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Classification of probability measures in terms of the canonically associated commutation relations

Talk given at the: 5-th Levy Conference (for the 120-th centenary of P. Levy)
Meijo University, Nagoya, December 15, 2006

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Abstract

The present paper is a survey of recent developments, mostly due to H. H. Kuo, A. Stan and myself on the new approach to the theory of multi–dimensional orthogonal polynomials based on Interacting Fock Spaces (IFS).

The starting point of these developments was the remark that some Interacting Fock Spaces are canonically associated to orthogonal polynomials in (in general infinitely) many variables and that, in the 1–variable case, the correspondence is one–to–one.

1 Interacting Fock Spaces: intuitive idea

An Interacting Fock Space (IFS) is any Hilbert space $\mathcal{H}$ with the following properties:

(i) $\mathcal{H}$ is the orthogonal sum of n–particle sub–spaces:

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$

(ii) there exist a vector space $\mathcal{H}_0$ and a linear map

$$A^+ : f \in \mathcal{H}_0 \to A_f^+$$

from $\mathcal{H}_0$ to densely defined operators, called creation operators,

$$A_f^+ : \mathcal{H}_n \to \mathcal{H}_{n+1}$$

(iii1) there is a unit vector $\Phi \in \mathcal{H}$ such that

$$\mathcal{H}_0 \equiv \mathbf{C} \cdot \Phi$$

(iii2) For each $n \in \mathbf{N}$ and $f_1, \ldots, f_n, f_{n+1} \in \mathcal{H}_0$

$$A_{f_n}^+ \cdot \ldots \cdot A_{f_1}^+ \Phi \in Dom(A_{f_{n+1}}^+)$$

(iii3) For each $n \in \mathbf{N}$ the vectors

$$\mathcal{N}_n := \{A_{f_n}^+ \cdot \ldots \cdot A_{f_1}^+ \Phi \ : \ f_1, \ldots, f_n \in \mathcal{H}_0\}$$

are total in $\mathcal{H}_n$

(iv) the adjoints of the creation operators, called annihilation operators, are defined on the (dense) sub–space

$$\mathcal{N} := \bigcup_{n\geq 0} \mathcal{N}_n$$

Notice that, introducing the orthogonal projections

$$P_n : \mathcal{H} \to \mathcal{H}_n$$

our assumptions imply that

$$A_f A_g^+ \ , \ A_g^+ A_f^+ \in \text{commutant of } = \{P_n \ : \ n \geq 0\}\{N := \sum_n nP_n\}'$$

2 Some criteria of interest for a theory

Let me briefly digress into a phylosophical consideration by listing some criteria of interest for a theory:

1) to be fruitful, i.e. to have applications outside its own field (not self–referential!): to say something new and unexpected on some old, well established objects and to achieve this goal by means of new, original, techniques
2) to create new interesting fields of investigation
3) to create a bridge among apparently far fields
4) to explain the fundamental origin of phenomenologically discovered objects
5) to show that some apparent mathematical coincidences have a universal nature

In what follows I will try to convince the reader that the notion of Interacting Fock spaces meets all these criteria.

1) To have applications outside its own field

– a new algebraic approach to the theory of orthogonal polynomials in (infinitely) many variables;

– a nontrivial generalization of the Jacobi coefficients,

– the idea to identify central limit measures by approximation of their Jacobi coefficients (Hashimoto, Obata, Tabei, Hora. We refer to the monograph see Hora–Obata monograph [HoraOb07] for this line of research.

2) To create new interesting fields of investigation:

– Monotone independence

– renormalized powers of white noise,

– use of quantum probabilistic techniques in the asymptotics of large graphs (Hora–Obata monograph),

– generalized Bargmann transform (initiated by Asai [Asai01] and fully realized in a paper by Asai, Kubo and Kuo [AsaKubKuo01])

– free evolutions in interacting Fock spaces (Das [Das03], [Das05]) [analogy with integrable systems],

– new notions of stochastic independence,

3) To create a bridge among apparently far fields:

– the quantum decomposition of a classical random variable,

– how to classify measures on $\mathbf{R}^d$ in terms of commutation relations

4) To explain the fundamental origin of phenomenologically discovered objects:

– from where do the Heisenberg commutation relations come from?

– which is the common theoretical root of the generalizations of the Kubo–Takenaka space of Hida distributions due to due to several authors: Oueridiane, Kondratev, Streit, Kuo, Cochran, Sengupta?

– Connetion with Haldane's exclusion statistics,

5) To show that some apparent mathematical coincidences have a universal nature:

– why the L^2–space of the d –dimensional Wiener process, the L^2–space of the d –dimensional Poisson process, and the Boson Fock space on $L^2(\mathbf{R}^d)$ can be canonically identified?

– Gaussianization of probability measures,

I will not survey the whole theory of IFS.

I only discuss some exciting recent development and several open problems related to them.

Before that however I will outline a quick history of the notion of IFS.

3 Quick history of IFS

Contrarily to the theory of IFS, the theory of Hilbert modules was well studied in mathematics.

However, at the beginning of the theory of IFS, the two notions emerged in an entangled way: the former as a development of the latter.

In other words the notion of IFS was abstracted from a notion of interacting Fock module which emerged in a concrete physical problem studied by Lu and Accard in the paper [AcLu92a], which contains, to our knowledge the first example in which an Hilbert module arises in a canonical way from a fundamental physical problem: quantum electrodynamics.

This was soon followed by the paper [AcLu92a] which contains, to our knowledge the first example (both in physics and in mathematics) of a non trivial Hilbert module, i.e. one in which there is a difference between left and right multiplication but between the two there is an explicit and non trivial relation.

In this paper Accardi and Lu formulated the following conjecture:

the Interacting Fock Functor can play for arbitrary probability measures the same role as the usual Fock Functor plays for the Gaussian measures

After 10 years we can say that, at least for measures with moments of all orders, the conjecture is correct.

In the paper [AcLu92a] Accardi and Lu defined the Fock Hilbert module over an abelian C^*–algebra. Pimsner Preprint [Pim93], of 1993, contains the first definition of Fock module over an arbitrary algebra.

As it is, it is not so easy to work with it. Skeide [Ske96b] proposed a compromise between the totally abstract generalization of Pimsner and the particular case discovered by Accardi and Lu. Skeide's definition of *centered Hilbert module* isolates a class of Hilbert modules for which one can develop a rich structure theory.

Skeide also proves that the renormalized square of white noise is not a centered Hilbert module.

The paper [AcLuVo97b] was the first attempt to summarize our understanding of the theory after five years of developments. It contains the first formal definition of IFS, the first steps towards a systematic treatment and a survey of results on IFS until 1997.

The fundamental identification beteween 1–mode interacting Fock space and orthogonal polynomialas in one variable was established in the paper [AcBo98]. The main theorem of this paper is nowadays known as *the quantum decomposition of a classical random variable.*

The preprint of this paper was delayed one year because Accardi and Bozeiko had written that the multi–dimensional case could be easily obtained from the one–dimensional one, but then they discovered tha this unfortunately was not the case.

However the manuscript circulated and at the end of 1997 Cabanal–Duvillard and Ionescu [Cab–DuIo97] published a paper where they proved the central limit analogue of the universal convolution introduced in the Accardi–Bozeiko's paper.

Their proof was combinatorial and could deal only with the symmetric case. A few years later, using the theory of one–mode–type interacting Fock spaces, the general case was solved by Accardi, Crismale and Lu [AcCrLu05] using the theory of IFS.

Th paper [Skei96b] contains the important discovery that, in the case of the QED Hilbert module, the interacting Fock space structure is a corollary of the Hilbert module structure, i.e. of the difference between left and right multiplication in the QED Hilbert module.

This difference is a consequence of the entangled commutation relations discovered by Accardi, Lu and Volovich.

The paper [AcSk98] is an attempt to generalize to every Hilbert module Skeide's result that the QED Hilbert module, the interacting Fock space structure is a corollary of the Hilbert module structure.

This can be done in abstrat terms, but up to now this abstract result has not been so fruitful as the more concrete idea of inteacting Fock space. Probably this is the same situation why the theory of usual Hilbert space is separated from the theory of Hilbert modules even if formally the former is a particular substructure of the latter.

4 years after the publication of the paper [AcBo98], the problem of understanding the connection between IFS and multi–vsariable orthogonal polynomials was solved by Accardi and Nahni [AcNah02] and the first intrinsic (i.e. basis independent) formulation of the multi–dimensional Jacobi relation was derived.

Starting from this, the paper [AKS04] inaugurates the program of a systematic classification of measures through their canonically associated IFS.

This program was pursued in the papers [AKS07], [AKS06], [AKS05] which are the main object of the present survey.

4 Multi–dimensional orthogonal polynomials

Let d be a positive integer and μ a Borel probability measure $\mathbb{R}^d$ with finite moments of any order

$$\int_{\mathbb{R}^d} |x_i|^p dx < \infty \qquad ; \qquad \forall p > 0 \ , \ \forall i \in \{1, 2, \ldots, d\}$$

where $x_i :=$ the i-th coordinate of the vector $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$.

For all non-negative integers n, we consider the space F_n of all polynomial functions $f(x_1, x_2, \ldots, x_d)$, of d-variables, such that $deg(f) \leq n$, where $deg(f)$ denotes the total degree of f. Because μ has finite moments of any order, it follows from Hölder's inequality that $F_n \subset L^2(\mathbb{R}^d, \mu)$, for all $n \geq 0$.

Moreover,

$F_n :=$ { all polynomial functions in the d-variables $x_1, x_2, \ldots, x_d$ with $deg(f) \leq n$}

$F_{-1} := \{0\}$

is a closed (finite dimensional) subspace of $L^2(\mathbb{R}^d, \mu)$ and

$G_n := F_n \ominus F_{n-1}$ (orthogonal complement in $L^2(\mathbb{R}^d, \mu)$

is an r_n^d–dimensional complex vector space with

$$r_n^d := \binom{n+d-1}{n} \tag{1}$$

So we see that, if $d > 1$, the dimension of this space grows like n^d. Denote:

$\mathcal{H} := \oplus_{n\geq 0} G_n$ (orthogonal sum)

$V :=$ { all polynomial functions in the d-variables x_1, x_2, …, x_d }

$X_i :=$ the operator of multiplication by the variable x_i $\forall i \in \{1, 2, \ldots, d\}$

$D(V) = V$

The following lemma leads us to a non trivial generalization of the usual notion of creation, preservation, and annihilation operators.

Lemma 1 *For all $n \geq 0$ and $i \in \{1, 2, \ldots, d\}$:*
$X_i G_n \subset G_{n-1} \oplus G_n \oplus G_{n+1}$ $\forall n \geq 0$, $\forall i \in \{1, 2, \ldots, d\}$

Theorem 1 (Recursion relations) *For any $j \in \{1, 2, \ldots, d\}$ and $n \geq 0$, the following equality holds:*

$$X_j P_n = P_{n+1} X_j P_n + P_n X_j P_n + P_{n-1} X_j P_n, \tag{2}$$

where $P_{-1} = 0$ by convention.

Summing over n this relation and using the relation $\sum_n P_n = 1$ we obtain the **quantum decomposition of the classical random variable** , i.e. $\forall i \in \{1, 2, \ldots, d\}$:

$$\begin{aligned} X_i &= \sum_n P_{n+1} X_j P_n + \sum_n P_n X_j P_n + \sum_n P_{n-1} X_j P_n \\ &=: a^+(i) + a^0(i) + a^-(i) \end{aligned} \tag{3}$$

More precisely, for each $j \in \{1, 2, \ldots, d\}$ and $n \geq 0$, we define the operators:

$$D_n^+(j) = P_{n+1} X_j P_n : \; G_n \longrightarrow G_{n+1}, \tag{4}$$

$$D_n^0(j) = P_n X_j P_n : \quad G_n \longrightarrow G_n, \tag{5}$$

$$D_n^-(j) = P_{n-1} X_j P_n : \; G_n \longrightarrow G_{n-1}, \tag{6}$$

where $G_{-1} = \{0\}$ by convention. Observe that

$$X_i | G_n \;\; = \;\; D_n^+(i) + D_n^0(i) + D_n^-(i), \tag{7}$$

for all $n \geq 0$ and $i \in \{1, 2, \ldots, d\}$. Extending these operators by linearity to the whole space V (polynomial functions) we get the quantum decomposition of X_i with

$$a^+(i) \;\; := \;\; \sum_{n=0}^{\infty} D_n^+(i) \tag{8}$$

$$a^0(i) \;\; := \;\; \sum_{n=0}^{\infty} D_n^0(i) \tag{9}$$

$$a^-(i) \;\; := \;\; \sum_{n=0}^{\infty} D_n^-(i) \tag{10}$$

$\forall i \in \{1, 2, \ldots, d\}$:
$a^+(i)$ is called a μ–*creation* operator,
$a^0(i)$ is called a μ–*preservation* operator,
$a^-(i)$ is called an μ–*annihilation* operator,
This naturally suggests the following program:
classify probability measures in terms of their quantum decomposition.
Let us give some examples to describe what this program means.

5 Quantum characterization of the polynomially factorisable probability measures on $\mathbb{R}^d$ in terms of creation, preservation and annihilation operators

.

Definition 1 *A probability measure μ on $\mathbb{R}^d$, having finite moments of any order, is called polynomially factorisable, if for any non-negative integers i_1, i_2, ..., i_d,*

$E[x_1^{i_1}x_2^{i_2}\cdots x_d^{i_d}] = E[x_1^{i_1}]E[x_2^{i_2}]\cdots E[x_d^{i_d}]$

A product measure $\mu = \mu_1 \otimes \mu_2 \ldots \otimes \mu_d$ on $\mathbb{R}^d$, is clearly polynomially factorisable but the converse is not true. Thus polynomial factorisability can be understood as a weak form of independence of the random variables X_1, X_2, ..., X_d.

The following theorems were proved in [AKS04].

Theorem 2 *A probability measure μ on $\mathbb{R}^d$, having finite moments of any order, is polynomially factorisable, if and only if for any j, $k \in \{1, 2, \ldots, d\}$, such that $j \neq k$, any operator from the set $\{a^-(j),\ a^0(j),\ a^+(j)\}$ commutes with any operator from the set $\{a^-(k),\ a^0(k),\ a^+(k)\}$.*

Theorem 3 *A probability measure μ is polynomially symmetric (i.e., all moments of odd degree vanish) if and only if the associated preservation operators $a^0(i)$, $1 \leq i \leq d$, are zero.*

6 Characterization of the standard Gaussian Probability Measure on $\mathbb{R}^d$ in terms of the preservation and commutators between the creation and annihilation operators

.

Theorem 4 *The standard Gaussian probability measure on $\mathbb{R}^d$, i.e., the probability measure given by the density function*

$$f(x_1, x_2, \ldots, x_d) = (2\pi)^{-\frac{d}{2}} e^{-\frac{x_1^2 + x_2^2 + \cdots + x_d^2}{2}},$$

is the only probability measure, on $\mathbb{R}^d$, having finite moments of all orders, such that for i, j, $k \in \{1, 2, \ldots, d\}$,

$$a^0(i) = 0 \tag{11}$$

$$[a^-(j), a^+(k)] = \delta_{j,k} I, \tag{12}$$

where $[a^-(j), a^+(k)] := a^-(j)a^+(k) - a^+(k)a^-(j)$ denotes the commutator of $a^-(j)$ and $a^+(k)$, and $\delta_{j,k} = 1$, if $j = k$, and $\delta_{j,k} = 0$, if $j \neq k$, is the Kronecker symbol. Here I denotes the identity operator of the space V of all polynomial functions of d real variables: x_1, x_2, $\ldots$, x_d. In equalities (11) and (12), the domain of $a^0(i)$, $a^-(j)$, and $a^+(k)$ is considered to be V.

The above characterization of the standard Gaussian probability measure is just a particular case of the following much more general result:

Theorem 5 *If μ and ν are two probability measures on $\mathbb{R}^d$, having finite moments of all orders, with the same preservation operators:*

$a_\mu^0(i) = a_\nu^0(i)$

and the same commutators between the annihilation and creation operators:

$[a_\mu^-(j), a_\mu^+(k)] = [a_\nu^-(j), a_\nu^+(k)]$*, for all i, j, $k \in \{1, 2, \ldots, d\}$*

then μ and ν have the same moments of all orders:

$E_\mu[x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}] = E_\nu[x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}]$

for all non-negative integers i_1, i_2, $\ldots$, i_d.

Summing up:

Assuming that a probability measure on $\mathbb{R}^d$ has finite moments of any order, its moments are completely determined by two family of operators. The first family is composed of the preservation operators.

The second family consists of the commutators between the annihilation and creation operators.

Notice that

$$D_n^+(j) \text{ is represented by an } r_n^d \times r_{n+1}^d \text{ matrix}$$

$$D_n^0(j) \text{ is represented by an } r_n^d \times r_n^d \text{ matrix}$$

$$D_n^-(j) \text{ is represented by an } r_n^d \times r_{n-1}^d \text{ matrix}$$

Lemma 2 *The operators $D_n^\epsilon(j)$ satisfy the following relations:*

$$D_{n+1}^+(i)\, D_n^+(j) \;=\; D_n^+(j)\, D_{n+1}^+(i) \tag{13}$$

$$D_{n+1}^0(i)\, D_n^+(j) \;+\; D_n^+(i)\, D_n^0(j) \;= \tag{14}$$

$$D_{n+1}^0(j)\, D_n^+(i) \;+\; D_n^+(j)\, D_n^0(i) \tag{15}$$

$$D_{n-1}^+(i)\, D_n^-(j) \;+\; D_n^0(i)\, D_n^0(j) \;+\; D_{n+1}^-(i)\, D_n^+(j) \;=$$

$$D_{n-1}^+(j)\, D_n^-(i) \;+\; D_n^0(i)\, D_n^0(j) \;+\; D_{n+1}^-(j)\, D_n^+(i) \tag{16}$$

for $i \neq j$, $1 \leq i,\, j \leq d$ and $n \geq 0$, where $D_{-1}^+(i) \;=\; 0$.

Theorem 6 *(Accardi–Nahni)*

These commutation relations characterize multi–variable orthogonal polynomials.

However there is a problem with this characterization because it is difficult to deal with rectangualr matrices.

7 The problem of existence

Problem: Let

$\{a_{i,j}\}_{1\leq i,j\leq d}$ and $\{b_k\}_{1\leq k\leq d}$

be two families of linear maps defined on the vector space V, of all polynomial functions in d variables, with values in the same space V.

What conditions must these two families of operators satisfy to ensure the existence of a probability measure μ on $\mathbb{R}^d$, having finite moments of any order, such that for all $i,\, j,\, k \in \{1, 2, \ldots, d\}$,

$[a_\mu^-(i), a_\mu^+(j)] = a_{i,j}$

$a_\mu^0(k) = b_k$?

The one dimensional case ($d = 1$) is well understood (Favard theorem).

In the multidimensional case ($d \geq 2$), at the moment, a full solution of this problem is not known: some necessary conditions are known in the general case and a full solution is known for special classes of measures.

To look for necessary conditions let us assume that there exists a Borel probability measure μ on $\mathbb{R}^d$, having finite moments of any order, such that for all i, j, $k \in \{1, 2, \ldots, d\}$,

$[a_\mu^-(i), a_\mu^+(j)] = a_{i,j}$

$a_\mu^0(k) = b_k$.

Recall that, at this moment, V is only a vector space because there is no inner product given on V since the probability measure μ has not been constructed yet.

V has an increasing sequence of well-defined vector subspaces:

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset V,$$

V_n = the vector space of all polynomial functions, in d variables, of degree $\leq n$.

If such a probability measure μ exists, then for all $n \geq 0$, V_n is invariant under the action of all the operators $a_{i,j}$ and b_k.

One of the necessary conditions is that all the spaces V_n, $n \geq 0$, be invariant under the action of each of the operators $a_{i,j}$ (commutator), $1 \leq i, j \leq d$, and b_k (preservation), $1 \leq k \leq d$:

$a_{i,j} V_n \subset V_n$

$b_k V_n \subset V_n$

are necessary conditions.

All other necessary conditions are deduced using the commutativity between the multiplication operators X_i and X_j, which is a purely algebraic condition i.e. independent of μ, and the relations

self–adjointness of the X_i's

$X_i = a_\mu^-(i) + a_\mu^0(i) + a_\mu^+(i)$ (quantum decomposition)

$(a_\mu^-(i))^* = a_\mu^+(i)$

$(a_\mu^0(i))^* = a_\mu^0(i)$

the positivity of the matrix $(X_i X_j)$ considered as an operator on $V \otimes \mathbb{C}^d$ which are all dependent of μ.

For example other important necessary conditions are

$$a_\mu^+(i) a_\mu^+(j) \quad = \quad a_\mu^+(j) a_\mu^+(i), \tag{17}$$

for all i, $j \in \{1, 2, \ldots, d\}$ and their adjoint:

$$a_\mu^-(j)a_\mu^-(i) \quad = \quad a_\mu^-(i)a_\mu^-(j), \tag{18}$$

for all i, $j \in \{1, 2, \ldots, d\}$.

We summarize now all the necessary conditions that we have found so far.

Theorem 7 *Let $\{a_{i,j}\}_{1 \le i,j \le d}$ and $\{b_k\}_{1 \le k \le d}$ be two families of linear maps from the vector space V, of all polynomial functions of d variables: x_1, x_2, $\ldots$, x_d, into itself.*

If there exists a probability measure μ on $\mathbb{R}^d$, having finite moments of any order, such that for all i, j, $k \in \{1, 2, \ldots, d\}$,

$[a_\mu^-(i), a_\mu^+(j)] = a_{i,j}$

$a_\mu^0(k) = b_k$,

then the following conditions hold:

For all $n \ge 0$ and all i, j, $k \in \{1, 2, \ldots, d\}$, $a_{i,j}V_n \subset V_n$ and $b_kV_n \subset V_n$, where V_n denotes the space of all polynomial functions of d variables, of degree at most n, for all i, $j \in \{1, 2, \ldots, d\}$,

$a_{i,j} - a_{j,i} = [b_j, b_i]$

This comes from the relation

$$[a_\mu^-(i), a_\mu^+(j)] - [a_\mu^-(j), a_\mu^+(i)] \quad = \quad [a_\mu^0(j), a_\mu^0(i)], \tag{19}$$

for all i, $j \in \{1, 2, \ldots, d\}$.

For all i, $j \in \{1, 2, \ldots, d\}$,

$[b_i, X_j] - [b_j, X_i] = 2[b_i, b_j]$

For all $n \ge 0$, the matrix

$(Tr(a_{i,j}|_{V_n}))_{1 \le i,j \le d}$

is positive semidefinite [this extends the requirement that $\omega_n \ge 0$, for all $n \ge 1$, from the case $d = 1$]

Comment

In the one dimensional all the V_n are one dimensional and the Szegö-Jacobi parameters are given by:

$\alpha_n = Tr(a^0|_{V_n})$

$\omega_{n+1} - \omega_n = Tr([a^-, a^+]|_{V_n}$

Thus the above relations can be interpreted as a multi-dimensional generalization of the the Szegö-Jacobi parameters.

8 IFS characterization of the support of probability measures

Definition 2 *A probability measure μ on $\mathbb{R}^d$ is said to have a* **square summable support** *if*

$$\mu = \sum_{n=1}^{\infty} p_n \delta_{x^{(n)}}, \tag{20}$$

for some sequence $\{p_n\}_{n\geq 1}$, of non-negative real numbers, such that

$$\sum_{n=1}^{\infty} p_n = 1,$$

and some sequence $\{x^{(n)}\}_{n\geq 1}$, of vectors in $\mathbb{R}^d$, such that

$$\sum_{n=1}^{\infty} |x^{(n)}|^2 < \infty, \tag{21}$$

where $|\cdot|$ denotes the euclidean norm of $\mathbb{R}^d$ and δ_x the Dirac delta measure at x, for any point x in $\mathbb{R}^d$.

Theorem 8 *A probability measure μ on $\mathbb{R}^d$ has a square summable support if and only if it has finite moments of any order and, for all $i \in \{1, 2, \dots, d\}$, the sequence $\{Tr((a^0(i)|_{F_n})^2)\}_{n\geq 0}$ is bounded and*

$$\sum_{n=0}^{\infty} Tr([a^-(i), a^+(i)]|_{F_n}) < \infty. \tag{22}$$

9 The operators a^0 and $[a^-, a^+]$ for some classical one-dimensional distributions

The tables below describe the Szegö-Jacobi parameters and the commutator $[a^-, a^+]$ for some classic one-dimensional distributions. Our program is to extend these tables to multi–dimensional distributions not of product type.

Measure	Polynomial P_n
Gaussian $N(0,\sigma^2)$	Hermite $H_n(x;\sigma^2)$ $= (-\sigma^2)^n e^{x^2/2\sigma^2}\partial_x^n e^{-x^2/2\sigma^2}$
Poisson $\mathrm{Poi}(a)$	Charlier $C_n(x;a) =$ $(-1)^n a^{-x}\Gamma(x+1)\Delta_{x+}^n\left[\frac{a^x}{\Gamma(x-n+1)}\right],$ $\Delta_{x+}f(x) = f(x+1) - f(x)$
Gamma $\Gamma(\alpha)$, $(\alpha > 0)$ $\frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}, \quad x > 0$	Laguerre $\mathcal{L}_n^{(\alpha-1)}(x)$ $= (-1)^n x^{-\alpha+1}e^x\partial_x^n[x^{n+\alpha-1}e^{-x}]$
Uniform on $[-1,1]$	Legendre $\widetilde{L}_n(x) = \frac{1}{2^n(2n-1)!!}\partial_x^n[(x^2-1)^n]$
Arcsine $\frac{1}{\pi\sqrt{1-x^2}}, \quad \lvert x\rvert < 1$	Chebyshev (1^{st} kind) $\widetilde{T}_0(x) = 1$ $\widetilde{T}_n(x) = \frac{1}{2^{n-1}}\cos(n\cos^{-1}x), n \geq 1$
Semicircle $\frac{2}{\pi}\sqrt{1-x^2}, \quad \lvert x\rvert < 1$	Chebyshev (2^{nd} kind) $\widetilde{U}_n(x) = \frac{1}{2^n}\frac{\sin[(n+1)\cos^{-1}x]}{\sin(\cos^{-1}x)}$
$\frac{1}{\sqrt{\pi}}\frac{\Gamma(\beta+1)}{\Gamma(\beta+\frac{1}{2})}(1-x^2)^{\beta-\frac{1}{2}}, \quad \lvert x\rvert < 1$ $\beta > -\frac{1}{2}, \quad \beta \neq 0,1$	Gegenbauer $\widetilde{G}_n^{(\beta)} = C_n^{(\beta)}(1-x^2)^{\frac{1}{2}-\beta}u(x),$ $u(x) = \partial_x^n[(1-x^2)^{n+\beta-\frac{1}{2}}]$ $C_n^{(\beta)} = \frac{(-1)^n 2^n\Gamma(2\beta+n)}{\Gamma(2\beta+2n)}$
Negative binomial $r > 0,\ 0 < p < 1$ $P(X=x) = p^r\binom{-r}{x}(-1)^x(1-p)^x,$ $x \in \mathbb{N}\cup\{0\}$	Meixner $M_n^{(r,p)}(x) = (-1)^n\frac{1}{p^n}\frac{\Gamma(x+1)}{\Gamma(x+r)}u(x),$ $u(x) =$ $(1-p)^{-x}\Delta_{x+}^n\left[\frac{\Gamma(x+r)}{\Gamma(x-n+1)}(1-p)^x\right]$

Measure	Szegö-Jacobi parameters
Gaussian $N(0,\sigma^2)$	$\alpha_n = 0$ $\omega_n = \sigma^2 n$ $(\lambda_n = \sigma^{2n} n!)$
Poisson Poi(a)	$\alpha_n = n + a$ $\omega_n = an$ $(\lambda_n = a^n n!)$
Gamma $\Gamma(\alpha)$, $(\alpha > 0)$ $\frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, \quad x > 0$	$\alpha_n = 2n + \alpha$ $\omega_n = n(n + \alpha - 1)$ $(\lambda_n = n!(n + \alpha - 1) \cdots \alpha)$
Uniform on $[-1, 1]$	$\alpha_n = 0$ $\omega_n = \frac{n^2}{(2n+1)(2n-1)}$ $(\lambda_n = \frac{(n!)^2}{[(2n-1)!!]^2 (2n+1)})$
Arcsine $\frac{1}{\pi\sqrt{1-x^2}}, \quad \lvert x\rvert < 1$	$\alpha_n = 0$ $\omega_n = \begin{cases} \frac{1}{2}, & n = 1 \\ \frac{1}{4}, & n \geq 2 \end{cases}$ $(\lambda_n = \frac{1}{2^{2n-1}})$
Semicircle $\frac{2}{\pi}\sqrt{1 - x^2}, \quad \lvert x\rvert < 1$	$\alpha_n = 0$ $\omega_n = \frac{1}{4}$ $(\lambda_n = \frac{1}{4^n})$
$\frac{1}{\sqrt{\pi}} \frac{\Gamma(\beta+1)}{\Gamma(\beta+\frac{1}{2})} (1 - x^2)^{\beta - \frac{1}{2}}, \quad \lvert x\rvert < 1$ $\beta > -\frac{1}{2}, \quad \beta \neq 0, 1$	$\alpha_n = 0$ $\omega_n = \frac{n(n+2\beta-1)}{4(n+\beta)(n+\beta-1)}$
Negative binomial $r > 0,\ 0 < p < 1$ $P(X = x) = p^r \binom{-r}{x} (-1)^x (1 - p)^x$, $x \in \mathbb{N} \cup \{0\}$	$\alpha_n = \frac{(2-p)n + r(1-p)}{p}$ $\omega_n = \frac{n(n+r-1)(1-p)}{p^2}$ $(\lambda_n = \frac{n!(1-p)^n (n+r-1) \cdots r}{p^{2n}})$

From the table below we can see that the operator a^0 is determined by the parameters $\{\alpha_n\}_{n\geq 0}$, while the commutator operator $[a^-, a^+]$ is determined by the numbers $\{\omega_n\}_{n\geq 1}$.

Measure	$[a^-, a^+]P_n$
Gaussian $N(0,\sigma^2)$	$\sigma^2 P_n$
Poisson Poi(a)	aP_n
Gamma $\Gamma(\alpha)$, $(\alpha > 0)$ $\frac{1}{\Gamma(\alpha)} x^{\alpha-1}e^{-x}$, $x > 0$	$(2n+\alpha)P_n$
Uniform on $[-1,1]$	$-\frac{1}{(2n+3)(2n+1)(2n-1)}P_n$
Arcsine $\frac{1}{\pi\sqrt{1-x^2}}$, $\|x\| < 1$	$\begin{cases} \frac{1}{2}P_0, & n=0 \\ -\frac{1}{4}P_1, & n=1 \\ 0, & n\geq 2 \end{cases}$
Semicircle $\frac{2}{\pi}\sqrt{1-x^2}$, $\|x\| < 1$	$\begin{cases} \frac{1}{4}P_0, & n=0 \\ 0, & n\geq 1 \end{cases}$
$\frac{1}{\sqrt{\pi}}\frac{\Gamma(\beta+1)}{\Gamma(\beta+\frac{1}{2})}(1-x^2)^{\beta-\frac{1}{2}}$, $\|x\| < 1$ $\beta > -\frac{1}{2}$, $\beta \neq 0,1$	$\frac{\beta^2-\beta}{2(n+1+\beta)(n+\beta)(n-1+\beta)}P_n$
Negative binomial $r > 0$, $0 < p < 1$ $P(X=x) = p^r\binom{-r}{x}(-1)^x(1-p)^x$, $x \in \mathbb{N}\cup\{0\}$	$\frac{(2n+r)(1-p)}{p^2}P_n$

10 Bibliography

[AKS07] Accardi, L., Kuo, H.-H., and Stan, A.:
Moments and commutators of probability measures,
Infinite Dimensional Analysis, Quantum Probability and Related Topics (2007) to appear

[AKS06] Accardi, L., Kuo, H.-H., and Stan, A.:
A combinatorial identity and its application to Gaussian measures
Levico Proc. to appear

[AKS05] Accardi, L., Kuo, H.-H., and Stan, A.:
Interacting Fock space characterization of probability measures

[AKS04] Accardi, L., Kuo, H.-H., and Stan, A.:
Characterization of probability measures through the cannonically associated Interacting Fock Spaces,
Infinite Dimensional Analysis, Quantum Probability and Related Topics 7 (2004) 485–505

[AcCrLu05] Accardi L., Crismale V., Lu Y.G.: Constructive universal central limit theorems based on interacting Fock spaces, Infinite Dimensional Analysis, Quantum Probability and Related Topics (IDA–QP) 8 (4) (2005) 631-650

[AcNah02] Accardi, L. and Nahni, M.:
Interacting Fock spaces and orthogonal polynomials in several variables;
in "Non-Commutativity, Infinite-Dimensionality and Probability at the Crossroads, N. Obata et al. (eds.), World Scientific (2002), 192–205.

[AcSk98]
Accardi L., Skeide M.:
Interacting Fock space versus full Fock module,
Volterra Preprint N. 328 (1998)

[AcBo98] Accardi, L. and Bożejko, M.:
Interacting Fock space and Gaussianization of probability measures,
Infinite Dimensional Analysis, Quantum Probability and Related Topics 1 (1998) 663–670

[AcLuVo97b] Accardi L., Lu Y.G., Volovich I.:
The QED Hilbert module and Interacting Fock spaces,
International institute for Advanced Studies (IIAS) Reports, Kyoto No. 1997-008 (1997)

[AcLu92b] Accardi L., Lu Y.G.:
The Wigner Semi–circle Law in Quantum Electro Dynamics,
Comm. Math. Phys. 180 (1996) 605–632, Volterra preprint N. 126 (1992) (same date as that of submission of the paper)

[AcLu92a] Accardi L., Lu Y.G.:
On the weak coupling limit for quantum electrodynamics,
Proceeding Intern. Workshop of Math. Phys., SIENA, F. Guerra, M. Loffredo (eds.) World Scientific (1992) 16–29 Volterra preprint N. 89 (1992)

[Cab–DuIo97] Cabanal–Duvillard T., Ionescu V.: Un théorème central limite pour des variables aléatoires non–commutatives, Probabilités/Probability Theory, C.R.Acad. Sci. Paris, t. 325, Série 1, pp. 1117–1120 (1997)

[Asai01] Asai N.: Analytic Characterization of One-mode Interacting Fock Space, IDA–QP 4 (2001) 409–415

[AsaKubKuo01] Asai N., Kubo I., Kuo H.–H.: Roles of log–concavity, log–convexity, and growth order in white noise analysis, Preprint (2001)

[Das05] Das P.K.: Dynamics of quantum mechanical feedback control system in interacting Fock space, Preprint (2005)

[Das03] Das P.K.: Evolution of the atom–field system in interacting Fock space, Preprint (2003) To appear in QP-25 conference Proceeding, World Scientific.

[HoraOb07] A. Hora and N. Obata:
Quantum Probability and Spectral Analysis of Graphs,
Springer Quantum Probability Programme (2007))

[Pim93] Pimsner, M.V.:
A class of C^*–algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$, axiomatization of a Fock Hilbert module in the abstract setting,
Preprint, Pennsylvania 1993, in: Fields Institute Communications (D.V. Voiculescu, ed.), Memoires of the American Mathematical Society

[Ske96b] M. Skeide: Hilbert modules in quantum electro dynamics and quantum probability, Commun. Math. Phys. 192 569–604 (1998) Volterra preprint N. 257 (1996)