Time reflected Markov processes

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Indice

Abstract

A classical stochastic process which is Markovian for its past filtration is also Markovian for its future filtration. We show with a counterexample based on quantum liftings of a finite state classical Markov chain that this property cannot hold in the category of expected Markov processes. Using a duality theory for von Neumann algebras with weights, developed by Petz on the basis of previous results by Groh and Kümmerer, we show that a quantum version of this symmetry can be established in the category of weak Markov processes in the sense of Bhat and Parthasarathy. Here time reversal is implemented by an anti–unitary operator and a weak Markov process is time reversal invariant if and only if the associated semigroup coincides with its Petz dual. This construction allows to extend to the quantum case, both for backward and forward processes, the Misra–Prigogine–Courbage *internal time operator* and to show that the two operators are intertwined by the time reversal anti–automorphism.

1 Introduction:

Classical Markov process are the prototype example of irreversible process yet they enjoy a weak form of time simmetry, in the sense that the classical Markov property, which is usually formulated in a non simmetric way with respect to past and future, has an equivalent formulation which is invariant under exchange of the past and future σ–algebras. Why this implies a weak form of reversibility is explained at the end of section (2.) below. A stronger form of statistical reversibility, corresponding to the fact that the original process and the time reversed one have the same joint probabilities (or, equivalently, that the time reversal $t \mapsto -t$ induces a measure preserving transformation) was studied by Hunt and also Nelson and is equivalent to the self–adjointness of the Markovian semigroup with respect to the invariant measure. This condition is called *detailed balance* in physics. In the quantum case if both the past and future algebras are expected (in the sense of existence of a Umegaki conditional expectation compatible with the state) the simmetry between past and future, with respect to the Markov property is preserved. However, in section (5.) below, we construct a counterexample showing that, even in the simplest situation of quantum liftings of finite state Markov chains, this property does not hold. This poses the following problem: *is it possible to establish a past–future simmetry in the general framework of quantum Markov processes?*

In the present paper we prove that the above mentioned problem has an affirmative solution in the framework of the weak Markov process of Bhat and Parthasarathy [BhP94], [BhP95]. Our construction is based on a duality theory for completely positive maps on von Neumann algebras developed by Petz [Pz88] (related results in a similar direction were obtained by Groh and Kümmerer [GroKü82]). We also prove that, in analogy with the classical case, the time reversal invariance of the process is equivalent to the self–adjointness of the Markov semi–group, but now the notion of adjoint has to be interpreted in the sense of Petz duality. A difference with the classical case is that, while here the time reversal is implemented by a unitary transformation, in our approach it is implemented by a anti–unitary transformation. This is in harmony with the usual description of time reversal in quantum field theory.

Our results have also an implication concerning the general notion of stochastic process in quantum probability. Nowadays it is usually accepted the idea that a quantum stochastic process is defined by a 1–parameter family of homomorphisms (not identity preserving in the case of weak processes) from a state algebra to a sample algebra [AFL82].

However we are able to establish a past–future simmetry for weak Markov processes only if the forward process is described by homomorphisms and the backward by anti–homomorphisms. This suggests that the very notion of stochastic process should be enlarged by putting on the same ground homomorphisms and anti–homomorphisms.

There seem to be no mathematical or physical obstructions to such an extension. In fact from the physical point of view this is quite natural because what we can really measure experimentally is the statistics of real valued observables, which correspond to real abelian subalgebras, where the two notions coincide. The implications of this extension are interesting and they shall be discussed elsewhere.

2 Classical Markov process

Consider a classical stochastic process indexed by a partially ordered set T (*time*). For each $t \in T$ let

$$\mathcal{A}_{t]} \quad , \quad \mathcal{A}_t \quad , \quad \mathcal{A}_{[t} \tag{1}$$

denote the algebras of all bounded measurable functions with respect to three σ–algebras of the process and assume that, for $s \leq t$,

$$\mathcal{A}_{s]} \subseteq \mathcal{A}_{t]} \quad ; \quad \mathcal{A}_{[t} \subseteq \mathcal{A}_{[s} \quad ; \quad \mathcal{A}_t \subseteq \mathcal{A}_{t]} \cap \mathcal{A}_{[t} \tag{2}$$

The conditional expectations of the process, associated to these three algebras, shall be respectively denoted

$$E_{t]} : \mathcal{A} \to \mathcal{A}_{t]} \quad ; \quad E_t ;\ \mathcal{A} \to \mathcal{A}_t \quad ; \quad E_{[t} : \mathcal{A} \to \mathcal{A}_{[t} \tag{3}$$

The process is said to be Markovian, with respect to the triples (121), if for each $t \in T$ one of the following equivalent properties:

$$E_{t]}(\mathcal{A}_{[t}) \subseteq \mathcal{A}_t \tag{4}$$

$$E_{[t}(\mathcal{A}_{t]}) \subseteq \mathcal{A}_t \tag{5}$$

$$E_t(a_{t]}a_{[t}) = E_t(a_{t]})E_t(a_{[t}) \quad ; \quad \forall a_{t]} \in \mathcal{A}_{t]} \ , \ a_{[t} \in \mathcal{A}_{[t} \tag{6}$$

is satisfied. By restricting the conditional expectations to the fixed time algebras we obtain the forward and the backward operators respectively

$$P^{t,s} := E_{s]} \quad | \quad \mathcal{A}_t \quad ; \quad \tilde{P}^{s,t} := E_{[t} \quad | \quad \mathcal{A}_s \tag{7}$$

which are completely positive linear maps

$$P^{t,s} : \mathcal{A}_t \to \mathcal{A}_s \quad ; \quad \tilde{P}^{s,t} : \mathcal{A}_s \to \mathcal{A}_t$$

satisfying the evolution equations, for $r < s < t$:

$$P^{s,r}P^{t,s} = P^{t,r} \quad ; \quad \tilde{P}^{s,t}\tilde{P}^{r,s} = \tilde{P}^{r,t} \tag{8}$$

The two evolutions are related by the identity:

$$\varphi_s(aP^{t,s}(b)) = \varphi_t(\tilde{P}^{s,t}(a)b) \quad , \quad s \leq t \in \mathbb{R} \quad , \quad a \in \mathcal{A}_s \quad , \quad b \in \mathcal{A}_t \tag{9}$$

In fact, using the backward Markov property one has, in the above notations:

$$\varphi(ab) = \varphi(aE_{s]}(b)) = \varphi_s(aP^{t,s}(b)) \tag{10}$$

and, using the forward Markov property:

$$\varphi(ab) = \varphi(E_{[t}(a)b) = \varphi_t(\tilde{P}^{s,t}(a)b) \tag{11}$$

This is a completely general fact: it does not use stationarity and only requires a partial order on T. In particular it holds also for Markov fields. Let now $t_o < t_1 < \ldots < t_n$ be a totally ordered chain in T and let $x_{t_\alpha} \in \mathcal{A}_{t_\alpha}$ then the time ordered correlation kernels $\varphi(x_{t_o} \ldots x_{t_n})$ are given by

$$\varphi_{t_o}(x_{t_o} P^{t_1,t_o}(x_{t_1} P^{t_2,t_1}(\ldots P^{t_n,t_{n-1}}(x_{t_n})\ldots))) \tag{12}$$

But, applying iteratively (116) we see that they can also be given by

$$\varphi_{t_n}(\tilde{P}^{t_{n-1},t_n}(\ldots \tilde{P}^{t_1,t_2}(\tilde{P}^{t_o,t_1}(x_{t_o})x_{t_1})x_{t_2} \ldots x_{t_{n-1}})x_{t_n}) \tag{13}$$

From (116) it follows that P^{t_2,t_o} extends uniquely to a Hilbert space contraction, still denoted P^{t_1,t_o}:

$$P^{t_1,t_o} : L^2(\mathcal{A}_{t_1}, \varphi_{t_1}) \to L^2(\mathcal{A}_{t_o}, \varphi_{t_o})$$

and $\tilde{P}^{t_o,t_1}$ is the restriction of the Hilbert space adjoint of the contraction (6) to $\mathcal{A}_{t_o}$. The identity of the expressions (119) and (120), follows, by induction, from (116). This implies that the past–future symmetry of a classical Markov process has its roots in the fact that the Hilbert space adjoint of a classical Markovian semigroup is still Markovian (i.e. positivity and identity preserving at the algebraic level). We shall see in the following that precisely this last property is broken at a quantum level, and this is the reason why at this level a more sophisticated analysis of the past–future symmetry is necessary. Putting, in (116), $a = 1$ we find, for $t_o < t_1$

$$\varphi_{t_o} \cdot P^{t_1,t_o} = \varphi_{t_1} \tag{14}$$

and putting, in (9), $b = 1$ we find

$$\varphi_{t_1} \cdot \tilde{P}^{t_o,t_1} = \varphi_{t_o} \tag{15}$$

In particular, for any choice of the initial state φ_{t_o},

$$\varphi_{t_o} \cdot P^{t_1,t_o} \cdot \tilde{P}^{t_o,t_1} = \varphi_{t_o} \quad ; \quad \varphi_{t_1} \cdot \tilde{P}^{t_o,t_1} \cdot P^{t_1,t_o} = \varphi_{t_1} \tag{16}$$

Notice that the relation (63) is compatible with the irreversibility of the evolutions $P^{t,s}$, $\tilde{P}^{s,t}$ because once given $P^{t,s}$ (resp. $\tilde{P}^{s,t}$) the adjoint semigroup $\tilde{P}^{s,t}$ (resp. $P^{t,s}$) depends on the initial state φ_{t_o} (resp. the final state $\varphi_{\tilde{t}_o}$). Nevertheless the identity (16) can be considered as a kind of reversibility of the evolutions $P^{t,s}$, $\tilde{P}^{s,t}$.

3 The reconstruction theorem: classical case

If T is totally ordered (as it will be assumed from now on) then:

i) if we fix an initial time t_o and an initial state φ_{t_o}, then we can reconstruct the process $t \geq t_o$ on using $P^{t,s}$ with $t_o \leq s \leq t$ ii) if we fix a final time $\tilde{t}_o$ and a final state $\varphi_{\tilde{t}_o}$ then we can reconstruct the process for all $t \leq \tilde{t}_o$ using $\tilde{P}^{s,t}$ with $s \leq t \leq \tilde{t}_o$.

Notice that the process in (i) is defined only for $t \geq t_o$ and the one in (ii) only for $t \leq t_o$. In the time interval $[t_o, \tilde{t}_o]$ both processes are defined and the correlation kernels (118) can be given either by expression (119) or (120). Summing up: if a Markov process indexed by $\mathbb{R}$ is given a priori, then there is complete symmetry between past and future, in the sense that, replacing the usual time order in $\mathbb{R}$ ($\leq$) by its opposite one ($\geq$) simply amounts to exchange the past and the future filtrations and the corresponding semigroups.

However from a constructive point of view this symmetry is broken because, given the forward evolution $P^{t,s}$ (resp. the backward evolution $\tilde{P}^{s,t}$), one has to give the whole past evolution $(\varphi_s)_{s \leq t_o}$ (resp. the whole forward evolution $(\varphi_s)_{s \geq \tilde{t}_o}$) in order to reconstruct process on $\mathbb{R}$.

This means that we can replace the interval $[t_o, \tilde{t}_o]$ by an interval of the form $[t_o, +\infty)$ or $(-\infty, \tilde{t}_o]$. However the fact that, for the reconstruction theorem, an initial (or final) state is required, represents an obstruction to a completely time symmetric construction of a Markov process. This obstruction can be overcome if one of the evolutions admits an invariant state. Suppose, and we shall always do so in the following, that all the $\mathcal{A}_s$ are isomorphic to a fixed algebra $\mathcal{A}_o$

$$\mathcal{A}_s \equiv \mathcal{A}_o \; ; \quad \forall\, s \in T \tag{17}$$

and let

$$j_s : \mathcal{A}_o \to \mathcal{A}_s \tag{18}$$

denote a fixed isomorphism. Then for $s \leq t$ we can identify $P^{t,s}$ and $\tilde{P}^{s,t}$ with operators from $\mathcal{A}_o$ to $\mathcal{A}_o$ and, with this identification one has

$$j_s \circ P^{t,s} = E_{s]} \circ j_t \qquad ; \qquad j_t \circ \tilde{P}^{s,t} = E_{[t} \circ j_s \tag{19}$$

Lemma 1 *In the above notations and under assumptions (121), (5), the following conditions are equivalent*

$$\varphi \cdot j_s = \varphi \circ j_t =: \varphi_o \in \mathcal{S}(\mathcal{A}_o) \quad ; \quad \forall\, s, t \in \mathbb{R} \tag{20}$$

$$\varphi_o \cdot P^{t,s} = \varphi_o \tag{21}$$

$$\varphi_o \cdot \tilde{P}^{s,t} = \varphi_o \tag{22}$$

If moreover $\varphi_o \in \mathcal{S}(\mathcal{A}_o)$*,* $P^{t,s}$ *and* $\tilde{P}^{s,t}$ *are such that*

$$\varphi_o(\tilde{P}^{s,t}(a)b) = \varphi_o(aP^{t,s}(b)) \ ; \ \forall s \le t\mathbb{R} \ , \ \forall a, b \in \mathcal{A}_o \tag{23}$$

then both (4) and (5) hold.

Proof. Using (5) we have

$$\varphi(j_s(a)j_t(b)) = \varphi(j_s(aP^{t,s}(b))) = \varphi(j_t(\tilde{P}^{s,t}(a)b)) \tag{24}$$

If (4) holds then, putting $b = 1$ in (6) we find

$$\varphi_o(a) = \varphi \cdot j_t(\tilde{P}^{s,t}(a)) = \varphi_o(\tilde{P}^{s,t}(a))$$

which is (21). Putting $a = 1$ in (113) we find

$$\varphi_o(b) = \varphi \cdot j_t(P^{t,s}(b)) = \varphi_o \cdot P^{t,s}(b)$$

which is (22). Conversly, if (21) holds, then putting $a = 1$ in (6) we get

$$\varphi_t(b) = \varphi_s(P^{t,s}(b)) = \varphi_o(P^{t,0}(b)) = \varphi_o(b)$$

and similarly for (22).

Lemma 2 *Let (18) be satisfied and let* φ_o*,* $P^{t,s}$*,* $\tilde{P}^{s,t}$ *satisfy condition (6). Then there exists a, unique up to stochastic equivalence, (symmetric) Markov process indexed by* T *over* $\mathcal{A}_o$ *such that, for any* $n \in \mathbb{N}$*,* $t_o < \ldots < t_n$ *and for any* $x_o, \ldots, x_n, y_o, \ldots, y_n \in \mathcal{A}_o$ *one has:*

$$\varphi(j_{t_o}(x_o^*) \ldots j_{t_n}(x_n^*) j_{t_n}(y_n) \ldots j_{t_o}(y_o)) =$$

$$= \varphi_o(x_o^* P^{t_1,t_o}(x_1^* P^{t_2,t_1}(\ldots P^{t_n,t_{n-1}}(x_n^* y_n) \ldots) y_1) y_o)$$
$$= \varphi_o(x_n^* \tilde{P}^{t_{n-1},t_n}(\ldots x_2^* \tilde{P}^{t_1,t_2}(x_1^* \tilde{P}^{t_o,t_1}(x_o^* y_o) y_1) y_2 \ldots) y_n) \tag{25}$$

Proof. Since $\mathcal{A}_o$ is commutative the identity (114) is equivalent to (**??**) or (**??**), and each of these ones is enough to construct a classical Markov process. For a stationary processes indexed by $\mathbb{R}$ it is known that the evolution is a semigroup:

$$P^{t,s} = P^{t-s} \qquad ; \qquad \tilde{P}^{s,t} = \tilde{P}^{t-s} \tag{26}$$

so that the relation (6) becomes:

$$\varphi_o(aP^t(b)) = \varphi_o(\tilde{P}^t(a)b) \tag{27}$$

In this case (114) becomes

$$\begin{aligned} &\varphi(j_{t_o}(x_o^*)\dots j_{t_n}(x_n^*)j_{t_n}(y_n)\dots j_{t_o}(y_o)) = \\ &= \varphi_o(x_o^* P^{t_1-t_o}(x_1^* P^{t_2-t_1}(\dots P^{t_n-t_{n-1}}(x_n^* y_n)\dots)y_1)y_o) \\ &= \varphi_o(x_n^* \tilde{P}^{t_n-t_{n-1}}(\dots x_2^* \tilde{P}^{t_2-t_1}(x_1^* \tilde{P}^{t_1-t_o}(x_o^* y_o)y_1)y_2\dots)y_n) \end{aligned} \tag{28}$$

Notice that this implies that, even if the semigroups $\tilde{P}^t, P^t$ are defined only for $t \geq 0$, the associated processes can be defined on the whole real line.

4 Classical time reflection invariance

From now on we shall suppose that the index set T is equal to $\mathbb{R}$. In this case the map

$$t \in \mathbb{R} \mapsto -t \in \mathbb{R} \tag{29}$$

is called *time reflection*. This induces the transformation

$$j_t \mapsto j_{-t} \tag{30}$$

of the stochastic process (j_t) and we want to know when the time reflected process (j_{-t}) is stochastically equivalent to the old one (j_t). We call this property: *statistical reversibility*. By definition of stochastic equivalence this means that for any $t_o < t_1 < \ldots < t_n$ the time ordered correlation kernels (which are refered to the increasing order of time):

$$\varphi(j_{t_o}(x_o^*)\dots j_{t_n}(x_n^*)j_{t_n}(y_n)\dots j_{t_o}(y_o)) \tag{31}$$

should be equal to the time reflected ones (which are refered to the decreasing order of time):

$$\varphi(j_{-t_o}(x_o^*)\dots j_{-t_n}(x_n^*)j_{-t_n}(y_n)\dots j_{-t_o}(y_o)) \tag{32}$$

Using the future conditional expectation, we see that (5) is equal to

$$\varphi_o(x_o^*\tilde{P}^{-t_o,-t_1}(x_1^*(\ldots\tilde{P}^{-t_1,-t_2}(x_{n-1}^*\tilde{P}^{-t_n,-t_{n-1}}(x_n^*y_n)y_{n-1})\ldots)y_1)y_o) \tag{33}$$

But we kow that the correlation kernels (4) are also given by (**??**). By comparing the two expressions we conclude that the time ordered correlation kernels of the time reflected (j_{-t}) of a backward Markov process (j_t) are obtained from those of the original process, i.e. (**??**), simply replacing the evolution $P^{t,s}$ by the evolution $\tilde{P}^{-t,-s}$. It follows that the two shall be equal if and only if $\forall\, x_o, x_1, y_o, y_1 \in \mathcal{A}_o$ one has

$$\varphi_o(x_o^*\tilde{P}^{-t_1,-t_0}(x_1^*y_1)y_o) = \varphi_o(x_o^*P^{t_1,t_o}(x_1^*y_1)y_o) \tag{34}$$

and this is equivalent to

$$P^{t_1,t_o} = \tilde{P}^{-t_1,-t_o} \tag{35}$$

where the identity is meant on the GNS representation space of φ_o. Finally, if the process is stationary, then also both evolutions $P^{t,s}$ and $\tilde{P}^{s,t}$ are semigroups and one has

$$P^{s,t} = P^{t-s} \qquad ; \qquad \tilde{P}^{-t_1,-t_o} = \tilde{P}^{t_1-t_o}$$

so that condition (114) becomes

$$P^t = \tilde{P}^t \tag{36}$$

Summing up the above discussion:

Theorem 1 *label1 The process (j_t) and its time reflection (j_{-t}) are stochastically equivalent if and only if (114) holds. In this case there exists a unitary operator U, in the GNS space of the process, such that*

$$Uj_tU^* = j_{-t} \tag{37}$$

In the stationary case this happens if and only if (115) holds, i.e. the if the semigroup P^t and its dual $\tilde{P}^t$, defined by the adjoint in the GNS representation space of φ_o, coincide.

5 Insufficiency of the expected Markov processes

As shown in [Ac78], [AFL82], if one postulates the existence of Umegaki conditional expectations and if one restricts one's attention to the time ordered correlation kernels, then the theory of classical Markov processes, as outlined in the preceeding sections, is translated without difficulties to the case in which all the algebras are interpreted as general von Neumann algebras. Following [Ac78] we shall call the processes obtained in this way *expected Markov processes*. This category of processes has been extensively studied by Kümmerer [Kü85] (cf. also [Sau86]). In particular the discussion of sections (5), (4) and (5) remains valid and this naturally leads to the notion of Markov semigroup on a von Neumann algebra.

Definition 1 *Let $\mathcal{A}_o$ be a von Neumann algebra acting on a complex separable Hilbert space $\mathcal{H}_o$. A one parameter family $(P^t,\ t \geq 0)$ of normal completely positive maps on $\mathcal{A}_o$ with the properties*

$$P^t(1) = 1 \quad ; \quad P^o = 1,\ P^s \circ P^t = P^{s+t} \quad ; \quad s, t \geq 0$$

is called a Markov semigroup. We say a state φ_o is **invariant** *for (P^t) if*

$$\varphi_o(P^t(x)) = \varphi_o(x) \qquad ; \qquad \forall t \geq 0$$

For the moment we keep the discussion at a purely algebraic level and do not introduce continuity properties on the t–dependence of the semigroup (these will be discussed in section (10)).

However in the noncommutative case the discussion of the previous sections, breaks down for two important reasons:
(i) If we want to reconstruct the process from its time ordered correlations, we need a further weakening of the notion of Markov process. This was done by Bhat and Parthasarathy [BhP94], [BhP95] and we shall discuss their construction in section (7.).
(ii) The category of expected Markov processes is too narrow because, in the largest class of examples of continuous time Markov processes presently known, those constructed by stochastic calculus, the generic situation is that the past filtration of the process is expected, but the future one is not. In

the following we shall discuss an example of this phenomenon.

From now on we shall only consider stationary processes $j_t : \mathcal{A}_o \to \mathcal{A}$ such that each j_t is injective so that, in particular:

$$\varphi \circ j_t = \varphi_o \qquad ; \quad \forall\, t \in \mathbb{R}$$

We shall assume from now on that $\mathcal{A}_o$ is a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}_o$ and that φ_o is a faithful normal state on $\mathcal{A}_o$. We know from Section (2) that, if both the past and the future filtration of the process j_t are expected then there exists a semigroup $\tilde{P}^t$ such that, for any a, $b \in \mathcal{A}_o$

$$\varphi_o(aP^t(b)) = \varphi_o(\tilde{P}^t(a)b) \tag{38}$$

But a theorem of [FGKV77] asserts that:

Theorem 2 *Condition (121) holds if and only if for each* $s, t \in \mathbb{R}$

$$P^t \sigma^s = \sigma^s P^t \tag{39}$$

where σ^s *is the modular group of* φ_o*.*

We use this theorem as follows:
(i) First we briefly describe the notion of *lifting* of a classical Markov process into a quantum one by means of quantum stochastic calculus.
(ii) Then, using the full classification of the possible GKSL forms of a bounded generator of a quantum Markovian semigroup on $\mathcal{B}(\mathcal{H})$, we prove the following

Proposition 1 *There exist finite state, stationary classical Markov chains whose quantum lifting, built by means of quantum stochastic calculus, exists but does not satisfy (5).*

From this we deduce *Corollary (4).* There exist quantum Markov processes

whose future filtration is not expected but whose past filtration is. **Proof**.

Take a quantum lifting of a classical Markov process as in Proposition (4). Then its past filtration is expected because any quantum lifting built by means of quantum stochastic calculus has this property, but the future filtration of this lifting cannot be expected otherwise, because of Theorems (5) and of the remark preceeding it, the identity (5) would hold, against our assumption. Before proving Proposition (4) we recall some basic notions which

shall be needed in the proof. The lifting of a classical Markov process into a quantum one by means of quantum stochastic calculus, depends on the choice of one explicit form of its generator. We shall only consider normal, norm continuous, Markov semigroups on the algebra of all bounded operators on a Hilbert space $\mathcal{H}_o$: $\mathcal{A}_o = \mathcal{B}(\mathcal{H}_o)$. According to the characterization of the generators of such semigroups, first obtained by [GKS76], [Li76] and later generalized by several other authors, if $\mathcal{L}$ is such a generator, its GKSL form is

$$\mathcal{L}(x) = Y^*x + xY + \sum_{k\geq 1} L_k^* x L_k \qquad ; \ x \in \mathcal{A}_o \tag{40}$$

where $Y \in \mathcal{A}_o$ is the generator of a norm continuous contractive semigroup on $\mathcal{H}_o$ and $L_k, \ k \geq 1$ is a family of bounded operators so that $\sum_k L_k^* x L_k \in \mathcal{A}_o$ whenever $x \in \mathcal{A}_o$ and the sum is meant in the sense of strong operator convergence. The choice of the operators $(Y, L_k, \ k \geq 1)$ is however not unique. A complete description of the possible choices is known in case $\mathcal{A}_o = B(\mathcal{H}_o)$ [Pa92], and we describe it in the following. If $Y', L'_k, \ k \geq 1$ is another family of operators satisfying (4), then there exist scalars $\lambda \in \mathbb{R}$ and $\lambda^i_j \in \mathbb{C}$ so that $Y' = Y + i\lambda$ and $L'_i = \sum_j \lambda^i_j L_j$ where $\Lambda = (\lambda^i_j)$ is an unitary matrix on $\ell^2(\mathbb{Z}_+)$. When $\mathcal{H}_o$ is finite dimensional, we may choose $L_k = 0$ except for finitely many $k \geq 0$. In such a case (P^t) commutes with (σ_t) if and only if there exists a family $(Y, L_k \ k, \geq 1)$ satisfying (4) so that

$$\sigma_t(Y) = Y \qquad ; \quad t \in \mathbb{R}$$

and

$$\sigma_t(L_k) = e^{i\omega_k t} L_k$$

for some scalars $\omega_k \in \mathbb{R}, \ k \geq 1$. **Proof of Proposition (4):** We consider

a continuous time Markov chain X with a finite state space S, which we identify with the subset $\{1, 2, \ldots n\}$ of the integers $\mathbb{Z}$, and Markov semigroup $P^t = (P^l_k(t), \ l, k \in S)$. So for each $l, k \in S$ the limit

$$q^l_k = lim_{t\to 0} \frac{1}{t}[P^l_k(t) - \delta^l_k]$$

exists and satisfies

$$q_k^l \geq 0 \ , \ \forall l \neq k \qquad ; \qquad q_k^l = -\sum_{k:k\neq l} q_k^l \leq 0$$

The Markov semigroup P^t acts on $\ell^\infty(S)$ by

$$P^t(\psi)(k) = \sum_l \psi(k) P_k^l(t)$$

and therefore its generator is $Q = (q_k^l)$. This generator can be lifted to a GKSL generator in $B(\ell^2(S))$ by the following prescriptions: choose elements $m_k^l \in \mathbb{C}$ so that

$$q_k^l = \begin{cases} |m_k^l|^2, \text{ if } & l \neq k \\ -|m_l^l|^2 \text{ if } & l = k \end{cases}$$

and consider the multiplication operators Y and the family of bounded operators $L_k, \ k \in \mathbb{Z}/\{0\}$ defined on $\ell^2(S)$ by

$$(Yf)_l = q_l^l f_l \qquad ; \qquad (L_k f)_l = m_{l+k}^l f_{l+k} \qquad ; \qquad \forall l \in S$$

where $m = (m_k^l, \ l, k \in \mathbb{Z})$ is the extension of m on $\mathbb{Z}\times\mathbb{Z}$ defined by periodicity in each variable. The unique Markov semigroup on $\mathcal{B}(\mathcal{H}_o)$, still denoted (P^t), with generator

$$\mathcal{L}(x) = xY + Y^* x + \sum_{k\neq 0} L_k^* x L_k \tag{41}$$

is a lifting of the original classical Markov semigroup in the sense that it reduces to it when restricted to $\ell^\infty(S)$, acting on $\mathcal{H}_o$ by multiplication. Several authors [Me90], [PaSi90], [Pa90], [MoSi90a], [MoSi90b], [Fa91] (cf. [Fa92] for a short description of these papers) have shown how to build, in a multiplicity of cases, starting from the right hand side of (5), a quantum Markov process with an expected past filtration which is a unitary dilation of the quantum Markov semigroup (P^t) and which, when restricted to the algebra $\ell^\infty(S)$ gives a classical process stochastically equivalent to the original Markov chain. In this sense one speaks of a *lifting* of a classical Markov chain into a quantum one by means of stochastic calculus. Now assume further that the Markov chain is irreducible and let $\pi = (\pi_k, \ 1 \leq k \leq n)$ be its invariant measure and denote $\Pi = ((\pi_j^i))$, with $\pi_j^i = \delta_j^i \pi_i$, its lifting to a density operator on $\mathcal{B}(\mathcal{H}_o)$

and σ_t the associated modular group ($\sigma_t(x) = \Pi^{it} x \Pi^{-it}$). Then Y, being a multiplication operator, is an invariant element for σ_t. We also note that

$$\sigma_t(L_k)f(l) = [\pi_{l+k}^{it}\pi_l^{-it}]m_{l+k}^l f_{l+k} \tag{42}$$

Moreover σ_t commutes with the lifting of P^t if and only if it commutes with its generator $\mathcal{L}$, given by (5), i.e.

$$\sigma_t \mathcal{L}(x) = \mathcal{L}\sigma_t(x) \ ; \quad \forall\, x \tag{43}$$

Since σ_t commutes with $\{Y, \cdot\}$, (113) holds if and only if there exists a unitary 1–parameter group of matrices $(\lambda_i^j(t))$ such that

$$\sigma_t(L_k) = \sum_j \lambda_j^k(t) L_j \ ; \quad \forall\, k \tag{44}$$

Since (114) is equivalent to

$$\sigma_t(L_k)f(l) = \sum_j \lambda_j^k(t) L_j f(l) = \sum_j \lambda_j^k(t) m_{l+j}^l f(l+j)$$

we have that (113) and (114) are compatible iff for any f, l, k

$$\pi_l^{it}\pi_{l+k}^{-it} m_{l+k}^l f(l+k) = \sum_j \lambda_j^k(t) m_{l+j}^l f(l+j) \tag{45}$$

Thus $\pi_l^{it}\pi_{-it}^{l+k} = \lambda_k^k(t)$ whenever $m_{l+k}^l \neq 0$. At this point for simplicity we also assume that each q_{l+1}^l is different from 0, thus also the m_{l+1}^l are $\neq 0$ and therefore the validity of (115) for any $f \in \ell^\infty(S)$ implies that $\pi_l^{it}\pi_{l+1}^{-it}$ is independent of $l = 1, 2, .., n-1$. Therefore the identity

$$\pi_1\pi_2^{-1} = \pi_2\pi_3^{-1} = ... = \pi_{n-1}\pi_n^{-1}$$

holds. This implies that, for some positive real number θ, one has

$$\pi_j = e^{-j\theta}\pi_1 \quad ; \quad \forall j > 1 \tag{46}$$

Summing over j we find

$$\frac{1-\pi_1}{\pi_1} = \frac{1}{\pi_1}\sum_{j=2}^n \pi_j = \sum_{i=2}^n e^{-j\theta} = e^{-\theta}\sum_{k=2}^n e^{-(j-1)\theta} = e^{-\theta}\left(\frac{1-e^{-n\theta}}{1-e^{-\theta}}\right)$$

which gives

$$\pi_1 = \left(\frac{1 - e^{-\theta}}{1 - e^{-(n+1)\theta}} \right) \tag{47}$$

It follows that, for any irreducible classical Markov chain with $q^l_{l+1} \neq 0$, having invariant measure not of the form (116), the semigroup P^t will not commute with the modular group. Ii is clear that there exist plenty of classical Markov chains whose invariant measure has not the form (116), (117). For example choosing $S = 1, 2, 3$ and any positive constants $c_l, l = 1, 2, 3$ we consider the Markov generator Q defined by $q^l_l = -c_l$ for $l = 1, 2, 3$, $q^l_{l+1} = c_l$ for $l = 1, 2$, $l^3_1 = c_3$ and all other entries are 0. Since π is an invariant measure, from the relation $(\pi_1, \pi_2, \pi_3)Q = 0$ we have $\pi_1 c_1 = \pi_2 c_2 = \pi_3 c_3$. Thus in order for π to be of the form (116), we must have $c_2^2 = c_1 c_2$. But, since c_1, c_2, c_3 are any arbitrary positive constants, we conclude that, already in dimensions three, many counterexamples can be constructed.

6 Petz's duality

Let φ_o be a faithful normal state on a von Neumann algebra $\mathcal{A}_o$. Without loss of generality we assume that $\mathcal{A}_o$ acts on a separable Hilbert space $\mathcal{H}_o$ with a cyclic and separating vector $\Phi_o \in \mathcal{H}_o$ such that $\varphi_o(x) =< \Phi_o, x\Phi_o >$ and we recall some basic facts of Tomita's theory which we are going to use.

The closure S, of the closable operator $S_o : x\Phi_o \to x^*\Phi_o$, posseses a polar decomposition $S = J\Delta^{1/2}$ and Tomita's theorem says that $\Delta^{it}\mathcal{A}_o\Delta^{-it} = \mathcal{A}_o,\ t \in \mathbb{R}$ and $J\mathcal{A}_oJ = \mathcal{A}'_o$, where $\mathcal{A}'_o$ is the commutant of $\mathcal{A}_o$. We denote $\sigma = (\sigma_t,\ t \in \mathbb{R})$ the modular automorphism group on $\mathcal{A}_o$ defined by

$$\sigma_t(x) = \Delta^{it} x \Delta^{-it} \tag{48}$$

One can prove that the map $t \mapsto \sigma_t$ admits an analytic continuation, for the strong topology on the GNS space, on the strip $\{z : |Im(z)| \leq 1\}$ and that the set of analytic elements, denoted $(\mathcal{A}_o)_z$, is a strongly dense $*$-subalgebra denoted $(\mathcal{A}_o)_z$. In particular, for $x \in (\mathcal{A}_o)_z$

$$\sigma_{i/2}(x) = \Delta^{-1/2} x \Delta^{1/2}\ ; \quad \sigma_{-i/2}(x) = \Delta^{1/2} x \Delta \tag{49}$$

Notice that

$$\sigma_{-i/2}(x^*) = \sigma_{i/2}(x)^* \tag{50}$$

$$\sigma_{i/2}(xy) = \sigma_{i/2}(x)\sigma_{i/2}(y) \tag{51}$$

$$\varphi_o(\sigma_\alpha x) = \varphi(x) \quad ; \quad \alpha \in \mathbb{C} \tag{52}$$

Recall further that for $x, y \in (\mathcal{A}_o)_z$, analytic elements of the modular automorphism group σ_t, we have

$$\varphi(x\sigma_{-i/2}(y)) =< x^*\Phi_o, Jy^*\Phi_o > \tag{53}$$

$$\varphi(\sigma_{i/2}(x)y) =< Jx\Phi_o, y\Phi_o > \tag{54}$$

Notice that, working in the GNS representation and using the right hand side of (113), (114), one can extend the maps $\sigma_{\pm i/2}$ to non necessarily analytic elements of $(\mathcal{A}_o)_z$. This shall be done in the following, but we keep the notation $\sigma_{\pm i/2}$ because it is more intuitive. The KMS relation is equivalently expressed by one of the following three conditions:

$$\varphi(xy) = \varphi(\sigma_{i/2}(y)\sigma_{-i/2}(x)) \tag{55}$$

$$\varphi_o(\sigma_{i/2}(x)z) = \varphi_o(\sigma_{i/2}(z)x) \tag{56}$$

$$\varphi_o(xz) = \varphi_o(z\sigma_{-i}(x)) \tag{57}$$

Proposition 2 *Let P be an identity preserving normal positive map on $\mathcal{A}_o$ with faithful normal invariant state φ_o. Then there exists a unique identity preserving normal positive map P' on $\mathcal{A}'_o$ so that*

$$\varphi_o(P(x)y') = \varphi_o(xP'(y')) \tag{58}$$

for all $x \in \mathcal{A}_o$ and $y \in \mathcal{A}'_o$. Moreover P' is completely positive if and only if P is completely positive and $(P')' = P$.

Proof: This is a known fact, for a proof we refer to [AcCe82].

Proposition 3 *In the above notations, let P be a completely positive identity preserving normal map on $\mathcal{A}_o$ with a faithful normal invariant state φ_o. Then the (completely positive identity preserving normal) map $\tilde{P}$, from $\mathcal{A}_o$ into itself, defined by*

$$\tilde{P}(x) := JP'(JxJ)J \qquad ; \qquad x \in \mathcal{A}_o \tag{59}$$

is characterized by one of the following equivalent conditions

$$\varphi_o(P(x)\sigma_{-i/2}(y)) = \varphi_o(\sigma_{i/2}(x)\tilde{P}(y)) \tag{60}$$

$$\varphi_o(P(x)y) = \varphi_o(\sigma_{i/2}(x)\tilde{P}(\sigma_{i/2}(y)) \tag{61}$$

Remark. We shall refer to any of these identities, first proved in [Pz84], as to *the Petz duality*. The map $\tilde{P}$ and the adjoint realtion (12) was introduced, in an abstract framework, by Groh and Kümmerer [GroKü82] and the relations (120), (111) were proved, in a particular case, by Accardi and Cecchini [AcCe82]. These relations, with P repalced by a Markov semigroup (P^t), will play a crucial role in the following. **Proof** . The following chain

of identities shows that (13) holds:

$$
\begin{aligned}
\varphi_0 \quad (P_t(x)\sigma_{-i/2}(y)) = \varphi_0(P_t(x)\Delta^{1/2}y\Delta^{-i/2}) \\
&= \langle P_t(x)^*\Phi_0, FJy\Phi_0\rangle = \langle P_t(x)\Phi_0^*, FJyJ\Phi_0\rangle \\
&= \langle P_t(x)^*\Phi_0, Fy'\Phi_0\rangle = \langle P_t(x)^*\Phi_0, y'^*\Phi_0\rangle \\
&= \varphi_0(P_t(x)y'^*) = \varphi_0(xP_t'(y'^*)) \\
&= \varphi_0(xP_t'(y')^*) = \langle x^*\Phi_0, FP_t'(y')\Phi_0\rangle \\
&= \langle x^*\Phi_0, (FJ)(JP_t'(JyJ)J\Delta^{-1/2}\Phi_0\rangle \\
&= \langle x^*\Phi_0, \Delta^{1/2}\tilde{P}_t(y)\Delta^{-1/2}\Phi_0\rangle = \varphi_0(x\sigma_{-i/2}\tilde{P}_t(y)) = \varphi_0
\end{aligned}
$$

The uniqueness of the adjoint map follows from the density of the analytic elements and it implies that $\tilde{P}_t$ is a semigroup. Since J is an isometry, it is a Markov semigroup on $\mathcal{A}_o$. Since $(\tilde{P}^t)$ is norm continuous if and only if $(\tilde{P}^t)$ is norm continuous, it is worth characterizing the class of Markov semigroup so that $P^t = \tilde{P}^t$. This problem does not seem to be simple and shall not be discussed in the following.

From Petz's duality we deduce the following two statements which will play a crucial role in the following. In the proof of the following formulas we assume that (P^t) preserves the analytic elements. This does not need to be true in general, but the resulting formulas are easily seen to hold in general by approximation arguments which here will be omitted.

Lemma 3 *Let $x, y, z \in (\mathcal{A})_z$ then, using the notation*

$$x' := \sigma_{i/2}(x) \tag{62}$$

one has

$$\varphi_o(x^*P^t(z)y) = \varphi_o(\sigma_{i/2}(z)\tilde{P}^t(\sigma_{i/2}(y)\sigma_{i/2}(x)^*) = \varphi_o(z'\tilde{P}^t(y'x'^*) \tag{63}$$

Proof. Using the KMS condition in the form (117) we have

$$\varphi_o(x^* P^t(z)y) = \varphi_o(P^t(z)y\sigma_{-i}(x^*)) \tag{64}$$

Using Petz duality the right hand side of (64) becomes

$$\varphi_o(\sigma_{i/2}(z)\tilde{P}^t(\sigma_{i/2}(y)\sigma_{-i/2}(x^*)) = \varphi_o(\sigma_{i/2}(z)\tilde{P}^t(\sigma_{i/2}(y)\sigma_{i/2}(x)^*))$$

and this proves (63). **Corollary (4)** In the above notations, the identity

$$\varphi_o(x_o^* P^{t_1-t_o}(x_1^* \dots P^{t_{n-1}-t_{n-2}}(x_{n-1}^* P^{t_n-t_{n-1}}(x_n^* y_n)y_{n-1}) \dots y_1)y_o)$$
$$= \varphi_o(y'_n \tilde{P}^{t_n-t_{n-1}}(y'_{n-1} \dots \tilde{P}^{t_2-t_1}(y'_1 \tilde{P}^{t_1-t_o}(y'_o x_o^{\prime *})x_1^{\prime *}) \dots x_{n-1}^{\prime *})x_n^{\prime *}) \tag{65}$$

holds for any $n \in \mathbb{N}$, $t_o < t_1 < \ldots < t_n \in \mathbb{R}$ and for any elements $x_o, \ldots, x_n$, $y_o, \ldots, y_n$ in $(\mathcal{A})_z$. **Proof**. From Lemma 4 the left hand side of (ref8) is equal to

$$\varphi_o(\sigma_{i/2}[x_1^* P^{t_2-t_1}(x_2^* \dots y_2)y_1]\tilde{P}^{t_1-t_o}(y'_o x_1^{\prime *})) =$$
$$= \varphi_o(\sigma_{i/2}(x_1^*)\sigma_{i/2}(P^{t_2-t_1}(x_2^* \dots y_2))\sigma_{i/2}(y_1)\tilde{P}^{t_1-t_o}(y'_o x'^*_o))$$

By the KMS condition in the form (10) this is equal to

$$\varphi_o(\sigma_{i/2}(P^{t_2-t_1}(x_2^* \dots y_2))\sigma_{i/2}(y_1)\tilde{P}^{t_1-t_o}(y'_o x'^*_o)\sigma_{-i/2}(x_1^*)$$

and, using the multiplicativity of $\sigma_{i/2}$ and the invariance of φ_o under $\sigma_{i/2}$, this is equal to

$$\varphi_o(P^{t_2-t_1}(x_2^* \dots y_2)[y_1\sigma_{-i/2}(\tilde{P}^{t_1-t_o}(y'_o x'^*_o))\sigma_{-i}(x_1^*)])$$

by Petz's duality this is equal to

$$\rho_o(\sigma_{i/2}[x_2^* P^{t_3-t_2}(x_3^* \dots y_3)y_2]\tilde{P}^{t_2-t_1}(y'_1 \tilde{P}^{t_1-t_o}(y'_o x'^*_o)x'^*_1))$$

By iterating this identity, the result follows. The following Proposition

[GroKü82] establishes a connection between the Hilbert space adjoint and the Petz adjoint of P^t

Proposition 4 *Let Q^{t*} denote the Hilbert space adjoint of P^t. Then, for any $y \in \mathcal{A}$, one has:*

$$\tilde{P}^t(\pi(y))\Phi_o = J_o Q^{t*} J_o \pi(y)\Phi_o \tag{66}$$

Proof. For $x, y \in (\mathcal{A}_o)_z$, i.e. analytic elements of the modular automorphism group σ_t, we have

$$\langle \varphi_o, \pi_o'(x)\pi_o(y)\Phi_o\rangle = \langle \pi_o(y)^*\Phi_o, J_o\pi_o(x)J_o\Phi_o\rangle = \langle J_o\Delta_o^{1/2}\pi_o(y)\Phi_o, J_o\pi_o(x)\Phi_o\rangle =$$

$$= \langle \pi_o(x)\Phi_o, \Delta_o^{1/2}\pi_o(y)\Phi_o\rangle = \langle \Delta_o^{1/4}\pi_o(x)\Phi_o, \Delta_o^{1/4}\pi_o(x)\Phi_o\rangle = (x|y)$$

$$\langle P^t(\pi(x))\Phi_o, \pi'(y)\Phi_o\rangle = \langle \pi(x)\Phi, P_t'(\pi'(y))\Phi_o\rangle = \langle \pi(x)\Phi_o, J_o\tilde{P}^t(\pi(y))J_o\Phi_o\rangle$$
$$= \langle \tilde{P}^t(\pi(y))\Phi_o, J_o\pi(x)J_o\Phi_o\rangle$$

This implies, in particular:

$$\langle Q^t\pi(x)\Phi_o, \pi'(y)\Phi_o\rangle = \langle \pi(x)\Phi_o, Q_t^*\pi'(y)\Phi = \langle J_oQ^{t*}J_o\pi(y)\Phi_o, J_o\pi(x)J_o\Phi_o\rangle$$

Therefore (66) follows. The following are simple examples semigroups which are Petz–self–adjoint. A non self–adjoint example has been dealt with in section (5). **Example (6)** A Wigner Weisskopf atom has state space $\mathcal{H} = \mathbb{C}^2$

so that $\mathcal{A} = \mathcal{B}(\mathcal{H})$ is the algebra of all complex 2×2 matrices and the states on $\mathcal{A}$ are the self-adjoint matrices with eigenvalues λ_o, λ_1 such that

$$\lambda_o \geq 0 \quad , \quad \lambda_1 \geq 0 \quad , \quad \lambda_o + \lambda_1 = 1$$

By definition the ground and the excited states of the system are respectively

$$P_o = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

the raising and lowering operators are respectively

$$A_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; A_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Consider the quantum mechanical Fokker-Planck equation on $\mathcal{A}$

$$x(t)' = -i[\omega_oP_o+\omega_1P_1, x]+c_o\{2A_+xA_- - A_+A_-x - xA_+A_-\}+c_1\{2A_-xA_+ - A_-A_+x - xA_-A_+\}$$

where $\omega_o, \omega_1 \in \mathbb{R}$ and $c_o, c_1 > 0$. One can verify that $\rho = \lambda_oP_o + \lambda_1P_1$ is an invariant state with eigenvalues $\lambda_o = c_o/(c_o + c_1)$ and $\lambda_1 = c_1/(c_o + c_1)$ and $\sigma_t(x) = \rho^{it}x\rho^{-it}$. Thus $\sigma_{i/2}(A_+) = (\lambda_o/\lambda_1)^{1/2}A_+$ and $\sigma_{i/2}(A_-) =$

$(\lambda_o/\lambda_1)^{1/2}A_-$. It is simple to check that the adjoint quantum mechanical Fokker-Planck equation on $\mathcal{A}$ is

$$x(t)' = i[\omega_o P_o + \omega_1 P_1, x] + c_1\{2A_-xA_+ - A_-A_+x - xA_-A_+\} + c_o\{2A_+xA_- - A_+A_-x - xA_+A_-\}$$

Thus the generator of the Fokker-Planck equation is Petz–symmetric if and only if $\omega_o = \omega_1 = 0$. **Example (7)** This is a simple generalization of the

example (6). We consider the Hilbert $\mathcal{H}_o = L^2(\mathbb{R}^3, dx)$, ($dx$ is the Lebesgue measure) and the skew-adjoint operators L_x, L_y, L_z on $\mathcal{H}_o$, the generators of the rotations about X, Y and Z axis respectively. As usual we set $L^{\pm} = 1/2[L_x \pm iL_y]$ the rising and lowering operators. We consider the formal generator $\mathcal{L}$ on $\mathcal{B}(\mathcal{H}_o)$ defined by formula (5.4) with $L_1 = L^+, L_2 = L^-,\ L_3 = L_z$ and $Y = -1/2\sum_{1\le k\le 3} L_k^* L_k$. That there exists a unique conservative σ-weakly continuous Markov semigroup on $\mathcal{B}(\mathcal{H}_o)$ with $\mathcal{L}$ as it's generator follows from the general criteria of Chebotarev and Fagnola [CheFa93].

This semigroup gives rise to the familiar Markov semigroup associated with spherical Brownian motion when we restrict to $L^\infty(\mathbb{R}^3, dx)$. However when we identify $\mathcal{H}_o$ with $\ell^2(\sigma(Y))$, where $S = \{|l,m>: l \ge 0, -l \le m \le l\}$ is the complete set of orthonormal eigenvectors of Y, $Y|l,m>= l|l,m>$ and consider the restriction of the Markov semigroup to $L^\infty(\sigma(Y))$, we get a continuous time Markov chain with state space indexed by S. A further analysis shows that the generator Q of the Markov chain has the following form:

$$Q_{|k,m'>,|l,m>} = \delta_k^l \begin{cases} 1/2(l+m)(l-m+1), & \text{if } m' = m-1 \\ -(l(l+1)-m^2), & \text{if } m' = m \\ 1/2(l+m+1)(l-m), & \text{if } m' = m+1 \\ 0, & \text{otherwise} \end{cases}$$

Let P_l, $l \ge 0$ be the family of eigen-subspaces of Y with eigenvalue l. Then for any scalars $\lambda_l \ge 0$, with $\sum_l (2l+1)\lambda_l = 1$, $\varphi = \sum_l \lambda_l P_l$ has trace equal to one and the associated state on $\mathcal{B}(\mathcal{H}_o)$ is invariant for (P^t). Once more we can check that the modular group commutes with (P^t) and that (P^t) is Petz-symmetric with respect to φ when $\lambda_l > 0$ for all $l \ge 0$.

7 The reconstruction theorem: forward process

In the following we give a simplified version of the Bhat–Parthasarathy construction of a weak Markov process associated to a Markov semigroup. We assume the existence of an invariant state because we want to define the process on the whole of $\mathbb{R}$ however, if one fixes an origin of time and one restricts to processes on the half–line (as Bhat and Parthasarathy do) then the construction below is not altered and even the semi–group assumption is not needed (only the Chapman–Kolmogorov equation (**??**)).

Theorem 3 *Let be given a Markov semigroup $(P^t,\ t \geq 0)$ on a von Neumann algebra $\mathcal{A}_o$, acting on a complex separable Hilbert space $\mathcal{H}_o$, and a (P^t)–invariant state φ_o on $\mathcal{A}_o$. Then there exist*
i) a Hilbert space $\mathcal{H}$, unique up to isomorphism
ii) a 1–parameter unitary group (S_t) on $\mathcal{H}$ ii) an (S_t)–invariant unit vector Φ in $\mathcal{H}$ iii) a 1–parameter family of injective $*$-homomorphisms (j_t), from $\mathcal{A}_o$ into $\mathcal{B}(\mathcal{H})$ which is a 1–cocycle for $S_t(\cdot)S_t^*$, i.e.

$$j_{t+s}(a) = S_t j_s(a) S_t^* \qquad ; \qquad \forall a \in \mathcal{A}_o \ ; \ \forall t \in \mathbb{R} \tag{67}$$

and, denoting

$$j_t(1) =: F_{t]} \tag{68}$$

one has, for any $-\infty < s \leq t < \infty$:

$$s \leq t \Rightarrow F_{s]} \leq F_{t]} \tag{69}$$

$$F_{s]} j_t(x) F_{s]} = j_s(P^{t-s}(x)) F_{s]} \tag{70}$$

moreover, if P_t is strongly continuous, then also $s \mapsto F_{s]}$ is such.
iv) the set of (time ordered) vectors

$$\{j_{t_n}(x_n) \dots j_{t_2}(x_2) j_{t_1}(x_1)\Phi \ : \ t_1 < t_2, \dots < t_n \in \mathbb{R} \ , \ x_1, x_2, \dots x_n \in \mathcal{A}_o \ , \ n = 1, 2, \dots\} \tag{71}$$

is total in $\mathcal{H}$.

Proof: Consider the $*$–algebra $\mathcal{M}$ of all $\mathcal{A}_o$ valued functions $\underline{x} : \mathbb{R} \to \mathcal{A}_o$ such that $x(r) =: x_r \neq 1$ at most for finitely many points with the pointwise operations $((\underline{x}\underline{y})_r = x_r y_r,\ (\underline{x}+\underline{y})_r = x_r + y_r,\ (\underline{x}^*)_r = x_r^*)$. The map $L : \mathcal{M}\times\mathcal{M}\to\mathbb{C}$, defined by

$$L(\underline{x},\underline{y}) = \varphi_o(x_{t_n}^* P^{t_{n-1}-t_n}(x_{t_{n-1}}^*(\ldots..x_{t_2}^* P^{t_1-t_2}(x_{t_1}^* y_{t_1})y_{t_2})\ldots y_{t_{n-1}})y_{t_n}) \qquad (72)$$

where $\underline{t} = (t_1, t_2, ..t_n)$ with $t_1 \leq t_2 < \ldots < t_n$ is the set of points t in $\mathbb{R}$ characterized by the property that either x_t or y_t are not equal to 1, is well defined and the complete positivity of (P^t) implies that it is a positive definite sesquilinear form, i.e. a pre–scalar product, on $\mathcal{M}$. By quotienting on the zero–norm sub–space and completing, we obtain a Hilbert space $\mathcal{H}$ and a map $\lambda : \mathcal{M}\to\mathcal{H}$ such that

$$< \lambda(\underline{x}), \lambda(\underline{y}) >= L(\underline{x},\underline{y}) \qquad (73)$$

Often in the following, when no confusion is possible, we will omit the symbol λ to simplify our notations. For any $t\in\mathbb{R}$ we denote

$$\mathcal{M}_{t]} := \{\underline{x}\in\mathcal{M},\ x_r = 1\ \forall r > t\}$$

and $F_{t]}$ the projection onto $\mathcal{H}_{t]}$, the closed linear span of $\{\lambda(\mathcal{M}_{t]})\}$. $\mathcal{M}_{t]}$ (resp. $\mathcal{H}_{t]}$) is an increasing filtration of subalgebras of $\mathcal{M}$ (resp. of subspaces of $\mathcal{H}$). So $F_{t]}$ is an increasing filtration of projections on $\mathcal{H}$ which, by construction, converges to the identity.

For each $t\in\mathbb{R}$ we define the operator $S_t : \mathcal{M}\to\mathcal{M}$ by the following prescription:

$$(S_t\underline{x})_r = x_{r+t} \qquad ; \qquad r\in\mathbb{R} \qquad (74)$$

Clearly (S_t) is a 1–parameter group of $*$–automorphisms of $\mathcal{H}$ and, by inspection of (113), one sees that it induces a 1–parameter unitary group of operators on $\mathcal{H}$, still denoted (S_t) and called *the shift* on $\mathcal{H}$. Denote $\Phi := \lambda(\underline{1})$ the unit vector in $\mathcal{H}$, associated with the constant function $\underline{1}$ (i.e. $(\underline{1})_r = 1$ for any $r\in\mathbb{R}$) and φ the associated state on $B(\mathcal{H})$:

$$\varphi(X) =< \Phi, X\Phi > \qquad (75)$$

Then Φ is invariant under the action of (S_t) and φ is invariant under the action of $(S_t(\cdot)S_t^*)$.

Now notice that, from the definition 113 of the scalar product, it immediately follows that, for any $t\in\mathbb{R}$ and for any element $\underline{x}\in\mathcal{M}$ such that

$x_r \neq 1$ for $r \in \{r_n > r_{n-1} > \ldots > t\}$, one has

$$(F_{t]}\underline{x})_r = \tag{76}$$
$$x_r, \quad ; \quad hboxif\ r < t \tag{77}$$
$$P^{r_k - t}(\ldots P^{r_{n-1}-r_{n-2}}(P^{r_n - r_{n-1}}(x_{r_n})x_{r_{n-1}})\ldots x_t), \quad ; \quad \text{if } r = t \tag{78}$$
$$1, \quad ; \quad \text{if } r > t \tag{79}$$

from this one easily checks that (4) holds. For any $t \in \mathbb{R}$ there is a natural $*$–homomorphism i_t from $\mathcal{A}_o$ to $\mathcal{M}_{t]} \subseteq \mathcal{M}$ defined by

$$x \in \mathcal{A}_o \mapsto i_t(x)_r = \tag{80}$$
$$x, \quad ; \quad \text{if } r = t \tag{81}$$
$$1, \quad ; \quad \text{otherwise} \tag{82}$$

This defines a $*$-homomorphism $j_t^o : \mathcal{A}_o \to \mathcal{B}(\mathcal{H}_{t]})$ by the prescription

$$j_t^o(x)\lambda(\underline{y}) = \lambda(i_t(x)\underline{y}) \tag{83}$$

for all $\underline{y} \in \mathcal{M}_{t]}$. That j_t^o is well defined follows from the inequality $P(z^*x^*xz) \leq \| x \|^2 \ P(z^*z)$ which implies that it maps zero norm elements into zero norm ones. Now we define $j_t : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by

$$j_t(x) = j_o^t(x)F_{t]} \tag{84}$$

Thus $j_t(x)$ is a $*$-homomorphism of $\mathcal{A}_o$ into $\mathcal{B}(\mathcal{H})$ and by construction (5) holds.

From the Definition (115) of the shift, it follows that, for any $s, t \in \mathbb{R}$ and for any element $a \in \mathcal{A}_o$ one has

$$S_t i_s(a) S_t^* = i_{s+t}(a) \tag{85}$$

This, together with (119), (120) and (86) implies that for any $t \in \mathbb{R}$ and for any $a \in \mathcal{A}_o$

$$j_t(a) = S_t j_o(a) S_t^*$$

which is equivalent to (121). ¿From the above construction it follows that, if $\underline{y} \in \mathcal{M}$ with $y_r = y_{r_i} \neq 1$ for $r_1 < r_2 < \ldots < r_n$, then

$$j_{r_n}(y_n)\ldots j_{r_2}(y_2)j_{r_1}(y_1)\Phi = \underline{y}$$

Thus the vectors of the form (6) are total in $\mathcal{H}$. The uniqueness, up to unitary equivalence, of a quadruple

$$\{\mathcal{H}, (S_t), \Phi, (j_t)\} \tag{86}$$

with these properties now follows from by standard arguments.
Theorem 121 justifies the following:

Definition 2 *A quadruple (86), satisfying conditions $(i), \ldots, (iii)$ of Theorem 121, is called a forward weak Markov process associated with the dynamical system $\{\mathcal{A}_o, P^t, \varphi_o\}$. If also condition (iv) is satisfied, then the process is called minimal.*

It is clear that two minimal forward weak Markov process, associated with the same dynamical system, are unitarily equivalent in the sense that there is a unitary isomorphism of the underlying spaces which intertwines the corresponding structures.

8 The reconstruction theorem: backward process

In the previous section we have shown how to associate, in a canonical way, a weak forward stationary Markov process to a Markov semigroup $(\tilde{P}^t)$ on a von Neumann algebra with an invariant state.

Our goal in this section is to construct a backward weak stationary Markov process with the following properties:
(i) The Petz adjoint $(\tilde{P}^t)$, of (P^t), is the canonical Markov semigroup associated with it
(ii) it acts on the same Hilbert space of the forward dilation
(iii) the time ordered correlation kernels of the forward and the backward process coincide (as it happens in the classical case).

The starting point for this construction shall be the identity (**??**) of Corollary (6.4).

Theorem 4 *Let be given a normal Markov semigroup $(P^t,\ t \geq 0)$ on a von Neumann algebra $\mathcal{A}_o$, acting on a complex separable Hilbert space $\mathcal{H}_o$, and a (P^t)–invariant faithful state φ_o on $\mathcal{A}_o$.*

Let $\{\mathcal{H},(S_t),\Phi,(j_t)\}$ be the forward weak Markov process, associated with this dynamical system as in Theorem **??**. Then there exist a 1–parameter family of injective $*$-anti-homomorphisms $\tilde{j}_t$ from $\mathcal{A}_o$ into $\mathcal{B}(\mathcal{H})$ which is a 1–cocycle for $S_t(\cdot)S_t^*$:

$$k_{t+s}(a)=S_tk_s(a)S_t^* \qquad ; \qquad \forall a\in\mathcal{A}_o\ ,\ s,t\in\mathbb{R} \tag{87}$$

and, denoting

$$k_t(1)=:F_{[t} \tag{88}$$

one has, for any $-\infty<s\le t<\infty$:

$$F_{[s}k_t(x)F_{[s}=k_s(\tilde{P}^{t-s}(x))F_{[s} \tag{89}$$

Moreover the set of (time anti–ordered) vectors

$$\{k_{t_n}(x_n)\dots k_{t_2}(x_2)k_{t_1}(x_1)\Phi\ :\ t_1<t_2,\dots<t_n\in\mathbb{R}\ ,\ x_1,x_2,\dots x_n\in\mathcal{A}_o\ ,\ n=1,2,\dots\} \tag{90}$$

is total in $\mathcal{H}$.

Proof. Let $\mathcal{H}$ be the Hilbert space defined in Theorem **??**. Define the future filtration $(F_{[t})$ in $\mathcal{H}$ as the family of projections onto $\mathcal{H}_{[t}$ where

$$\mathcal{H}_{[t}=\text{closed span}\{\lambda(\underline{y}):\underline{y}\in\mathcal{M},\ y_r=1\ \forall r<t\} \tag{91}$$

Define the map $*\ k_o^0$, of $\mathcal{A}_o$ into $\mathcal{B}(H_{[0})$, by:

$$k_o^0(x)\underline{y}:=\underline{y}i_o(\sigma_{-i/2}(x))=\underline{y}i_o(x^{'*}) \qquad ; \qquad \forall\underline{y}\in\mathcal{M}_{[0} \tag{92}$$

where, in the last equality we have used the notation (**??**). Then $k_o^0(x)$ is well defined for $x\in(\mathcal{A}_o)_z$ and linear. Since the map $x\mapsto x^{'}$ is multiplicative, it follows that j_o^0 is an anti–homomorphism. Moreover using the basic identity (**??**), we see that the norm of $k_o^0(x)\underline{y}$ is

$$\varphi_o(y'_n\tilde{P}^{t_n-t_{n-1}}(y'_{n-1}\dots\tilde{P}^{t_2-t_1}(y'_1\tilde{P}^{t_1-t_o}(y'_ox_ox_o^*y_o^{'*})y_1^{'*})\dots y_{n-1}^{'*})y_n^{'*})$$

and since $P^{t_1-t_o}(y'_ox_ox_o^*y_o^{'*})\le\|x_o\|^2\ P^{t_1-t_o}(y'_oy_o^{'*})$ it follows that $\|k_o^0(x)\underline{y}\|\le\|x_o\|$. Since the analytic elements are dense in the σ-weak operator topology, we extend the map k_o^0 to the whole of $\mathcal{A}_o$ by density arguments. With the same method we check that k_o^0 is a $*$–anti–homomorphism, i.e. $k_o^0(x^*)=(k_o^0(x))^*$. Now we set

$$k_o(x):=k_o^0(x)F_{[0} \tag{93}$$

and define the backward process (k_t) by

$$k_t(x) = S_t k_o(x) S_t^* \tag{94}$$

which implies (1). For any $r_1 \leq r_2 \leq \ldots r_n$ and $y_{r_i} \in (\mathcal{A}_o)_z$ we also check that

$$k_{r_1}(y_{r_1}) k_{r_2}(y_{r_2}) \ldots \tilde{j}_{r_n}(y_{r_n})\Phi = \lambda(\underline{\sigma_{-i/2}(y)})) \tag{95}$$

where $\underline{\sigma_{-i/2}(y)}_r = \sigma_{-i/2}(y_r)$ with $y_r = y_{r_i}$ if $r = r_i$ otherwise 1. Once more we appeal to (**??**), to the density of the analytic elements in the σ-weak operator topology and to the fact that the map $(t, x) \to P^t(x)$ is sequentially jointly continuous in the σ–weak operator topology, to conclude that the time–anti–ordered vectors (116) are also total in $\mathcal{H}$. To prove that for all $-\infty < s \leq t < \infty$

$$F_{[t} k_s(x) F_{[t} = k_t(\tilde{P}^{t-s}(x)) F_{[t}$$

we choose elements $\underline{y} = (y_{r_\alpha}), \underline{y'} = (y'_{r_\alpha}) \in M_{[t}$ and check that

$$< \underline{y}, k_s(x)\underline{y'} >=< \underline{y}, \underline{y'} i_s(\sigma_{-i/2}(x) >$$

$$\varphi_o(\tilde{P}^{t-s}(y_s^* \tilde{P}^{s-r_n}(y_{r_n}^* \tilde{P}^{r_{n-1}-r_n}(y_{r_{n-1}}^*(\ldots y_{r_2}^* \tilde{P}^{r_1-r_2}(y_{r_1}^* y'_{r_1}) y'_{r_2}) \ldots y'_{r_{n-1}}) y'_{r_n}) y'_s) \sigma_{-i/2}(x))$$

$$\varphi_o(\sigma_{i/2}(y_s^* \tilde{P}^{s-r_n}(y_{r_n}^* \tilde{P}^{r_{n-1}-r_n}(y_{r_{n-1}}^*(\ldots y_{r_2}^* \tilde{P}^{r_1-r_2}(y_{r_1}^* y'_{r_1}) y'_{r_2}) \ldots y'_{r_{n-1}}) y'_{r_n}) y'_s)) \tilde{P}^{t-s}(x))$$

$$=< \underline{y}, \underline{y'} i_t(\sigma_{-i/2}(\tilde{P}^{t-s}(x)) >=< \underline{y}, \underline{y'} j_t(\tilde{P}^{t-s}(x) >$$

where we have used the relation (**??**).

In analogy with Definition **??** we summarize the above construction of backward weak Markov processes in the following:

Definition 3 *A quadruple $\{\mathcal{H}, (S_t), \Phi, (k_t)\}$, such that each k_t is a $*$-anti-homomorphisms from $\mathcal{A}_o$ into $\mathcal{B}(\mathcal{H})$ satisfying the conditions (121), ..., (5) of Theorem 121, is called a minimal backward weak Markov process associated with the dynamical system $\{\mathcal{A}_o, \tilde{P}^t, \varphi_o\}$*

9 Time reversal in weak Markov processes

So far we have described forward and backward weak Markov process in the framework of non-commutative probability theory. The reader might have noticed that unlike in the classical Markov process, the backward and the forward processes are not same in the framework of weak Markov processes because of the conditions

$$j_t(1) = F_{t]} \qquad , \qquad k_t(1) = F_{[t} \tag{96}$$

Nevertheless, both processes are described in the same minimal Hilbert space $\mathcal{H}$ associated with the dynamical system $\{\mathcal{A}_o, P^t, \varphi_o\}$. Similarly we now consider the minimal Hilbert space $\tilde{\mathcal{H}}$ associated with the dynamical system $\{\mathcal{A}_o, \tilde{P}^t, \varphi_o\}$ and denote associated forward process by $(\tilde{j}_t, \tilde{F}_{t]})$ with expected semigroup $(\tilde{P}_t)$ and backward process by $(\tilde{k}_t, \tilde{F}_{[t})$ with expected semigroup (P_t). In this section we shall investigate the connection between the two family $(\mathcal{H}, j_t, F_{t]}, k_t, F_{[t})$ and $(\tilde{\mathcal{H}}, \tilde{j}_t, \tilde{F}_{t]}, \tilde{F}_{[t})$. To this goal, in the notations of Section 7, for any fixed $r \in \mathbb{R}$, we define the time reflection operator $U_r : \mathcal{H} \to \tilde{\mathcal{H}}$, with respect to time r, by

$$(U_r\underline{x})_s = \sigma_{-i/2}(x^*_{2r-s}) = \sigma_{i/2}(x_{2r-s})^* = x_{2r-s})'^* \tag{97}$$

Lemma 4 *For any $r \in \mathbb{R}$, U_r is an anti-unitary operator i.e.*

$$< U_r\underline{z}, U_r\underline{y} >_{\tilde{\mathcal{H}}} = < \underline{y}, \underline{z} >_{\mathcal{H}} \tag{98}$$

Moreover

$$U_r = S^*_r U_o S_r$$

Proof: We appeal to (**??**) and to the remark that $t_{n-1} - t_n = (2r - t_{n-1}) - (2r - t_n)$ to verify the following identities:

$$< \underline{y}, \underline{z} >_{\mathcal{H}} = \varphi_o(y^*_{t_n} P^{t_{n-1}-t_n}(y^*_{t_{n-1}}(\dots y^*_{t_2} P^{t_1-t_2}(y^*_{t_1} z_1) z_2 \dots) z_{n-1}) z_n) =$$

$$= \varphi_o(z'_{t_1} \tilde{P}^{t_1-t_2}(z'_{t_2} \tilde{P}^{t_2-t_3} z'_{t_3}) \dots \tilde{P}^{t_{n-1}-t_n}(z'_{t_n} y'^*_{t_n}) y'^*_{t_{n-1}}) \dots) y'^*_{t_2}) y'^*_{t_1})$$
$$= < \lambda(2r - \underline{t}, \sigma_{-i/2}(\underline{z}^*), \lambda(2r - \underline{t}, \sigma_{-i/2}(\underline{y}^*)) >_{\tilde{\mathcal{H}}} = < U_r\underline{z}, U_r\underline{y} >_{\tilde{\mathcal{H}}}$$

Thus U_r is well defined and anti unitary operator for each $r \in \mathbb{R}$. The second statement is a direct verification. **Corollary (2)** There exists a unique

anti-unitary operator $U_r : \mathcal{H} \to \tilde{\mathcal{H}}$ so that the following holds:

$$U_r \Phi_{\mathcal{H}} = \Phi_{\tilde{\mathcal{H}}} \tag{99}$$

$$U_r^* \tilde{F}_{t]} U_r = F_{[2r-t} \qquad ; \qquad U_r^* \tilde{F}_{[t} U_r = F_{2r-t]} \tag{100}$$

$$U_r^* \tilde{S}_t U_r = S_{-t} \tag{101}$$

$$U_r^* \tilde{k}_t(x) U_r = j_{2r-t}(x^*) \tag{102}$$

$$U_r^* \tilde{j}_t(x) U_r = k_{2r-t}(x^*) \tag{103}$$

Proof. That there exists such an U_r is clear from the last follows from last Lemmata. In order to show the uniqueness, we use (a) and one of the last two identitity to deduce (5).

Theorem 5 *There exists a unique anti-unitary involution $U_r : \mathcal{H} \to \mathcal{H}$ satisfying the following identities*

$$U_r \Phi = \Phi \tag{104}$$

$$U_r^* F_{s]} U_r = F_{[2r-s} \tag{105}$$

$$j_t(x) = U_r k_{2r-t}(x^*) U_r \tag{106}$$

$$U_r S_t U_r = S_{-t} \tag{107}$$

for all $s, r \in \mathbb{R}$, if and only if $\tilde{P}^t = P^t$ for all $t \geq 0$.

Proof. It is a simple consequence of Corollary 2.

10 Ergodic properties of semi–groups and process

A normal Markov $(P^t,\ t \geq 0)$ semigroup on $\mathcal{A}_o$ is said to be **weak*** continuous if for each fixed $y \in \mathcal{A}_o$ the map $t \to P^t(y)$ is continuous with respect to the σ-weak topology. Unlike strong continuity on a Banach space, weak* continuity need not imply that the map $(t, y) \to P^t(y)$ is jointly continuous, however the general theory [Ru] guarantees that it is sequentially jointly continuous i.e. $t_n \to t$ and $y_n \to y$ implies that $P^{t_n}(y_n) \to P^t(y)$. We recall [Sa71] that the weak (strong) operator topology and the $\sigma-$ weak ($\sigma-$ strong) operator topology coincide on bounded sets. Moreover the σ-weak operator continuity of (P^t) is equivalent to σ-strong operator continuity.

Theorem 6 *Let (P^t) be a normal Markov semigroup on $\mathcal{A}_o$ and φ be a faithful normal invariant state for (P^t). Then the following statements are equivalent:*

(a1) $(P^t,\ t \geq 0)$ is σ-weakly-continuous
(a2) $(\tilde{P}^t)$ is σ-weakly continuous.
(b1) φ_o is ergodic for (P^t) (b2) φ_o is ergodic for $(\tilde{P}^t)$.
(b) The map $t \to S_t, (\ t \in \mathbb{R})$ is continuous in the strong operator topology.

Moreover in such a case, i.e. when either (a) or (b) holds, the following statements are also equivalent:
(c) If $y \in \mathcal{A}_o$ is such that $P^t(y) = y$ for all $t \geq 0$ then y is a constant multiple of the identity
(d1) (P^t) is σ–weakly–φ_o–ergodic, i.e. for all $y \in \mathcal{A}_o$ one has

$$\sigma - weak - \lim_{t\to\infty} \frac{1}{t}\int_o^t P^s(y)ds = \varphi_o(y)1 \tag{108}$$

(d2) $(\tilde{P}^t)$ is σ–weakly–φ_o–ergodic
(e) (S_t) is ergodic.

Proof. We first show that (a) implies (b). By a standard result in semigroup theory, strong continuity of (S_t) is equivalent to weak continuity of S_t. Since the family S_t is contractive, it is enough to check that the map $t \to< h_1, S_t h_2 >$ is continuous for all h_1, h_2 in any total set. Since $\lambda(\mathcal{M})$ is total in $\mathcal{H}$ and

$$< \underline{y}, S_t\underline{y'} > \tag{109}$$

$$= \varphi_o(y^*_{r_n(t)}P^{r_{n-1}(t)-r_n(t)}(y^*_{r_{n-1}(t)}(\ldots .y^*_{r_2(t)}P^{r_1(t)-r_2(t)}(y^*_{r_1(t)}y'_{r_1(t)})y'_{r_2(t)}\ldots y'_{r_{n-1}(t)})y'_{r_n(t)})$$

where $\underline{r}(t) = \{\underline{r}\} \cup \{\underline{r'}+t\}$. Since the map $t \to r_k(t)$ is continuous in t, the sequential joint continuity of (P^t) implies that the map $t \to< \lambda(\underline{r},\underline{y}), S_t\lambda(\underline{r'},\underline{y'}) >$ is continuous. This completes the proof that (a) implies (b).

To prove the converse we first remark that, being (P^t) a contractive family, for each fixed $x \in \mathcal{A}_o$, $P^t(x)$ is uniformly bounded, hence the map $t \to P^t(x)$ is continuous in σ -weak topology whenever it is continuous in the weak operator topology. This continuity follows once we verify the map $t \to< f_1, P^t(x)f_2 >$ is continuous for all f_1, f_2 in a total set in $\mathcal{H}_o$. Since φ_o is a faithful normal state, without loss of generality we can identify $\mathcal{A}_o$ with its

GNS representation. From (5) with $n = 2$ we see that the strong continuity of (S_t) implies that the map $t \to < y_1\Phi_o, P^t(x)y_2\Phi_o >$ is continuous.

For the equivalence of (a1) and (a2) we note that it is enough if we show that for each $x' \in \mathcal{A}'_o$ the map $t \to P'_t(x')$ is continuous whenever (P^t) is σ-weakly continuous. For that purpose without loss of generality we assume that $\varphi_o(x) =< \Phi_o, x\Phi_o >$ where Φ_o is cyclic and separating. Since (P'_t) is contractive, hence uniformly norm bounded, it is enough if we verify that the map $t \to < f, P'_t(x')g >$ is continuous for all f, g in a dense subset of $\mathcal{H}_o$. To that goal we note that for any $x, y \in \mathcal{A}_o$ $< x\Phi_o, P'_t(x')y\Phi >=< \Phi_o, P^t(x^*y)x'\Phi >$ and thus Φ_o being a cyclic vector for $\mathcal{A}_o$ we obtain the required continuity from that of (P^t).

This completes the first part of the theorem.

That (c) and (d) are equivalent is well known, we refer to Frigerio [Fr78] for a proof. We will show (d) and (e) are equivalent. First we recall von Neumann's criterium [Par81] that (S_t) is ergodic if and only if it is weakly mixing, i.e. the following holds:

$$1/t \int_o^t < h_1, S_r h_2 > dr \to < h_1, 1 >< 1, h_2 > \tag{110}$$

as $t \to \infty$. Again we note that it is enough if we verify (4) for all h_1, h_2 in a total set of $\mathcal{H}$. So we once more choose $h_1 = \lambda(\underline{y})$, $h_2 = \lambda(\underline{y'})$, two elements in $\lambda(\mathcal{M})$ as in (5) and check that for all $t \geq r_n - r'_1$ we have

$$< h_1, S_t h_2 >= \tag{111}$$

$$= \varphi_\mu([y_1(P^{r_2-r_1}(y_2(\ldots P^{r_n-r_{n-1}}(y_n)\ldots)))]P^{t+r'_1-t_n}[P^{r'_2-r'_1}(\ldots P^{r'_m-r'_{m-1}}(y'_m)\ldots)y'_2)y'_1])$$

Hence (4) is equivalent to (5). The equivalence between (d1) and (d2) is analogous to that between (b1) and (b2), we omit the details. **Remark** Here we

note that the assumption that ψ_o is a faithful state, is essential in proving (b) implies (a). **Remark** The above theorem shows that it is natural to relate

the various notions of mixing (weak, strong...) for the semigroup to those of the associated weak Markov process. The following Proposition collects some known facts and a few simple observations in this direction:

Proposition 5 *Let* $(\mathcal{A}_o, P^t, \varphi_o)$ *be a* $\sigma-$ *weakly continuous dynamical semi-group with a faithful normal invariant state* φ_o *and let* $(\mathcal{H}, S_t, J_t, \Phi)$ *be its minimal Markov shift.*

(i) The following statements are equivalent: (a) For all $h_1, h_2 \in \mathcal{H}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_o^T | < h_1, S_t h_2 > - < h_1, \Phi >< \Phi, h_2 > | dt \ = 0 \tag{112}$$

(b) For all $x, y \in \mathcal{A}_o$,

$$\lim_{T\to\infty} \frac{1}{T} \int_o^T |\varphi_o(xP^t(y)) - \varphi_o(x)\varphi_o(y)| dt \ = 0 \tag{113}$$

(c) The spectrum of (S_t) *in the orthocomplement of* $\mathbb{C}\Phi$ *is continuous.*

(ii) Moreover the following statements are also equivalent: (d) For all $h_1, h_2 \in \mathcal{H}$,

$$\lim_{T\to\infty} < h_1, S_t h_2 >=< h_1, \Phi >< \Phi, h_2 > \tag{114}$$

(e) For all $x, y \in \mathcal{A}_o$,

$$\lim_{T\to\infty} \varphi_o(xP^t(y)) = \varphi_o(x)\varphi_o(y) \tag{115}$$

Proof. We shall only sketch the proof, omitting the details. That (a) implies (b) follows once we choose $f = \lambda(i_o(x^*))$ and $g = \lambda(i_o(y))$. To show that (b) implies (a), we first note that it is enough if we prove (a) for elements of the form $h_1 = \lambda(\underline{y}), h_2 = \lambda(\underline{y'})$ and this is done in the same way as in Theorem 121. That (a) and (c) are equivalent follows with the same method outlined in Theorem 3.2.4 in [Par81]. That (d) and (e) are equivalent also follows with arguments similar as in Theorem 121. Motivated by the well known notion,

Kolmogorov shift, in ergodic theory, we introduce the following definition.

Definition 4 *The dynamical system* $(\mathcal{A}_o, P^t, \varphi_o)$ *is called a* **(forward) weak K-shift** *if the minimal weak Markov process* $(\mathcal{H}, F_{t]}, \Phi, S_t)$ *associated with it satisfies*

$$\bigwedge_{t\in\mathbb{R}} F_{t]} = \lim_{t\to-\infty} F_{t]} =: p_{-\infty} = \mathbb{C}\Phi \tag{116}$$

We shall need the following well known fact that we prove for completeness.

Lemma 5 *Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space with a cyclic vector Φ and let $\varphi = \langle \Phi, (\cdot)\Phi \rangle$. Then*

$$\textit{supp } \varphi = P_\varphi := \textit{ orthogonal projection onto } [\mathcal{A}'\Phi] \tag{117}$$

Proof. If $p \in \mathcal{A}$ is the support of φ, i.e. it is such that $p^\perp \Phi = 0$, then

$$0 = a' p^\perp \Phi = p^\perp a' \Phi \ ; \quad \forall\, a' \in \mathcal{A}'$$

so that

$$0 = p^\perp P_\varphi \Leftrightarrow p P_\varphi = P_\varphi \Leftrightarrow P_\varphi \le p \tag{118}$$

But we know that $P_\varphi \in \mathcal{A}$ and $P_\varphi \Phi = \Phi$ so that

$$\varphi(a) = \langle \Phi, a\Phi \rangle = \varphi(P_\varphi a P_\varphi) \ ; \quad \forall\, a \in \mathcal{A}$$

Therefore, if p is the support of φ, then

$$P_\varphi \ge p \tag{119}$$

From (118) and (119) the thesis follows.

Lemma 6 *The following are equivalent: (i) (P^t) is ergodic (ii) $\{\mathcal{H},\ (F_{t]}),\ (S_t)\}$ is a K–shift (iii) the algebra $\mathcal{A}$, generated by $\{j_t(\mathcal{A}_0) : t \in \mathbb{R}\}$ coincides with $\mathcal{B}(\mathcal{H})$ (iv) $P_\varphi = \mid \Phi \rangle \langle \Phi \mid$*

Proof. (i) $\Rightarrow$ (ii). Because of the covariance property the projection $p_{-\infty}$, defined by (116) commutes with (S_t). Since Theorem (1) implies that (S_t) is ergodic, it follows that

$$p_{-\infty} = |\Phi >< \Phi| \tag{120}$$

so (S_t) is a K–shift.

(ii) $\Rightarrow$ (iii). Let P_φ denote the support of the state φ, obtained by restriction of $\langle \Phi, (\cdot)\Phi \rangle$ on $\mathcal{A}$. Then since $F_{t]}\Phi = \Phi$ for any $t \in \mathbb{R}$ it follows that $p_{-\infty} \ge P_\varphi$ so, if $p_{-\infty}$ has the form (), this implies $P_\varphi = |\Phi >< \Phi|$.

But P_φ is the projection onto the cyclic space $[\mathcal{A}'\Phi]$ of $\mathcal{A}'$ and this space can be 1–dimensional if and only if $\mathcal{A}' = \mathbb{C}1$ or equivalently $\mathcal{A} = \mathcal{B}(\mathcal{H})$

That (iii) is equivalent to (iv) is clear in view of Lemma (4). Finally we prove that
(iii) $\Rightarrow$ (i). Let $p_{-\infty}$ be as in (116) and suppose that $p_{-\infty} \ne |\Phi >< \Phi|$. Then, since $p_{-\infty} \ge |\Phi >< \Phi|$, it follows that

$$P_\varphi := \text{ supp } \varphi \ge p_{-\infty} > |\Phi >< \Phi|$$

But, since from Lemma (4) we know that P_φ is the projection onto $[\mathcal{A}'\Phi]$, this implies that $[\mathcal{A}'\Phi]$ is not one dimensional against our assumption that $\mathcal{A} = \mathcal{B}(\mathcal{H})$.

11 Quantum Prigogine–Misra–Courbage internal times

In this section we suppose that the semigroup (P^t) is strongly continuous so that, by Theorem **??** the same is true for the maps $t \mapsto F_{t]}$. Theorem () also implies that the same is true for the semigroup $(\tilde{P}^t)$ and the family of projections $F_{[t}$.

Let the algebra $\mathcal{A}$ be as in Lemma 6. Since $F_{t]} = j_t(1)$ is in $\mathcal{A}$ the same is true for $1 - F_{t]}$. Therefore, since $(1 - F_{t]})\Phi = 0$, it follows that Φ is not a separating vector for $\mathcal{A}$ even if φ_o is faithful. So the support of the state φ on $\mathcal{A}$ is the projection $P_\varphi \in \mathcal{A}$ projecting onto $[\mathcal{A}'\Phi]$ where $\mathcal{A}'$ is the commutant of $\mathcal{A}$. Since $(S_t(\cdot)S_t^*)$ preserves $\mathcal{A}$, it also preserves $\mathcal{A}'$ and P_φ is an invariant element $(S_t(\cdot)S_t^*)$.

In general the family of projections $Q_{t]}$ defined by

$$Q_{t]} := F_{t]} - p_{-\infty}$$

is a spectral family for the Hilbert space $\mathcal{K} := \mathcal{H} \ominus p_{-\infty}\mathcal{H}$ such that $S_t Q_{s]} S_t^* = Q_{s+t]} \ \ \forall s, t \in \mathbb{R}$. So by Mackey's imprimitivity theorem $(\mathcal{K}, Q_{t]}, S_t)$ is unitarily isomorphic to the bilateral shift on the space $L^2(\mathbb{R}; \mathcal{L})$, of the square integrable functions from $\mathbb{R}$ to some separable Hilbert space $\mathcal{L}$ or, equivalently, to the direct sum of a countable family of bilateral shifts on the Hilbert space $L^2(\mathbb{R})$. In particular (S_t) admits only infinite Lebesgue spectrum in the orthocomplement of $p_{-\infty}\mathcal{H}$. So when $(S_t, F_{t]})$ is a K-shift, the self-adjoint operator

$$T = \int_{\mathbb{R}} s dF_{s]}$$

on $\mathcal{K}$ satisfies $S_t T S_t^* = T + t1$ on $\mathcal{K}$. In particular, denoting P the generator of the shift $(S_t = e^{itP})$, we have the canonical commutation relation $[P, T] = i1$ on a dense set in $\mathcal{K}$. Following Prigogine, Misra and Courbage we call T **internal forward time**.

By applying the same argument to the backward process we say that $(\mathcal{H}, S_t, F_{[t})$ is a **(backward) weak K-shift** when

$$\bigwedge_{t \in \mathbb{R}} F_{[t} = \lim_{t \to +\infty} F_{[t} =: p_{+\infty} = \mathbb{C}\Phi \tag{121}$$

We thus obtain another imprimitivity system on $\mathbb{R}$ based on the Hilbert space $\mathcal{K} = \mathcal{H} \ominus p_{+\infty}\mathcal{H}$. It is natural to call the time operator associated to the backward imprimitivity system, i.e.

$$\tilde{T} = \int_{\mathbb{R}} t dF_{[t}$$

a **backward internal time**. By the Stone-von Neumann theorem, the two imprimitivity systems, so in particular T and $\tilde{T}$, are unitarily equivalent up to multiplicity.

In particular, if they are both K–shifts, then $p_{-\infty} = p_{+\infty} = |\Phi >< \Phi|$ and they are unitarily equivalent. Moreover one easily verifies that this unitary equivalence intertwines the stochastic processes j_t and k_t thus establishing a symmetry between past and future. However this symmetry will not be, in general, the time reversal. Since the backward and forward processes are completely symmetric, it is natural to conjecture that this unitary equivalence hods in general, i.e. independently of the fact that they are K–shifts. However a proof of this conjecture is not available at the moment. [I HAD NO TIME TO THINK CAREFULLY ABOUT THIS POINT!] More generally, to investigate the probabilistic meaning of the operators T and $\tilde{T}$, in analogy with the results of [], [], is an interesting but we shall not discuss it here.

Finally Theorem **??** also tells us that $(S_t, F_{t]})$ is a forward K-shift if and only if $(S_t, F_{[t})$ is a backward K-shift.

The results of section (9.) show that, when (P^t) is symmetric for the Petz duality, irrespective of whether $(S_t, F_{t]})$ is a K-shift or not, Theorem (9.3) shows that there exists an anti-unitary operator intertwining T and $\tilde{T}$. In such a case in fact, in the notations of section (9), $\tilde{T} = U_o^* T U_o$. That $U_o^*(\cdot)U_o$ is an anti-homomorphism is reflected in the property **??**. For example, in the context of Example **??**, **??** merely means that the forward procees associated with spin up corresponds to the backward process associated with spin down.

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