# A Linking Type Method to solve a Class of Semilinear Elliptic Variational Inequalities

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**Abstract**

The aim of this paper is to study the existence of a nontrivial solution of the following semilinear elliptic variational inequality

$$\begin{cases} u \in H_0^1(\Omega), \quad u \leq \psi \quad \text{on} \quad \Omega \\ \displaystyle\int_\Omega \nabla u(x) \nabla(v(x) - u(x))dx - \lambda \int_\Omega u(x)(v(x) - u(x))dx \geq \\ \qquad\qquad\qquad\qquad \geq \displaystyle\int_\Omega p(x, u(x))(v(x) - u(x))dx \\ \forall v \in H_0^1(\Omega), \quad v \leq \psi \quad \text{on} \quad \Omega \end{cases}$$

where $\Omega$ is an open bounded subset of $\mathbb{R}^N$ $(N \geq 1)$, $\lambda$ is a real parameter, with $\lambda \geq \lambda_1$, the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$, $\psi$ belongs to $H^1(\Omega)$, $\psi_{|\partial\Omega} \geq 0$ and $p$ is a Carathéodory function on $\Omega \times \mathbb{R}$, which satisfies some general superlinearity growth conditions at zero and at infinity. The method of finding the solutions is based on the consideration of a family of penalized equations associated with the variational inequality. A solution of any penalized equation is found through a Linking theorem, using some suitable conditions connecting $\psi$ with the eigenfunctions related to the eigenvalues of $-\Delta$. Some $H_0^1(\Omega)$-estimates from above and from below for these solutions allow, by a suitable passage to the limit as the penalization parameter $\epsilon$ goes to zero, to exhibite a nontrivial solution for the variational inequality. We note that the estimate from above is got by assuming some further regularity properties on $\psi$, which is moreover required to be a subsolution of a suitable Dirichlet problem.

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# 1 Introduction

Let us consider the following variational inequality

$$\begin{cases} u \in K_\psi = \{v \in H_0^1(\Omega) : v(x) \leq \psi(x) \text{ on } \Omega\} \text{ such that } \forall v \in K_\psi, \\ \displaystyle\int_\Omega \nabla u(x) \nabla (v(x) - u(x))dx - \lambda \int_\Omega u(x)(v(x) - u(x))dx \geq \\ \qquad\qquad\qquad\qquad \geq \displaystyle\int_\Omega p(x, u(x))(v(x) - u(x))dx \end{cases} \tag{1}$$

where $\Omega$ is an open bounded subset of $\mathbb{R}^N$ $(N \geq 1)$ with a sufficiently smooth boundary, $\psi \in H^1(\Omega), \psi_{|\partial\Omega} \geq 0$, $\lambda$ is a real parameter and $p(\cdot, \cdot)$ is a real function on $\Omega \times \mathbb{R}$ such that $p(\cdot, v(\cdot)$ belongs to $L^2(\Omega), \forall v \in H_0^1(\Omega)$.

In the case $\lambda < \lambda_1$, the first eigenvalue of the operator $-\Delta$ on $H_0^1(\Omega)$, an extensive literature was developed concerning various existence and multiplicity results, even with $K_\psi$ replaced by

$$K^\psi = \{v \in H_0^1(\Omega) : v(x) \geq \psi(x) \text{ on } \Omega\}, (\psi_{|\partial\Omega} \leq 0)$$

(see the papers [4, 5, 6, 7, 9, 10] and the introduction of [3] for a short discussion of the relative results).
Still in case $\lambda < \lambda_1$, an existence result was proved in [3] (actually presented with the choice $\lambda = 0$, but trivially extendible to the general case $\lambda < \lambda_1$), in case that $p$ has a suitable superlinear growth at zero and at infinity with respect to the second variable (i.e. $p(x, t)$ is of the type $t \mid t \mid^{\beta - 2}$, with $\beta > 2$). In [3] a penalization method and some estimates for the Mountain Pass type solutions found for the penalized equations yield a nonnegative not identically zero solution of (1).

The case $\lambda \geq \lambda_1$ was firstly studied in [2] with the choice $\lambda = \lambda_1$. In [10] Szulkin proved various significant existence, nonexistence and multiplicity results even in case where $\lambda > \lambda_1$ with the constraint set $K_\psi$ replaced by $K^\psi$ with $\psi = 0$. The methods used in [10] are based on a general minimax theory for a large class of variational inequalities.
Always for general $\lambda \geq \lambda_1$, Passaseo studied in [7] various cases with $p(x, t)$ independent of $t$ (that is linear case), by using some interesting methods of subsolutions and supersolutions for the equation related to (1).
Other important results were obtained in [9].

The aim of the present paper is to extend the idea of [3] based on the penalization method to the general case $\lambda \geq \lambda_1$. In this situation one gives some conditions connecting the obstacle $\psi$ with the eigenfunctions related to the eigenvalues of $-\Delta$ which are less or equal to $\lambda$. Under these assumptions one proves the existence of a family $(u_\epsilon)_{\epsilon > 0}$ of Linking type solutions for the penalized equations associated with (1) (here $\epsilon$ denotes the penalization parameter). Still as in [3] some estimates for $\parallel u_\epsilon \parallel_{H_0^1(\Omega)}$ allow to obtain a solution $u \not\equiv 0$ of (1), by passing to the limit as $\epsilon \to 0$. We point out that the proof of the estimate from above is rather delicate and requires, in particular, that the obstacle $\psi$ belongs to $H^2(\Omega) \cap L^q(\Omega)$, for a suitable $q$ (see condition $(H1)$) and that $\psi$ is a subsolution of a Dirichlet problem depending on $\lambda$ (see condition $(H3)$). Finally we observe that, in this case, one cannot expect, as in case $\lambda < \lambda_1$, the nonnegativity of $u$, which could change sign as any $u_\epsilon$, because a

Linking theorem, as well known, does not guarantee at all the nonnegativity of related critical point (see e.g. [8], remark 5.19).

## 2 The existence result of a nontrivial solution

Let us consider the following variational inequality

$$\begin{cases} u \in H_0^1(\Omega), \quad u \leq \psi \quad \text{on} \quad \Omega \\ \displaystyle\int_\Omega \nabla u(x) \nabla (v(x) - u(x)) dx - \lambda \int_\Omega u(x)(v(x) - u(x)) dx \geq \\ \qquad\qquad\qquad\qquad \displaystyle\geq \int_\Omega p(x, u(x))(v(x) - u(x)) dx \\ \forall v \in H_0^1(\Omega), \quad v \leq \psi \quad \text{on} \quad \Omega \end{cases} \tag{2}$$

where $\Omega$ is an open bounded subset of $\mathbb{R}^N$ $(N \geq 3)$ with a sufficiently smooth boundary, $H_0^1(\Omega)$ is the usual Sobolev space on $\Omega$ obtained as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\| v \| = \left( \int_\Omega | \nabla v(x) |^2 \, dx \right)^{\frac{1}{2}},$$

$\psi$ belongs to $H^1(\Omega)$, with $\psi_{|\partial\Omega} \geq 0$, $\lambda$ is a real parameter and $p : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

$(P1)$ $p(x, \xi)$ is measurable in $x \in \Omega$ and continuous in $\xi \in \mathbb{R}$;

$(P2)$ $| p(x, \xi) | \leq a_1 + a_2 | \xi |^s$ for some $a_1, a_2 > 0$ a.e. $x \in \Omega, \forall \, \xi \in \mathbb{R}$,

with $1 < s < \frac{N+2}{N-2}$;

$(P3)$ $p(x, \xi) = o(| \xi |)$ as $\xi \to 0$ a.e. $x \in \Omega$.

Moreover, putting

$$P(x, \xi) := \int_0^\xi p(x, t) dt, \quad \text{a.e. } x \in \Omega, \quad \forall \, \xi \in \mathbb{R},$$

we assume that

$(P4)$ for all $\xi \in \mathbb{R} \setminus \{0\}$, one has

$$0 < (s+1) P(x, \xi) \leq \xi p(x, \xi) \quad \text{a.e. } x \in \Omega.$$

Note that $(P4)$ easily yields

$(P5)$ $P(x, \xi) \geq a_3 | \xi |^{s+1} - a_4$ for some $a_3, a_4 > 0$, a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}$.

Finally, let $0 < \lambda_1 < \lambda_2 \leq ... \lambda_j \leq ...$ the divergent sequence of the eigenvalues of the operator $-\Delta$ on $H_0^1(\Omega)$, where each $\lambda_i$ has finite multiplicity coinciding with the number of its different indexes. Thus, for $\lambda_k < \lambda_{k+1}$, the space

$V_k$ related to $\{\lambda_1, ..., \lambda_k\}$ has finite dimension given exactly by $k$. Let us denote by $\{e_1, ..., e_k\}$ an $L^2(\Omega)$-orthonormal base of $V_k$, where $e_i$ is an eigenfunction related to $\lambda_i$.

In case that $\psi(x) \geq 0$ on $\overline{\Omega}$, it is obvious that $u_0 \equiv 0$ is a trivial solution of problem (2).

One can state the following

**Theorem 1** *Let $k \in \mathbb{N}$ such that $\lambda \in [\lambda_k, \lambda_{k+1}[$ in (2). Let (P1), (P2), (P3), (P4) and the following hypotheses hold*

$$(H1) \quad \psi \in H_0^1(\Omega) \cap H^2(\Omega) \cap L^q(\Omega), \quad \textit{where} \quad q = \left(\frac{2^*}{s}\right)' ;$$

$$(H2) \quad \psi(x) \geq (k+1)\bar{x} \max_{i=1,\dots,k+1} \mid e_i(x) \mid \quad \textit{a.e.} \quad x \in \Omega,$$

*where $\bar{x}$ is the positive zero of $f(x) = x^2 - a_3 c_k x^{s+1} + a_4 \mid \Omega \mid$*

*and $c_k$ is a suitable positive constant depending on $k$;*

$$(H3) \quad \Delta\psi + \lambda\psi > 0 \quad \textit{on} \quad \Omega;$$

$$(H4) \quad s < 2 \quad \textit{in} \quad (P2) \quad \textit{and} \quad (P4).$$

*Then there exists a nontrivial solution $u$ of problem (2).*

**Remark 1** Let us observe that the assumption '$\psi \in H^2(\Omega)$' may imply '$\psi \in L^q(\Omega)$', where $q = \left(\frac{2^*}{s}\right)'$ for some choices of $N$ and $s$. In particular one can easily check that , if $N = 3$, then $\psi \in H^2(\Omega) \Rightarrow \psi \in L^\infty(\Omega)$ and, if $N = 4$, then $\psi \in H^2(\Omega) \Rightarrow \psi \in L^r(\Omega)$ for all $r \in [1, \infty[$. In case that $N \geq 5$, the calculus of the number $(2^*)^*$ yields that $\psi \in H^2(\Omega) \Rightarrow \psi \in L^q(\Omega)$ if $N \leq \frac{2s+4}{s-1}$ $\left(2^* \text{ is the critical exponent i.e. } 2^* = \frac{2N}{N-2}\right)$.

**Remark 2** *Note that $\bar{x}$ exists and is unique. Further any $y > \bar{x}$ verifies the relation $f(y) < 0$.*

**Remark 3** Actually the cases $N = 1, 2$ can be considered too, even under simpler assumptions than those required in theorem 1. [1] We have decided to present our result only for $N \geq 3$ as, for $N = 1, 2$, the proof is the same, even using easier arguments in some step of the proof.

The method of finding the solution $u$ relies on the consideration of a family of 'penalized' equations associated, in a standard way, with (2) (see [1]). Indeed, one can prove that any penalized equation possesses a solution of 'Linking type', and that a sequence chosen in this family actually converges to a nontrivial solution $u$ of (2), by suitably using some estimates from below and from above for the $H_0^1(\Omega)$-norm of the solutions of the penalized equations. As mentioned before, we apply the following Linking theorem (see [8]):

[1] In particular one only requires $s \in (1, 2)$ and $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$.

**Saddle Point Theorem** *Let $E$ be a real Banach space with $E = V \bigoplus X$, where $V$ is finite dimensional. Suppose $I \in C^1(E, \mathbb{R})$ satisfies the following conditions:*
*(PS) for any $(u_n)_n \in E$ such that $(I(u_n))_n$ is bounded and $I^{'}(u_n) \to 0$ in the dual space of $E$ as $n \to \infty$, there exists a subsequence of $(u_n)_n$ strongly converging in $E$;*
*($I_1^{'}$) there are constants $\rho, \alpha > 0$ such that $I_{|\partial B_\rho \cap X} \geq \alpha$, where $B_\rho$ is the ball of center $0$ and radius $\rho$;*
*($I_5$) there are an element $e \in \partial B_1 \cap X$ and some $R > \rho$ such that, if $Q = (\overline{B}_R \cap V) \bigoplus \{re : 0 \leq r \leq R\}$, then $I_{|\partial Q} \leq 0$.*
*Then $I$ possesses a critical value $c \geq \alpha$ which can be characterized as*

$$c \equiv \inf_{h \in \Gamma} \max_{u \in Q} I(h(u)),$$

*where*

$$\Gamma = \{h \in C(\overline{Q}, E) : h = id \ \ on \ \ \partial Q\}.$$

First of all, let us introduce the 'penalized' problem associated with (2), that is, for any $\epsilon > 0$, the weak equation

$$\begin{cases} u_\epsilon \in H_0^1(\Omega) \ \text{ such that} \\ \displaystyle\int_\Omega \nabla u_\epsilon(x) \nabla v(x) dx \ - \ \lambda \int_\Omega u_\epsilon(x) v(x) dx \ + \\ \qquad\qquad + \ \displaystyle\frac{1}{\epsilon} \int_\Omega (u_\epsilon - \psi)^+(x) v(x) dx \ = \ \int_\Omega p(x, u_\epsilon(x)) v(x) dx \\ \forall v \in H_0^1(\Omega), \end{cases} \tag{3}$$

where $g^+$ denotes the positive part of the function $g$. Let us note that the last integral is well defined for all $v \in H_0^1(\Omega)$ as a consequence of $(P2)$ and of the continuous embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$.
Actually in order to look for solutions of (3), we study the critical points of the functional

$$I_\epsilon(v) = \frac{1}{2} \int_\Omega | \nabla v(x) |^2 dx - \frac{\lambda}{2} \int_\Omega v^2(x) dx + \frac{1}{\epsilon} \int_\Omega \int_0^{v(x)} (t - \psi(x))^+ dt dx - \int_\Omega P(x, v(x)) dx,$$

$\forall v \in H_0^1(\Omega)$.

Indeed one can easily check that $I_\epsilon$ belongs to $C^1(H_0^1(\Omega))$ and that the pairing $\langle I_\epsilon^{'}(u_\epsilon), v \rangle$ between $H_0^1(\Omega)$ and its dual space coincides with the difference between the first and the second member in (3).
At this point, to prove theorem 1, we verify that the functional $I_\epsilon$ satisfies all the hypotheses of the Saddle Point Theorem where $E \equiv H_0^1(\Omega), V \equiv V_k \equiv$ span

$\{e_1, ..., e_k\}$ and $X \equiv \overline{span\{e_j : j \geq k+1\}}$ (i.e. $X \equiv V^{\perp}$).

**Proof:** (of theorem 1) Let us proceed by steps.

**Step 1.** *The functional $I_\epsilon$ verifies, for any $\epsilon > 0$, the conditions*

$$I_\epsilon(0) = 0 \tag{4}$$

$$I_{\epsilon\ |\partial B_\rho \cap X} \geq \alpha \ \text{ for some } \ \rho, \alpha > 0. \tag{5}$$

*Proof.* Property (4) is trivial. As for (5), let us note that the positivity of $\psi$ on $\Omega$ yields

$$\int_\Omega \int_0^{u(x)} (t - \psi(x))^+ dt dx = \int_{\{x \in \Omega : u(x) \geq \psi(x)\}} \int_{\psi(x)}^{u(x)} (t - \psi(x)) dt dx \geq 0, \tag{6}$$

for all $u \in H_0^1(\Omega)$.

On the other hand, as a consequence of $(P2)$, $(P3)$, one gets that

$$\forall \delta > 0 \ \ \exists\, c(\delta) > 0 \ \text{ such that } \ P(x, \xi) \leq \frac{\delta}{2} \mid \xi \mid^2 + c(\delta) \mid \xi \mid^{s+1} \tag{7}$$

a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}$.

Then, by using (6),(7), the variational characterization of the eigenvalue $\lambda_{k+1}$ and by choosing $\rho > 0$ such that $c(\delta)\ c_s\ \rho^{s-1} < \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}} - \delta\right)$ ($c_s$ denoting the embedding Sobolev constant of $H_0^1(\Omega)$ into $L^{s+1}(\Omega)$), for all $u \in \partial B_\rho \cap X$, we have

$$\begin{aligned}
I_\epsilon(u) \ &\geq \ \frac{1}{2} \int_\Omega \mid \nabla u(x) \mid^2 dx - \frac{\lambda}{2} \int_\Omega u^2(x) dx - \int_\Omega P(x, u(x)) dx \geq \\
&\geq \ \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \parallel u \parallel^2_{H_0^1(\Omega)} - \frac{\delta}{2} \parallel u \parallel^2_{H_0^1(\Omega)} - c(\delta) c_s \rho^{s-1} \parallel u \parallel^2_{H_0^1(\Omega)} = \\
&= \ \left(\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}} - \delta\right) - \ c(\delta) c_s \rho^{s-1}\right) \parallel u \parallel^2_{H_0^1(\Omega)} = \\
&= \ \left(\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}} - \delta\right) - \ c(\delta) c_s \rho^{s-1}\right) \rho^2.
\end{aligned}$$

So step 1 follows from the fact that $\lambda < \lambda_{k+1}$.

**Remark 4** Note that the positive constant $\alpha$ does not depend on $\epsilon$ and this fact will be used in the proof of theorem 1.

**Step 2.** *There exist an element $e \in \partial B_1 \cap X$ and some $R > \rho$ such that $I_{\epsilon\ |\partial Q} \leq 0$, where $Q = (\overline{B}_R \cap V) \bigoplus \{re : 0 \leq r \leq R\}$.*

*Proof.* Let us choose $e = \frac{e_{k+1}}{\|e_{k+1}\|_{H_0^1(\Omega)}}$ and $R > 0$ such that $R_1 \leq R \leq R_2$, with

$$R_1 = \bar{x} \quad \text{as in } (H2)$$

and

$$R_2 = \frac{1}{k+1} \inf_{x \in \Omega} \frac{\psi(x)}{max\{| e_i(x) | \ : i = 1, ..., k+1\}}.$$

We observe that $R_1 \leq R_2$ (see hypothesis $(H2)$).

Actually one notes that $\partial Q \subset A_1 \cup A_2$, where

$$A_1 = \{v \in V : \| v \| \leq R\}$$

and

$$A_2 = \{v \in V \oplus span\{e\} : R \leq \| v \| \leq \sqrt{2} R\}.$$

So it is enough to prove that

$$I_\epsilon(v) \leq 0 \ \text{ for all } \ v \in A_1 \tag{8}$$

and

$$I_\epsilon(v) \leq 0 \ \text{ for all } \ v \in A_2. \tag{9}$$

First of all, by $(H2)$ and the fact that $R \leq R_2$, it follows that $v \leq \psi$ on $\Omega$ for all $v \in A_1 \cup A_2$. So

$$\int_\Omega \int_0^{v(x)} (t - \psi(x))^+ dt dx = 0, \tag{10}$$

for all $v \in A_1 \cup A_2$.
Let $v$ be an element of $A_1$. By hypothesis $(P4)$ and from the fact that $\lambda \geq \lambda_i$ for all $i = 1, ..., k$, we obtain

$$\frac{1}{2} \int_\Omega | \nabla v(x) |^2 dx - \frac{\lambda}{2} \int_\Omega v^2(x) dx - \int_\Omega P(x, v(x)) dx \leq 0. \tag{11}$$

By (10) and (11), one deduces relation (8).
Now let $v$ be an element of $A_2$. From hypothesis $(P5)$, the choice of $R$ $(\geq R_1)$ and remark 2, one easily deduces the relation

$$\begin{aligned} &\frac{1}{2} \int_\Omega | \nabla v(x) |^2 dx - \frac{\lambda}{2} \int_\Omega v^2(x) dx - \int_\Omega P(x, v(x)) dx \leq \\ &\leq R^2 - \int_\Omega \big( a_3 \mid v(x) \mid^{s+1} - a_4 \big) dx \leq \\ &\leq R^2 - a_3 c_k R^{s+1} + a_4 \mid \Omega \mid \leq 0, \end{aligned} \tag{12}$$

for a suitable $c_k$ that exists as $V \oplus span\{e\}$ is finite dimensional. By (10) and (12) one gets relation (9). Thus $I_{\epsilon \ |\partial Q} \leq 0$ and step 2 is proved.

**Remark 5** Note that (10) is true not only for $v \in A_1 \cup A_2$, but also for $v \in Q$.

**Step 3.** *For any $\epsilon > 0$, $I_\epsilon$ satisfies the Palais-Smale condition, i.e.*

$$(PS)\quad \begin{array}{l} \textit{for any } (u_n)_n \in H_0^1(\Omega) \textit{ such that } (I_\epsilon(u_n))_n \textit{ is bounded and} \\ I_\epsilon^{'}(u_n) \to 0 \textit{ in the dual space of } H_0^1(\Omega) \textit{ as } n \to \infty, \textit{there exists} \\ \textit{a subsequence of } (u_n)_n \textit{ strongly converging in } H_0^1(\Omega). \end{array}$$

*Proof.* Let us fix $\beta \in \left(\frac{1}{s+1}, \frac{1}{2}\right)$. By the properties of $(u_n)_n$ one deduces

$$I_\epsilon(u_n) - \beta\langle I_\epsilon^{'}(u_n), u_n\rangle \leq K_1 + \| u_n \|_{H_0^1(\Omega)} \tag{13}$$

and, by definition of $I_\epsilon$ and $I_\epsilon^{'}$, one gets

$$\begin{aligned} & I_\epsilon(u_n) - \beta\langle I_\epsilon^{'}(u_n), u_n\rangle = \\ & = \frac{1}{\epsilon}\int_{\{x\in\Omega: u_n(x)\geq\psi(x)\}} \left[\left(\frac{1}{2}-\beta\right)u_n^2(x) + \frac{1}{2}\psi^2(x) + (\beta-1)\psi(x)u_n(x)\right]dx + \\ & + \left(\frac{1}{2}-\beta\right)\| u_n \|^2_{H_0^1(\Omega)} - \lambda\left(\frac{1}{2}-\beta\right)\| u_n \|^2_{L^2(\Omega)} - \int_\Omega P(x,u_n(x))dx + \\ & + \beta\int_\Omega p(x,u_n(x))u_n(x)dx, \end{aligned} \tag{14}$$

for all $n \in \mathbb{N}$, where $K_1$ is a positive constant independent of $n$.
Actually, as for the integral multiplied by $\frac{1}{\epsilon}$, some obvious calculations and Hölder inequality yield

$$\begin{aligned} & \frac{1}{\epsilon}\int_{\{x\in\Omega: u_n(x)\geq\psi(x)\}} \left[\left(\frac{1}{2}-\beta\right)u_n^2(x) + \frac{1}{2}\psi^2(x) + (\beta-1)\psi(x)u_n(x)\right]dx \geq \\ & \geq -K_2 \| u_n \|_{H_0^1(\Omega)}, \end{aligned} \tag{15}$$

for all $n \in \mathbb{N}$, where $K_2$ is a positive constant depending on $\epsilon, \| \psi \|_{L^2(\Omega)}$, but not on $n$.
As for the other terms in (14), by using $(P4)$, the fact that $\beta$ is greater than $\frac{1}{s+1}$ and the continuous embedding of $L^{s+1}(\Omega)$ into $L^2(\Omega)$ one easily gets

$$\begin{aligned} & \left(\frac{1}{2}-\beta\right)\| u_n \|^2_{H_0^1(\Omega)} - \lambda\left(\frac{1}{2}-\beta\right)\| u_n \|^2_{L^2(\Omega)} - \int_\Omega P(x,u_n(x))dx + \\ & \qquad + \beta\int_\Omega p(x,u_n(x))u_n(x)dx \geq \\ & \geq \left(\frac{1}{2}-\beta\right)\| u_n \|^2_{H_0^1(\Omega)} - \lambda\left(\frac{1}{2}-\beta\right)\| u_n \|^2_{L^2(\Omega)} + \\ & \qquad + (s+1)\left(\beta - \frac{1}{s+1}\right)a_3 \| u_n \|^{s+1}_{L^{s+1}(\Omega)} - K_3 \geq \end{aligned} \tag{16}$$

$$\geq \left(\frac{1}{2} - \beta\right) \parallel u_n \parallel^2_{H^1_0(\Omega)} - \ \lambda \left(\frac{1}{2} - \beta\right) \parallel u_n \parallel^2_{L^2(\Omega)} +$$

$$+ (s+1)\left(\beta - \frac{1}{s+1}\right) \tilde{a}_3 \parallel u_n \parallel^{s+1}_{L^2(\Omega)} - \ K_3,$$

for all $n \in \mathbb{N}$, where $\tilde{a}_3$ and $K_3$ are positive constants independent of $n$.
Finally, combining (13), (14), (15), (16), one gets

$$\parallel u_n \parallel^2_{H^1_0(\Omega)} \leq K_4 \parallel u_n \parallel_{H^1_0(\Omega)} + K_5,$$

for all $n \in \mathbb{N}$, for suitable positive constants $K_4, K_5$ independent of $n$. Thus $(u_n)_n$ is bounded in $H^1_0(\Omega)$. At this point, step 3 easily follows from a standard argument based on the compact embedding of $H^1_0(\Omega)$ into $L^2(\Omega)$.

**Step 4.** *For any $\epsilon > 0$, there exists a solution $u_\epsilon$ of problem (3) such that*

$$I_\epsilon(u_\epsilon) = \inf_{h \in \Gamma} \max_{u \in Q} I_\epsilon(h(u)),$$

*where $\Gamma = \{h \in C(\overline{Q}; H^1_0(\Omega)) : h = id \ \ on \ \ \partial Q\}$.*
*Moreover*

$$I_\epsilon(u_\epsilon) \geq \alpha.$$

*Proof.* It is a consequence of steps 1,2,3 and of the Saddle Point Theorem.

**Step 5.** *There exists a constant $c_1 > 0$ such that $I_\epsilon(u_\epsilon) \leq c_1$ for any $\epsilon > 0$.*

*Proof.* By remark 5 it follows that

$$\int_\Omega \int_0^{u(x)} (t - \psi(x))^+ dt dx = 0,$$

for all $u \in Q$.

Moreover, by step 4 with $h = id_{\overline{Q}}$ and $(P5)$, one deduces

$$I_\epsilon(u_\epsilon) \leq \max_{u \in Q} I_\epsilon(u) \leq \max_{u \in Q} \left\{ \frac{1}{2} \parallel u \parallel^2_{H^1_0(\Omega)} + a_4 \mid \Omega \mid \right\}$$

and step 5 is proved as the right member of the previous relation is independent of $\epsilon$.

**Step 6.** *There exists a constant $c_2 > 0$ such that $\parallel u_\epsilon \parallel_{H^1_0(\Omega)} \geq c_2$ for any $\epsilon > 0$.*

*Proof.* By definition of a solution of problem (3), it follows, in particular,

$$\int_\Omega \mid \nabla u_\epsilon(x) \mid^2 dx - \lambda \int_\Omega u^2_\epsilon(x) dx + \frac{1}{\epsilon} \int_\Omega (u_\epsilon - \psi)^+(x) u_\epsilon(x) dx \quad = \quad = \int_\Omega p(x, u_\epsilon(x)) u_\epsilon(x) dx. \tag{17}$$

We can have two possible cases.
*First one*:

$$\begin{cases} \text{let } \ \epsilon > 0 \ \text{ such that} \\ \\ \dfrac{1}{\epsilon}\displaystyle\int_\Omega (u_\epsilon - \psi)^+(x) u_\epsilon(x) dx - \lambda \int_\Omega u_\epsilon^2(x) dx \geq 0. \end{cases} \tag{18}$$

Thus, for any $\epsilon > 0$ which satisfies (18), by (17), we obtain

$$\int_\Omega \mid \nabla u_\epsilon(x) \mid^2 dx \leq \int_\Omega p(x, u_\epsilon(x)) u_\epsilon(x) dx. \tag{19}$$

On the other hand, as a consequence of $(P2)$ and $(P3)$, one gets that

$$\forall \delta > 0 \, \exists c(\delta) > 0 \, \text{such that } \mid \xi p(x,\xi) \mid \leq \delta \mid \xi \mid^2 + c(\delta) \mid \xi \mid^{s+1} \ \text{ a. e. } x \in \Omega, \ \forall \xi \in \mathbb{R}$$

which yields, using (19), the arbitrarity of $\delta$ and the continuous embedding of $L^{s+1}(\Omega)$ into $L^2(\Omega)$, the relation

$$\int_\Omega \mid \nabla u_\epsilon(x) \mid^2 dx \leq \tilde{c} \int_\Omega \mid u_\epsilon(x) \mid^{s+1} dx,$$

where $\tilde{c}$ is a positive constant. Thus step 6 easily follows from the continuous embedding of $H_0^1(\Omega)$ into $L^{s+1}(\Omega)$ and the assumption $s+1 > 2$, for all $\epsilon > 0$ which satisfies (18).
*Second one*:

$$\begin{cases} \text{let } \ \epsilon > 0 \ \text{ such that} \\ \\ \dfrac{1}{\epsilon}\displaystyle\int_\Omega (u_\epsilon - \psi)^+(x) u_\epsilon(x) dx - \lambda \int_\Omega u_\epsilon^2(x) dx < 0. \end{cases} \tag{20}$$

By $(P4)$, (20) and by using the fact that $I_\epsilon(u_\epsilon) \geq \alpha$ ( note that $\alpha$ is independent of $\epsilon$ (see remark 4)), it follows

$$\frac{1}{\epsilon}\int_\Omega \int_0^{u_\epsilon(x)} (t - \psi(x))^+ dt dx \geq \alpha + \frac{1}{2\epsilon}\int_\Omega (u_\epsilon - \psi)^+(x) u_\epsilon(x) dx - \frac{1}{2}\int_\Omega \mid \nabla u_\epsilon(x) \mid^2 dx. \tag{21}$$

Putting $\Omega_\epsilon = \{x \in \Omega : u_\epsilon(x) > \psi(x)\}$, one deduces from (21)

$$\begin{aligned} &\frac{1}{2\epsilon}\int_{\Omega_\epsilon} u_\epsilon^2(x) dx - \frac{1}{\epsilon}\int_{\Omega_\epsilon} u_\epsilon(x)\psi(x) dx + \frac{1}{2\epsilon}\int_{\Omega_\epsilon} \psi^2(x) dx \geq \\ &\geq \alpha + \frac{1}{2\epsilon}\int_{\Omega_\epsilon} (u_\epsilon^2(x) - u_\epsilon(x)\psi(x)) dx - \frac{1}{2}\int_\Omega \mid \nabla u_\epsilon(x) \mid^2 dx \end{aligned} \tag{22}$$

so

$$\frac{1}{2}\int_\Omega \mid \nabla u_\epsilon(x) \mid^2 dx \geq \alpha + \frac{1}{2\epsilon}\int_{\Omega_\epsilon} u_\epsilon(x)\psi(x)dx - \frac{1}{2\epsilon}\int_{\Omega_\epsilon}\psi^2(x)dx >$$

$$> \alpha + \frac{1}{2\epsilon}\int_{\Omega_\epsilon}\psi^2(x)dx - \frac{1}{2\epsilon}\int_{\Omega_\epsilon}\psi^2(x)dx = \alpha.$$

Thus step 6 follows for all $\epsilon > 0$ which satisfies (20). Then step 6 is true for all $\epsilon > 0$.

**Step 7.** *There exists a constant $c_3 > 0$ such that $\| u_\epsilon \|_{L^2(\Omega)} \leq c_3$ for any $\epsilon > 0$.*

*Proof.* First of all, let us prove that, for any $\epsilon > 0$,

$$meas\ \ \widetilde{\Omega}_\epsilon = meas\ \ \{x \in \Omega : u_\epsilon(x) < -\psi(x)\} = 0. \tag{23}$$

We note that $\widetilde{\Omega}_\epsilon \not\equiv \Omega$. Indeed, let, by contradiction, $\widetilde{\Omega}_\epsilon \equiv \Omega$.
Let $v_1$ be a positive eigenfunction related to $\lambda_1$. By the fact that $u_\epsilon$ solves (3) and the definition of $\lambda_1$ and $\widetilde{\Omega}_\epsilon$, one obtains

$$(\lambda_1 - \lambda)\int_\Omega u_\epsilon(x)v_1(x)dx = \int_\Omega p(x, u_\epsilon(x))v_1(x)dx,$$

which is a contradiction because the first member is positive and the second is negative (see($P4$)). So $\widetilde{\Omega}_\epsilon \not\equiv \Omega$.
Hence, let $\widetilde{\Omega}_\epsilon \not\equiv \Omega$ and let us prove (23).
Let, by contradiction, $meas\ \ \widetilde{\Omega}_\epsilon > 0$. Then let us define $\mathcal{U}_{\epsilon,\psi}$ in this way

$$\mathcal{U}_{\epsilon,\psi}(x) \quad = \quad \begin{cases} u_\epsilon(x) + \psi(x) & \text{in } \widetilde{\Omega}_\epsilon \\ \varphi_\epsilon(x) & \text{in } \widetilde{\Omega}_\epsilon{}' \setminus \widetilde{\Omega}_\epsilon \\ 0 & \text{in } \Omega \setminus \widetilde{\Omega}_\epsilon{}', \end{cases}$$

where $\widetilde{\Omega}_\epsilon{}'$ is a suitable open set with $\widetilde{\Omega}_\epsilon \subset \widetilde{\Omega}_\epsilon{}' \subset \Omega$ and $\varphi_\epsilon$ is a suitable regular function to be chosen in such a way that $\mathcal{U}_{\epsilon,\psi}$ belongs to $H^2(\Omega)$.
Actually, on one side, by the definition of $\mathcal{U}_{\epsilon,\psi}$ and $\lambda_1$, one has

$$\begin{aligned} -\int_\Omega \Delta\mathcal{U}_{\epsilon,\psi}(x)v_1(x)dx &= -\int_\Omega \mathcal{U}_{\epsilon,\psi}(x)\Delta v_1(x)dx = \\ &= \lambda_1\int_{\widetilde{\Omega}_\epsilon}(u_\epsilon(x) + \psi(x))v_1(x)dx + \lambda_1\int_{\widetilde{\Omega}_\epsilon{}' \setminus\ \widetilde{\Omega}_\epsilon}\varphi_\epsilon(x)v_1(x)dx, \end{aligned} \tag{24}$$

on the other side, by the fact that $u_\epsilon$ solves (3) and $\mathcal{U}_{\epsilon,\psi} \in H^2(\Omega)$, one gets

$$\begin{aligned} -\int_\Omega \Delta\mathcal{U}_{\epsilon,\psi}(x)v_1(x)dx &= \int_{\widetilde{\Omega}_\epsilon} p(x, u_\epsilon(x))v_1(x)dx + \lambda\int_{\widetilde{\Omega}_\epsilon}(u_\epsilon(x) + \psi(x))v_1(x)dx + \\ &- \int_{\widetilde{\Omega}_\epsilon}(\Delta\psi(x) + \lambda\psi(x))v_1(x)dx - \int_{\widetilde{\Omega}_\epsilon{}' \setminus\ \widetilde{\Omega}_\epsilon}\Delta\varphi_\epsilon(x)v_1(x)dx. \end{aligned} \tag{25}$$

Thus (24) and (25) yield

$$
\begin{aligned}
&(\lambda-\lambda_1)\int_{\widetilde{\Omega}_\epsilon}(u_\epsilon(x)+\psi(x))v_1(x)dx+\int_{\widetilde{\Omega}_\epsilon}p(x,u_\epsilon(x))v_1(x)dx=\\
&=\int_{\widetilde{\Omega}_\epsilon}(\Delta\psi(x)+\lambda\psi(x))v_1(x)dx+\int_{\widetilde{\Omega}_\epsilon{}'\setminus\,\widetilde{\Omega}_\epsilon}(\Delta\varphi_\epsilon(x)+\lambda_1\varphi_\epsilon(x))v_1(x)dx.
\end{aligned}
\tag{26}
$$

At this point, if one assumes

$$
\Big|\int_{\widetilde{\Omega}_\epsilon{}'\setminus\,\widetilde{\Omega}_\epsilon}(\Delta\varphi_\epsilon(x)+\lambda_1\varphi_\epsilon(x))v_1(x)dx\Big|
$$

sufficently small, (26) yields a contradiction with hypothesis $(H3)$, since the first member of (26) is negative and $meas\ \ \widetilde{\Omega}_\epsilon>0$. Thus $meas\ \ \widetilde{\Omega}_\epsilon=0$ and (23) is proved.

On the other hand, by using (23) it follows the obvious relation

$$
\int_{\Omega\setminus(\Omega_\epsilon\cup\widetilde{\Omega}_\epsilon)}u_\epsilon^2(x)dx=\int_{\Omega\setminus\Omega_\epsilon}u_\epsilon^2(x)dx=\int_{\{x\in\Omega:|u_\epsilon(x)|\le\psi(x)\}}u_\epsilon^2(x)dx\le\parallel\psi\parallel^2_{L^2(\Omega)}.
\tag{27}
$$

Moreover, by step 5, $(P4)$ and (27), one gets, for any $\epsilon>0$,

$$
\begin{aligned}
&\left(\frac{1}{2}-\frac{1}{s+1}\right)\int_\Omega|\nabla u_\epsilon(x)|^2\,dx\le\\
&\le\lambda\left(\frac{1}{2}-\frac{1}{s+1}\right)\int_\Omega u_\epsilon^2(x)dx+\frac{1}{(s+1)\epsilon}\int_\Omega(u_\epsilon-\psi)^+(x)u_\epsilon(x)dx+\\
&\quad-\frac{1}{\epsilon}\int_\Omega\int_0^{u_\epsilon(x)}(t-\psi(x))^+dtdx+c_1\le\\
&\le\lambda\left(\frac{1}{2}-\frac{1}{s+1}\right)\int_{\Omega_\epsilon}u_\epsilon^2(x)dx+\frac{1}{\epsilon}\left(\frac{1}{s+1}-\frac{1}{2}\right)\int_{\Omega_\epsilon}u_\epsilon^2(x)dx+\\
&\quad+\frac{1}{\epsilon}\left(1-\frac{1}{s+1}\right)\int_{\Omega_\epsilon}u_\epsilon(x)\psi(x)dx+\parallel\psi\parallel^2_{L^2(\Omega)}+c_1\le\\
&\le\lambda\left(\frac{1}{2}-\frac{1}{s+1}\right)\int_{\Omega_\epsilon}u_\epsilon^2(x)dx+\frac{1}{\epsilon}\left(\frac{1}{s+1}-\frac{1}{2}\right)\int_{\Omega_\epsilon}u_\epsilon^2(x)dx+\\
&\quad+\frac{1}{\epsilon}\left(1-\frac{1}{s+1}\right)\parallel\psi\parallel_{L^2(\Omega)}\parallel u_\epsilon\parallel_{L^2(\Omega_\epsilon)}+\ \parallel\psi\parallel^2_{L^2(\Omega)}+c_1.
\end{aligned}
\tag{28}
$$

At this point, if $\left(\int_\Omega u_\epsilon^2(x)dx\right)_\epsilon$ was unbounded, then, by (27), even $\left(\int_{\Omega_\epsilon}u_\epsilon^2(x)dx\right)_\epsilon$ should be unbounded. Then (28) easily would yield an absurdum, since the last member of (28) would be unbounded from below as $\epsilon\to0$, while the first member is positive for any $\epsilon>0$. So step 7 is proved.

**Step 8.** *There exists a constant* $c_4 > 0$ *such that* $\| u_\epsilon \|_{H_0^1(\Omega)} \leq c_4$ *for any* $\epsilon > 0$.

*Proof.* By step 5 and by hypothesis $(P4)$, one gets, for any $\epsilon > 0$,

$$\frac{1}{2}\int_\Omega \mid \nabla u_\epsilon(x) \mid^2 dx - \frac{\lambda}{2}\int_\Omega u_\epsilon^2(x)dx + \frac{1}{\epsilon}\int_\Omega \int_0^{u_\epsilon(x)} (t-\psi(x))^+ dtdx \leq$$

$$\leq c_1 + \frac{1}{s+1}\int_\Omega p(x, u_\epsilon(x))u_\epsilon(x)dx.$$

Thus, as $u_\epsilon$ solves (3), one gets

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{s+1}\right)\int_\Omega \mid \nabla u_\epsilon(x) \mid^2 dx \leq \\
&\leq \lambda\left(\frac{1}{2} - \frac{1}{s+1}\right)\int_\Omega u_\epsilon^2(x)dx + \frac{1}{(s+1)\epsilon}\int_\Omega (u_\epsilon - \psi)^+(x)u_\epsilon(x)dx + \\
&\quad - \frac{1}{\epsilon}\int_\Omega\int_0^{u_\epsilon(x)} (t-\psi(x))^+ dtdx + c_1 \leq \\
&\leq \lambda\left(\frac{1}{2} - \frac{1}{s+1}\right)\int_\Omega u_\epsilon^2(x)dx + \frac{1}{(s+1)\epsilon}\int_{\Omega_\epsilon} (u_\epsilon(x) - \psi(x))u_\epsilon(x)dx + \\
&\quad - \frac{1}{2\epsilon}\int_{\Omega_\epsilon} (u_\epsilon - \psi)^2(x)dx + c_1 \leq \\
&\leq \lambda\left(\frac{1}{2} - \frac{1}{s+1}\right)\int_\Omega u_\epsilon^2(x)dx + \frac{1}{2\epsilon}\int_{\Omega_\epsilon} (u_\epsilon(x) - \psi(x))u_\epsilon(x)dx + \\
&\quad - \frac{1}{2\epsilon}\int_{\Omega_\epsilon} (u_\epsilon - \psi)^2(x)dx + c_1 \leq \\
&\leq \lambda\left(\frac{1}{2} - \frac{1}{s+1}\right)\int_\Omega u_\epsilon^2(x)dx + \frac{1}{2\epsilon}\int_{\Omega_\epsilon} (u_\epsilon(x) - \psi(x))\psi(x)dx + c_1.
\end{aligned} \tag{29}$$

At this point, taking $v = \psi$ in (3) and using $(P2)$, one gets

$$\begin{aligned}
&\frac{1}{2\epsilon}\int_{\Omega_\epsilon} (u_\epsilon(x) - \psi(x))\psi(x)dx = -\frac{1}{2}\int_\Omega \nabla u_\epsilon(x)\nabla\psi(x)dx + \\
&\quad + \frac{\lambda}{2}\int_\Omega u_\epsilon(x)\psi(x)dx + \frac{1}{2}\int_\Omega p(x, u_\epsilon(x))\psi(x)dx \leq \\
&\leq -\frac{1}{2}\int_\Omega \nabla u_\epsilon(x)\nabla\psi(x)dx + \frac{\lambda}{2}\int_\Omega u_\epsilon(x)\psi(x)dx + \\
&\quad + \frac{1}{2}\int_\Omega (a_1\psi(x) + a_2 \mid u_\epsilon(x) \mid^s \psi(x))\, dx.
\end{aligned} \tag{30}$$

By (29),(30),$(H1)$ and the continuous embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$, one obtains

$$\begin{aligned} \| u_\epsilon \|_{H_0^1(\Omega)}^2 &\leq M_1 \| u_\epsilon \|_{L^2(\Omega)}^2 + M_2 \| u_\epsilon \|_{L^{2^*}(\Omega)}^s + M_3 \leq \\ &\leq M_1 \| u_\epsilon \|_{L^2(\Omega)}^2 + M_4 \| u_\epsilon \|_{H_0^1(\Omega)}^s + M_3, \end{aligned}$$

where $M_1, M_2, M_3, M_4$ are positive constants depending only on $\lambda, s, \psi$, but not on $\epsilon$.
Thus the statement of step 8 easily follows from step 7 and $(H4)$.

**Step 9.** *There exists a constant $c_5 > 0$ such that*

$$\| (u_\epsilon - \psi)^+ \|_{L^2(\Omega)} \leq c_5 \sqrt{\epsilon}$$

*for any $\epsilon > 0$.*

*Proof.* Since $u_\epsilon$ is a solution of problem (3), in particular one gets

$$\begin{aligned} &\frac{1}{\epsilon} \int_\Omega (u_\epsilon - \psi)^+(x) u_\epsilon(x) dx = \\ &= \int_\Omega p(x, u_\epsilon(x)) u_\epsilon(x) dx - \int_\Omega | \nabla u_\epsilon(x) |^2 \, dx + \lambda \int_\Omega u_\epsilon^2(x) dx. \end{aligned} \tag{31}$$

Thus, by the positivity of $\psi$, it follows

$$\begin{aligned} &\frac{1}{\epsilon} \int_\Omega \left( (u_\epsilon - \psi)^+ \right)^2 (x) dx \leq \\ &\leq \int_\Omega p(x, u_\epsilon(x)) u_\epsilon(x) dx - \int_\Omega | \nabla u_\epsilon(x) |^2 \, dx + \lambda \int_\Omega u_\epsilon^2(x) dx. \end{aligned}$$

By $(P2)$, step 8 and the continuous embedding of $H_0^1(\Omega)$ into $L^{s+1}(\Omega)$ one deduces the thesis.

**Step 10.** *There exists a sequence $(\epsilon_n)_n$ converging to $0$ as $n$ goes to $\infty$ such that $(u_{\epsilon_n})_n$ weakly converges in $H_0^1(\Omega)$ to some $u \not\equiv 0$.*

*Proof.* First of all, by step 8, there exists a sequence $(u_{\epsilon_n})_n$ weakly converging in $H_0^1(\Omega)$ to some $u$ as $\epsilon_n$ goes to 0. We claim that $u$ is not identically zero. Indeed, $u \equiv 0$ would imply an absurdum deduced by step 6 and by passing to the limit as $\epsilon_n$ goes to 0 in the following relation (due to the fact that $u_{\epsilon_n}$ is a solution of problem (3) with $\epsilon = \epsilon_n$)

$$\begin{aligned} &\int_\Omega | \nabla u_{\epsilon_n}(x) |^2 \, dx - \lambda \int_\Omega u_{\epsilon_n}^2(x) dx + \frac{1}{\epsilon_n} \int_\Omega (u_{\epsilon_n} - \psi)^+(x) u_{\epsilon_n}(x) dx = \\ &= \int_\Omega p(x, u_{\epsilon_n}(x)) u_{\epsilon_n}(x) dx. \end{aligned}$$

**Step 11.** *The element $u$ given by Step 10 is a nontrivial solution of problem(2).*

*Proof.* First of all, $u_{\epsilon_n}$ verifies these two convergence properties:

$$u_{\epsilon_n} \to u \quad \text{in} \quad L^2(\Omega)$$

and

$$(u_{\epsilon_n} - \psi)^+ \to 0 \quad \text{in} \quad L^2(\Omega),$$

as $\epsilon_n$ goes to 0. So $u \leq \psi$ on $\Omega$.
From the fact that $u_{\epsilon_n} \to u$ weakly in $H_0^1(\Omega)$ as $\epsilon_n$ goes to 0, one deduces

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_{\epsilon_n}(x)|^2 \, dx \geq \int_\Omega |\nabla u(x)|^2 \, dx \tag{32}$$

and, by using hypothesis $(P2)$,

$$\int_\Omega p(x, u_{\epsilon_n}(x)) u_{\epsilon_n}(x) dx \to \int_\Omega p(x, u(x)) u(x) dx. \tag{33}$$

Finally, as $u_{\epsilon_n}$ is a solution of problem (3) with $\epsilon = \epsilon_n$, one gets

$$\int_\Omega \nabla u_{\epsilon_n}(x) \nabla (v(x) - u_{\epsilon_n}(x)) dx - \lambda \int_\Omega u_{\epsilon_n}(x)(v(x) - u_{\epsilon_n}(x)) dx +$$

$$+ \frac{1}{\epsilon_n} \int_\Omega (u_{\epsilon_n} - \psi)^+(x)(v(x) - u_{\epsilon_n}(x)) dx = \int_\Omega p(x, u_{\epsilon_n}(x))(v(x) - u_{\epsilon_n}(x)) dx, \tag{34}$$

$\forall v \in H_0^1(\Omega), \quad v \leq \psi.$

By (32) and (33) and passing to the limit as $\epsilon_n$ goes to 0 in (34), one easily gets that $u$ is a nontrivial solution of problem (2).

□

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