Interacting Fock Spaces and Orthogonal Polynomials in several variables

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Abstract

We extend to polynomials in several variables the Accardi–Bozeiko canonical isomorphism between 1–mode interacting Fock spaces and orthogonal polynomials in one variable. This gives a constructive rule to write down easily the quantum decomposition, as a sum of creation, annihilation and number operators, of an arbitrary vector valued random variable with moments of any order. In the multi–mode case not all interacting Fock spaces are canonically isomorphic to spaces of orthogonal polynomials. We characterize those which enjoy this property in terms of a sequence of quadratic commutation relations among finite dimensional matrices.

1 Introduction

In the past years the theory of interacting Fock spaces has been used in a multiplicity of different contexts (cf. [Hash97], [HaHoOb01],[HaObTa01], [Asai01], [Das01a], [Das01b]). In many of these papers the natural correspondence between interacting Fock spaces and orthogonal polynomials played a relevant role.

This correspondence was proved, for one–mode interacting Fock spaces by Accardi and Bozeiko who shewed that the theory of 1–mode interacting Fock spaces is canonically isomorphic to the theory of orthogonal polynomials in one variable, i.e. with respect to a probability measure on the real line with finite moments of any order. The canonical feature of this isomorphism is exhibited by the fact that it maps the multiplication operator by the independent variable into a linear combination of the creation, annihilation and number operators of the corresponding interacting Fock space.

The problem to extend this isomorphism to polynomials in several variables has been open for a few years. In the present paper we discuss a solution of this problem. The main new feature with respect to the one–mode case is that, in the multi–mode case, case not all interacting Fock spaces are canonically isomorphic to spaces of orthogonal polynomials. We characterize those which enjoy this property in terms of a sequence of quadratic commutation relations among finite dimensional matrices. This gives in particular a constructive rule to write down easily the quantum decomposition, as a sum of creation, annihilation and number operators, of an arbitrary vector valued random variable with moments of any order.

Let $d \in \mathbf{N}$. We denote μ a probability measure on $\mathbf{R}^d$ with finite moments of any order; $x = (x_1, ..., x_d)$ any element of $\mathbf{R}^d$; x_j denote the coordinates in the canonical basis; $X^0 = (X_1^0, ..., X_d^0) : (\mathbf{R}^d, \mu) \to \mathbf{R}^d$ the $\mathbf{R}^d$-valued coordinate random variable, characterized by:

$$X_j^0(x) = x_j \quad ; \quad j = 1, ..., d$$

Definition 1 *The complex $*$-algebra $\mathcal{P} = \mathcal{P}^d$, with identity, generated (algebraically) by the X_j^0 ($j = 1, ..., d$) and the constant functions with pointwise addition and multiplication and $f^*(x) = \bar{f}(x)$-complex conjugate) is called the "polynomial algebra in d indeterminates".*

In the following, the dimension d will be fixed, so we will frequently omit the super–script d and write, for example, $\mathcal{P}$ instead of $\mathcal{P}^d$.

The μ-integral defines a state on $\mathcal{P}$ and the GNS construction, applied to the pair $\{\mathcal{P}, \mu\}$ [Sak71] gives a Hilbert space $\mathcal{H}_\mu$, a representation

$$\pi_\mu \, : \, \mathcal{P} \to \mathcal{B}(\mathcal{H}_\mu)$$

(the bounded linear operators on $\mathcal{H}_\mu$) and a unit vector 1_μ, cyclic for $\pi(\mathcal{P})$. $\mathcal{H}_\mu$ is a closed sub-space of $L^2(\mathbf{R}^d, \, \mu)$ and coincides with it if and only if the measure μ is uniquely determined by its moments. Thus the space $\mathcal{H}_\mu$ gives in some sense a measure of the non uniqueness of the moment problem for μ.

Definition 2 *We will say that μ is generic if*
  (i) the union of the coordinate hyperplanes $x_j = 0$ $(j = 1, \ldots, d)$ has μ–measure zero
  (ii) π_μ is injective (notice that this implies that also the map $p \in \mathcal{P} \to \pi_\mu(p) \cdot 1_\mu$ is injective).

This is surely the case if π_μ has a density with respect to the Lebesgue measure. In the generic case the $*$–algebra

$$\mathcal{P}_\mu := \pi(\mathcal{P}) \tag{1}$$

is isomorphic to $\mathcal{P}$ and also the elements of $\mathcal{P}_\mu$ will be called polynomials. They are bounded operators on $\mathcal{H}_\mu$ if and only if μ has compact support.
  Define the coordinate multiplication operators

$$X_j^0 \, f(x) \, := \, x_j \, f(x); \quad j = 1, ..., d$$

and their action on $\mathcal{H}_\mu$:

$$\pi(X_j^0)|\mathcal{H}_\mu =: X_j \qquad ; \qquad j = 1, ..., d \tag{2}$$

Condition (i) above implies that the operators X_j $(j = 1, ..., d)$ are invertible in $\mathcal{H}_\mu$. In the following we will identify the elements of $\mathcal{H}_\mu$ to elements of $L^2(\mathbf{R}^d, \, \mu)$ so that, for each $j = 1, ..., d$, X_j acts by multiplication on $\mathcal{H}$:

$$X_j \, f(x) \, = \, x_j \, f(x); \, f \, \in \, \mathcal{H} \tag{3}$$

and defines a symmetric pre-closed operator on $\mathcal{H}$ on the dense invariant domain $\mathcal{P}_\mu \cdot 1_\mu$. The algebra $\mathcal{P}_\mu$ is generated by the monomials

$$X_1^{n_1}...X_d^{n_d}; \quad \forall \, n_1, ..., n_d \in \mathbf{N} \tag{4}$$

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and the vectors

$$X_1^{n_1}...X_d^{n_d} \cdot 1_\mu \tag{5}$$

are total in $\mathcal{H}_\mu$ by construction.

In the following, when no confusion can arise, we shall denote with the same symbol both the vectors (5) and the corresponding multiplication operators (4) in $\mathcal{P}_\mu$.

Definition 3 *For each $n \in \mathbf{N}$ define*

$$\mathcal{P}_n^d := \text{ linear span of } \{X_1^{n_1}...X_d^{n_d}; \ n_j \in \mathbf{N}, \ \sum_{j=1}^{d} n_j \leq n \}$$

It is clear that $\mathcal{P}_n^d$ is the vector space of all polynomials in the variables $X_1, ..., X_d$ of degree at most n, where the degree of the monomial $X_1^{n_1}...X_d^{n_d}$ is $\sum_{j=1}^{d} n_j$ and the degree of a polynomial P is defined to be the highest, among the degrees of the monomials which appear in P with a non zero coefficient. Notice that, for each $j = 1, ..., d$,

$$X_j \mathcal{P}_n^d \subseteq \mathcal{P}_{n+1}^d. \tag{6}$$

Define inductively

$$V_0 = \mathbf{C} \cdot 1_\mu = \mathcal{P}_0^d \cdot 1_\mu \tag{7}$$

$$V_{n+1}^d := (\bigoplus_{k=0}^{n} V_k^d)^\perp \cap \mathcal{P}_{n+1}^d \cdot 1_\mu \tag{8}$$

Lemma 1 . *For each $n \in \mathbf{N}$:*

$$\mathcal{P}_n^d \cdot 1_\mu = \bigoplus_{k=0}^{n} V_k^d$$

In particular:

$$V_{n+1}^d = (\bigoplus_{k=0}^{n} V_k^d)^\perp \cap \{ \text{ linear span of } X_1^{n_1}...X_d^{n_d} \text{ with } : n_j \in \mathbf{N}, \ (j=1,...,d), \sum_{j=1}^{d} n_j = n+1 \}$$

Proof. Clear.

For $n_1 + ... + n_d = n$, denote $: X_1^{n_1}... X_d^{n_d} :$ the orthogonal projection of $X_1^{n_1}... X_d^{n_d} \cdot 1_\mu$ in $\bigoplus_{k<n} V_k = \mathcal{P}_{n-1} \cdot 1_\mu$. By linearity the symbol $: p_n(X) :$ is defined for any polynomial p_n of degree n.

The vectors $: X_1^{n_1}... X_d^{n_d} :$ span V_n but in general, they are not orthogonal.

Lemma 2 . *(Linear independence) For any fixed n, the vectors $: X_1^{n_1}... X_d^{n_d} :$, with $\sum_j^d n_j = n$, are linearly independent. In particular $V_n^d \equiv C^{r_n^d}$ with*

$$r_n^d := \binom{n+d-1}{n} \tag{9}$$

and the vectors $: X_1^{n_1}... X_d^{n_d} :$ are a basis of V_n^d.

Proof. Fix $n \in \mathbf{N}$. The genericity of μ implies that the map $p_n \mapsto : p_n :$ is injective (p_n polynomial of degree n). Therefore, $\forall n_j \in \mathbf{N}$ with $\sum_j^d n_j = n$, the $: X_1^{n_1}... X_d^{n_d} :$ are linearly independent because, by the genericity condition, such are the $X_1^{n_1}... X_d^{n_d} \cdot 1_\mu$. Therefore the cardinality of the set

$$\{: X_1^{n_1}... X_d^{n_d} :, n_j \in \mathbf{N}, \sum_j^d n_j = n\}$$

is $\binom{n+d-1}{n}$. Since $\dim \mathcal{P}_n^d = \binom{n+d}{d}$ [KrSh67], using the identity

$$\binom{n+d}{d} - \binom{n+d-1}{d} = \binom{n+d-1}{n}$$

we can conclude that

$$\dim V_n^d = \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-1}^d = \binom{n+d-1}{n}$$

This proves the lemma, since the set $\{: X_1^{n_1}... X_d^{n_d} :, n_j \in \mathbf{N}, \sum_j^d n_j = n\}$ generates V_n^d.

Remark. Since the elements of any orthonormal basis of V_n are in one-to-one correspondence with the solutions (in $\mathbf{N}$) of the equation

$$n_1 + ... + n_d = n$$

we will often use the notation

$$|\bar{n}\rangle = |n_1, ..., n_d\rangle \tag{10}$$

to denote an arbitrary orthonormal basis of V_n. When confusion can arise we will use the more explicit notation

$$\phi_{\bar{n}} = \phi_{n_1,...,n_d}$$

The recurrence relations among orthogonal polynomials are consequences of the following result:

Theorem 1 . *For any $n \in \mathbf{N}$, if $k \notin \{n-1, n, n+1\}$ then for each $j = 1, ..., d$,*

$$X_j V_n \perp V_k$$

Proof. Let $|\bar{n}\rangle \in V_n$ and $f \in \mathcal{P}_{n-2}$, then for each $j = 1, ..., d$, $X_j f \in P_{n-1}^d$ hence, by the symmetry of X_j and (6):

$$\langle X_j \bar{n}, f \rangle = \langle \bar{n}, X_j f \rangle = 0$$

because by assumption $|\bar{n}\rangle \in V_n \perp \mathcal{P}_{n-1}$.

If $f \in V_k$ with $k > n+1$, then $f \perp X_j |\bar{n}\rangle$ because $X_j |\bar{n}\rangle$ is a polynomial of degree $n+1$. This proves the statement.

Denote P_m the orthogonal projection on V_m. The following is a multidimensional generalization of the Jacobi relations for 1-dimensional orthogonal polynomials.

Corollary 1 *(Recurrence relations) Let us fix, for any $n \in \mathbf{N}$, an orthonormal basis $\{|\bar{n}\rangle = |n_1, ..., n_d\rangle; \sum_j^d n_j = n\}$ of V_n. For each $\bar{n} = (n_1, ..., n_d) \in \mathbf{N}^d$ with $\sum_j^d n_j = n$, and for each $j = 1, ..., d$ we have*

$$X_j P_n = P_{n+1} X_j P_n + P_n X_j P_n + P_{n-1} X_j P_n \tag{11}$$

Proof. We know from Theorem 1 that for each $j = 1, ..., d$, $n \in \mathbf{N}$, $|\bar{n}\rangle \in V_n$,

$$X_j |\bar{n}\rangle \in V_{n+1} \oplus V_n \oplus V_{n-1} \tag{12}$$

Since $P_{n+1} + P_n + P_{n-1}$ is the orthogonal projection on $V_{n+1} \oplus V_n \oplus V_{n-1}$ (12) implies that

$$X_j \, |\bar{n}> = \; P_{n+1} \, X_j \, |\bar{n}> \; + P_n \, X_j \, |\bar{n}> \; + P_{n-1} \, X_j \, |\bar{n}>;$$

and, since $|\bar{n}> \in V_n$ is arbitrary, this is equivalent to (11).

Now define the following operators:

$$D_n^+(j) := \; P_{n+1} \, X_j \, P_n \; |_{V_n} \in \mathcal{B}(V_n, \, V_{n+1}) \tag{13}$$

$$D_n^0(j) := \; P_n \, X_j \, P_n \; |_{V_n} \in \mathcal{B}(V_n, \, V_n) \; = \; \mathcal{B}(V_n) \tag{14}$$

$$D_n^-(j) := \; P_{n-1} \, X_j \, P_n \; |_{V_n} \in \mathcal{B}(V_n, \, V_{n-1}) \tag{15}$$

Given any orthonormal basis $(|\bar{n}>)$ of V_n, we can write the finite dimensional operators (13), (14), (15) as matrices:

$$D_n^0(j) = \sum_{\bar{m}, \bar{n}} |\bar{m}><\bar{m}|D_n^0(j)\,\bar{n}><\bar{n}| \; =: \sum_{|\bar{m}|=|\bar{n}|=n} D_{\bar{m}, \bar{n}}^0(j) \, |\bar{m}><\bar{n}| \tag{16}$$

$$D_n^+(j) = \sum_{\bar{n}, \bar{m}} |\bar{m}><\bar{m}|D_n^+(j)\,\bar{n}><\bar{n}| \; =: \sum_{|\bar{n}|=n, \, |\bar{m}|=n+1} D_{\bar{n}, \bar{m}}^+(j) \, |\bar{m}><\bar{n}|$$

$$\tag{17}$$

$$D_n^-(j) = \sum_{\bar{n}, \bar{m}} |\bar{m}><\bar{m}|D_n^-(j)\,\bar{n}><\bar{n}| \; =: \sum_{|\bar{n}|=n, \, |\bar{m}|=n-1} D_{\bar{n}, \bar{m}}^-(j) \, |\bar{m}><\bar{n}|$$

$$\tag{18}$$

where, if $\overline{n} = (n_1, \ldots, n_d)$, we use the notation $|\overline{n}| = n_1 + \ldots + n_d$.

We note that

$$D_n^+(j) \text{ is represented by an } r_n^d \times r_{n+1}^d \text{ matrix}$$

$$D_n^0(j) \text{ is represented by an } r_n^d \times r_n^d \text{ matrix}$$

$$D_n^-(j) \text{ is represented by an } r_n^d \times r_{n-1}^d \text{ matrix}$$

Lemma 3 . *The operators (13), (14), (15) satisfy the following relations:*

$$D_{n+1}^+(i)\, D_n^+(j) \;=\; D_n^+(j)\, D_{n+1}^+(i) \tag{19}$$

$$D_{n+1}^0(i)\, D_n^+(j) \;+\; D_n^+(i)\, D_n^0(j) \;=\; D_{n+1}^0(j)\, D_n^+(i) \;+\; D_n^+(j)\, D_n^0(i) \tag{20}$$

$$D_{n-1}^+(i)\, D_n^-(j) \;+\; D_n^0(i)\, D_n^0(j) \;+\; D_{n+1}^-(i)\, D_n^+(j) \;=$$

$$D_{n-1}^+(j)\, D_n^-(i) \;+\; D_n^0(i)\, D_n^0(j) \;+\; D_{n+1}^-(j)\, D_n^+(i) \tag{21}$$

for $i \neq j$, $1 \le i, j \le d$ and $n \ge 0$, where $D_{-1}^+(i) \;=\; 0$.

Proof. We have

$$D_{n+1}^+(j)\, D_n^+(i) = P_{n+2}\, X_j\, P_{n+1}\, |_{V_{n+1}} \; P_{n+1}\, X_i\, P_n\, |_{V_n} = P_{n+2}\, X_j\, P_{n+1}\, X_i\, P_n\, |_{V_n} \tag{22}$$

Thus, exchanging i and j:

$$D_{n+1}^+(i)\, D_n^+(j) \;=\; P_{n+2}\, X_i\, P_{n+1}\, X_j\, P_n\, |_{V_n} \tag{23}$$

On the other hand, since $X_i\, X_j \;=\; X_j\, X_i$, it follows that

$$P_{n+2}\, X_i\, X_j\, P_n \;=\; P_{n+2}\, X_j\, X_i\, P_n \tag{24}$$

Using (11) we see that the left hand side of (24) is equal to

$$P_{n+2}\, X_i\, P_{n+1}\, X_j\, P_n + P_{n+2}\, X_i\, P_n\, X_j\, P_n + P_{n+2}\, X_i\, P_{n-1}\, X_j\, P_n$$

and the right hand side to

$$P_{n+2}\, X_j\, P_{n+1}\, X_i\, P_n + P_{n+2}\, X_j\, P_n\, X_i\, P_n + P_{n+2}\, X_j\, P_{n-1}\, X_i\, P_n$$

Since

$$P_{n+2}\, X_i\, P_n\, X_j\, P_n = P_{n+2}\, X_j\, P_{n-1}\, X_i\, P_n = P_{n+2}\, X_j\, P_n\, X_i\, P_n = P_{n+2}\, X_i\, P_{n-1}\, X_j\, P_n \;= 0$$

this is equivalent to

$$P_{n+2}\, X_i\, P_{n+1}\, X_j\, P_n \;=\; P_{n+2}\, X_j\, P_n\, X_i\, P_n$$

Therefore (22) and (23) are equal, which proves (19).

In a similar way we observe that

$$P_{n+1}\, X_i\, X_j\, P_n \;=\; P_{n+1}\, X_j\, X_i\, P_n$$

is equivalent to

$$P_{n+1}\, X_i\, P_{n+1}\, X_j\, P_n \;+\; P_{n+1}\, X_i\, P_n\, X_j\, P_n \;+\; P_{n+1}\, X_i\, P_{n-1}\, X_j\, P_n \;=$$

$$P_{n+1}\, X_j\, P_{n+1}\, X_i\, P_n \;+\; P_{n+1}\, X_j\, P_n\, X_i\, P_n \;+\; P_{n+1}\, X_j\, P_{n-1}\, X_i\, P_n \qquad (25)$$

Since

$$P_{n+1}\, X_i\, P_{n-1}\, X_j\, P_n \;=\; P_{n+1}\, X_j\, P_{n-1}\, X_i\, P_n \;=\; 0$$

the relation (25) implies

$$D_{n+1}^0(i)\, D_n^+(j) \;+\; D_n^+(i)\, D_n^0(j) \;=\; D_{n+1}^0(j)\, D_n^+(i) \;+\; D_n^+(j)\, D_n^0(i)$$

which proves (20).

To prove (21) we apply the recurrence relation to the two sides of the equality

$$P_n\, X_i\, X_j\, P_n \;=\; P_n\, X_j\, X_i\, P_n$$

and argue as above.

Lemma 4 . *The operators* $D_n^+(j)$, $D_n^-(j)$, $D_n^0(j)$ *are such that*

$$(D_n^+(j))^* \;=\; D_{n+1}^- \quad ; \quad (D_n^0(j))^* \;=\; D_n^0(j) \qquad (26)$$

Proof.

$$(D_n^0(j))^* \;=\; (P_n\, X_j\, P_n)^* \;=\; P_n\, X_j\, P_n \;=\; D_n^0(j)$$

$$(D_n^+(j))^* \;=\; (P_{n+1}\, X_j\, P_n)^* \;=\; P_n\, X_j\, P_{n+1} \;=\; D_{n+1}^-$$

Lemma 5 . *If μ is generic, then $D_n^+(j)$ is injective for any $n \in \mathbf{N}$ and $j = 1, ..., d$.*

Proof. If there exists $\bar{n} \in V_n$ such that

$$P_{n+1} X_j \bar{n} = 0$$

then

$$X_j \bar{n} = P_n X_j \bar{n} + P_{n-1} X_j \bar{n}$$

But the left hand side of this equality is a polynomial of degree $n + 1$ and the right hand side is a polynomial of degree n. Because of the genericity of μ they can be equal if and only if they are both zero. Also because of genericity X_j is invertible in $\mathcal{H}_\mu$, hence

$$X_j \bar{n} = 0 \implies \bar{n} = 0$$

It follows that $P_{n+1} X_j P_n = D_n^+(j)$ is injective.

Lemma 6 . *For each $n \in \mathbf{N}$, the family $D_n^+(1), ..., D_n^+(d) : V_n \longrightarrow V_{n+1}$ is surjective, in the sense that the linear span of the ranges of $D_n^+(j)$ is the whole of V_{n+1}.*

Proof. Let $\xi \in V_{n+1}$ be an element orthogonal to the range of all $D_n^+(j)$'s $(j = 1, ..., d)$. Then for any $\bar{n} \in V_n$ and for any $j = 1, ..., d$

$$0 = < \xi, D_n^+(j) \bar{n} > = < \xi, P_{n+1} X_j \bar{n} > = < \xi, X_j \bar{n} >$$

But, by definition of V_n, it also follows that $\xi \perp X_j V_k$ for any $k \leq n - 1$. By taking linear combinations we see that ξ is orthogonal to all polynomials of the form $X_j q_n$ where $q_n \in P_n$. But, taking linear combinations of these, we obtain all possible polynomials of degree $n + 1$ without constant term. This contradicts the fact that $\xi \in V_{n+1}$.

2   Interacting Fock spaces

Let us recall the definition of interacting Fock space. We use here the definition introduced by [AcSk99] which, although equivalent to the original one of [AcLu92], [AcLuVo97], is more suitable for our purposes.

Definition 4 *Let $\mathcal{H}_1^0$ be a pre–Hilbert space. An interacting Fock space based on $\mathcal{H}_1^0$ is an Hilbert space $\mathcal{H}$, with a gradation indexed by $\mathbf{N}$*

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n = C \cdot \Phi \oplus \bigoplus_{n \geq 1} \mathcal{H}_n \qquad (27)$$

and with the following property.

$\forall\, v \in \mathcal{H}_1^0$, *there exists a densely defined linear operator $a^+(v)$ on $\mathcal{H}$ such that:*

(i) the map $\quad v \in \mathcal{H}_1^0 \mapsto a^+(v)$ *is complex linear.*

(ii) $\forall\, n \in \mathbf{N}$ *the set*

$$\{a^+(v_n)...a^+(v_1)\,\Phi \;:\; v_1,\,...,\,v_n \in \mathcal{H}_1^0\} \qquad (28)$$

is contained in the domain $D(a^+(v))$ of $a^+(v)$. For fixed $n \in \mathbf{N}$, we will denote $\mathcal{N}_n$ the linear subspace (algebraically) generated by the vectors (28).
(iii) The union of all the $\mathcal{N}_n$ ($n \in \mathbf{N}$) is a dense subspace of $\mathcal{H}$.
(iv) $a^+(v)$ maps $\mathcal{N}_n$ onto $\mathcal{N}_{n+1}$.
(v) Each $a^+(v)$ has an adjoint $a(v)$ defined on $\mathcal{N} := \bigcup_n \mathcal{N}_n$.

Remark. The direct sum in (27) is in the Hilbert space sense. We will denote

$$\mathcal{H}^0 = \overset{algebraic}{\underset{n \geq 0}{\bigoplus}} \mathcal{H}_n \qquad ; \qquad \mathcal{H}_0 = C \cdot \Phi \qquad (29)$$

the algebraic direct sum, i.e. the subspace of $\mathcal{H}$ of the vectors which have a non zero component only on a finite number of spaces $\mathcal{H}_n$.

Now let us fix the following choices:

$$\mathcal{H} = \mathcal{H}_\mu \qquad ; \qquad \mathcal{H}_\mu^0 = \overset{algebraic}{\underset{n}{\bigoplus}} \mathcal{H}_\mu \qquad ; \qquad \mathcal{H}_1^0 = C^d \;\; ; \;\; \mathcal{H}_n = V_n^d$$

where the operators a_j^+ are defined by the following:

Proposition 1 *In the notation (10) let, for $n \in \mathbf{N}$, $|\bar{n} > = |n_1, ..., n_d >$ denote an arbitrary orthonormal basis of V_n and let $D_{\bar{m},n}^\epsilon(j)$ denote the matrices in this basis, of the operators $D_n^\epsilon(j)$ ($\epsilon = +, -, 0$, $n \in \mathbf{N}$) as defined by (16) , (17) , (18). Then:*

(i) For each $j = 1, ..., d$ the operator a_j^+, defined on the vectors $|n_1, ..., n_d >$ by

$$a_j^+ |n_1, ..., n_d > = D_n^+(j) |\bar{n} > = \sum_{|\bar{m}_+|=n+1} D_{\bar{n}, \bar{m}_+}^+(j) |\bar{m}_+ > \qquad (30)$$

has an adjoint, also defined on the vectors $|n_1, ..., n_d >$ by

$$a_j^- |0, ..., 0 > = 0 \qquad (31)$$

$$a_j^- |\bar{n} > = D_n^-(j)\,\bar{n} = \sum_{|\bar{m}_-|=n-1} D_{\bar{n}, \bar{m}_-}^-(j) |\bar{m}_- > \qquad (32)$$

(ii) The operators a_j^0 ($j = 1, ..., d$) defined by

$$a_j^0 |\bar{n} > = \sum_{|\bar{m}|=n} D_{\bar{n},\bar{m}}^0(j) |\bar{m} > \qquad (33)$$

is symmetric on H_μ^0.

(iii) The following identity holds on H_μ^0:

$$X_j := a_j^+ + a_j^0 + a_j^-, \; j = 1, ..., d \qquad (34)$$

Remark. Notice that that definition of the operators a_j^+ depends on the choice of the orthonormal basis $|n_1, ..., n_d >$.

An intrinsic definition is possible, but makes the intuitive connection with ordinary multiplication operators more obscure.

Proof. Since, for each $\bar{n} \neq \bar{0}$, $a_j^+ |\bar{n} >$ is always in some V_k with $k \geq 1$, one has

$$< \bar{0}, a_j^+ \bar{n} > = 0$$

Moreover since, by definition,

$$a^+(j) |_{V_n} = D_n^+(j) = P_{n+1} X_j P_n$$

and

$$a^-(j) |_{V_{n+1}} = D_{n+1}^-(j)$$

it follows that
$$(D_n^+(j))^* = P_n X_j P_{n+1} = a_{n+1}^-(j), \qquad (35)$$
which means that $a^-(j)$ is the adjoint of $a^+(j)$ on H_μ^0. Similarly
$$a^0(j)|_{V_n} = D_n^0(j) = P_n X_j P_n = (P_n X_j P_n)^* = (D_n^0(j))^* \qquad (36)$$

Therefore the operator a_j^0 is symmetric on H_μ^0. For each $n \in \mathbf{N}$ the identity
$$X_j P_n = (a_j^+ + a_j^0 + a_j^-) P_n$$
follows from the definition of the $D_n^\epsilon(j)$, $\epsilon = +, -, 0$ and this implies (34).

Conversely:

Theorem 2 *Let be given a Hilbert space $\mathcal{H}$ with a $\mathbf{N}$-gradation*
$$\mathcal{H} = \bigoplus_{n \in \mathbf{N}} V_n \quad ; \quad V_0 = C \cdot \Phi$$
such that
$$V_n \equiv C^{r_n^d} \quad \forall n \in \mathbf{N}$$
and let be given:

(i)   d sequences of (finite dimensional) operators
$$D_n^+(j) \in \mathcal{B}(V_n, V_{n+1})$$
$$D_n^0(j) \in \mathcal{B}(V_n, V_n)$$
$$D_n^-(j) \in \mathcal{B}(V_n, V_{n-1})$$
satisfying the conditions (19), (20), (21).

(ii) For each $n \in \mathbf{N}$ an orthonormal basis of V_n, denoted
$$|\bar{n}> = |n_1, ... n_d> \quad ; \quad |\bar{n}| := \sum_{j=1}^d n_j = n$$

Then, on $\mathcal{H}$, there exists a structure of interacting Fock space over C^d, with creation and annihilation operators given by (30), (31), (33) respectively.

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If, in addition the commutativity relations (19), (20), (21) are satisfied, then the operators

$$X_j := a_j^+ + a_j^0 + a_j^- \quad ; \quad j = 1, \ldots, d \qquad (37)$$

are a commuting family of symmetric operators satisfying the relations (11), P_n being the orthonogonal projection onto V_n. Moreover Φ is a cyclic vector for the polynomial algebra generated by the X_j and the identity.

Proof. Let us fix arbitrarily, for each $n \in \mathbf{N}$, an orthonormal basis $(|\bar{n} >)$ of V_n. Define the operators a_j^+, a_j^-, a_j^0 by (30), (32), (33), $(j = 1, \ldots, d)$ respectively.

The symmetry of the a_j^0 comes from the fact that the $D_n^0(j)$ are symmetric and the fact that a_j^- is the adjoint of a_j^+ comes from the relation $D_n^-(j) = (D_n^+(j))^*$ as in the proof of Proposition (1). This implies that the operators X_j, defined by (37) are symmetric on H_μ^0 and the relation (11) holds.

The commutativity of the family $(X_1, \ldots, X_d)$ comes from the relations (19), (20), (21) as follows.

For each $i, j = 1, \ldots, d$ and $n \in \mathbf{N}$ we have

$$X_i \, X_j \, |\bar{n} >=$$

$$(a^+(i) + a^0(i) + a^-(i)) \, (a^+(j) + a^0(j) + a^-(j)) \, |\bar{n} >=$$

$$(a^+(i) \, a^+(i) + a^+(i) \, a^+(0) + a^+(i) \, a^+(-)) \, |\bar{n} > +$$

$$(a^0(i) \, a^+(i) + a^0(i) \, a^+(0) + a^0(i) \, a^+(-)) \, |\bar{n} > +$$

$$(a^-(i) \, a^+(i) + a^-(i) \, a^+(0) + a^-(i) \, a^+(-)) \, |\bar{n} >=$$

$$(D_{n+1}^+(i) \, D_n^+(j) + D_n^+(i) \, D_n^0(j) + (D_{n-1}^+(i) \, D_n^-(j)) \, |\bar{n} > +$$

$$(D_{n+1}^0(i) \, D_n^+(j) + D_n^0(i) \, D_n^0(j) + (D_{n-1}^0(i) \, D_n^-(j)) \, |\bar{n} > +$$

$$(D_{n+1}^-(i) \, D_n^+(j) + D_n^-(i) \, D_n^0(j) + (D_{n-1}^-(i) \, D_n^-(j)) \, |\bar{n} >=$$

$$D_{n+1}^+(i)\, D_n^+(j)\, |\bar{n}> \; + (D_{n-1}^+(i)\, D_n^+(j) + D_n^0(i)\, D_n^0(j) + D_{n+1}^-(i)\, D_n^+(j))\, |\bar{n}> \; +$$

$$(D_n^-(i)\, D_n^0(j) \; + \; D_{n-1}^0(i)\, D_n^-(j) \; + \; D_{n-1}^-(i)\, D_n^-(j))\, |\bar{n}> \; +$$

$$(D_{n+1}^0(i)\, D_n^+(j) \; + \; D_n^+(i)\, D_n^0(j))\, |\bar{n}>$$

The relations (19), (20), (21) imply that the above expression is equal to

$$X_j\, X_i\, |\bar{n}>$$

which proves the commutativity.

By assumption, each $D_n^+(j)$ is injective. Therefore the operators a_j^+ are injective.

By assumption, for each $n \in \mathbf{N}$ the linear span of the ranges of the operators $D_n^+(j)$ ($j = 1, ..., d$) is the whole of V_n. Therefore the vectors of the form (28) are total in $\mathcal{H}$. Finally denote $\mathcal{P}_D$ the (polynomial) algebra generated by the X_j and the identity. Then $\Phi \in \mathcal{P}_D \cdot \Phi$ and since, due to (31)

$$(a_j^+ \; + \; a_j^0 \; + \; a_j^-)\, \Phi \; = \; a_j^+\, \Phi \; + \; D_0^0(j)\, \Phi$$

$D_0^0(j)$ is a constant, also the vectors of the form $a_j^+\, \Phi$ are in $\mathcal{P}_D \cdot \Phi$. Now suppose, by induction, that

$$\bigoplus_{k=0}^n V_k \; \subseteq \mathcal{P}_D \cdot \Phi$$

Then, because of the relation (11), for any $|\bar{n}> \in V_n$, one has

$$X_j\, |\bar{n}> = \; P_{n+1}\, X_j\, |\bar{n}> \; + P_n\, X_j\, |\bar{n}> \; + P_{n-1}\, X_j\, |\bar{n}>$$

and, by the induction assumption

$$P_n\, X_j\, |\bar{n}> \; + P_{n-1}\, X_j\, |\bar{n}> \in \bigoplus_{k=0}^n V_k \; \subseteq \mathcal{P}_D \cdot \Phi$$

It follows that also

$$X_j \, |\bar{n}> \, - \, (P_n \, X_j \, |\bar{n}> \, + P_{n-1} \, X_j \, |\bar{n}> \, = a_j^+ \, |\bar{n}> \, = \, D_n^+(j) \, |\bar{n}> \, \in \, \mathcal{P}_D \cdot \Phi$$

The surjectivity condition then implies that $V_{n+1} \subseteq \mathcal{P}_D \cdot \Phi$ and therefore the thesis follows by induction.

The interacting Fock space defined by the theorem (**??**) will be denoted

$$\mathcal{H}_D \tag{38}$$

Theorem 3 *Suppose all the conditions of Theorem 2 are fullfilled. Then there exists a state μ on the *-algebra $\mathcal{A}$ of the polynomials in d variables such that, denoting $L^2(\mathbf{R}^d, \mu)$ the GNS space of the pair $\{\mathcal{A}, \mu\}$, there exists an unitary isomorphism*

$$U \,:\, L^2(\mathbf{R}^d, \mu) \longrightarrow \mathcal{H}_D$$

such that for any $j = 1, ..., d$

$$U \, X_j \, U^* \,=\, a_j^+ \,+\, a_j^0 \,+\, a_j^- \tag{39}$$

$$U \, 1 \,=\, \Phi$$

Proof: We define

$$\mu(P) \,=\, <0, P(X_1, ..., X_d) \,|0>, \quad \text{for each polynomial} \quad P \in \mathcal{A}$$

It is clear that μ is a state on $\mathcal{A}$. Define

$$U \,:\, L^2(\mathbf{R}^d, \mu) \longrightarrow \mathcal{H}_D$$

by

$$U(P) \,=\, P(X_1, ..., X_d)\bar{0}, \quad \text{for each polynomial} \quad P \in \mathcal{A}$$

It is easy to verify that U is an unitary isomorphism wich satisfies (39).

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