**NONEQUILIBRIUM STEADY STATES FOR A HARMONIC OSCILLATOR INTERACTING WITH TWO BOSE FIELDS**
**- Stochastic limit approach and C* algebraic approach -**

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Recently, nonequilibrium steady states (NESS) are intensively studied by several methods. In this article, NESS of a harmonic oscillator interacting with two free-boson reservoirs at different temperatures and/or chemical potentials are studied by the stochastic limit approach and the C* algebraic approach. Their interrelation is investigated as well.

## 1 Introduction

The understanding of irreversible phenomena including nonequilibrium steady states (NESS) is a longstanding problem of statistical mechanics. Various theories have been developed so far[1]. One of promising approaches deals with infinitely extended dynamical systems[2,3,4]. Not only equilibrium properties, but also nonequilibrium properties have been rigorously investigated. The latter include analytical studies of nonequilibrium steady states, e.g., of harmonic crystals[5,6], a one-dimensional gas[7], unharmonic chains[8], an isotropic XY-chain[9], systems with asymptotic abelianness[10,11], a one-dimensional quantum conductor[12], an interacting fermion-spin system[13] and fermionic junction systems[14]. Entropy production has been rigorously studied as well (see also Refs.15-19 and the references therein).

The stochastic limit approach[20,21] is a generalization of the scaling limit theory[22]. There, the reduced density matrices obey Pauli's master equation and the external degrees of freedom turn out to be quantum white noises. Recently, this approach was extended to systems arbitrarily far from equilibrium[23,24] and would be a new promising tool to deal with nonequilibrium properties. However, the relation between the two appraches is not clear and, in this paper, we compare them for a harmonic oscillator linearly coupled with two harmonic reservoirs at different temperatures and/or chemical potentials.

The paper is arranged as follows: In the next section, the model is described. In Sec. 3, the results of the stochastic limit approach is reviewed. In Sec. 4, nonequilibrium steady states (NESS) are derived with the aid of the

C*-algebraic approach. In Sec. 5, their weak coupling limits are investigated. The last section is devoted to the summary.

## 2 Model

To illustrate the relation between the C*-algebraic and stochastic limit approaches, we consider a harmonic oscillator linearly coupled with two harmonic reservoirs[24], which is a bosonic junction system.

The model is defined on a tensor product of a Hilbert space $L^2(\mathbf{R})$ and two Fock spaces $\mathcal{H}_n$ $(n = 1, 2)$, both of which are constructed from $L^2(\mathbf{R}^3)$: $\mathcal{H} \equiv L^2(\mathbf{R}) \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$. In terms of standard annihilation operators $a$ and $a_{n,k}$ satisfying canonical commutation relations:

$$[a, a^\dagger] = 1, \quad [a_{n,k}, a^\dagger_{n',k'}] = \delta_{n,n'}\delta(k - k') \ , \tag{1}$$

the Hamiltonian is given by

$$H = H_0 + \lambda \sum_{n=1,2} V_n \tag{2}$$

where

$$H_0 = \Omega a^\dagger a + \sum_{n=1,2} \int dk \omega_k a^\dagger_{n,k} a_{n,k} \ , \tag{3}$$

$$V_n = \int dk \left( g_n^*(k) a^\dagger a_{n,k} + g_n(k) a a^\dagger_{n,k} \right) \ . \tag{4}$$

In the above, $\omega_k = |k|^2$, $k$-integrations are taken over $\mathbf{R}^3$ and the functions $g_n(k)$ $(n = 1, 2)$ are square integrable. Strictly speaking, the free field parts $\int dk \omega_k a^\dagger_{n,k} a_{n,k}$ $(n = 1, 2)$ should be understood as the second quantization of the multiplicative self-adjoint operator $\varphi(k) \to \omega_k \varphi(k)$ defined on the Fourier transform of $L^2(\mathbf{R}^3)$ to the Fock spaces $\mathcal{H}_n$ $(n = 1, 2)$.

At initial states, the two fields and the harmonic oscillator are assumed to be independent of each other. Moreover, the two fields are assumed to obey Gibbsian distributions with different temperatures $\beta_1^{-1}, \beta_2^{-1}$ and chemical potentials $\mu_1, \mu_2$. Namely, one has

$$\omega_0(a^\dagger_{n,k} a_{n',k'}) = \delta_{n,n'} \mathcal{N}_n(\omega_k) \delta(k - k') \ , \tag{5}$$

where

$$\mathcal{N}_1(\omega_k) = \frac{1}{e^{\beta_1(\omega_k - \mu_1)} - 1}, \quad \mathcal{N}_2(\omega_k) = \frac{1}{e^{\beta_2(\omega_k - \mu_2)} - 1}. \tag{6}$$

In this paper, we only consider the case where $\mathcal{N}_n(\omega_k)$ $(n = 1, 2)$ are bounded.

Provided $g_n(-k) = g_n(k)^*$, the system is invariant under the time reversal operation $\iota$, which is an antilinear *-morphism such that $\iota^2$ is identity and it satisfies

$$\iota(a) = a \ , \quad \iota(a_{n,k}) = a_{n,-k} \ . \tag{7}$$

Under the time evolution generated by $H$, the mass and energy are conserved. Indeed, in the Heisenberg representation, one has

$$\frac{d}{dt} a^\dagger a = \sum_{n=1,2} J_n \ , \tag{8}$$

where $J_n$ $(n = 1, 2)$ are given by

$$J_n = i\lambda \int dk \left( g_n(k) a^\dagger_{n,k} a - g_n(k)^* a^\dagger a_{n,k} \right) \ , \tag{9}$$

and they correspond to the mass flows from the reservoirs since formally $J_n = -\frac{d}{dt} \int dk a^\dagger_{n,k} a_{n,k}$ holds. Similarly, the energy conservation holds:

$$\frac{d}{dt} \Omega a^\dagger a = \sum_{n=1,2} \Omega J_n \ , \tag{10}$$

$$\frac{d}{dt} V_n = J^\epsilon_n - \Omega J_n \ , \tag{11}$$

where $J^\epsilon_n$ $(n = 1, 2)$ are given by

$$J^\epsilon_n = i\lambda \int dk \omega_k \left( g_n(k) a^\dagger_{n,k} a - g_n(k)^* a^\dagger a_{n,k} \right) \ , \tag{12}$$

and they correspond to the energy flows from the reservoirs since formally $J^\epsilon_n = -\frac{d}{dt} \int dk \omega_k a^\dagger_{n,k} a_{n,k}$ holds. From Eqs.(10) and (11), one finds that $J^\epsilon_1$, $\Omega J_1$, $-\Omega J_2$ and $-J^\epsilon_2$ correspond, respectively, to the energy flows from the first reservoir to the first joint, from the first joint to the oscillator, from the oscillator to the second joint and from the second joint to the second reservoir.

## 3 Stochastic Limit Approach

Here we summarize the results of the stochastic limit approach obtained by Accardi, Imafuku and Lu[24].

In the stochastic limit approach[20,21], one studies the rescaled evolution operator in the interaction picture

$$U^{(\lambda)}_{t/\lambda^2} \equiv \exp\left(itH_0/\lambda^2\right) \exp\left(-itH/\lambda^2\right) \ , \tag{13}$$

which is shown to converge, in the sense of correlations under appropriate assumptions on the model[21], to the solution of

$$\frac{d}{dt}U_t = -ih_t U_t, \quad U(0) = 1 \tag{14}$$

where

$$h_t = \lim_{\lambda\to 0}\frac{1}{\lambda}\exp\left(itH_0/\lambda^2\right)\sum_{n=1,2} V_n \exp\left(-itH_0/\lambda^2\right) \ . \tag{15}$$

In the present case, two steps are necessary to write down (14). Firstly, in order to deal with the finite temperature situation, field operators $a_{n,k}$ are represented in terms of thermal fields $\{\xi_k^{(n)}, \tilde{\xi}_k^{(n)}\}$[21] as

$$a_{n,k} = \sqrt{1+\mathcal{N}_n(\omega_k)}\xi_k^{(n)} + \sqrt{\mathcal{N}_n(\omega_k)}\tilde{\xi}_k^{(n)\ \dagger} \tag{16}$$

$$a_{n,k}{}^\dagger = \sqrt{1+\mathcal{N}_n(\omega_k)}\xi_k^{(n)\ \dagger} + \sqrt{\mathcal{N}_n(\omega_k)}\tilde{\xi}_k^{(n)} \tag{17}$$

where $\xi_k^{(n)}$ and $\tilde{\xi}_k^{(n)}$ satisfy the commutation relations:

$$[\xi_k^{(n)}, \xi_{k'}^{(n')\ \dagger}] = \delta_{n,n'}\delta(k-k') \ , \quad [\tilde{\xi}_k^{(n)}, \tilde{\xi}_{k'}^{(n')\ \dagger}] = \delta_{n,n'}\delta(k-k') \ ,$$

and the initial state is represented as the vacuum state with respect to $\xi_k^{(n)}$ and $\tilde{\xi}_k^{(n)}$ $(n = 1, 2)$. Secondly, the stochastic limit is taken and one obtains the evolution equation:

$$\frac{d}{dt}U_t = -i\sum_{n=1,2}\left\{a\left(c_t^{(n)\ \dagger} + d_t^{(n)}\right) + a^\dagger\left(c_t^{(n)} + d_t^{(n)\ \dagger}\right)\right\}U_t \ , \tag{18}$$

where $c_t^{(n)}$ and $d_t^{(n)}$ are quantum white noises defined by

$$c_t^{(n)} = \lim_{\lambda\to 0}\frac{1}{\lambda}\int dk\ \sqrt{1+\mathcal{N}_n(\omega_k)}g_n(k)\xi_k^{(n)}e^{-i\frac{t}{\lambda^2}(\omega_k-\Omega)} \tag{19}$$

$$d_t^{(n)} = \lim_{\lambda\to 0}\frac{1}{\lambda}\int dk\ \sqrt{\mathcal{N}_n(\omega_k)}g_n^*(k)\tilde{\xi}_k^{(n)}e^{+i\frac{t}{\lambda^2}(\omega_k-\Omega)}. \tag{20}$$

Based on the evolution equation (18), Accardi, Imafuku and Lu[24] studied nonequilibrium steady states and found

(i) The reduced density matrix of NESS is given by a function of the system Hamiltonian $\Omega a^\dagger a$:

$$\rho_{\rm sys} = \frac{1}{Z}e^{-\beta'\Omega a^\dagger a} \tag{21}$$

where the parameter $\beta'$ is given by

$$\beta' = \frac{1}{\Omega} \log \left( \frac{\gamma^{(1)} \frac{e^{\beta_1(\Omega-\mu_1)}}{e^{\beta_1(\Omega-\mu_1)}-1} + \gamma^{(2)} \frac{e^{\beta_2(\Omega-\mu_2)}}{e^{\beta_2(\Omega-\mu_2)}-1}}{\gamma^{(1)} \frac{1}{e^{\beta_1(\Omega-\mu_1)}-1} + \gamma^{(2)} \frac{1}{e^{\beta_2(\Omega-\mu_2)}-1}} \right) , \qquad (22)$$

and

$$\gamma^{(n)} = \pi \int dk |g_n(k)|^2 \delta\left(\omega_k - \Omega\right) \ . \qquad (23)$$

(ii) NESS carries nonvanishing mass and energy flows, which are consistent with the second law of thermodynamics. Particularly, the mass flow is

$$\langle \hat{J}(+\infty) \rangle = 2 \frac{\gamma^{(1)}\gamma^{(2)}}{\gamma^{(1)} + \gamma^{(2)}} \left( \frac{1}{e^{\beta_1(\Omega-\mu_1)} - 1} - \frac{1}{e^{\beta_2(\Omega-\mu_2)} - 1} \right) , \qquad (24)$$

where the mass flow operator $\hat{J}$ is defined as[a]

$$\hat{J}(t) = \frac{1}{2} \frac{d}{dt} \left\{ U_{-t}(N_2 - N_1) U_{-t}^\dagger \right\} , \qquad (25)$$

with $N_n$ $(n = 1, 2)$ the number operator corresponding to $\int dk a_{n,k}^\dagger a_{n,k}$.

As the system Hamiltonian is time reversal symmetric and the flows are not, the finding (i) implies that the reduced NESS is time-reversal symmetric, while the second finding (ii) asserts that NESS is not. However, as we will see, the two observations are both valid and imply the necessity of a careful treatment of observables in the stochastic limit.

## 4 Nonequilibrium Steady States in C*-algebraic Approach

In the C*-algebraic approach, the Heisenberg time evolution is considered and steady states are obtained as the weak limits of the initial states[10]. Although the transports of fermionic junction systems were studied in detail by Fröhlich, Merkli and Ueltschi[14], the argument cannot be applied to the bosonic case. To proceed, we additionally assume the followings:

(A) The initial state satisfies

$$|\omega_0(a^{\natural_1} a^{\natural_2} \cdots a^{\natural_n})| \leq n! K_n \qquad (26)$$

where $a^{\natural_j} = a$ or $a^\dagger$ and $K_n(> 0)$ satisfies $\lim_{n\to\infty} K_{n+1}/K_n = 0$.

[a] In Ref. 24, $2\hat{J}(t)$ was identified with the mass flow. However, because of the mass conservation (8), $\hat{J}(t)$ should be identified with the mass flow. See the arguments in the next section.

(B) The form factor $g_n(k)$ is a function of $|k|$ and

$$\frac{\gamma^{(n)}(\omega)}{\pi} = \int dk|g_n(k)|^2\delta(\omega_k-\omega) = \begin{cases} 2\pi\sqrt{\omega}|g_n(\sqrt{\omega})|^2 & (\omega \geq 0) \\ 0 & (\omega < 0) \end{cases} , \qquad (27)$$

is in $L^2(\mathbf{R})$ and uniformly Hölder continuous with index $\alpha \in (0,1)$, i.e.,

$$\left|\gamma^{(n)}(x)-\gamma^{(n)}(y)\right| \leq K|x-y|^\alpha \quad ({}^\exists K>0)$$

(C) There exists no real solution for $\eta(z)=0$, where

$$\eta(z) \equiv z-\Omega-\lambda^2\sum_{n'=1,2}\int dk'\frac{|g_{n'}(k')|^2}{z-\omega_{k'}} , \qquad (28)$$

and $1/\eta_-(\omega) \equiv 1/\eta(\omega-i0)$ $(\omega \in \mathbf{R})$ is bounded.

The assumptions (A) and (B) are posed in order to simplify the investigation, while the first half of the assumption (C) plays an essential role as it guarantees the existence of the steady states.

First we note that the Hilbert space $\mathcal{H}$, where our model is defined, is a boson Fock space over a Hilbert space

$$\ell^2 \equiv \left\{ f = \begin{pmatrix} c \\ \psi_1(k) \\ \psi_2(k) \end{pmatrix} : |c|^2 + \sum_{n=1,2}\int dk|\psi_n(k)|^2 < +\infty \right\} ,$$

equipped with the inner product

$$(f,\widetilde{f}) = c^*\widetilde{c} + \sum_{n=1,2}\int dk\psi_n(k)^*\widetilde{\psi}_n(k) .$$

The CCR algebra $\mathcal{A}$ is, then, generated by Weyl operators (see Theorem 5.2.8 of Ref. 2 ):

$$W(f) = \exp(i\Phi(f)) \qquad (29)$$

where $\Phi$ is a map from the Hilbert space $\ell^2(\ni f = (c,\psi_1,\psi_2))$ to the space $(\ni \Phi(f))$ of (unbounded) operators on $\mathcal{H}$ and is formally defined as $\Phi(f) = \Phi_S(f) + \sum_{n=1,2}\Phi_{b,n}(f)$ with

$$\Phi_S(f) = \frac{1}{\sqrt{2}}\overline{\{c^*a+ca^\dagger\}} \qquad (30)$$

$$\Phi_{b,n}(f) = \overline{\int\frac{dk}{\sqrt{2}}\left(\psi_n(k)^*a_{n,k}+\psi_n(k)a^\dagger_{n,k}\right)} . \qquad (31)$$

In the above, the overline stands for the closure. In other words, any element of $\mathcal{A}$ can be approximated with arbitrary precision by a finite linear combination of finite products of Weyl operators. On the other hands, by repeatedly applying the identity (Theorem 5.2.4 of Ref. 2 ): $W(f)W(g) = W(f + g)\exp(-i\mathrm{Im}(f,g))$, a product of Weyl operators is found to reduce to a single Weyl operator. Therefore, any element of $\mathcal{A}$ can be approximated with arbitrary precision by a finite linear combination of Weyl operators and, in order to investigate the time evolution of the states, it is enough to investigate the average values of the Weyl operators.

One, then, obtains the following proposition and corollary, which are proved in the rest of this section.

**Proposition**: For the Weyl operator, we have

$$\begin{aligned}\lim_{t\to+\infty}\omega_0\left\{\tau_t\left(W(f)\right)\right\} &= \exp\left(-\frac{1}{2}\sum_{n=1,2}\int dk\,|\varphi_n(k,f)|^2\left\{\mathcal{N}_n(\omega_k)+\frac{1}{2}\right\}\right)\\ &\equiv \omega_{+\infty}\left(W(f)\right)\end{aligned}\tag{32}$$

where $\varphi_n(k,f)$ is given by

$$\varphi_n(k,f) = \psi_n(k) + \frac{\lambda g_n(k)}{\eta_-(\omega_k)}\left\{c + \lambda\sum_{n'=1,2}\int dk'\,\frac{g^*_{n'}(k')\psi_{n'}(k')}{\omega_k-\omega_{k'}-i0}\right\}\ . \tag{33}$$

This implies that NESS $\omega_{+\infty}$ exists and that it is quasi-free with a two-point function:

$$\omega_{+\infty}\left(\Phi(f)^2\right) = \sum_{n=1,2}\int dk\,|\varphi_n(k,f)|^2\left\{\mathcal{N}_n(\omega_k)+\frac{1}{2}\right\}\ . \tag{34}$$

**Corollary**

(i) The average values of the harmonic oscillator variables are

$$\omega_{+\infty}(a^\dagger a) = \sum_{n=1,2}\int dk\frac{\lambda^2|g_n(k)|^2}{|\eta_+(\omega_k)|^2}\mathcal{N}_n(\omega_k)\ ,\qquad \omega_{+\infty}(aa)=0\ .$$

This implies that the reduced state is described by the density matrix:

$$\begin{aligned}\rho^\lambda_{\mathrm{sys}} &\equiv \frac{1}{Z_\lambda}e^{-\beta_\lambda\Omega a^\dagger a}\\ \beta_\lambda &= \frac{1}{\Omega}\log\frac{\sum_{n=1,2}\int dk\frac{\lambda^2|g_n(k)|^2}{|\eta_+(\omega_k)|^2}\left\{\mathcal{N}_n(\omega_k)+1\right\}}{\sum_{n=1,2}\int dk\frac{\lambda^2|g_n(k)|^2}{|\eta_+(\omega_k)|^2}\mathcal{N}_n(\omega_k)}\end{aligned}\tag{35}$$

where $Z_\lambda$ is the normalization constant.

(ii) The average mass flow is

$$\omega_{+\infty}(J_1) = -\omega_{+\infty}(J_2)$$
$$= 2\pi\lambda^4 \int dk \int dk' \frac{|g_1(k)|^2 |g_2(k')|^2}{|\eta_+(\omega_k)|^2} \delta(\omega_k - \omega_{k'}) \{\mathcal{N}_1(\omega_k) - \mathcal{N}_2(\omega_k)\} \ .$$

### *4.1 Time Evolution of Weyl Operators*

Note that the Hamiltonian $H$ is the second quantization of the Hamiltonian $h$ densely defined on $\ell^2$:

$$hf \equiv h \begin{pmatrix} c \\ \psi_1(k) \\ \psi_2(k) \end{pmatrix} = \begin{pmatrix} \Omega c + \lambda \sum_{n=1,2} \int dk g_n(k)^* \psi_n(k) \\ \omega_k \psi_1(k) + \lambda g_1(k) c \\ \omega_k \psi_2(k) + \lambda g_2(k) c \end{pmatrix} \ .$$

The group $\tau_t$ of time-evolution automorphisms generated by $H$ satisfies (cf. the argument before Proposition 5.2.27 of Ref. 2 )

$$\tau_t\left(W(f)\right) = W\left(e^{iht} f\right) \ ,$$

and, under the condition (C), one has

$$e^{iht} f = \begin{pmatrix} c(t) \\ \psi_1(k,t) \\ \psi_2(k,t) \end{pmatrix} \ , \tag{36}$$

where

$$c(t) = \sum_{n=1,2} \int dk \frac{\lambda g_n^*(k)}{\eta_+(\omega_k)} e^{i\omega_k t} \varphi_n(k,f)$$
$$\psi_n(k,t) = e^{i\omega_k t} \varphi_n(k,f) + \sum_{n'=1,2} \int dk' \frac{\lambda^2 g_n(k) g_{n'}^*(k') e^{i\omega_{k'} t} \varphi_{n'}(k',f)}{\eta_+\left(\omega_{k'}\right)\left(\omega_{k'} - \omega_k + i0\right)}$$

Then, since the two fields and the harmonic oscillator are independent at the initial state, the average value of the Weyl operator at time $t$ is evaluated as

$$\omega_0\left(\tau_t\left(W(f)\right)\right) = \omega_0\left(\exp\left(i\Phi_S(e^{iht} f)\right)\right) \prod_{n=1,2} \omega_0\left(\exp\left(i\Phi_{b,n}(e^{iht} f)\right)\right) \tag{37}$$

### 4.2 *On the limit:* $\lim_{t\to\pm\infty}\omega_0\left(\exp\left(i\Phi_S(e^{iht}f)\right)\right)$

As shown in Appendix A, $\eta_+(\omega)$ is continuous and $\varphi_n(k,f)\in L^2(\mathbf{R}^3)$ under the assumption (B). Because of $\omega_k=k^2$ and $g_n(k)=g_n(\sqrt{\omega})$, $c(t)$ can be rewritten as

$$c(t)=\frac{\lambda}{2}\sum_{n=1,2}\int_0^\infty d\omega e^{i\omega t}v_n(\omega)\ , \tag{38}$$

$$v_n(\omega)=\frac{\sqrt{\omega}g_n^*(\sqrt{\omega})}{\eta_+(\omega)}\left[\int d\hat{k}\ \varphi_n(k,f)\right]_{|k|=\sqrt{\omega}}\ , \tag{39}$$

where $d\hat{k}$ stands for the angular integral. Since $g_n(k)$ and $\varphi_n(k,f)$ are $L^2(\mathbf{R}^3)$, one has

$$\begin{aligned}\bar{c}_n &\equiv \int_0^\infty d\omega\,|v_n(\omega)| \\ &\leq \sup_{0<\omega}\frac{2}{|\eta_+(\omega)|}\sqrt{\int dk|g_n(k)|^2}\sqrt{\int dk|\varphi_n(k,f)|^2}<+\infty\ ,\end{aligned}$$

or $v_n\in L^1(\mathbf{R}^+)$. Hence the Riemann-Lebesgue theorem[25] leads to $\lim_{t\to+\infty}c(t)=0$.

The assumption (A) implies

$$\left|\omega_0\left(\Phi_S(e^{iht}f)^m\right)\right|\leq m!(2|c(t)|)^mK_m\ , \tag{40}$$

and, because $|c(t)|\leq\lambda(\bar{c}_1+\bar{c}_2)/2\equiv\bar{c}$,

$$\begin{aligned}&\left|\omega_0\left(\exp\left(i\Phi_S(e^{iht}f)\right)\right)-1\right|\leq\sum_{m=1}^\infty\frac{1}{m!}\left|\omega_0\left(\Phi_S(e^{iht}f)^m\right)\right| \\ &\leq\sum_{m=1}^\infty(2|c(t)|)^mK_m\leq\frac{|c(t)|}{\bar{c}}\sum_{m=1}^\infty(2\bar{c})^mK_m\to 0\quad(\text{as } t\to+\infty)\ .\end{aligned}$$

The sum in the right-hand-side is finite as a result of $\lim_{j\to+\infty}K_{j+1}/K_j=0$.

### 4.3 *On the limit:* $\lim_{t\to\pm\infty}\omega_0\left(\exp\left(i\Phi_{b,n}(e^{iht}f)\right)\right)$

Since the initial state $\omega_0$ is quasi-free with respect to reservoir degrees of freedom, we have

$$\omega_0\left(\exp\left(i\Phi_{b,n}(e^{iht}f)\right)\right)=\exp\left\{-\omega_0\left(\Phi_{b,n}(e^{iht}f)^2\right)/2\right\}\ , \tag{41}$$

and

$$\omega_0\left(\Phi_{b,n}(e^{iht}f)^2\right) = \int dk\,|\varphi_n(k,f)|^2\left\{\mathcal{N}_n(\omega_k)+\frac{1}{2}\right\}$$
$$+2\lambda^2\mathrm{Re}I_n^{(1)}(t)+\lambda^4 I_n^{(2)}(t)\ ,$$

where

$$I_n^{(1)}(t) \equiv \int dk\varphi_n(k,f)^* g_n(k) I(\omega_k;t)\left\{\mathcal{N}_n(\omega_k)+\frac{1}{2}\right\}$$
$$I_n^{(2)}(t) \equiv \int dk|g_n(k)|^2|I(\omega_k;t)|^2\left\{\mathcal{N}_n(\omega_k)+\frac{1}{2}\right\}\ ,$$
$$I(\omega_k;t) = \sum_{n'=1,2}\int dk'\,\frac{g_{n'}^*(k')e^{i(\omega_{k'}-\omega_k)t}\varphi_{n'}(k',f)}{\eta_+(\omega_{k'})(\omega_{k'}-\omega_k+i0)}\ .$$

The time-dependent terms $I_n^{(j)}(t)$ $(j=1,2)$ are shown to vanish in the limit of $t\to+\infty$ with the aid of the following Lemma. Its proof is given in Appendix B.

**Lemma 1** If $w_n\in L^1(\mathbf{R}^+)\cap L^2(\mathbf{R}^+)$ $(n=1,2)$, then

$$\lim_{t\to+\infty}\int_0^\infty d\omega w_1^*(\omega)\int_0^\infty d\omega'\,\frac{w_2(\omega')e^{i(\omega'-\omega)t}}{\omega-\omega'-i0} = 0 \tag{42}$$

As easily seen, one has

$$I_n^{(1)}(t) = -\frac{1}{4}\sum_{m=1,2}\int_0^\infty d\omega u_n(\omega)^*\int_0^\infty d\omega'\,\frac{v_m(\omega')e^{i(\omega'-\omega)t}}{\omega-\omega'-i0}$$

where $v_m(\omega)$ is given by (39) and

$$u_n(\omega) = \sqrt{\omega}\ g_n^*(\sqrt{\omega})\left\{\mathcal{N}_n(\omega)+\frac{1}{2}\right\}\left[\int d\hat{k}\ \varphi_n(k,f)\right]_{|k|=\sqrt{\omega}}$$

In the previous subsection, we have seen $v_m\in L^1(\mathbf{R}^+)$. On the other hand,

$$\int_0^\infty d\omega|v_m(\omega)|^2 \le \sup_{\omega>0}\frac{4}{|\eta_+(\omega)|}\left(\sup_{\omega>0}\frac{\gamma^{(m)}(\omega)}{\pi}\right)\int dk|\varphi_m(k,f)|^2 < +\infty$$

or $v_m\in L^2(\mathbf{R}^+)$. Exactly in the same way, one has $u_n\ \in\ L^1(\mathbf{R}^+)\cap L^2(\mathbf{R}^+)$. Hence, Lemma 1 implies $I_n^{(1)}(t)\to 0$ (as $t\to+\infty$).

Moreover, one has

$$I_n^{(2)}(t) = 2\pi \int_0^\infty d\omega \sqrt{\omega} \left| g_n(\sqrt{\omega}) \right|^2 \left\{ \mathcal{N}_n(\omega) + \frac{1}{2} \right\} |I(\omega;t)|^2$$
$$\leq \sup_{0<\omega} \frac{\gamma^{(n)}(\omega)}{\pi} \sup_{0<\omega} \left\{ \mathcal{N}_n(\omega) + \frac{1}{2} \right\} \int_{-\infty}^{+\infty} d\omega \, |I(\omega;t)|^2$$

and

$$\int_{-\infty}^{+\infty} d\omega \, |I(\omega;t)|^2 = \frac{\pi}{2i} \sum_{n,m=1,2} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \frac{v_m(\omega_1) v_n^*(\omega_2)}{\omega_2 - \omega_1 - i0} e^{i(\omega_1 - \omega_2)t} \ .$$

Then, because of $v_m, \ v_n \ \in \ L^1(\mathbf{R}^+) \ \cap \ L^2(\mathbf{R}^+)$ and Lemma 1, we have

$$\lim_{t \to +\infty} \int_{-\infty}^{+\infty} d\omega \, |I(\omega;t)|^2 = 0$$

and, thus, $I_n^{(2)}(t) \to 0$ (for $t \to +\infty$).

In short, we derive

$$\lim_{t \to +\infty} \omega_0 \left( \Phi_{b,n}(e^{iht} f)^2 \right) = \int dk \, |\varphi_n(k,f)|^2 \left\{ \mathcal{N}_n(\omega_k) + \frac{1}{2} \right\} \ , \tag{43}$$

which implies the desired results.

## 5 Weak coupling limits

In order to compare the results of the C*-algebraic approach with those of the stochastic limit approach, we consider weak limits of the former with the aid of the following lemma:

**Lemma 2** Suppose $\sum_{n=1,2} \gamma^{(n)}(\omega)$ is uniformly Hölder continuous on $\mathbf{R}$, square integrable on $\mathbf{R}^+$ and $\sum_{n=1,2} \gamma^{(n)}(\Omega) > 0$, then, for any continuous bounded function $F(\omega)$,

$$\lim_{\lambda \to 0} \sum_{n=1,2} \int dk \frac{\lambda^2 |g_n(k)|^2}{|\eta_-(\omega_k)|^2} F(\omega_k) = F(\Omega) \tag{44}$$

Lemma 2 gives

$$\lim_{\lambda \to 0} \int \frac{\lambda^2 |g_n(k)|^2}{|\eta_+(\omega_k)|^2} \mathcal{N}_n(\omega_k) = \lim_{\lambda \to 0} \sum_{m=1,2} \int \frac{\lambda^2 |g_m(k)|^2}{|\eta_+(\omega_k)|^2} \frac{|g_n(k)|^2 \mathcal{N}_n(\omega_k)}{\sum_{m'=1,2} |g_{m'}(k)|^2}$$
$$= \frac{\gamma^{(n)}}{\gamma^{(1)} + \gamma^{(2)}} \mathcal{N}_n(\Omega) \ ,$$

where $|g_n(k)|^2_{k^2=\Omega} = |g_n(\sqrt{\Omega})|^2 = \gamma^{(n)}/(2\pi^2\sqrt{\Omega})$ is used. Then, Corollary (i) leads to

$$\lim_{\lambda\to 0}\rho^\lambda_{\rm sys} = \frac{1}{Z}e^{-\beta'\Omega a^\dagger a}$$
$$\beta' = \lim_{\lambda\to 0}\beta_\lambda = \frac{1}{\Omega}\log\frac{\sum_{n=1,2}\gamma^{(n)}\left\{\mathcal{N}_n(\Omega)+1\right\}}{\sum_{n=1,2}\gamma^{(n)}\mathcal{N}_n(\Omega)}$$
$$Z = \lim_{\lambda\to 0}Z_\lambda\ ,$$

which agrees with the the reduced distribution (21)-(22) derived by the stochastic limit approach.

Next we consider the flows. From Corollary (ii), one finds that the limit of $\lambda \to 0$ leads to a contradictory result, i.e., the absence of the flow

$$\lim_{\lambda\to 0}\omega_{+\infty}(J_1) = 0\ . \tag{45}$$

This is, however, consistent with the physical situation: When the system-reservoir interactions are vanishingly weak, the flows induced by them are vanishingly small and the accumulation over a longer time interval is necessary to have observable values. Indeed, well-defined weak-coupling limit of the flow can be obtained after dividing them by a factor of $\lambda^2$:

$$\lim_{\lambda\to 0}\omega_{+\infty}(J_1)/\lambda^2 = \frac{2\gamma^{(1)}\gamma^{(2)}}{\gamma^{(1)}+\gamma^{(2)}}\left\{\mathcal{N}_1(\Omega)-\mathcal{N}_2(\Omega)\right\}\ . \tag{46}$$

This limit agrees with NESS flow (24) derived by the stochastic limit approach. Note that the division by $\lambda^2$ precisely corresponds to the scaling of the time variable: $t \to \lambda^2 t$.

In short, the stochastic limit approach successfully gives the weak coupling limits of NESS averages of approriately rescaled observables.

## 6 Summary and Discussion

We have compared the nonequilibrium steady states obtained by the stochastic limit approach and the C*-algebraic approach and have shown that the stochastic limit approach does lead to the weak coupling limits of NESS averages of appropriately rescaled observables.

We note that the apparent inconsistency of the results addressed at the end of Sec.3 is not a problem. Indeed, as shown in Corollary (i), the exact reduced density matrix can be time-reversal symmetric even though the original state is not.

The reason of the time-reversal symmetry of the exact reduced density matrix $\rho_{\rm sys}^\lambda$ can be understood as follows. From Corollary, the steady state $\omega_{+\infty}$ and, hence, the reduced state are invariant under the gauge transformation. As the reduced state is quasi-free, it can be expressed as an exponential function of a bilinear form of the creation and annihilation operators of the oscillator. On the other hand, $a^\dagger a$ is the only quadratic operator which is invariant under the gauge transformation. Therefore, the reduced density matrix $\rho_{\rm sys}^\lambda$ should be an exponential function of $a^\dagger a$ and, thus, is time-reversal symmetric.

Here a remark is in order. Generally speaking, the reduced density matrix *is not* time-reversal symmetric when the original state is not, but its weak coupling limit *is* time-reversal symmetric[24]. For example, when one reduces the state onto the subalgebra generated by $a$ and

$$b \equiv \frac{1}{\alpha}\int dk g_1^*(k) a_{1,k} \qquad \left(\alpha^2 \equiv \int dk |g_1(k)|^2 : \text{ the normalization constant}\right) ,$$

the reduced density matrix is given by

$$\rho_{\rm red} = \frac{1}{Z_{\rm red}} \exp\left(-\beta_1 a^\dagger a - \beta_2 b^\dagger b - \beta_{12} a^\dagger b - \beta_{12}^* b^\dagger a\right)$$

where $Z_{\rm red}$ is the normalization constant and the real parameters $\beta_1$, $\beta_2$ and a complex parameter $\beta_{12}$ are related to the correlation matrix $C$ as

$$\begin{pmatrix} \beta_1 \ , & \beta_{12} \\ \beta_{12}^* \ , & \beta_2 \end{pmatrix} = \log(E + C^{-1}) \ ,$$
$$C \equiv \begin{pmatrix} \omega_{+\infty}(a^\dagger a) \ , & \omega_{+\infty}(a^\dagger b) \\ \omega_{+\infty}(b^\dagger a) \ , & \omega_{+\infty}(b^\dagger b) \end{pmatrix} ,$$

with $E$ the 2×2 unit matrix. The matrix elements of $C$ are given by

$$\omega_{+\infty}(a^\dagger a) = \sum_{m=1,2} \int dk \frac{\lambda^2 |g_m(k)|^2}{|\eta_+(\omega_k)|^2} \mathcal{N}_m(\omega_k)$$
$$\omega_{+\infty}(b^\dagger a) = \sum_{m=1,2} \int dk \frac{\lambda |g_m(k)|^2}{\alpha \eta_+(\omega_k)} \mathcal{N}_m(\omega_k) \left\{\delta_{1,m} + \frac{\lambda^2}{\eta_-(\omega_k)} \int dk' \frac{|g_1(k')|^2}{\omega_k - \omega_{k'} - i0}\right\}$$
$$\omega_{+\infty}(b^\dagger b) = \sum_{m=1,2} \int dk \frac{|g_m(k)|^2}{\alpha^2} \mathcal{N}_m(\omega_k) \left|\delta_{1,m} + \frac{\lambda^2}{\eta_-(\omega_k)} \int dk' \frac{|g_1(k')|^2}{\omega_k - \omega_{k'} - i0}\right|^2$$

Since the symmetry of $g_1(k)$ leads to $\iota(b) = b$, the reduced density matrix is time-reversal symmetric if and only if $\beta_{12}$ is real. As easily seen, this condition is equivalent to

$$\beta_{12}^* - \beta_{12} = \omega_{+\infty}(b^\dagger a) - \omega_{+\infty}(a^\dagger b) = -i\frac{\omega_{+\infty}(J_1)}{\lambda\alpha} = 0$$

where $J_1$ stands for the mass flow operator (9). Therefore, when the steady state admits nonvanishing flow, the reduced state $\rho_{\rm red}$ is not time-reversal symmetric. However, in the weak coupling limit, the correlation matrix reduces to

$$\lim_{\lambda\to 0} C = \begin{pmatrix} 1/\{e^{\beta'\Omega} - 1\} \ , & 0 \\ 0 \ , & \int dk |g_1(k)|^2 \mathcal{N}_1(\omega_k)/\alpha^2 \end{pmatrix}$$

which corresponds to the time-reversal symmetric reduced state:

$$\widetilde{\rho}_{\rm red} = \frac{1}{\widetilde{Z}_{\rm red}} \exp\left(-\beta'\Omega a^\dagger a - \tilde{\beta}_2 b^\dagger b\right)$$

where $\widetilde{Z}_{\rm red}$ is the normalization constant, $\beta'$ is given by (22) and

$$\tilde{\beta}_2 = \log \frac{\int dk |g_1(k)|^2 \{\mathcal{N}_1(\omega_k) + 1\}}{\int dk |g_1(k)|^2 \mathcal{N}_1(\omega_k)} \ .$$

The present results suggest that, when one considers the weak coupling limit, it is necessary to classify observables into the ones such as $a^\dagger a$ possessing nonvanishing $\lambda \to 0$ limits, the ones such as $J_1$ which should be divided by $\lambda^2$ before taking the $\lambda \to 0$ limit, and so on. The average values of the former observables can be characterized by the reduced density matrix in the weak coupling limit. However, we do not know reduced states which provide the average values of the latter observables. This aspect will be investigated elsewhere.

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## Appendix A: On functions $\eta_+(\omega)$, $\varphi_n(k,f)$ and $\psi_n(k,t)$

Here we show that $\varphi_n(k,f)$, $\psi_n(k,t) \in L^2(\mathbf{R}^3)$ and that $\eta_+(\omega) - \omega + \Omega$ is uniformly Hölder continuous and square integrable on the positive real line.

Since $g_n(k)$ is spherical symmetric, in terms of the function $\gamma^{(n)}(\omega)$ defined by (27), one has

$$\eta(\omega + i\epsilon) - \omega - i\epsilon + \Omega = -\frac{\lambda^2}{\pi} \sum_{n=1,2} \int_0^\infty d\omega' \frac{\gamma^{(n)}(\omega')}{\omega - \omega' + i\epsilon} \ .$$

Then, as $\gamma^{(n)}(\omega)$ is square integrable and uniformly Hölder continuous (cf. assumption B), Theorem 106 of Section 5.15 of Ref. 25 implies that the limit of the right-hand side for $\epsilon \to 0+$ exists and is again square integrable and uniformly Hölder continuous with the same index $\alpha$ as $\gamma^{(n)}$. Because of the same theorem, one finds

$$\lim_{\omega\to+\infty} \gamma^{(n)}(\omega) = 0 \ , \tag{47}$$

$$\lim_{\omega\to+\infty} \{\omega - \Omega - \eta_+(\omega)\} = \lim_{\omega\to+\infty} \frac{\lambda^2}{2\pi} \sum_{n=1,2} \int_0^\infty d\omega' \frac{\gamma^{(n)}(\omega')}{\omega - \omega' + i0} = 0 \ . \tag{48}$$

Now we show that $\varphi_n(k,f)$ given by (33) is square integrable. Since $\psi_n(k), g_n(k) \in L^2(\mathbf{R}^3)$ and $1/\eta_-(\omega_k)$ is bounded, $\varphi_n(k,f) \in L^2(\mathbf{R}^3)$ follows if

$$\frac{g_n(k)}{\eta_-(\omega_k)} \int dk' \frac{g^*_{n'}(k')\psi_{n'}(k')}{\omega_k - \omega_{k'} - i0} \tag{49}$$

is square integrable. On the other hand, if $w(\omega) \in L^2(\mathbf{R}^+)$, then,

$$\frac{g_n(k) w(\omega_k)}{\eta_-(\omega_k)} \in L^2(\mathbf{R}^3) \ .$$

Indeed, (47) implies $\sup_{0<\omega} \gamma^{(n)}(\omega) < +\infty$ and one has

$$\begin{aligned} \int dk \left| \frac{g_n(k) w(\omega_k)}{\eta_-(\omega_k)} \right|^2 &= \frac{1}{\pi} \int_0^\infty d\omega \frac{\gamma^{(n)}(\omega)}{|\eta_+(\omega)|^2} |w(\omega)|^2 \\ &\le \frac{1}{\pi} \sup_{\omega>0} \gamma^{(n)}(\omega) \ \sup_{\omega>0} \frac{1}{|\eta_+(\omega)|^2} \ \int_0^\infty d\omega |w(\omega)|^2 < +\infty \end{aligned}$$

Therefore, it is enough to show that the integral in (49):

$$\int dk' \frac{g^*_{n'}(k')\psi_{n'}(k')}{\omega - \omega_{k'} - i0} = \frac{1}{2}\int_0^\infty d\omega' \frac{\sqrt{\omega'}g^*_{n'}(\sqrt{\omega'})}{\omega - \omega' - i0}\left[\int d\hat{k}'\psi_{n'}(k')\right]_{|k'|=\sqrt{\omega'}}$$

defines a square integrable function of $\omega \in \mathbf{R}^+$. This is the case because of Theorem 101 of Section 5.10 of Ref. 25 and

$$\int_0^\infty d\omega' \left|\sqrt{\omega'}g^*_{n'}(\sqrt{\omega'})\int d\hat{k}'\psi_{n'}(k')\right|^2 \leq \frac{4}{\pi}\sup_{0<\omega}\gamma^{(n')}(\omega)\int dk|\psi_{n'}(k)|^2 < +\infty\,.$$

Exactly in the same way, one can show that $\psi_n(k,t)$ defined after (36) is in $L^2(\mathbf{R}^3)$.

## Appendix B: Proof of Lemma 1

We have

$$\begin{aligned}
&\int_0^\infty d\omega w_1^*(\omega)\int_0^\infty d\omega' \frac{w_2(\omega')e^{i(\omega'-\omega)t}}{\omega-\omega'-i0}\\
&= \lim_{\epsilon\to 0+}\int_0^\infty d\omega w_1^*(\omega)\int_0^\infty d\omega' \frac{w_2(\omega')e^{i(\omega'-\omega)t}}{\omega-\omega'-i\epsilon}\\
&= \lim_{\epsilon\to 0+}\frac{1}{i}e^{\epsilon t}\int_0^\infty d\omega w_1^*(\omega)\int_0^\infty d\omega' w_2(\omega')\int_0^\infty ds e^{i(\omega'-\omega+i\epsilon)(s+t)}\\
&= \lim_{\epsilon\to 0+}\frac{1}{i}e^{\epsilon t}\int_t^\infty ds e^{-\epsilon s}\widetilde{w}_1^*(s)\widetilde{w}_2(s) = \frac{1}{i}\int_t^\infty ds\widetilde{w}_1^*(s)\widetilde{w}_2(s)
\end{aligned}\tag{50}$$

where $\widetilde{w}_n(s) \equiv \int_0^\infty d\omega w_n(\omega)\exp(i\omega s)$ stands for the Fourier transform.

In the above, the first equality holds because the $\omega'$-integral exists for $w_2 \in L^2(\mathbf{R}^+)$ as an element of $L^2(\mathbf{R}^+)$ (cf. Theorem 101 of Section 5.10 of Ref. 25 ).

The second and third equalities follow from Fubini's theorem and the absolute integrability of the integrand (remind that $w_1, w_2 \in L^1(\mathbf{R}^+)$).

Since $w_1, w_2 \in L^2(\mathbf{R}^+)$, their Fourier transformations are square integrable as well. Thus, $\widetilde{w}_1^*(s)\widetilde{w}_2(s)$ is integrable and Lebesgue's dominated convergence theorem leads to the last equality.

The desired result also follows from the integrability of $\widetilde{w}_1^*(s)\widetilde{w}_2(s)$:

$$\begin{aligned}
&\lim_{t\to+\infty}\int_0^\infty d\omega w_1^*(\omega)\int_0^\infty d\omega' \frac{w_2(\omega')e^{i(\omega'-\omega)t}}{\omega-\omega'-i0}\\
&\qquad = \lim_{t\to+\infty}\frac{1}{i}\int_t^\infty ds\widetilde{w}_1^*(s)\widetilde{w}_2(s) = 0
\end{aligned}\tag{51}$$

## Appendix C: Proof of Lemma 2

Let $\gamma(\omega) \equiv \sum_{n=1,2} \gamma^{(n)}(\omega)$, then, since $\gamma(\Omega) > 0$ and $\gamma(\omega)$ is continuous, there exists $\kappa > 0$ such that $\gamma(\omega) \geq \gamma(\Omega)/2$ for $|\omega - \Omega| \leq \kappa$. As shown in Appendix A, the integral involved in $\eta_+(\omega)$

$$I(\omega) \equiv \frac{1}{\pi} \int_0^\infty d\omega' \frac{\gamma(\omega')}{\omega - \omega' + i0}$$

is bounded and, thus, for sufficient small $\lambda$,

$$|\mathrm{Re}\eta_+(\omega) - \omega + \Omega| = \lambda^2 |I(\omega)| \leq \frac{\kappa}{2} .$$

Then, we divide the integral into four terms:

$$\begin{aligned}
\int_0^\infty d\omega F(\omega) \frac{\lambda^2 \gamma(\omega)}{|\eta_+(\omega)|^2} &= \int_{\Omega-\kappa}^{\Omega+\kappa} d\omega F(\omega) \frac{\lambda^2 \gamma(\Omega)}{(\omega - \Omega - \Delta_\lambda)^2 + \lambda^4 \gamma(\Omega)^2} \\
&+ \int_0^{\Omega-\kappa} d\omega F(\omega) \frac{\lambda^2 \gamma(\omega)}{|\eta_+(\omega)|^2} + \int_{\Omega+\kappa}^\infty d\omega F(\omega) \frac{\lambda^2 \gamma(\omega)}{|\eta_+(\omega)|^2} \\
&+ \int_{\Omega-\kappa}^{\Omega+\kappa} d\omega F(\omega) \left\{ \frac{\lambda^2 \gamma(\omega)}{|\eta_+(\omega)|^2} - \frac{\lambda^2 \gamma(\Omega)}{(\omega - \Omega - \Delta_\lambda)^2 + \lambda^4 \gamma(\Omega)^2} \right\} . \qquad (52)
\end{aligned}$$

where $\Delta_\lambda = \lambda^2 \mathrm{Re} I(\Omega)$.

In the standard way, one can easily show

$$\lim_{\lambda \to 0} \int_{\Omega-\kappa}^{\Omega+\kappa} d\omega F(\omega) \frac{\lambda^2 \gamma(\Omega)}{(\omega - \Omega - \Delta_\lambda)^2 + \lambda^4 \gamma(\Omega)^2} = \pi F(\Omega) . \qquad (53)$$

When $\omega \leq \Omega - \kappa$, one has

$$|\eta_+(\omega)| \geq |\mathrm{Re}\eta_+(\omega)| \geq \Omega - \omega - |\mathrm{Re}\eta_+(\omega) - \omega + \Omega| \geq \Omega - \omega - \frac{\kappa}{2}$$

and the second integral of (52) is evaluated as

$$\begin{aligned}
\left| \int_0^{\Omega-\kappa} d\omega F(\omega) \frac{\lambda^2 \gamma(\omega)}{|\eta_+(\omega)|^2} \right| &\leq \lambda^2 \sup_{0<\omega} |F(\omega)| \sup_{0<\omega} |\gamma(\omega)| \int_{-\infty}^{\Omega-\kappa} \frac{d\omega}{(\Omega - \omega - \kappa/2)^2} \\
&= \frac{2\lambda^2}{\kappa} \sup_{0<\omega} |F(\omega)| \sup_{0<\omega} |\gamma(\omega)| \to 0 \quad (\text{for } \lambda \to 0)
\end{aligned}$$

The third integral of (52) can be evaluated in the same way:

$$\lim_{\lambda \to 0} \int_{\Omega+\kappa}^\infty d\omega F(\omega) \frac{\lambda^2 \gamma(\omega)}{|\eta_+(\omega)|^2} = 0 .$$

The last integral is rewritten as

$$\int_{\Omega-\kappa}^{\Omega+\kappa} d\omega F(\omega)\left\{\frac{\lambda^2\gamma(\omega)}{|\eta_+(\omega)|^2}-\frac{\lambda^2\gamma(\Omega)}{(\omega-\Omega-\Delta_\lambda)^2+\lambda^4\gamma(\Omega)^2}\right\}$$

$$=\lambda^4\int_{\Omega-\kappa}^{\Omega+\kappa} d\omega F(\omega)\gamma(\omega)\ \frac{2\mathrm{Re}\eta_+(\omega)\mathrm{Re}\{I(\omega)-I(\Omega)\}}{|\eta_+(\omega)|^2\left[(\omega-\Omega-\Delta_\lambda)^2+\lambda^4\gamma(\Omega)^2\right]}$$

$$+\lambda^6\int_{\Omega-\kappa}^{\Omega+\kappa} d\omega F(\omega)\gamma(\omega)\ \frac{2\mathrm{Re}I(\omega)\mathrm{Re}\{I(\omega)-I(\Omega)\}+|I(\omega)|^2-|I(\Omega)|^2}{|\eta_+(\omega)|^2\left[(\omega-\Omega-\Delta_\lambda)^2+\lambda^4\gamma(\Omega)^2\right]}$$

$$+\int_{\Omega-\kappa}^{\Omega+\kappa} d\omega F(\omega)\left\{\frac{\gamma(\omega)}{\gamma(\Omega)}-1\right\}\frac{\lambda^2\gamma(\Omega)}{(\omega-\Omega-\Delta_\lambda)^2+\lambda^4\gamma(\Omega)^2}\ .$$

Then, because of $|\eta_+(\omega)|\geq\lambda^2\gamma(\omega)\geq\lambda^2\gamma(\Omega)/2$ and $|\mathrm{Re}\eta_+(\omega)|/|\eta_+(\omega)|\leq 1$, one has

$$\left|\int_{\Omega-\kappa}^{\Omega+\kappa} d\omega F(\omega)\left\{\frac{\lambda^2\gamma(\omega)}{|\eta_+(\omega)|^2}-\frac{\lambda^2\gamma(\Omega)}{(\omega-\Omega-\Delta_\lambda)^2+\lambda^4\gamma(\Omega)^2}\right\}\right|$$

$$\leq\int_{\Omega-\kappa}^{\Omega+\kappa} d\omega H(\omega)\frac{\lambda^2\gamma(\Omega)}{(\omega-\Omega-\Delta_\lambda)^2+\lambda^4\gamma(\Omega)^2}\ , \tag{54}$$

where

$$H(\omega)=\frac{4\gamma(\omega)}{\gamma(\Omega)^3}|F(\omega)|\left|2\mathrm{Re}I(\omega)\mathrm{Re}\{I(\omega)-I(\Omega)\}+|I(\omega)|^2-|I(\Omega)|^2\right|$$
$$+\frac{4\gamma(\omega)}{\gamma(\Omega)^2}|F(\omega)|\left|\mathrm{Re}\{I(\omega)-I(\Omega)\}\right|+|F(\omega)|\left|\frac{\gamma(\omega)}{\gamma(\Omega)}-1\right|\ .$$

Since $H(\omega)$ is continuous and $H(\Omega)=0$, (53) implies that the second integral of (52) vanishes in the $\lambda\to 0$ limit.

$$\lim_{\lambda\to 0}\left|\int_{\Omega-\kappa}^{\Omega+\kappa} d\omega F(\omega)\left\{\frac{\lambda^2\gamma(\omega)}{|\eta_+(\omega)|^2}-\frac{\lambda^2\gamma(\Omega)}{(\omega-\Omega-\Delta_\lambda)^2+\lambda^4\gamma(\Omega)^2}\right\}\right|$$
$$\leq\pi H(\Omega)=0\ .$$

In short, one obtains

$$\lim_{\lambda\to 0}\int_0^\infty d\omega F(\omega)\frac{\lambda^2\gamma(\omega)}{|\eta_+(\omega)|^2}=\pi F(\Omega)\ ,$$

and, thus,

$$\lim_{\lambda\to 0}\sum_{n=1,2}\int dk F(\omega_k)\frac{\lambda^2|g_n(k)|^2}{|\eta_-(\omega_k)|^2}=\lim_{\lambda\to 0}\int_0^\infty d\omega F(\omega)\frac{\lambda^2\gamma(\omega)}{\pi|\eta_+(\omega)|^2}=F(\Omega)\ .$$

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