

PRODUCTS OF SEQUENTIALLY PSEUDOCOMPACT SPACES

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ABSTRACT. We show that the product of any number of sequentially pseudocompact topological spaces is still sequentially pseudocompact. The definition of sequential pseudocompactness can be given in (at least) two ways: we show their equivalence.

Some of the results of the present note already appeared in [DPSW].

According to [AMPRT, Definition 1.8], a topological space X is *sequentially pseudocompact* if and only if, for any sequence $(O_n)_{n \in \omega}$ of pairwise disjoint nonempty open sets of X , there are an infinite set $J \subseteq \omega$ and a point $x \in X$ such that every neighborhood of x intersects all but finitely many elements of $(O_n)_{n \in J}$.

In [AMPRT] X is assumed to be a Tychonoff space; however, here we assume no separation axiom, if not otherwise stated. Thus, in the present note, a topological space is *sequentially pseudocompact* if and only if it satisfies the above condition, regardless of the separation axioms it satisfies.

In this note we present some results about sequential pseudocompactness. Some of them are known, at least for one of the possible definitions of sequential pseudocompactness: see [DPSW].

In fact, in [CKS, DPSW] a slightly different definition of sequential pseudocompactness is given, not requesting the sets $(O_n)_{n \in \omega}$ to be pairwise disjoint. This notion is called *sequential feeble compactness* in [DPSW]. In the next proposition we show that the two definitions are actually equivalent. This turns out to be somewhat useful.

Proposition 1. *For every topological space X , the following conditions are equivalent.*

- (1) *X is sequentially pseudocompact.*

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- (2) For any sequence $(O_n)_{n \in \omega}$ of nonempty open sets of X , there are an infinite set $J \subseteq \omega$ and a point $x \in X$ such that, for every neighborhood U of x , the set $\{n \in J \mid U \cap O_n = \emptyset\}$ is finite.

Proof. (2) \Rightarrow (1) is trivial.

For the converse, suppose that X is sequentially pseudocompact, and let $(O_n)_{n \in \omega}$ be a sequence of nonempty open sets of X . Suppose by contradiction that

(*) for every infinite set $J \subseteq \omega$ and every point $x \in X$ there is some neighborhood $U(J, x)$ of x such that $N(J, x) = \{n \in J \mid U(J, x) \cap O_n = \emptyset\}$ is infinite. Without loss of generality, we can assume that $U(J, x)$ is open.

We shall construct by induction a sequence $(m_i)_{i \in \omega}$ of distinct natural numbers, a sequence $(J_i)_{i \in \omega}$ of infinite subsets of ω , and a sequence of pairwise disjoint nonempty open sets $(U_i)_{i \in \omega}$ such that

- (a) $U_i \subseteq O_{m_i}$ for every $i \in \omega$,
- (b) $U_i \cap O_n = \emptyset$, for every $n \in \omega$, and $j \in J_i$, and
- (c) $J_i \supseteq J_h$, whenever $i \leq h \in \omega$.

Put $m_0 = 0$ and pick $x_0 \in O_0$ (this is possible, since O_0 is nonempty). Apply (*) with $J = \omega$ and $x = x_0$, and let $U_0 = U(\omega, x_0) \cap O_0 \subseteq O_0 = O_{m_0}$ and $J_0 = N(\omega, x_0)$. U_0 is nonempty, since $x_0 \in U(\omega, x_0) \cap O_0$. By (*), J_0 is infinite, and Clause (b) is satisfied for $i = 0$. Hence the basis of the induction is completed.

Suppose now that $0 \neq i \in \omega$, and that we have constructed finite sequences $(m_k)_{k < i}$, $(J_k)_{k < i}$, and $(U_k)_{k < i}$ satisfying the desired properties. Let m_i be any element of J_{i-1} . Since J_{i-1} is infinite, we can choose m_i distinct from all the m_k 's, for $k < i$ (however, this would follow automatically from (a) and (b)). Let x_i be any element of the nonempty O_{m_i} . Apply (*) with $J = J_{i-1}$ and $x = x_i$, and let $U_i = U(J_{i-1}, x_i) \cap O_{m_i} \subseteq O_{m_i}$ and $J_i = N(J_{i-1}, x_i)$. As above, U_i is nonempty, since $x_i \in U(J_{i-1}, x_i) \cap O_{m_i}$. By the definition of $N(J_{i-1}, x_i)$, we have that $J_i \subseteq J_{i-1}$, hence Clause (c) holds, by the inductive hypothesis. By (*), J_i is infinite, and moreover Clause (b) is satisfied for i . It remains to show that U_i is disjoint from U_k , for $k < i$. Since, by construction, $m_i \in J_{i-1}$, then, by (c) of the inductive hypothesis, for every $k < i$, we have that $m_i \in J_k$, hence, by (b), $U_k \cap O_{m_i} = \emptyset$, hence also $U_k \cap U_i = \emptyset$, since by construction $U_i \subseteq O_{m_i}$. The induction step is thus complete.

Having constructed sequences satisfying the above properties, we can apply sequential pseudocompactness to the sequence $(U_i)_{i \in \omega}$ of nonempty pairwise disjoint open sets, getting some $J \subseteq \omega$ and some

$x \in X$ such that every neighborhood of x intersects all but finitely many elements of $(U_i)_{i \in J}$.

If we put $J^* = \{m_i \mid i \in J\}$, then every neighborhood of x intersects all but finitely many elements of $(O_n)_{n \in J^*}$, because of Clause (a). We have reached a contradiction, thus the proposition is proved. \square

We now state the main result of the present note. In case sequential pseudocompactness is defined according to Condition (2) in Proposition 1, it is Theorem 4.1 in [DPSW]. If not otherwise specified, a product of topological spaces is always endowed with the Tychonoff product topology (the coarsest topology which makes the projections continuous).

Theorem 2. *A product of topological spaces is sequentially pseudocompact if and only if each factor is sequentially pseudocompact.*

Proof. The only-if part is trivial, since any continuous image of a sequentially pseudocompact space is sequentially pseudocompact.

In order to prove the converse, we shall use the equivalent formulation given by Condition (2) in Proposition 1. We first show:

(A) *A product of topological spaces is sequentially pseudocompact if and only if every subproduct of $\leq \omega$ factors is sequentially pseudocompact.*

The proof is similar to the classical argument showing that a topological space is pseudocompact if and only if every subproduct of $\leq \omega$ factors is pseudocompact.

Again, one implication is trivial. To prove the converse, assume that $X = \prod_{h \in H} X_h$, and that $\prod_{h \in C} X_h$ is sequentially pseudocompact, for every $C \subseteq H$ such that $|C| \leq \omega$.

Let $(O_n)_{n \in \omega}$ be a sequence of nonempty open sets of X . For every $n \in \omega$ there is a nonempty open set $O'_n \subseteq O_n$ such that O'_n has the form $\prod_{h \in H} O'_{n,h}$, where each $O'_{n,h}$ is a (nonempty) open set of X_h , and $O'_{n,h} = X_h$, for all but finitely many h 's. The set $K = \{h \in H \mid O'_{n,h} \neq X_h, \text{ for some } n \in \omega\}$ is countable, being a countable union of finite sets. For every $n \in \omega$, the set $\prod_{h \in K} O'_{n,h}$ is an open set of $\prod_{h \in K} X_h$. By assumption, $\prod_{h \in K} X_h$ is sequentially pseudocompact, hence, by Proposition 1(2), there are $J \subseteq \omega$ and $(x_h)_{h \in K} \in \prod_{h \in K} X_h$ such that, for every neighborhood U of $(x_h)_{h \in K}$ in $\prod_{h \in K} X_h$, the set $\{n \in J \mid U \cap \prod_{h \in K} O'_{n,h} = \emptyset\}$ is finite. Extend the sequence $(x_h)_{h \in K}$ to a sequence $(x_h)_{h \in H}$ by picking some arbitrary x_h , for each $h \in H \setminus K$. Then, for every neighborhood U' of $(x_h)_{h \in H}$, the set $\{n \in J \mid U' \cap \prod_{h \in H} O'_{n,h} = \emptyset\}$ is finite, since the image U of U' under the natural projection onto $\prod_{h \in K} X_h$ is a neighborhood of $(x_h)_{h \in K}$, and, for every

$n \in \omega$, $U' \cap \prod_{h \in H} O'_{n,h} = \emptyset$ if and only if $U \cap \prod_{h \in K} O'_{n,h} = \emptyset$, since $O'_{n,h} = X_h$, for every $n \in \omega$ and every $h \in H \setminus K$. According to Proposition 1(2), $\prod_{h \in H} X_h$ is sequentially pseudocompact, as witnessed by $(x_h)_{h \in H}$ and J , and (A) is proved.

In view of (A), in order to prove the theorem, it is enough to prove that a product of $\leq \omega$ sequentially pseudocompact spaces is sequentially pseudocompact. This is similar to the proof that a countable product of sequentially compact spaces is sequentially compact.

Suppose that $X = \prod_{i \in \omega} X_i$ is a product of sequentially pseudocompact spaces, and that $(O_n)_{n \in \omega}$ is a sequence of nonempty open sets of X . For every $n \in \omega$ there is a nonempty open set $O'_n \subseteq O_n$ such that O'_n has the form $\prod_{i \in \omega} O'_{n,i}$, where each $O'_{n,i}$ is a (nonempty) open set of X_i .

We shall construct a sequence of elements $(x_i)_{i \in \omega}$, where each x_i belongs to X_i , and a sequence $(J_i)_{i \in \omega}$ of infinite subsets of ω such that $J_i \supseteq J_j$ for $i \leq j < \omega$, and such that, for every $i \in \omega$, and every neighborhood U_i of x_i in X_i , the set $\{n \in J_i \mid U_i \cap O'_{n,i} = \emptyset\}$ is finite.

Since X_0 is sequentially pseudocompact, and the $O'_{n,0}$'s are nonempty, by Proposition 1(2) there are an infinite set $J_0 \subseteq \omega$ and some $x_0 \in X_0$ such that, for every neighborhood U_0 of x_0 , the set $\{n \in J_0 \mid U_0 \cap O'_{n,0} = \emptyset\}$ is finite.

Going on, suppose that we have constructed x_i and J_i . By applying the sequential pseudocompactness of X_{i+1} to the (infinite) sequence $(O'_{n,i+1})_{n \in J_i}$, we get an infinite set $J_{i+1} \subseteq J_i$ and $x_{i+1} \in X_{i+1}$ such that, for every neighborhood U_{i+1} of x_{i+1} , the set $\{n \in J_{i+1} \mid U_{i+1} \cap O'_{n,i+1} = \emptyset\}$ is finite. The desired sequences are thus constructed.

Pick some $n_0 \in J_0$, and, for each $i \in \omega$, pick some $n_i \in J_i \setminus \{n_0, \dots, n_{i-1}\}$, and let $J = \{n_i \mid i \in \omega\}$. Notice that, for every $i \in \omega$, $J_i \setminus J$ is finite, since the sequence $(J_i)_{i \in \omega}$ is decreasing with respect to inclusion.

Let U be any neighborhood of $(x_i)_{i \in \omega}$ in $\prod_{i \in \omega} X_i$. Thus, U contains some product $U' = \prod_{i \in \omega} U_i$, where $F = \{i \in \omega \mid U_i \neq X_i\}$ is finite, and each U_i is a neighborhood of x_i in X_i . If $i \in \omega$ and $i \notin F$, then $U_i = X_i$, hence $U_i \cap O'_{n,i} \neq \emptyset$, for every $n \in \omega$, since each $O'_{n,i}$ is nonempty. If $i \in F$, then, since U_i is a neighborhood of x_i , we get by the above construction that the set $\{n \in J_i \mid U_i \cap O'_{n,i} = \emptyset\}$ is finite. Since all members of J belong to J_i , except possibly for a finite number of elements, then also $N_i(U') = \{n \in J \mid U_i \cap O'_{n,i} = \emptyset\}$ is finite. Let $N(U')$ be the finite set $\bigcup_{i \in F} N_i(U')$. We have proved that if $i \in \omega$ and $n \in J \setminus N(U')$, then $U_i \cap O'_{n,i} \neq \emptyset$, hence, if $n \in J \setminus N(U')$, then $U \cap O_n \supseteq (\prod_{i \in \omega} U_i) \cap (\prod_{i \in \omega} O'_{n,i}) = \prod_{i \in \omega} (U_i \cap O'_{n,i}) \neq \emptyset$.

In conclusion, we have showed that, for every sequence $(O_n)_{n \in \omega}$ of nonempty open sets of X , there are $J \subseteq \omega$ and an element $(x_i)_{i \in \omega} \in X = \prod_{i \in \omega} X_i$ such that, for every neighborhood U of $(x_i)_{i \in \omega}$, the set $\{n \in J \mid U \cap O_n = \emptyset\}$ is finite (being a subset of $N(U')$). This is equivalent to sequential pseudocompactness of X , according to Condition (2) in Proposition 1. \square

Both $\beta\omega$ and D^c are classical examples of compact non sequentially compact spaces. As noticed on [AMPRT, p. 7], $\beta\omega$ is not sequentially pseudocompact. On the other hand, D^c is sequentially pseudocompact, by Theorem 2. Thus, compactness together with sequential pseudocompactness do not necessarily imply sequential compactness. In particular, normality and sequential pseudocompactness do not imply sequential compactness (thus the result that normality and pseudocompactness imply countable compactness cannot be generalized in the obvious way). Also, a compact subspace of a compact sequentially pseudocompact space is not necessarily sequentially pseudocompact, since $\beta\omega$ can be embedded in D^c . In particular, a closed subspace of a sequentially pseudocompact space is not necessarily sequentially pseudocompact.

Remark 3. A remark on terminology is now needed, since we are assuming no separation axiom, while usually pseudocompactness is considered in conjunction with the Tychonoff separation axiom. There are many conditions which are equivalent to pseudocompactness in the class of Tychonoff spaces, but which are in general distinct, for spaces satisfying weaker separation axioms. See, e. g., [S]. A topological space is *feebly compact* if and only if, for any sequence $(O_n)_{n \in \omega}$ of nonempty open sets of X , there is a point $x \in X$ such that $\{n \in \omega \mid U \cap O_n \neq \emptyset\}$ is infinite, for every neighborhood U of x . Feeble compactness is also equivalent to a notion called *weak initial ω -compactness*. See, e. g., [L, Remark 3], and further references there. For Tychonoff spaces, feeble compactness is well-known to be equivalent to pseudocompactness.

Corollary 4. *Let \mathcal{P} be any property of topological spaces such that every feebly compact (resp., Tychonoff pseudocompact) topological space satisfying \mathcal{P} is sequentially pseudocompact.*

Then the product of any family of feebly compact (resp., Tychonoff pseudocompact) spaces all of which satisfy \mathcal{P} is feebly compact (resp., pseudocompact).

Proof. Immediate from Theorem 2, since sequential pseudocompactness trivially implies feeble compactness, and hence, in the (productive) class of Tychonoff spaces, also pseudocompactness. \square

In [AMPRT] many properties \mathcal{P} are shown to satisfy the assumption in Corollary 4, among them, the properties of being a topological group, of being Tychonoff and scattered, or first countable, or ψ - ω -scattered.

In particular, from Theorem 2 and [AMPRT, Proposition 1.10], we get another proof of the classical result by Comfort and Ross that any product of pseudocompact topological groups is pseudocompact. However, it is not clear whether this is a real simplification: it might be the case that any proof that every pseudocompact topological group is sequentially pseudocompact already contains enough sophistication to be easily converted into a direct proof of Comfort and Ross Theorem.

Corollary 5. *If $X = \prod_{h \in H} X_h$ is a product of Tychonoff pseudocompact topological spaces and, for each $h \in H$, X_h is either scattered, or first countable, or, more generally, ϕ - ω -scattered, or allows the structure of a topological group, then X is pseudocompact (actually, sequentially pseudocompact).*

Proof. By the mentioned results from [AMPRT], each X_h is sequentially pseudocompact, hence X is sequentially pseudocompact by Theorem 2. \square

Problems 6. (1) Find other properties \mathcal{P} satisfying the assumption in Corollary 4, besides those described in [AMPRT].
 (2) Is there some significant part of the (topological) theory of pseudocompact topological groups which follows already from the assumption of sequential pseudocompactness? More precisely, are there other theorems holding for pseudocompact topological groups which can be generalized to sequentially pseudocompact topological spaces (with no algebraic structure on them)?

The proof of Theorem 2 actually shows a little more. If $(X_h)_{h \in H}$ is a family of topological spaces, the ω -box topology on $\prod_{h \in H} X_h$ is defined as the topology a base of which consists of the set of the form $\prod_{h \in H} O_h$, where each O_h is an open set of X_h , and $|\{h \in H \mid O_h \neq X_h\}| \leq \omega$.

Proposition 7. *Suppose that $(X_h)_{h \in H}$ is a family of sequentially pseudocompact topological spaces. If $(O_n)_{n \in \omega}$ is a sequence of nonempty open sets in the ω -box topology on $\prod_{h \in H} X_h$, then there are an infinite set $J \subseteq \omega$ and a point $x \in \prod_{h \in H} X_h$ such that $\{n \in J \mid U \cap O_n = \emptyset\}$ is finite, for every neighborhood U of x in the Tychonoff product topology on $\prod_{h \in H} X_h$.*

Proof. Same as the proof of Theorem 2. Indeed, the set K in the proof of (A) is countable anyway, this time being the countable union of a family of countable sets. Thus the proposition holds if and only if it

holds for every countable H . Now we can argue as in the second part of the proof of Theorem 2. Notice that there we have only used the assumption that, for each n , O'_n has the form $\prod_{i \in \omega} O'_{n,i}$, for open sets $O'_{n,i}$ of X_i , with no need of assuming that (for fixed n) $O'_{n,i} = X_i$, for all but finitely many $i \in \omega$. \square

Of course, in the statement of Proposition 7, the neighborhoods U of x have to be considered in the Tychonoff product topology. The statement would turn out to be false allowing U vary among the neighborhoods of x in the ω -box topology. Indeed, for example, a discrete two-element space is vacuously sequentially pseudocompact; however, its ω^{th} power in the ω -box topology is a discrete space of cardinality \mathfrak{c} , hence the conclusion of Proposition 7 would fail.

We shall present now a version of Theorem 2 in which the assumption that all factors are sequentially pseudocompact can be relaxed (with a correspondingly weaker conclusion, of course).

Proposition 8. (1) *The product of a Tychonoff sequentially pseudocompact topological space with a Tychonoff pseudocompact space is pseudocompact.*
 (2) *More generally, the product of a sequentially pseudocompact topological space with a feebly compact space is feebly compact.*
 (3) *If all factors of some product are sequentially pseudocompact, except possibly for one factor which is feebly compact, then the product is feebly compact.*

Proof. We shall prove (2), which is more general than (1), by the results recalled in Remark 3, and since the product of two Tychonoff spaces is still Tychonoff.

Suppose that X is sequentially pseudocompact, and Y is feebly compact, and let $(O_n)_{n \in \omega}$ be a sequence of nonempty open sets of $X \times Y$. Thus, there are nonempty opens sets $(A_n)_{n \in \omega}$ in X and nonempty open sets $(B_n)_{n \in \omega}$ in Y such that $O_n \supseteq A_n \times B_n$, for every $n \in \omega$.

Since X is sequentially pseudocompact, then, by Proposition 1, there are an infinite set $J \subseteq \omega$ and a point $x \in X$ such that every neighborhood of x intersects all but finitely many elements of $(A_n)_{n \in J}$. By applying the feeble compactness of Y to the sequence $(B_n)_{n \in J}$, we get $y \in Y$ such that every neighborhood of y intersects infinitely many elements of $(B_n)_{n \in J}$. The above conditions imply that every neighborhood of (x, y) in $X \times Y$ intersects infinitely many elements of $(A_n \times B_n)_{n \in J}$, hence infinitely many elements from the original sequence $(O_n)_{n \in \omega}$. Feeble compactness of $X \times Y$ is thus proved.

(3) is immediate from (2) and Theorem 2 (grouping together the sequentially pseudocompact factors). \square

Condition (1) in Proposition 8 is also a consequence of [AMPRT, Proposition 1.9]. The same arguments on [AMPRT, p. 7], together with [V, Theorem 1.2] (which uses no separation axiom) furnish another proof of (2). Condition (3) is apparently new. The particular case of Proposition 8 in which the assumption of sequential pseudocompactness is strengthened to sequential compactness is well known [S].

The following notions might deserve some study, in particular when α is a cardinal.

Definition 9. For α an infinite limit ordinal, we say that a topological space X is *sequentially α -pseudocompact* if and only if, for any sequence $(O_\beta)_{\beta \in \alpha}$ of nonempty open sets of X , there are some $x \in X$ and a subset Z of α such that Z has order type α , and, for every neighborhood U of x , there is $\beta < \alpha$ such that $U \cap O_\gamma \neq \emptyset$, for every $\gamma \in Z$ such that $\gamma > \beta$.

If we modify the above definition by further requesting that the (O_β) 's are pairwise disjoint, we say that X is *d-sequentially α -pseudocompact*. Clearly, for every α , sequential α -pseudocompactness implies d-sequential α -pseudocompactness, and for $\alpha = \omega$, both notions are equivalent (and equivalent to sequential pseudocompactness), by Proposition 1.

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