ON THE VON NEUMANN ALGEBRA OF TOEPLITZ OPERATORS WITH AUTOMORPHIC SYMBOL

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ABSTRACT. We prove an affirmative answer for a question of V. Jones concerning the structure of the commutant of the representations obtained by restriction to $PSL(2,\mathbb{Z})$ of the representations in the discrete series of $PSL(2,\mathbb{R})$. We prove that the commutant algebra is generated by the Toeplitz operators having cusp forms as symbols.

In this paper we describe the von Neumann algebra of Toeplitz operators with automorphic symbol with respect to a discrete fuchsian subgroup Γ of $PSL(2,\mathbb{R})$ ($\cong SU(1,1)$) of the first kind. The Toeplitz operators are acting on the Hilbert space of square summable analytic functions on the unit disk D (or the upper half plane) endowed with the measure corresponding to the m–th element π_m in the discrete series of unitary representations of $PSL(2,\mathbb{R})$.

We show that the commutant of the image of the group $PSL(2,\mathbb{Z})$ in the representation $\pi_m\,|_{PSL(2,\mathbb{Z})}$ coincides with the algebra of Toeplitz operators with automorphic symbol. In this identification the (eventually unbounded and square summable) elements in the commutant of $PSL(2,\mathbb{Z})$ are in a bijective correspondence with dif-

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ferentials

$$k = k(z,w)(dz)^m(d\overline{w})^m, z, w \in D$$

that are invariant under the diagonal action of Γ on $D \times D$ and that are square summable over a fundamental domain $\mathcal{D}$ for Γ in $D \times D$. This correspondence also extends to the linear space of intertwiners between the corresponding representations of $PSL(2,\mathbb{Z})$.

In this way we find an affirmative answer to a question of V. Jones ([VJ1], [GHJ, par. 2.4]).

By using the von Neumann algebra of Toeplitz operators with automorphic symbol, we get an alternative description of the type II_1 factors associated to free groups. We hope that eventually, using this new representation, a solution to R. V. Kadison's question (see also [Sa], problem 4.4.44) on the isomorphism class of the von Neumann algebras of free groups, could be found.

Let us recall ([GeGr], [Be]) that the Lie group $PSL(2,\mathbb{R})$ has a discrete (analytic) series of representations π_m, $m \in \mathbb{N}$, $m \geq 2$, in the space H_m of analytic functions in $L^2(D,\nu_m)$, where ν_m is the measure on the unit disk D that is absolutely continuous with respect to the Lebesgue measure λ and has density $(1-|z|^2)^{m-2}$, $m \geq 2$. The representation π_m acts by the formula $\pi_m(\gamma)f = f|_m[\gamma]$. Recall that for each $k \in \mathbb{N}$, $k \geq 2$, one has a natural action of $PSL(2,\mathbb{R})$ on functions on D, acting unitarily on $L^2(D,\nu_k)$, defined by the formula

$$(f|_k[\alpha])(z) = j(\alpha,z)^{-k}f(\alpha z), \ z \in D, \ \alpha \in PSL(2,\mathbb{R}),$$

where $j(\alpha,z)$ is the modular factor. Also recall that the modular factor for a 2×2 matrix α in $PSL(2,\mathbb{R})$ with entries (c,d) on the second line is defined by the formula

$j(\alpha, z) = (cz + d), \ z \in D$.

Let Γ be a lattice subgroup in $PSL(2, \mathbb{R})$ of finite covolume that is fuchsian of the first kind ([Shi]). By [GHJ], [VJ2] the algebra of bounded operators on H_m that commute with $\pi_m(\Gamma)$ is a type II_1 factor $\mathcal{A}_m$. The coupling constant ([MvN]) for $\mathcal{A}_m \subseteq B(H_m)$, i.e., the real dimension of H_m as left module over $\mathcal{A}_m$ (see [GHJ]), depends linearly on the covolume of Γ in $PSL(2, \mathbb{R})$ (see [GHJ], [VJ2], [AS], [Co]).

Recall from [GHJ], par 2.4, that basic estimates in the theory of automorphic operators ([My], p.42) show that the multiplication operator on H_m by a cusp form f with respect to Γ, of positive weight p, is a bounded intertwiner T_f^m between the representations of Γ on H_m and H_{m+p} obtained by restriction of π_m and respectively of π_{m+p} to Γ, for all $m, p \in \mathbb{N}, m \geq 2$.

Our main result is

Theorem 1. *The linear space of all bounded intertwiners between the representations $\pi_m|_\Gamma$ and $\pi_{m+p}|_\Gamma$ is the closure (in the weak operator topology) of the linear span of the operators $(T_g^m)^*(T_f^m)$ for all cusp forms f, g with respect to Γ of weights q and respectivelys $q - p$ and for all positive integer numbers $q \geq p$. Here, when $q = p$ we let g be a constant function.*

The proof of this theorem will be essentially a consequence of Proposition 3 which identifies intertwiners for the representations $\pi_m|\Gamma$ and $\pi_{m+p}|\Gamma$ with automorphic kernels $k = k(z, w)$ on $D \times D$ that are analytic in the first variable and antianalytic in the second.

Recall ([Ogg], [Shi], [My]) that an automorphic cusp form of order p for Γ is an analytic function f on D that vanishes at the cusps of Γ and is invariant under the action of $\Gamma \subseteq PSL(2, \mathbb{R})$ on functions on D (i.e. $f|_p[\gamma] = f$ for all $\gamma \in \Gamma$). Recall that for $PSL(2,\mathbb{Z})$ there exists only one cusp that corresponds to $0 \in D$ (or the point at the infinity if we use the upper half plane formalism).

We will first mention some consequences of our main theorem.

Corollary 2. *Let as above $\mathcal{A}_m$ be the von Neumann algebra of all linear bounded opesrators on the Hilbert space H_m that commute with $\pi_m(\Gamma)$. Then $\mathcal{A}_m$ is the closed linear span (in the weak operator topology) of the operators of the form $(T_f^m)^* T_g^m$ on H_m, where f, g are cusp forms for Γ having the same order p (when $p = 0$ we let f, g be constant functions).*

Note that by Theorem 3.3.4 in [GHJ], (see also [AS]), $\mathcal{A}_m$ is isomorphic to

$$(\mathcal{A}_m)_t, \text{ where } t = (d_{\pi_m})^{-1} covol(\Gamma),$$

and $(d_{\pi_m})^{-1}$ is the inverse of the formal Plancherel dimension ([Dix]) of π_m as an element of the discrete representation series of $PSL(2,\mathbb{R})$. Recall that for any type II_1 factor M we denote by M_t the isomorphism class of the type II_1 factor eMe where e is any projection in M of trace t.

In particular, when $\Gamma = PSL(2,\mathbb{Z})$, we get by ([Vo], [Dy], [Ra]) that $\mathcal{L}(\Gamma) \cong M_6(\mathbb{C}) \otimes \mathcal{L}(F_7)$ and it follows that $\mathcal{A}_m$ is isomorphic to a certain type II_1 factor $\mathcal{L}(F_t)$ associated to the reduced algebra of the type II_1 factor of a free group (see [Dy], [Ra]).

In this way, using the von Neumann algebra of Toeplitz operators with automorphic symbol, we get an alternative description of the type II_1 factors associated to free groups.

A different and independent approach to the fact that the operators T_f^m, where f is any L^∞ function on D automorphic with respect to Γ, are generating the commutant algebra $\mathcal{A}_m$ has been discovered by Curt McMullen.

The identification mentioned in the introduction of the elements commuting with $\pi_m(\Gamma)$ with differentials $k = k(z,w)(dz)^m(d\overline{w})^m, z, w \in D$ that are invariant under the diagonal action of Γ on $D \times D$ is the content of the following proposition. We are making use of an idea which probably goes back to Gelfand and Vilenkin [GV] and Bargmann [Ba], which consists to associate certain two variables kernels to the elements in the commutant of a representation by using point evaluations elements. The set of all these kernels then completely exhausts the set of intertwiners.

Proposition 3. *Let π_m, $m \geq 2$, $m \in \mathbb{N}$, be the (analytic) discrete series representation of $PSL(2,\mathbb{R})$. Let Γ be any lattice subgroup of $PSL(2,\mathbb{R})$ which is fuchsian of the first kind (e.g. the modular group $PSL(2,\mathbb{Z})$). Let $\mathcal{D}$ and respectively F be the fundamental domains for the action of Γ in $D \times D$ and respectively D. Let also $\mathcal{A}_p$ be the type II_1 factor consisting of all elements that commute with $\pi_p|_\Gamma$ in $B(H_p)$ and let $tr_{\mathcal{A}_p}$ be the normalized trace of this type II_1 factor ($p \geq 2$, $p \in \mathbb{N}$).*

For $z \in D$, $p \geq 2$, let also $e_z^p \in H_p$ be the point evaluation function, i.e.,

$$f(z) = < f, e_z^p >_p = \int_D f(w)\overline{e_z^p(w)}d\nu_p(z) \text{ for all } f \in H_p.$$

Let A be any element that intertwines two representations $\pi_m|_\Gamma$, $\pi_n|_\Gamma$ ($n, m \geq 2$, $n, m \in \mathbb{N}$) (i.e., $A \in L(H_n, H_m)$, $A\pi_n(g) = \pi_m(g)A$, $g \in \Gamma$). Let

$$k(z,w) = < Ae_w^n, e_z^m >_m, \ z, w \in D,$$

be the kernel defined by A. Then k is analytic in the first variable, antianalytic in the second variable, and

(*i*). $\qquad k(\gamma z, \gamma w) = j(\gamma, z)^n \overline{j(\gamma, w)^m} k(z, w), z, w \in D,$

(ii). $\iint_D |k(z,w)|^2 d\nu_n(z) d\nu_m(w) = c_{n,m} tr_{\mathcal{A}_n}(A^*A) = c_{m,n} tr_{\mathcal{A}_m}(AA^*) < \infty$,

(iii). *If* $A \in \mathcal{A}_n$ *then* $tr_{\mathcal{A}_n}(A) = d_n \int_F k(z,z) d\nu_n(z)$,

(iv). $(Af)(z) = P_n(k(z,\cdot)f(\cdot))(z), f \in H_n, z \in D$.

Here, $c_{n,m}, d_n$ *are constants.*

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