangian degree of saturation (Coussy *et al.*, 2010), degree of saturation of macrostructure (Alonso *et al.*, 2010). As pointed out by Coussy *et al.* (2010), all these concepts are theoretically justified and differ uniquely by the assumption considered for the relationship between changes in porosity, water content and air content under straining.

The aim of this note is to put forward an objective procedure that relaxes the latter assumption and allows assessing directly χ parameter from the $\delta e - \delta e_w$ relationship obtained from the change in Pore Size Distribution measured in Mercury Intrusion Porosimetry experiments. The method is supported by experimental results obtained for three compacted materials: a low plasticity sandy clay, a low plasticity silt and a highly active clay.

General derivation of Bishop's χ parameter

According to Gens (1995), a combination of total stresses, water and air pressures is an effective stress if any changes in these variables that cause the same change in effective stress traduces into the same mechanical response of the material. A preliminary condition is that the work performed by the effective stress must be equal to that produced by the total stresses and the fluid pressures. Houlsby (1997) showed that the total work input into a unsaturated soil sums up components related to the mechanical work, the work necessary to change water and air contents in the soil and the work of water and air filtration forces (Eq. 16; Houlsby, 1997). Under isotropic loading conditions, neglecting the work done by filtration forces, and identifying the divergences of water and air Darcy's velocities with the changes in air and water volumes per unit volume of the material, the expression provided by Houlsby (1997) reduces to:

$$\delta w = -p\frac{\delta V}{V} + p_w\frac{\delta V_w}{V} + p_a\frac{\delta V_a}{V} \qquad (1)$$

where w is the work input per unit volume of material, V the material total volume, p the total mean stress (positive in compression), p_a the air pressure, p_w the water pressure, V_w and V_a the air and water volumes and δ means the increment of the variable. δV is the change of volume of the sample whereas δV_w and δV_a are the volumes of air and water expelled by it. Under the restriction of solid incompressibility, the change in volume δV is equal to the change in pore volume δV_V, in turn equal to the sum of the change of water and air volume contained in the pores:

$$\delta V = \delta V_V = \delta V_w + \delta V_a \qquad (2)$$

Although not explicitly stated in the initial proposals, the different concepts for Bishop's type effective stress can all be formally derived by considering that δV_w can be split into two independent components $\delta V_{w1}$and δV_{w2}, one of them (δV_{w1}) proportional to the change in pore volume:

$$\delta V_w = \delta V_{w1} + \delta V_{w2} = \chi \delta V_V + \delta V_{w2} \qquad (3)$$

where χ is a proportionality coefficient between δV_{w1} and δV_V. Substitution of (3) and (2) into (1) leads to the following expression for the work input:

$$\delta w = -(p - \chi p_w - (1 - \chi) p_a) \frac{\delta V_V}{V} + (p_a - p_w) \frac{\delta V_{w2}}{V} \qquad (4)$$

or, in terms of volumetric strain increment $\delta \varepsilon_V$:

$$\delta w = (p - \chi p_w - (1 - \chi) p_a) \delta \varepsilon_V - (p_a - p_w) \frac{\delta V_{w2}}{V} \qquad (5)$$

Equation (5) states that Bishop's effective stress, with Bishop's parameter equal to the proportionality coefficient introduced in Eq. 3, is work-conjugated to $\delta \varepsilon_V$ provided that suction $(p_a - p_w)$ is the intensive variable associated to changes in $-\delta V_{w2}/V$.

The physical meaning of the partition stated by Equation 3 is illustrated in Fig. 1 for an incremental path involving a generic change of suction (δs) and void ratio (δe) that may represent, depending on the magnitude and sign considered for δs and δe, all the possible combinations between drying, wetting, swelling, compression and collapse. Water volume change is expressed in the figure by the change in water ratio $\delta e_w = \delta V_w / V_s$ where V_s is the volume of solid phase, considered incompressible for the sake of simplicity.

According to the figure, the total change in water volume $V_s \delta e_w$ can be decomposed into the sum of two components:

1) the change of water volume ($V_s \delta e_{w2}$) due to application of suction increment at constant volume (path OA). During this process, material state moves along the $e_w - s$ curve at fixed void ratio e, and, since δV_V is equal to zero in Equation 3, all the change in water volume is imputable to change in δV_{W2}.
2) the change of water volume ($V_s \delta e_{w1}$) due to the application of an increment in void ratio at constant suction (path AB). During this process, $e_w - s$ curve shifts to a new position at $e + \delta e$, and water volume changes to maintain constant suction. This change is caused only by deformation and corresponds to the term $\chi \delta V_V = \chi V_s \delta e$ in Equation 3.

According to this decomposition, χ coefficient is defined as the ratio:

$$\chi = \frac{\delta V_{w1}}{\delta V_V} = \frac{\delta e_{w1}}{\delta e} \tag{6}$$

where δe_{w1} is the water ratio exchanged during straining while maintaining constant the suction at which χ is computed and δe is the change in void ratio resulting from the straining process. Equation (6) emphasizes that Bishop's parameter has to be related to the change with strain of the soil-water storage curve rather than to averaging variables of gas and liquid phase volumes. In a comparable approach, Mašín (2013) proposes to uses Bishop's parameter to describe the strain dependency of the water retention curve.

Another expression for Bishop's parameter is obtained by substituting water ratio by degree of saturation (S_r) in Equation (6):

$$\chi = \left.\frac{\delta e_w}{\delta e}\right|_{s=Cnst} = \left.\frac{\delta (eS_r)}{\delta e}\right|_{s=Cnst} = S_r + e\left.\frac{\delta S_r}{\delta e}\right|_{s=Cnst} \quad (7)$$

Equation (7) evidences that the common assumption that Bishop's parameter is equal to degree of saturation (Jommi & Di Prisco, 1994; Lewis & Schrefler, 1998; Houlsby, 1997; Jommi, 2000; Sheng et al., 2004; Pereira et al., 2005; Casini, 2008) only applies to materials provided with $S_r - s$ curves independent of void ratio. Fig. 2.a depicts schematically such a curve expressed in term of water ratio *vs* suction. Since, for any suction, $e_{w0}/e_0 = S_r = e_{w1}/e_1$, the curve at e_1 is simply obtained by scaling the whole curve at e_0 by factor e_1/e_0. On the other hand, $-\delta V_{w2}/V = -(\delta V_w/V - \chi dV/V) = S_r dV/V - \delta V_w/V$ can be readily shown to be equal to $-n\delta S_r$. The work-conjugated variables proposed by Houlsby (1997) are thus valid for $S_r - s$ curves independent of void ratio.

Alonso et al. (2010) relax partly the latter assumption by considering that the material may have a fixed volume of water trapped in the microstructure (V_{wm}) and propose to equate Bishop's coefficient with the effective degree of saturation (S_{rM}) defined as:

$$S_{rM} = \frac{e_w - e_{wm}}{e - e_{wm}} \tag{8}$$

where $e_{wm} = V_{wm}/V_S$ is a material constant giving the microstructural water ratio. Bases for such a proposal can be explored by substituting Eq. (8) into Eq. (6), which leads to the expression:

$$\chi = S_{rM} + (e - e_{wm}) \frac{\delta S_{rM}}{\delta(e - e_{wm})}\bigg|_{s=Cnst} \tag{9}$$

According to Equation (9), the assumption $\chi = S_{rM}$ is thus valid for $S_r - s$ curves that are independent of strain over the range of effective degree of saturation. A schematic example of $e_w - s$ curve leading to Alonso et al.'s proposal for Bishop's coefficient is shown in Fig. 2b. e_1-curve is identical to e_0-curve for water ratio below e_{wm}. For higher water ratio, it is obtained as the e_0-curve scaled by $(e_1 - e_{wm})/(e_0 - e_{wm})$. In that case, the work conjugated variable to suction $-\delta V_{w2}/V$ can be readily found equal to $-n_M \delta S_{rM}$ where $n_M = (e - e_{wm})/(1 + e - e_{wm})$ is the porosity of the macrostructure.

In the most general case, $S_r - s$ curve experiments a change in shape during straining, often governed by the change of air entry pressure with density and the ability of smallest pores to experiment volume changes. A schematic example is presented in Fig. 2.c.

Fig. 2.d summarized the variation of Bishop's coefficient obtained by the present approach for the different types of water retention curve. It is equal to degree of saturation in case a), effective degree of saturation in case b) and takes other values in case c). In this last case, χ coefficient needs to be evaluated by a direct procedure without any *a priori* assumption about possible dependency on degree of saturation.

Direct evaluation of Bishop's parameter

Procedure of evaluation

Determination of water retention curves at different void ratios may result in a long process if addressed through the classical methods based on water transfer. An alternative way consists in obtaining them from Mercury Intrusion Porosimetry, by recognizing that intrusion of a non-wetting fluid like mercury obeys to the same principles as water drying process (Durner, 1994; Romero & Simms, 2008; Casini et al., 2012). According to this possibility, χ coefficient can be directly estimated from the change in Pore Size Distribution (PSD) under straining.

The procedure consists in the following steps:

1) Obtain the pore size distributions at two different void ratios e and $e + \Delta e$. Each distribution must have been scaled such that the area computed below it from the lowest (R_{min}) to the highest (R_{max}) pore entrance radii gives directly the value of void ratio at which it had been determined. The difference between both areas gives thus the change in void ratio Δe.

2) Select a value of suction s. According to Laplace and Wasburn laws, s is inversely proportional to the radius (R) of the pore able to sustain it, through a proportionality coefficient μ (0.484 N/m at 25 ºC) that includes effects of water and mercury contact angles and surface tensions. Thus, during application of suction s, all pores with entrance radius lower than R are filled by water and the area below the PSD between R_{min} and R gives the water ratio e_w of the material at suction s. The change in water ratio Δe_w during application of a change in void ratio Δe can thus be computed as the difference of the PSDs at e and $e + \Delta e$ from R_{min} to R.
3) Compute χ as the ratio Δe_w over Δe.

Example of computation

The procedure is tested on Mercury Intrusion Porosimetry tests obtained on three materials: a sandy clay (Lean clay, CF = 11%, w_L = 48%, IP = 18%; Li & Zhang 2009); a low plasticity silt (Jossigny silt, CF = 25%, w_L = 32%, IP = 15%; Vicol, 1990; Casini et al., 2012) and an active clay (FEBEX bentonite, CF = 67%, w_L = 102%, IP = 49%; Vilar, 2002; Lloret & Vilar, 2007) composed by 90% Montmorillonite and provided by a Cation Exchange Capacity equal to 111 meq/100 g.

Figure 2 shows the pore size distributions obtained on two samples of Jossigny silt compacted at different void ratios ($e_1 = 0.8$ and $e_1 = 0.7$) and at a constant water content equal to 15%. They evidence bi-modal distributions typical of materials compacted in a relatively dry state. According to the procedure described in the last section, the change in water ratio during a compaction process at constant suction s is computed as the area of the hatched zone in Fig. 3:

$$\Delta e_w(R) = \sum_{r_i=R_{min}}^{R} [PSD_1(r_i) - PSD_2(r_i)]\delta \log_{10}(r_i) \tag{10}$$

where R is equal to μ/s . Value of χ at suction s is thus computed as the ratio $\Delta e_w(R)/\Delta e$.

Fig. 4 show the values of Bishop's coefficient obtained from the change in Pore Size Distribution depicted in Fig. 3. It is expressed as a function of degree of saturation, computed by convention at the reference state $e = 0.8$. S_r is equal to e_w/e, where e_w is obtained as the area below the PSD for $e = 0.8$ and target radius R (shaded area in Figure 2). χ is close to zero at low degree of saturation, when water is stored only in the smallest pores. For this range of pores radii, the PSD appears to suffer little change under the applied compaction load and there is thus only a small amount of expelled water. For degrees of saturation between 0.3 and 0.42, χ exhibits a low increase associated to pore volume reduction for the lowest mode of the PSD (entrance radii between 0.4 and 1 µm). For degrees of saturation above 0.42, a strong increase of χ is observed, due to the fact that there is an important amount of water expelled during compaction by pores with entrance radii between 1 and 15 µm. χ reaches value of 1 for degree of saturation equal to 0.95, as no volume change is detected for pore entrance radii above 15 µm. χ parameter obtained in this manner matches quite well the effective degree of saturation computed by considering $e_{wm} = 0.3$ and proposed by Alonso et al. (2010) from back-analyses of the variation of elastic stiffness with suction.

Similar trends can be observed for the variation of Bishop's coefficient obtained on Lean and FEBEX clays (Fig. 5 and 6). PSDs of both materials evidence a pronounced double structure characterized by a low deformability of the micropores. As a result, most of the volume change experimented by the samples during compaction is imputable to the deformation of the largest pores and Bishop's parameter takes null values for degrees of saturation typically below 60%. For higher degree of saturation, χ lies close to effective degree of saturation for the Lean clay. For the FEBEX clay, values obtained with the direct procedure matches quite well the power law $\chi = S_r^{\alpha}$ proposed by Alonso et al. (2010).

Conclusions

A novel and general derivation of Bishop's parameter has been presented and gives to it with a new significance: it provides, for any saturation state of the material, the ratio between changes in water volume and total volume during a loading process at constant suction. This ratio is shown to be equal to the total or effective degree of saturation only for retention curve have a specific strain dependency.

According to this, a procedure is proposed to estimate the change in water content from the Pore Size Distribution of samples compacted at two different densities. Method is tested on results obtained on a low plasticity silt by Mercury Intrusion Porosimetry. Values of χ appear to be comparable with published results obtained by back-analysis of the change in elastic stiffness with suction.

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Notation

e	Void ratio
e_w	Water ratio
e_{wm}	Microstructural water ratio
n_M	Porosity of the macrostructure
p	Mean total stress
p_a	Air pressure
p_w	Water pressure
r_i	Pore entrance radius of i^{th}-family
R	Pore entrance radius
R_{min}	Lowest pore entrance radius of the PSD
R_{max}	Highest pore entrance radius of the PSD
s	Suction
S_r	Degree of saturation
S_{rM}	Degree of saturation of the macrostructure
V	Soil volume
V_a	Pore air volume
V_S	Solid grain volume
V_V	Pore volume

V_w Pore water volume

V_{wm} Volume of water trapped in soil microstructure

w Work per soil unit volume

α Power coefficient in the power law relating χ to degree of saturation

χ Bishop's parameter

$\Delta e, \delta e$ Change in void ratio

δ_{ij} Kronecker symbol

$\Delta e_w, \delta e_w$ Change in water ratio

δS_r Change in degree of saturation

δV Change in soil volume

δV_a Change in pore air volume

δV_V Change in pore volume

δV_w Change in pore water volume

δV_{w1} Change in pore water volume at constant suction

δV_{w2} Change in pore water volume at constant void ratio

μ Proportionality coefficient between suction and pore entrance radius

σ_{ij} Total stress tensor

σ''_{ij} Net stress tensor

References

Alonso, E.E., Pereira, J.M., Vaunat, J. and Olivella, S. (2010). A microstructurally based effective stress for unsaturated soils. Géotechnique, **60** (12): 913–925. doi:10.1680/geot.8.P.002.

Casini, F. (2008). Effetti del grado di saturazione sul comportamento meccanico di un limo, PhD Thesis, Universitá degli Studi di Roma "La Sapienza", Italy.

Casini, F., Vaunat, J., Romero, E. & Desideri, A. (2012). Consequences on water retention properties of double porosity features in a compacted silt. Acta Geotechnica, **7**(2): 139-150.

Coussy, O., Pereira, J.M. & Vaunat, J. (2010). Revisiting the thermodynamics of hardening plasticity for unsaturated soils. Computers and Geotechnics, **37**: 207-215.

Cui, Y.-J. (1993). "Etude du comportement d'un limon non saturé et de sa modélisation dans un cadre élasto-plastique" PhD Thesis, Ecole Nationale des Ponts et Chaussées, Paris, France.

Cui, Y.-J., Delage, P. (1996). Yielding and plastic behavior of an unsaturated compacted silt. Géotechnique, **46** (2): 291-311.

Durner, W. (1994). Hydraulic conductivity estimation for soils with heterogeneous pore structures. Water Ressources Research, **30** (2): 211-223.

Fredlund, D.G., Morgenstern, N.R. (1977). Stress state variables for unsaturated soils, J. Geotech. Eng. Div., 103 (GT5): 447-466.

Gens, A. (1995). Constitutive modelling : application to compacted soils. Unsaturated Soils, Balkema, Rotterdam, 3, 1179-1200.

Houlsby, G.T. (1997). The Work Input to an Unsaturated Granular Material, Géotechnique, **47** (1): 193-196.

Jommi, C. & di Prisco, C. (1994). A simple theoretical approach for modelling the mechanical behaviour of unsaturated granular soils (in Italian). In "Il ruolo dei fluidi in ingegneria geotecnica", Proc. Italian Conf. Mondovì, **1, No.** II, 167-188.

Jommi, C. (2000). Remarks on the constitutive modelling of unsaturated soils. In Experimental Evidence and Theoretical Approaches in Unsat. Soils, Tarantino A, Mancuso C (eds). Proc. of the Intern. Workshop on Unsat. Soils, Trento, Balkema: Rotterdam, 139-153.

Khalili, N., Geiser, F., and Blight, G. (2004). Effective stress in unsaturated soils: review with new evidence. International Journal of Geomechanics, **4** (2): 115–126. doi:10.1061/(ASCE)1532-3641.

Loret, B. & Khalili, N. (2000). A three phase model for unsaturated soils. International Journal for Numerical and Analytical Methods in Geomechanics, **24**: 893-927.

Lewis, R.W. & Schrefler, B.A. (1998). The Finite Element Method in the Static and Dynamic Deformation and Consolidation in Porous Media, J. Wiley, Chichester, 1998.

Lloret, A. & Vilar, M.-V. (2007). Advances on the knowledge of the thermo-hydro-mechanical behaviour of heavily compacted FEBEX bentonite. Physics and Chemistry of the Earth, **32**: 701-715.

Pereira, J.-M., Wong, H., Dubujet, P. & Dangla, P. (2005). Adaptation of existing behaviour models to unsaturated states: application to CJS model. International Journal for Numerical and Analytical Methods in Geomechanics, **29**(11): 1127-1155.

Romero, E., Simms, P.H. (2008). Microstructure Investigation in Unsaturated Soils: A Review with Special Attention to Contribution of Mercury Intrusion Porosimetry and Environmental Scanning Electron Microscopy. Geotechn and Geol Eng, **26(**6): 705-727.

Sheng, D., Sloan, S.W. & Gens, A. (2004). A constitutive model for unsaturated soils: thermomechanical and computational aspects. Computational Mechanics **33**, 453-465.

Tarantino, A., Mongiovì, L., Bosco, G. (2000). An experimental investigation on the independent isotropic stress variables for unsaturated soils. Géotechnique, **50**(3): 275-282.

Vicol, T. (1990) Comportement hydraulique et mécanique d'un sol fin non saturé. Application à la modélisation. PhD Thesis, Ecole Nationale des Ponts et Chaussées, Paris, France.

Vilar, M.-V. (2002). Thermo-hydro-mechanical characterization of a bentonite from Cabo de Gata. A study applied to the use of bentonite as a sealing material in high level radioactive waste repositories. Technical Publication ENRESA 01/2002, Madrid, 258 p.

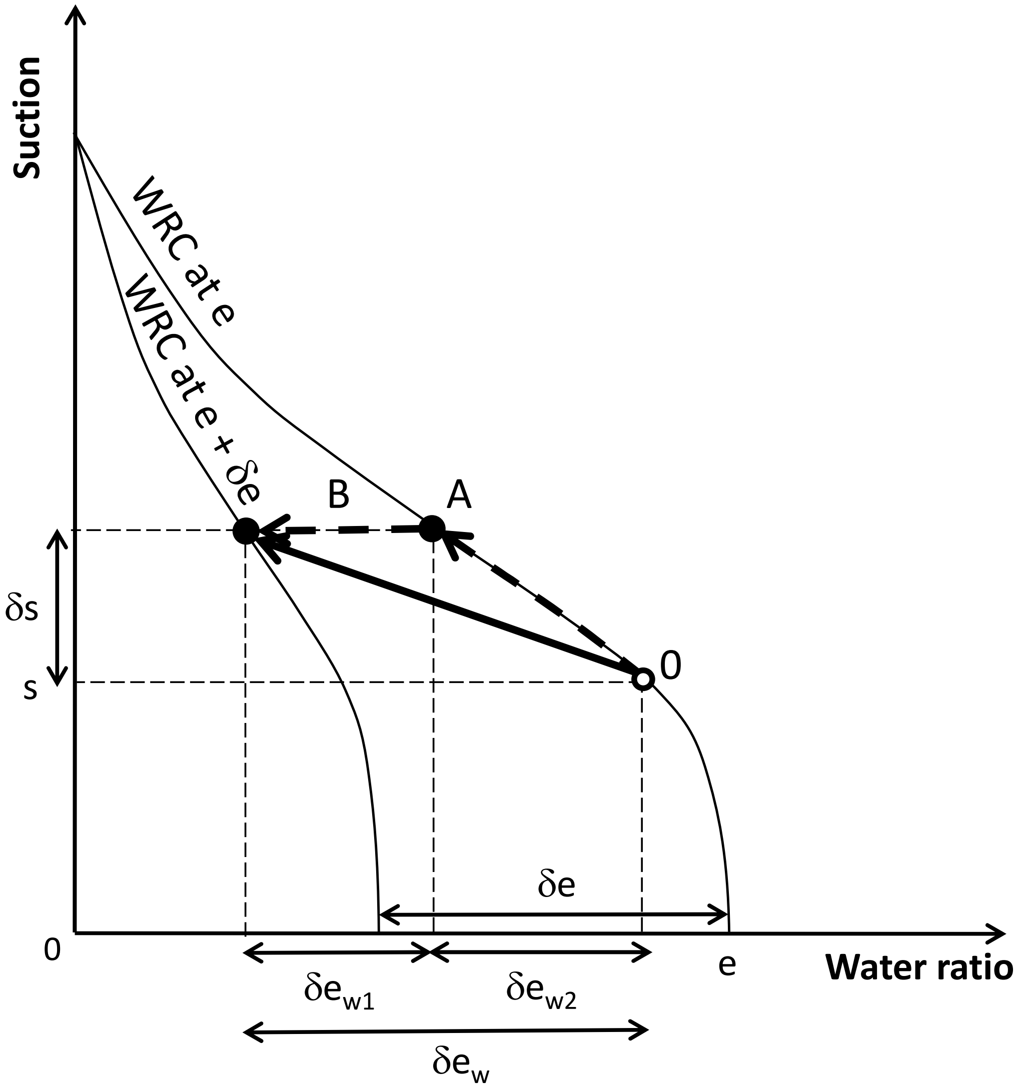


Figure 1. Partition of total water volume change into components due to suction and deformation only.

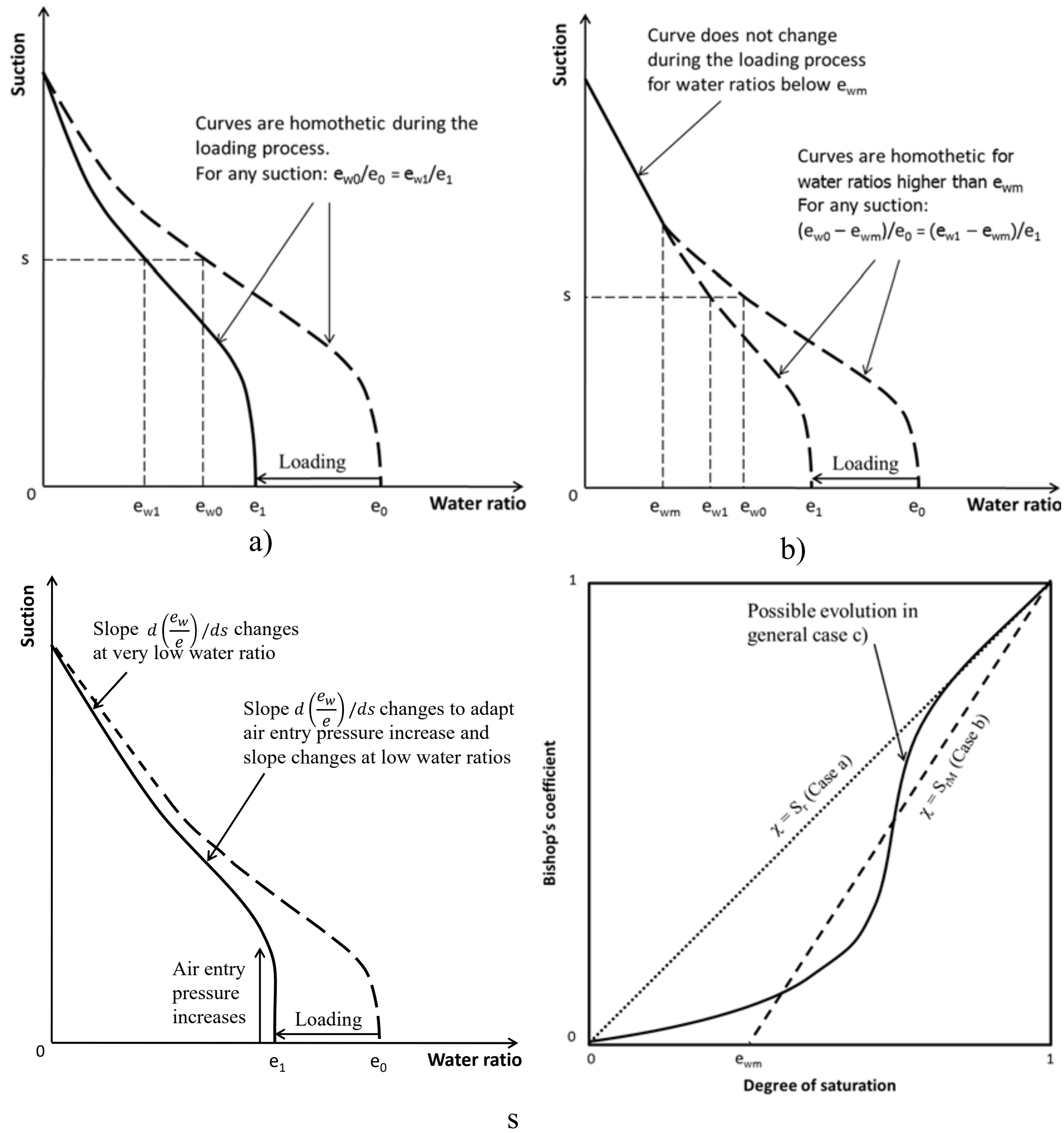


Figure 2. Schematic variation of Bishop's coefficient for different types of water retention curve strain dependency: a) water ratio – suction curve that preserves its shape over the whole range of degree of saturation during straining; b) water ratio – suction curve that preserves its shape over the range of macrostructural degree of saturation and remain constant in the microstructural range; c) water ratio – suction curve that changes in shape during straining; d) variation of Bishop's coefficient with degree of saturation for cases a), b) and c) .

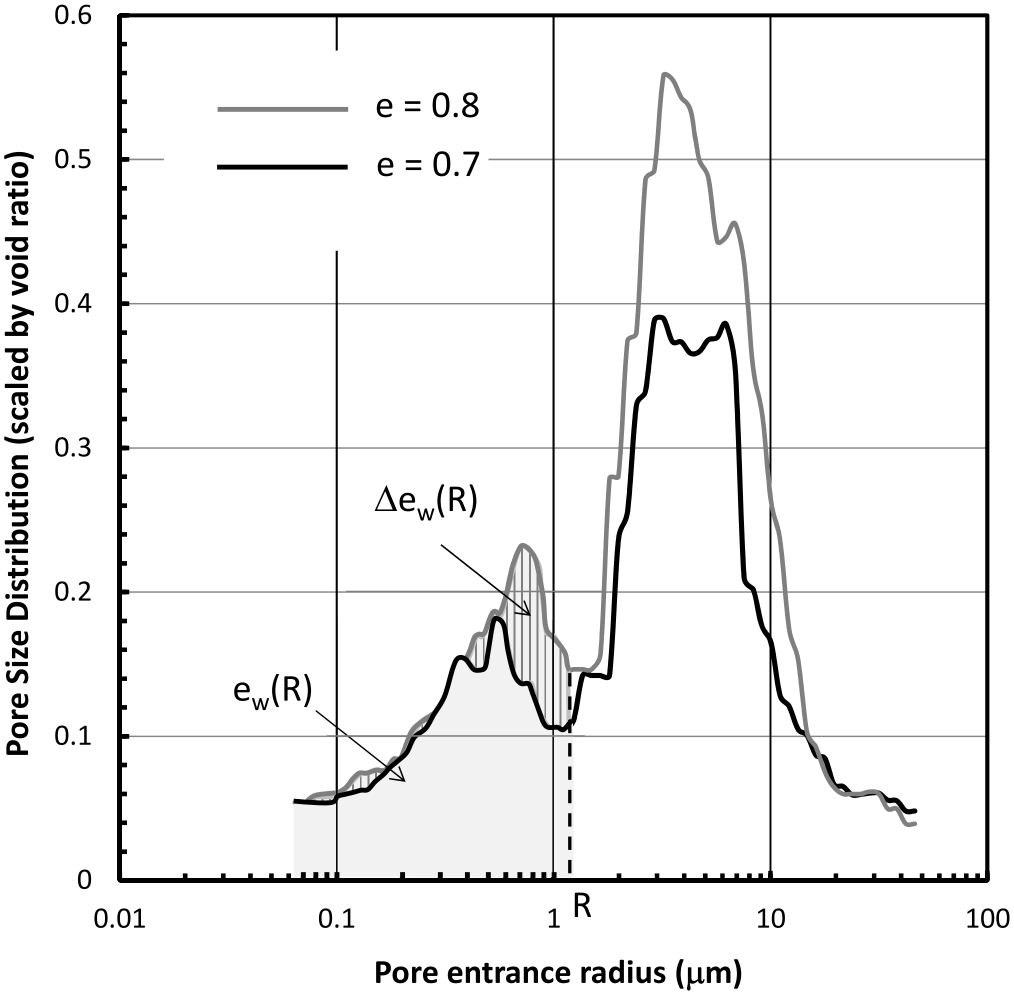


Figure 3. Pore Size Distribution of two samples of Jossigny silt compacted at void ratios equal to 0.8 and 0.7 (the PSDs has been scaled by the respective value of void ratios).

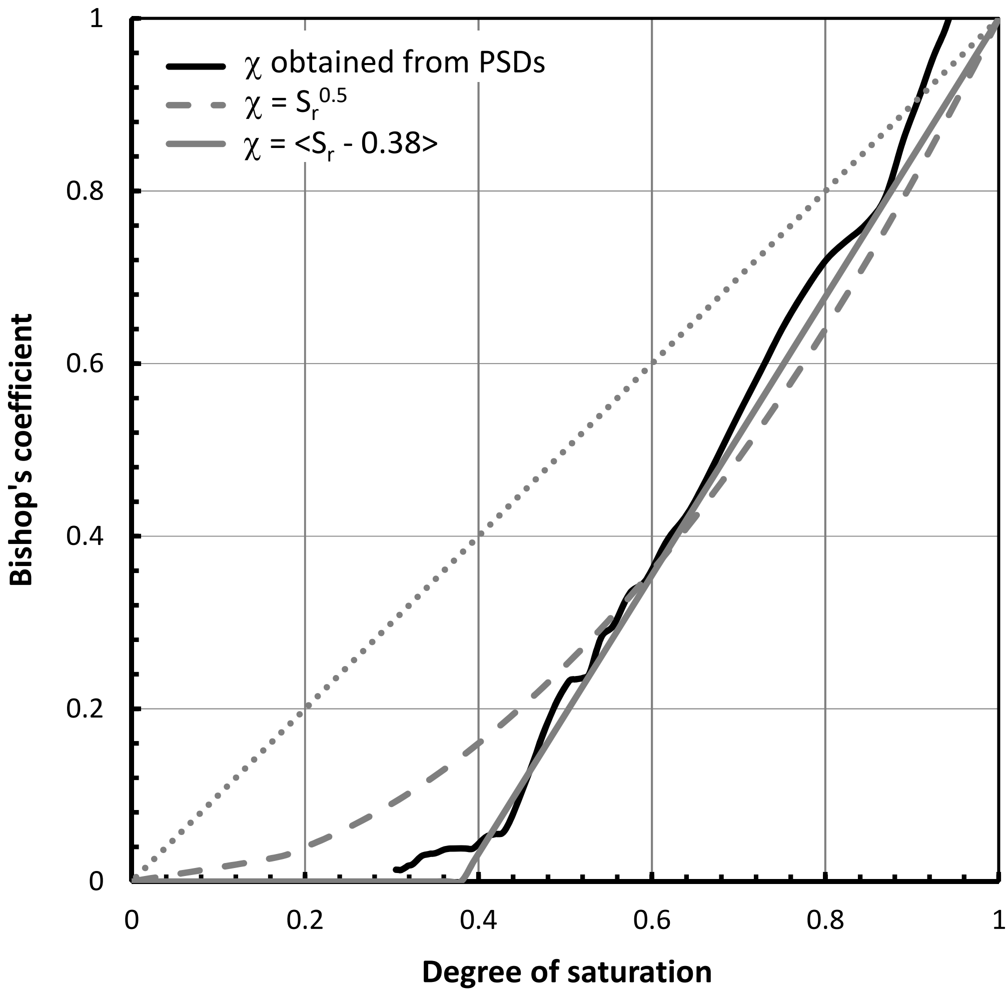
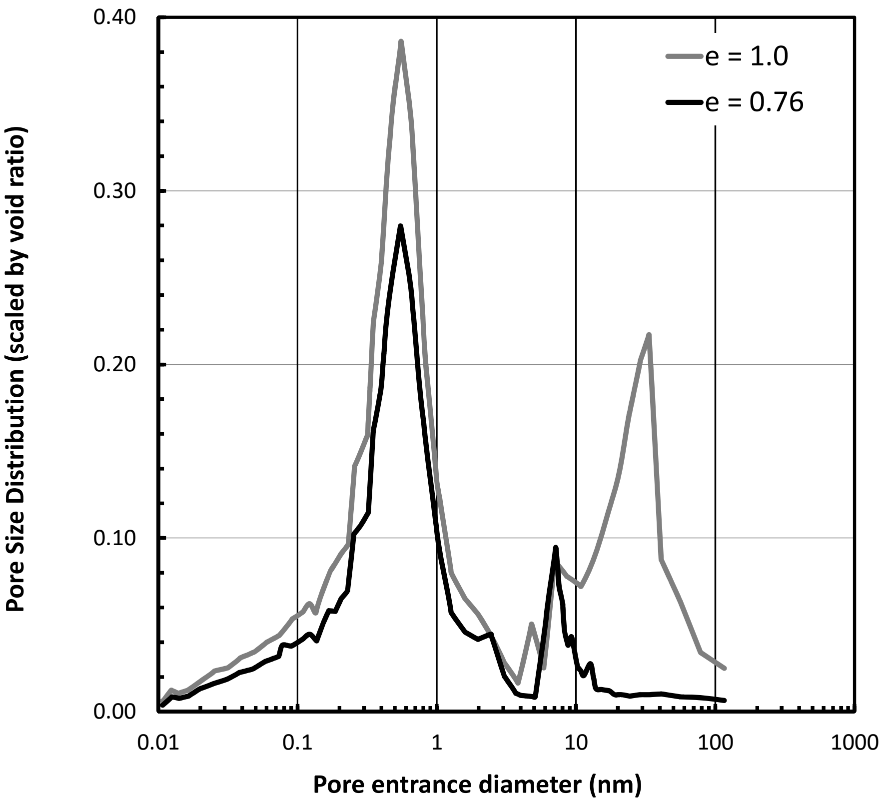
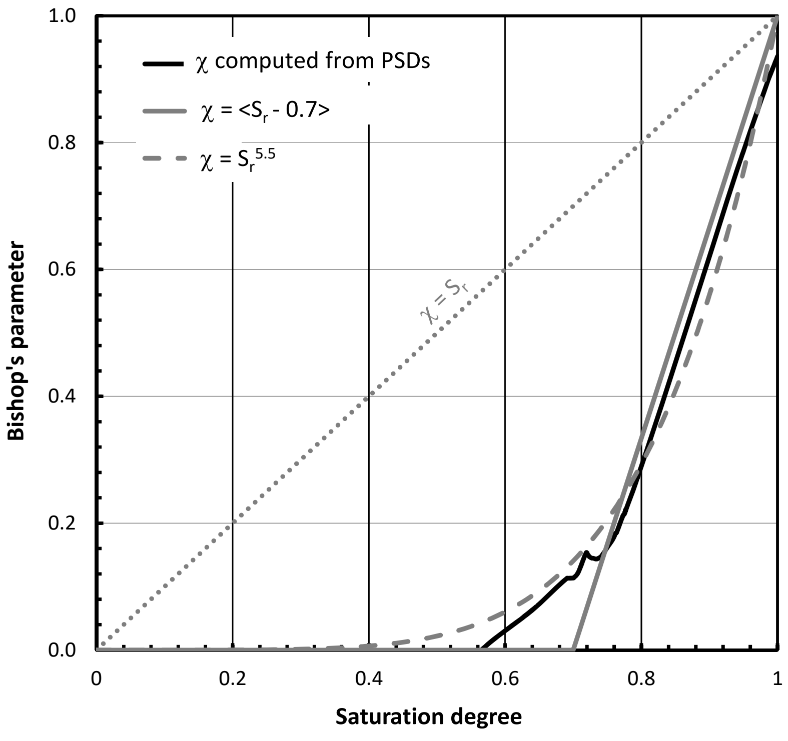


Figure 4. Variation of χ with degree of saturation for samples of Jossigny silt compacted at void ratio equal to 0.8.

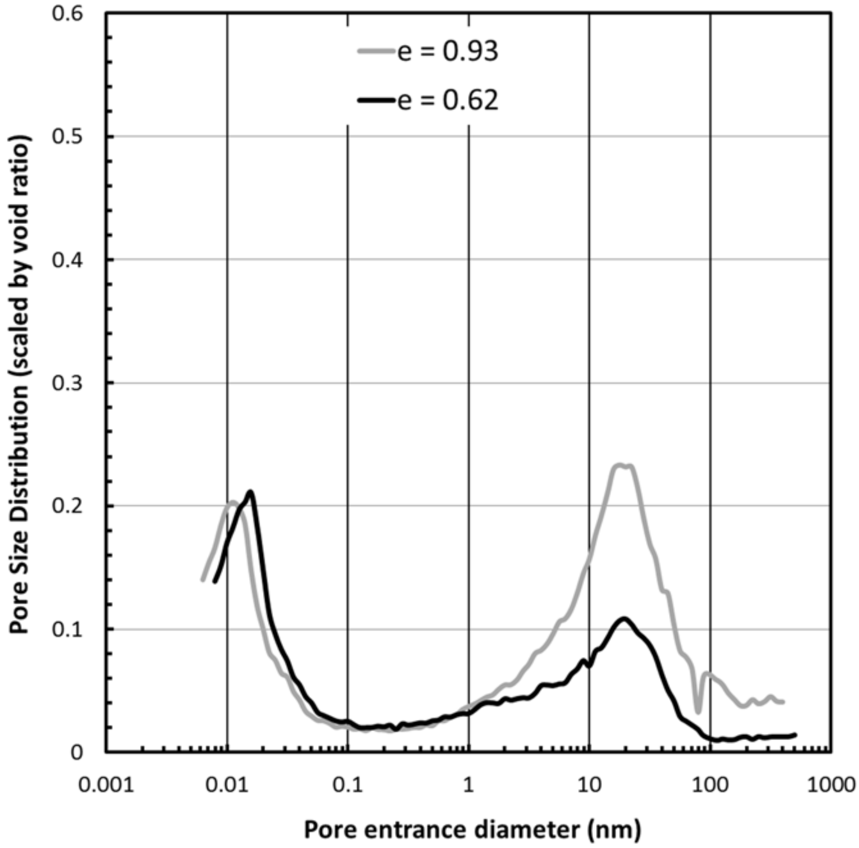


a)



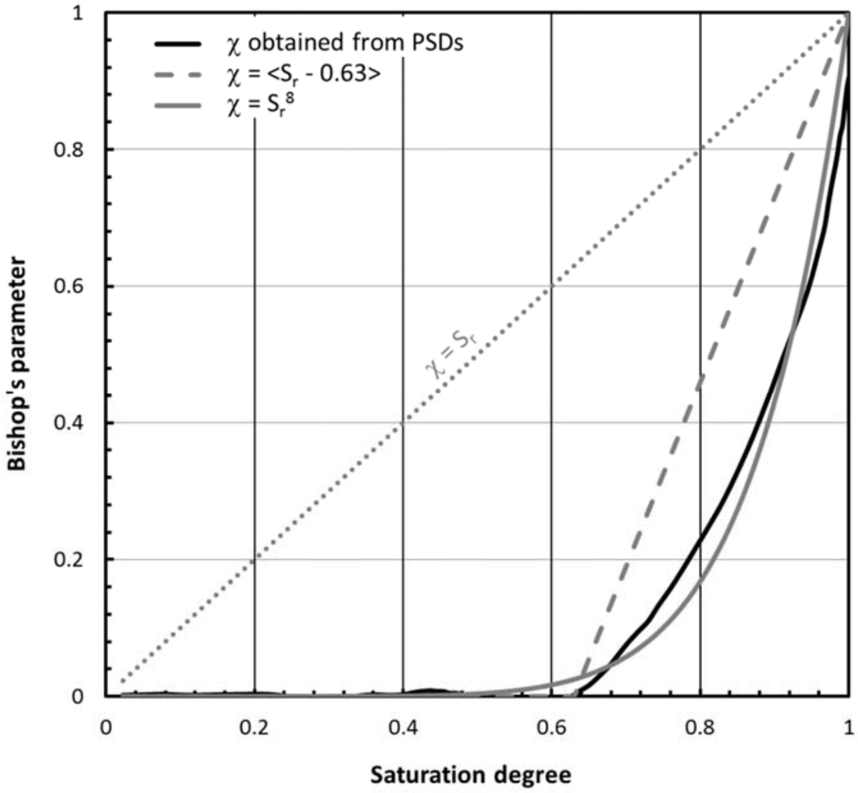
b)

Fig. 5 – Determination of Bishop's coefficient for low plasticity decomposed granite: a) Pore Size Distribution; b) Variation of Bishop's coefficient with degree of saturation



e = 0.93
e = 0.62
Pore Size Distribution (scaled by void ratio)
Pore entrance diameter (nm)

a)



χ obtained from PSDs
χ = <S_r - 0.63>
χ = S_r^8
χ = S_r
Bishop's parameter
Saturation degree

b)