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Existence of Static Solutions of the Semilinear Maxwell Equations

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Abstract In this paper we study a model which describes the relation of the matter and the electromagnetic field from a *unitarian standpoint* in the spirit of the ideas of Born and Infeld. This model, introduced in [1], is based on a semilinear perturbation of the Maxwell equation (SME). The particles are described by the finite energy solitary waves of SME whose existence is due to the presence of the nonlinearity. In the magnetostatic case (i.e. when the electric field $\mathbf{E} = 0$ and the magnetic field \mathbf{H} does not depend on time) the semilinear Maxwell equations reduce to the following semilinear equation

$$\nabla \times (\nabla \times \mathbf{A}) = f'(\mathbf{A}) \quad (1)$$

where “ $\nabla \times$ ” is the *curl* operator, f' is the gradient of a smooth function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $\mathbf{A} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the gauge potential related to the magnetic field \mathbf{H} ($\mathbf{H} = \nabla \times \mathbf{A}$). The presence of the curl operator causes (1) to be a strongly degenerate elliptic equation. The existence of a nontrivial finite energy solution of (1) having a kind of cylindrical symmetry is proved. The

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proof is carried out by using a variational approach based on two main ingredients: the *Principle of symmetric criticality* of Palais, which allows to avoid the difficulties due to the curl operator, and the concentration-compactness argument combined with a suitable minimization argument.

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1 Introduction

The study of the relation of matter and the electromagnetic field is a classical, intriguing problem both from physical and mathematical point of view. In the framework of a classical relativistic theory, particles must be considered pointwise. However charged pointwise particles have infinite energy and therefore infinite inertial mass. This fact gives rise to well known difficulties (see for example [4, 5, 9]). The use of nonlinear equations in classical electrodynamics permits in some situations to avoid these difficulties. In a pioneering paper ([3]) Born and Infeld introduced a nonlinear formulation of the Maxwell equations. This theory avoids the divergences, however it is not *unitarian*, i.e. the nonlinearity they introduce does not allow the existence of a self-induced electromagnetic field and an external source is needed (see chapter 12 in [10]).

Following these lines of thought, in [1] a unitarian field theory has been introduced. This theory is based on a semilinear perturbation of the Maxwell equations. More precisely the usual Maxwell action for the gauge potentials $\mathbf{A} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$, $\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}$

$$S_M(\mathbf{A}, \varphi) = \frac{1}{2} \int \int \left(\left| \frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi \right|^2 - |\nabla \times \mathbf{A}|^2 \right) dxdt$$

is modified as follows:

$$\mathcal{S}(\mathbf{A}, \varphi) = \frac{1}{2} \int \int \left(\left| \frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi \right|^2 - |\nabla \times \mathbf{A}|^2 + W(|\mathbf{A}|^2 - \varphi^2) \right) dxdt \quad (2)$$

where $W : \mathbb{R} \rightarrow \mathbb{R}$ and “ $\nabla \times$ ” denotes the *curl* operator.

The argument of W is $|\mathbf{A}|^2 - |\varphi|^2$ in order to make the action invariant for the Poincaré group and the equations consistent with Special Relativity.

Making the variation of \mathcal{S} with respect to $\delta \mathbf{A}$, $\delta \varphi$ respectively, we get the equations

$$\frac{\partial}{\partial t} \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi \right) + \nabla \times (\nabla \times \mathbf{A}) = W'(|\mathbf{A}|^2 - \varphi^2) \mathbf{A}, \quad (3)$$

$$-\nabla \cdot \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi \right) = W'(|\mathbf{A}|^2 - \varphi^2) \varphi. \quad (4)$$

If we set

$$\rho = \rho(\mathbf{A}, \varphi) = W'(|\mathbf{A}|^2 - \varphi^2)\varphi, \quad (5)$$

$$\mathbf{J} = \mathbf{J}(\mathbf{A}, \varphi) = W'(|\mathbf{A}|^2 - \varphi^2)\mathbf{A}, \quad (6)$$

equations (3) and (4) are formally the Maxwell equations in the presence of matter if we interpret $\rho(\mathbf{A}, \varphi)$ as charge density and $\mathbf{J}(\mathbf{A}, \varphi)$ as current density. Notice that ρ and \mathbf{J} are not assigned functions representing external sources: they depend on the gauge potentials, so that we are in the presence of an unitarian theory. We make the following assumptions on W :

(W1) $W \in C^1(\mathbb{R}, \mathbb{R})$; $W(0) = 0$;

(W2) there exists $\xi > 0$ such that $W(\xi) > 0$;

(W3) there exist positive constants c, p, q with $2 < p < 6 < q$ such that

$$\begin{aligned} |W'(s)| &\leq c|s|^{p/2-1} && \text{for } |s| \geq 1, \\ |W'(s)| &\leq c|s|^{q/2-1} && \text{for } |s| \leq 1. \end{aligned}$$

The set

$$\Omega_t = \left\{ x \in \mathbb{R}^3 : 1 \leq \left| |\mathbf{A}(x, t)|^2 - |\varphi(x, t)|^2 \right| \right\}$$

is interpreted as the region of the space filled with matter at time t (see section 2 of [1]). Observe that the above assumptions allow to take $W(s) = 0$ for $|s| \leq 1 - \varepsilon$ ($\varepsilon > 0$), so that ρ and \mathbf{J} vanish outside a neighbourhood of Ω_t and in this region equations (3) and (4) reduce to the Maxwell equations in the empty space.

Equations (3) and (4) have been extensively studied in [1] where, among other things, the existence of a finite energy (magnetostatic) solution $(\mathbf{A}, 0)$, with \mathbf{A} depending only on the space variable x , has been stated. However the proof contains a gap, which will be overcome by Theorem 1 below.

In the magnetostatic case, equations (3) and (4) reduce to

$$\nabla \times (\nabla \times \mathbf{A}) = W'(|\mathbf{A}|^2)\mathbf{A}. \quad (7)$$

In this paper we study equation (7) and we prove the existence of a nontrivial, finite energy solution $\mathbf{A} = (A_1, A_2, A_3)$ whose components are related to each others by some kind of cylindrical symmetry. More precisely the following theorem holds:

Theorem 1 *Assume that hypotheses (W1), (W2), (W3) hold. Then equation (7) has a nontrivial, weak solution \mathbf{A} having the following form:*

$$\mathbf{A}(x) = A\left(\sqrt{x_1^2 + x_2^2}, x_3\right)(-x_2, x_1, 0)$$

where $A : (0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$. Moreover \mathbf{A} has finite energy, i.e.

$$\int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 dx < +\infty.$$

The main difficulty in dealing with the equation (7) lies in the fact that the energy functional related to it

$$\mathcal{E}[\mathbf{A}] = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2 dx - \int_{\mathbb{R}^3} W(|\mathbf{A}|^2) dx \quad (8)$$

is, in general, strongly indefinite in the sense that it is not bounded from below or from above and any possible critical point has infinite Morse index; namely the second variation of (8) is negative definite (if W is strongly convex) on the infinite dimensional space

$$\{\mathbf{A} = \nabla \varphi : \varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{R})\}.$$

To overcome this difficulty, in section 2 we introduce a suitable space $\mathcal{D}_{\mathcal{F}}^1$ whose elements are divergence free, so that for $\mathbf{A} \in \mathcal{D}_{\mathcal{F}}^1$ we have

$$\int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2 dx = \int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 dx.$$

It can be shown that $\mathcal{D}_{\mathcal{F}}^1$ is a natural constraint for (8), so that we are reduced to look for critical points of $\mathcal{E}|_{\mathcal{D}_{\mathcal{F}}^1}$. Furthermore the maps in $\mathcal{D}_{\mathcal{F}}^1$ have a sort of cylindrical symmetry and the functional $\mathcal{E}|_{\mathcal{D}_{\mathcal{F}}^1}$ has a lack of compactness due to its invariance under the translations along the x_3 axis. The proof of the existence of critical points for $\mathcal{E}|_{\mathcal{D}_{\mathcal{F}}^1}$ is carried out in section 4 combining a concentration-compactness type result, proved in section 3, with a suitable minimization argument.

2 The Variational Setting

In this section we collect some preliminary results concerning the variational structure of the system (7). Let $C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ be the set of the C^∞ vector fields $\mathbf{A} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ having compact support. Then let $\mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3)$ denote the completion of $C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ with respect to the norm

$$\|\mathbf{A}\|_{\mathcal{D}^1}^2 = \int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 dx, \quad \mathbf{A} \in \mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3).$$

$\mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3)$ is a Hilbert space with the scalar product

$$\int_{\mathbb{R}^3} (\nabla \mathbf{A} | \nabla \mathbf{B}) dx, \quad \mathbf{A}, \mathbf{B} \in \mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3),$$

where $\mathbf{A} = (A_1, A_2, A_3)$, $\mathbf{B} = (B_1, B_2, B_3)$ and $(\nabla \mathbf{A} | \nabla \mathbf{B}) = \sum_{i=1}^3 \nabla A_i \cdot \nabla B_i$, being “ \cdot ” the scalar product in \mathbb{R}^3 . By the Sobolev inequalities, $\mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3)$ is continuously embedded into $L^6(\mathbb{R}^3, \mathbb{R}^3)$:

$$\mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3, \mathbb{R}^3). \quad (9)$$

Consequently, for every $\Omega \subset \mathbb{R}^3$ open and bounded we have

$$\mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3) \hookrightarrow H^1(\Omega, \mathbb{R}^3) \quad (10)$$

with continuous embedding.

The functional associated to (7) is

$$\mathcal{E}[\mathbf{A}] = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} W(|\mathbf{A}|^2) dx.$$

Observe that, according to assumption (W3), we have

$$|W(s^2)| \leq \frac{c}{3} |s|^6, \quad |W'(s^2)| \leq c |s|^4. \quad (11)$$

Then, by (9), it is easy to prove that \mathcal{E} is well defined on $\mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3)$ and belongs to the class $C^1(\mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3), \mathbb{R})$. Hence critical points of \mathcal{E} correspond to solutions of (7).

The main difficulty in dealing with the functional \mathcal{E} lies in its strongly indefinite nature: indeed, if W is positive, it is negatively definite on the infinite-dimensional subspace

$$\{\mathbf{A} = \nabla \phi \mid \phi \in C_0^\infty(\mathbb{R}^3, \mathbb{R})\}.$$

Hence it does not exhibit a mountain pass geometry. In order to remove this indefiniteness, we are going to restrict our functional to a suitable subspace of $\mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3)$. More precisely consider the following space

$$\mathcal{F} = \left\{ \mathbf{A} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \mid \begin{array}{l} \exists A : (0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R} \text{ s.t.} \\ \mathbf{A}(x) = A(r, x_3)(-x_2, x_1, 0) \text{ a.e. in } \mathbb{R}^3 \end{array} \right\}$$

where

$$r = r_x = |(x_1, x_2)| = \sqrt{x_1^2 + x_2^2}, \quad (12)$$

and set

$$\mathcal{D}_{\mathcal{F}}^1 = \mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3) \cap \mathcal{F}.$$

It is obvious that $\mathcal{D}_{\mathcal{F}}^1$ is a closed subspace of $\mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3)$. Furthermore, for all $\mathbf{A} \in \mathcal{D}_{\mathcal{F}}^1$ we have that $\operatorname{div} \mathbf{A} = 0$, by which

$$\nabla \times (\nabla \times \mathbf{A}) = -\Delta \mathbf{A} \quad \forall \mathbf{A} \in \mathcal{D}_{\mathcal{F}}^1; \quad (13)$$

hence the restricted functional $\mathcal{E}|_{\mathcal{D}_{\mathcal{F}}^1}$ has the following form

$$\mathcal{E}|_{\mathcal{D}_{\mathcal{F}}^1}[\mathbf{A}] = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} W(|\mathbf{A}|^2) dx. \quad (14)$$

The first object is to prove that $\mathcal{D}_{\mathcal{F}}^1$ is a *natural* constraint for \mathcal{E} , i.e. a suitable subspace where to find solutions of (7). To this aim we recall the following Principle of symmetric criticality of Palais ([8]):

Principle of symmetric criticality. *Assume that there exists a topological group of transformations \mathcal{G} which acts isometrically on a Hilbert space X and define*

$$\operatorname{Fix} \mathcal{G} := \{\mathbf{A} \in X \mid G\mathbf{A} = \mathbf{A} \quad \forall G \in \mathcal{G}\}. \quad (15)$$

If $\mathcal{J} \in C^1(X, \mathbb{R})$ is invariant under \mathcal{G} , i.e.

$$\mathcal{J}(G\mathbf{A}) = \mathcal{J}(\mathbf{A}) \quad \forall G \in \mathcal{G}, \forall \mathbf{A} \in X, \quad (16)$$

and if \mathbf{A} is a critical point of $\mathcal{J}|_{\text{Fix } \mathcal{G}}$, then \mathbf{A} is a critical point of \mathcal{J} .

We are going to apply the above principle to our functional \mathcal{E} .

Indeed, let $\mathcal{O}(N)$ denote the orthogonal group of the rotation matrices in \mathbb{R}^N ; in particular consider

$$\mathcal{O}(2) = \left\{ \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \mid \alpha \in [0, 2\pi) \right\}.$$

For any $g \in \mathcal{O}(2)$ define the following action \mathcal{T}_g on $\mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3)$:

$$\mathcal{T}_g \mathbf{A}(x) = \tilde{g}^{-1} \mathbf{A}(\tilde{g}x) \in \mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3), \quad \tilde{g} = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{O}(3).$$

Now we set

$$\text{Fix } \mathcal{O}(2) = \{ \mathbf{A} \in \mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3) \mid \mathcal{T}_g \mathbf{A} = \mathbf{A} \quad \forall g \in \mathcal{O}(2) \}.$$

It is immediate that the action of $\mathcal{O}(2)$ on $\mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3)$ is isometric. Furthermore $\nabla \times \mathcal{T}_g \mathbf{A}(x) = \tilde{g}^{-1}(\nabla \times \mathbf{A})(\tilde{g}x)$ and $|\mathcal{T}_g \mathbf{A}(x)| = |\mathbf{A}(\tilde{g}x)|$, then it is easily deduced that \mathcal{E} is invariant with respect to $\mathcal{O}(2)$. According to the Principle of symmetric criticality every critical point \mathbf{A} of $\mathcal{E}|_{\text{Fix } \mathcal{O}(2)}$ is a critical point of \mathcal{E} , and, consequently, a weak solution to equation (7). Observe that $\mathcal{D}_{\mathcal{F}}^1 \subset \text{Fix } \mathcal{O}(2)$.

Next we will introduce a new group \mathcal{G} acting on $\text{Fix } \mathcal{O}(2)$ and we will apply again the Principle of symmetric criticality to the functional $\mathcal{E}|_{\text{Fix } \mathcal{O}(2)}$. Since, as we will show, $\text{Fix } \mathcal{G} = \mathcal{D}_{\mathcal{F}}^1$, to solve equation (7) it will be sufficient to look directly for critical points of $\mathcal{E}|_{\mathcal{D}_{\mathcal{F}}^1}$.

In order to define \mathcal{G} we have to use a decomposition of the functions in $\text{Fix } \mathcal{O}(2)$ provided by the following Lemma.

Lemma 1 *For every $\mathbf{A} \in \text{Fix } \mathcal{O}(2)$ there exist three functions $\mathbf{A}_\rho, \mathbf{A}_\tau, \mathbf{A}_\zeta \in \text{Fix } \mathcal{O}(2)$ such that $\mathbf{A} = \mathbf{A}_\rho + \mathbf{A}_\tau + \mathbf{A}_\zeta$ and*

$$\begin{aligned} \mathbf{A}_\rho &= A_\rho(r, x_3)(x_1, x_2, 0), \\ \mathbf{A}_\tau &= A_\tau(r, x_3)(-x_2, x_1, 0), \\ \mathbf{A}_\zeta &= A_\zeta(r, x_3)(0, 0, 1) \end{aligned}$$

for some $A_\rho, A_\tau, A_\zeta : (0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$. Furthermore for a.e. $x \in \mathbb{R}^3$:

$$(\nabla \mathbf{A}_\rho(x) | \nabla \mathbf{A}_\tau(x)) = (\nabla \mathbf{A}_\rho(x) | \nabla \mathbf{A}_\zeta(x)) = (\nabla \mathbf{A}_\tau(x) | \nabla \mathbf{A}_\zeta(x)) = 0. \quad (17)$$

Proof. Set $\mathbf{A} = (A_1, A_2, A_3) \in \text{Fix } \mathcal{O}(2)$ and set

$$\mathbb{R}_{x_3} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2 = 0\}.$$

Denote by $\mathbf{A}_\rho, \mathbf{A}_\tau, \mathbf{A}_\zeta$ the vector fields such that, for every $x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \mathbb{R}_{x_3}$, $\mathbf{A}_\rho(x), \mathbf{A}_\tau(x)$ and $\mathbf{A}_\zeta(x)$ are the projections of the vector $\mathbf{A}(x)$ along

the directions $\boldsymbol{\rho}(x) = (x_1, x_2, 0)$, $\boldsymbol{\tau}(x) = (-x_2, x_1, 0)$ and $\boldsymbol{\zeta}(x) = (0, 0, 1)$. By some computations we get $\mathbf{A}_\rho = A_\rho \boldsymbol{\rho}$, $\mathbf{A}_\tau = A_\tau \boldsymbol{\tau}$, $\mathbf{A}_\zeta = A_\zeta \boldsymbol{\zeta}$ where

$$A_\rho(x) = \frac{A_1 x_1 + A_2 x_2}{r^2}, \quad A_\tau(x) = \frac{-A_1 x_2 + A_2 x_1}{r^2}, \quad A_\zeta = A_3. \quad (18)$$

By construction we have $\mathbf{A} = \mathbf{A}_\rho + \mathbf{A}_\tau + \mathbf{A}_\zeta$. We are going to prove that A_ρ , A_τ , A_ζ have cylindrical symmetry, i.e. $A_\rho = A_\rho(r, x_3)$, $A_\tau = A_\tau(r, x_3)$, $A_\zeta = A_\zeta(r, x_3)$. For every $x \in \mathbb{R}^3 \setminus \mathbb{R}_{x_3}$ consider

$$\theta_x = \frac{1}{r^2} \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} \in \mathcal{O}(2).$$

Observe that, since $\tilde{\theta}_x \mathbf{A}(x) = (A_\rho(x), A_\tau(x), A_\zeta(x))$, it is enough to show that the vector field $x \in \mathbb{R}^3 \rightarrow \tilde{\theta}_x \mathbf{A}(x)$ is cylindrically symmetric, that is $\tilde{\theta}_{\tilde{g}x} \mathbf{A}(\tilde{g}x) = \tilde{\theta}_x \mathbf{A}(x)$ for every $g \in \mathcal{O}(2)$. Indeed, since $\tilde{\theta}_{\tilde{g}x} = \tilde{\theta}_x \tilde{g}^{-1}$ for every $g \in \mathcal{O}(2)$, then we get

$$\tilde{\theta}_{\tilde{g}x} \mathbf{A}(\tilde{g}x) = \tilde{\theta}_x \tilde{g}^{-1} \mathbf{A}(\tilde{g}x) = \tilde{\theta}_x \mathcal{T}_g \mathbf{A}(x) = \tilde{\theta}_x \mathbf{A}(x).$$

It remains to prove that

$$\mathbf{A}_\rho, \mathbf{A}_\tau, \mathbf{A}_\zeta \in \mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3), \quad (19)$$

which is not immediate because of the presence of the singular term $\frac{1}{r^2}$ in the definitions (18). Notice that, once we have proved (19), the conclusion follows immediately since \mathbf{A}_ρ , \mathbf{A}_τ , \mathbf{A}_ζ are fixed points for the action $\mathcal{O}(2)$. By the definition of \mathbf{A}_ρ , \mathbf{A}_τ , \mathbf{A}_ζ we immediately have $\mathbf{A}_\rho, \mathbf{A}_\tau, \mathbf{A}_\zeta \in L^6(\mathbb{R}^3, \mathbb{R}^3) \cap H_{loc}^1(\mathbb{R}^3 \setminus \mathbb{R}_{x_3}, \mathbb{R}^3)$. Denote by $\nabla \mathbf{A}_\rho|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}}$, $\nabla \mathbf{A}_\tau|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}}$ and $\nabla \mathbf{A}_\zeta|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}}$ the gradient in the sense of the distributions of \mathbf{A}_ρ , \mathbf{A}_τ , \mathbf{A}_ζ in $\mathbb{R}^3 \setminus \mathbb{R}_{x_3}$, and let $\nabla \mathbf{A}_\rho$, $\nabla \mathbf{A}_\tau$ and $\nabla \mathbf{A}_\zeta$ be the functions defined a.e. in \mathbb{R}^3 representing such distributions. A direct computation shows that for a.e. $x \in \mathbb{R}^3$:

$$(\nabla \mathbf{A}_\rho(x) | \nabla \mathbf{A}_\tau(x)) = (\nabla \mathbf{A}_\rho(x) | \nabla \mathbf{A}_\zeta(x)) = (\nabla \mathbf{A}_\tau(x) | \nabla \mathbf{A}_\zeta(x)) = 0.$$

Indeed the equalities $(\nabla \mathbf{A}_\rho(x) | \nabla \mathbf{A}_\zeta(x)) = (\nabla \mathbf{A}_\tau(x) | \nabla \mathbf{A}_\zeta(x)) = 0$ are immediate and

$$\begin{aligned} (\nabla \mathbf{A}_\rho(x) | \nabla \mathbf{A}_\tau(x)) &= -\nabla(A_\rho x_1) \cdot \nabla(A_\tau x_2) + \nabla(A_\rho x_2) \cdot \nabla(A_\tau x_1) \\ &= -(x_1 \nabla A_\rho + (A_\rho, 0, 0)) \cdot (x_2 \nabla A_\tau + (0, A_\tau, 0)) \\ &\quad + (x_2 \nabla A_\rho + (0, A_\rho, 0)) \cdot (x_1 \nabla A_\tau + (A_\tau, 0, 0)) \\ &= -x_1 A_\tau \frac{\partial A_\rho}{\partial x_2} - x_2 A_\rho \frac{\partial A_\tau}{\partial x_1} + x_2 A_\tau \frac{\partial A_\rho}{\partial x_1} + x_1 A_\rho \frac{\partial A_\tau}{\partial x_2} = 0 \end{aligned}$$

since A_ρ and A_τ have a cylindrical symmetry. This implies

$$|\nabla \mathbf{A}|^2 = |\nabla \mathbf{A}_\rho|^2 + |\nabla \mathbf{A}_\tau|^2 + |\nabla \mathbf{A}_\zeta|^2 \text{ a.e. in } \mathbb{R}^3 \quad (20)$$

and then

$$\nabla \mathbf{A}_\rho|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}}, \nabla \mathbf{A}_\tau|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}}, \nabla \mathbf{A}_\zeta|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}} \in L^2(\mathbb{R}^3, \mathbb{R}^3). \quad (21)$$

Denoting by $\frac{\partial}{\partial x_i} \Big|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}}$ the distributional derivative in $\mathbb{R}^3 \setminus \mathbb{R}_{x_3}$, from (20) we deduce that

$$\frac{\partial \mathbf{A}_\rho}{\partial x_i} \Big|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}}, \frac{\partial \mathbf{A}_\tau}{\partial x_i} \Big|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}}, \frac{\partial \mathbf{A}_\zeta}{\partial x_i} \Big|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}} \in L^2(\mathbb{R}^3, \mathbb{R}^3), \quad (22)$$

so (19) will follow if we show that $\frac{\partial \mathbf{A}_\rho}{\partial x_i} \Big|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}}, \frac{\partial \mathbf{A}_\tau}{\partial x_i} \Big|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}}, \frac{\partial \mathbf{A}_\zeta}{\partial x_i} \Big|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}}$ actually coincide with the distributional derivatives of $\mathbf{A}_\rho, \mathbf{A}_\tau, \mathbf{A}_\zeta$ in the whole \mathbb{R}^3 : in other words, considering the component \mathbf{A}_ρ (the computations for the other components are similar), we have to show that, for any $\mathbf{B} \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$, it results

$$\int_{\mathbb{R}^3} \frac{\partial \mathbf{A}_\rho}{\partial x_i} \Big|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}} \cdot \mathbf{B} \, dx = - \int_{\mathbb{R}^3} \mathbf{A}_\rho \cdot \frac{\partial \mathbf{B}}{\partial x_i} \, dx. \quad (23)$$

Observe that, since $\mathbf{A}_\rho \in L^6(\mathbb{R}^3, \mathbb{R}^3)$ and $\frac{\partial \mathbf{A}_\rho}{\partial x_i} \Big|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$, then the above integrals are both finite. Now for all $\varepsilon > 0$ consider a function $\eta_\varepsilon \in C^\infty(\mathbb{R}^3, \mathbb{R})$ such that

$$\eta_\varepsilon = 0 \text{ for } |r| \leq \frac{\varepsilon}{2}, \quad \eta_\varepsilon = 1 \text{ for } |r| \geq \varepsilon, \quad 0 \leq \eta_\varepsilon \leq 1, \quad |\nabla \eta_\varepsilon| \leq \frac{4}{\varepsilon}.$$

Set $\mathbf{B}_\varepsilon(x) = \mathbf{B}(x)\eta_\varepsilon(x) \in C_0^\infty(\mathbb{R}^3 \setminus \mathbb{R}_{x_3})$. For all $\varepsilon > 0$ we have

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{\partial \mathbf{A}_\rho}{\partial x_i} \cdot \mathbf{B}_\varepsilon \, dx &= - \int_{\mathbb{R}^3} \mathbf{A}_\rho \cdot \frac{\partial \mathbf{B}_\varepsilon}{\partial x_i} \, dx \\ &= - \int_{\mathbb{R}^3} \eta_\varepsilon \mathbf{A}_\rho \cdot \frac{\partial \mathbf{B}}{\partial x_i} \, dx - \int_{\mathbb{R}^3} \frac{\partial \eta_\varepsilon}{\partial x_i} \mathbf{A}_\rho \cdot \mathbf{B} \, dx. \end{aligned} \quad (24)$$

Now, by Lebesgue's Theorem

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3} \frac{\partial \mathbf{A}_\rho}{\partial x_i} \cdot \mathbf{B}_\varepsilon \, dx &= \int_{\mathbb{R}^3} \frac{\partial \mathbf{A}_\rho}{\partial x_i} \Big|_{\mathbb{R}^3 \setminus \mathbb{R}_{x_3}} \cdot \mathbf{B} \, dx, \\ \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3} \eta_\varepsilon \mathbf{A}_\rho \cdot \frac{\partial \mathbf{B}}{\partial x_i} \, dx &= \int_{\mathbb{R}^3} \mathbf{A}_\rho \cdot \frac{\partial \mathbf{B}}{\partial x_i} \, dx. \end{aligned} \quad (25)$$

Let $R > 0$ be such that $\mathbf{B} = 0$ for $|x| \geq R$ and set $\Omega_\varepsilon := B(0, R) \cap \{r \leq \varepsilon\}$. Observe that $meas(\Omega_\varepsilon) \leq 2\pi R\varepsilon^2$, so

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \frac{\partial \eta_\varepsilon}{\partial x_i} \mathbf{A}_\rho \cdot \mathbf{B} \, dx \right| &\leq \|\mathbf{B}\|_{L^\infty} \frac{4}{\varepsilon} \int_{\Omega_\varepsilon} |\mathbf{A}_\rho| \, dx \\ &\leq \|\mathbf{B}\|_{L^\infty} \frac{4}{\varepsilon} (meas(\Omega_\varepsilon))^{\frac{5}{6}} \left(\int_{\Omega_\varepsilon} |\mathbf{A}_\rho|^6 \, dx \right)^{1/6} \rightarrow 0 \end{aligned} \quad (26)$$

as ε goes to 0. Letting ε go to 0 in (24) and using (25) and (26), the equality (23) follows.

Now we are ready to prove the following proposition, which is the key result of this section: it shows how the introduction of the functional set $\mathcal{D}_{\mathcal{F}}^j$ has a crucial role in dealing with the strong-indefiniteness of the functional \mathcal{E} .

Proposition 1 *Let $\mathbf{A} \in \mathcal{D}_{\mathcal{F}}^1$ be a critical point of $\mathcal{E}|_{\mathcal{D}_{\mathcal{F}}^1}$. Then \mathbf{A} is a weak solution to (7).*

According to Lemma 1 let \mathcal{S} be the action on $\text{Fix } \mathcal{O}(2)$ defined by:

$$\mathcal{S}\mathbf{A} = \mathcal{S}(\mathbf{A}_\rho + \mathbf{A}_\tau + \mathbf{A}_\zeta) = -\mathbf{A}_\rho + \mathbf{A}_\tau - \mathbf{A}_\zeta.$$

We set \mathcal{G} the group generated by \mathcal{S} ; since $\mathcal{S}^2 = id$, we have $\mathcal{G} \approx \mathbb{Z}_2$. According to (17) we get

$$\begin{aligned} & \int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 dx \\ &= \int_{\mathbb{R}^3} |\nabla \mathbf{A}_\rho|^2 dx + \int_{\mathbb{R}^3} |\nabla \mathbf{A}_\tau|^2 dx + \int_{\mathbb{R}^3} |\nabla \mathbf{A}_\zeta|^2 dx = \int_{\mathbb{R}^3} |\nabla \mathcal{S}\mathbf{A}|^2 dx, \end{aligned}$$

then the action of \mathcal{G} on $\text{Fix } \mathcal{O}(2)$ is isometric. It results $\mathcal{D}_{\mathcal{F}}^1 = \text{Fix } \mathcal{G} := \{\mathbf{A} \in \text{Fix } \mathcal{O}(2) \mid \mathcal{S}\mathbf{A} = \mathbf{A}\}$: indeed the inclusion $\mathcal{D}_{\mathcal{F}}^1 \subset \text{Fix } \mathcal{G}$ is obvious; on the other hand, if $\mathbf{A} \in \text{Fix } \mathcal{G}$, then the invariance under \mathcal{S} implies $\mathbf{A}_\rho = \mathbf{A}_\zeta = 0$ and, consequently, $\mathbf{A} = \mathbf{A}_\tau \in \mathcal{D}_{\mathcal{F}}^1$.

The conclusion will follow from the principle of symmetric criticality just taking $X = \text{Fix } \mathcal{O}(2)$ and $\mathcal{J} = \mathcal{E}|_{\text{Fix } \mathcal{O}(2)}$ and proving that (16) holds with respect to the group \mathcal{G} generated by \mathcal{S} . If $\mathbf{A} \in \text{Fix } \mathcal{O}(2)$, then $\mathbf{A}_\tau \in \mathcal{D}_{\mathcal{F}}^1$, which implies by (13) $\nabla \times (\nabla \times \mathbf{A}_\tau) = -\Delta \mathbf{A}_\tau$ and consequently by (17)

$$\begin{aligned} \int_{\mathbb{R}^3} (\nabla \times \mathbf{A}_\tau \mid \nabla \times (\mathbf{A}_\rho + \mathbf{A}_\zeta)) dx &= \int_{\mathbb{R}^3} (\nabla \times (\nabla \times \mathbf{A}_\tau) \mid \mathbf{A}_\rho + \mathbf{A}_\zeta) dx \\ &= \int_{\mathbb{R}^3} (\nabla \mathbf{A}_\tau \mid \nabla (\mathbf{A}_\rho + \mathbf{A}_\zeta)) dx = 0. \end{aligned}$$

Then

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla \times (\mathcal{S}\mathbf{A})|^2 dx &= \int_{\mathbb{R}^3} |\nabla \times \mathbf{A}_\tau|^2 dx + \int_{\mathbb{R}^3} |\nabla \times (\mathbf{A}_\rho + \mathbf{A}_\zeta)|^2 dx \\ &= \int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2 dx. \end{aligned}$$

Therefore the functional $\mathbf{A} \in \text{Fix } \mathcal{O}(2) \mapsto \int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2 dx$ is invariant with respect to \mathcal{G} . Finally we immediately compute that $|\mathbf{A}|^2 = |\mathbf{A}_\rho|^2 + |\mathbf{A}_\tau|^2 + |\mathbf{A}_\zeta|^2$ which leads to $\int_{\mathbb{R}^3} W(|\mathcal{S}\mathbf{A}|^2) dx = \int_{\mathbb{R}^3} W(|\mathbf{A}|^2) dx$ and this concludes the proof of (16).

According to the previous proposition we can solve equation (7) by looking directly for critical points of $\mathcal{E}|_{\mathcal{D}_{\mathcal{F}}^1}$; in this way we have avoided the strong indefiniteness of \mathcal{E} and we will deal with the functional $\mathcal{E}|_{\mathcal{D}_{\mathcal{F}}^1}$ which has the form (14) and then it can be treated with standard methods of nonlinear analysis.

3 A Compactness Result

Because of the action of the translations the Sobolev embeddings for the space $\mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3)$ are non compact. In order to recover some compactness, following the ideas of P.L. Lions (see [6]-[7]), we will show that if a sequence of functions is such that the nonlinear term $\int_{\mathbb{R}^3} W(|\mathbf{A}|^2)dx$ does not vanish, then it keeps away from zero in some suitable sense.

Lemma 2 *Suppose that $(\mathbf{A}_n)_n$ is bounded in $\mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3)$ and there is $R > 0$ such that*

$$\lim_{n \rightarrow +\infty} \sup_{x \in \mathbb{R}^3} \int_{B(x,R)} |\mathbf{A}_n|^2 dx = 0.$$

Then

$$\int_{\mathbb{R}^3} W(|\mathbf{A}_n|^2) dx \rightarrow 0.$$

Proof. During this proof we will often use the symbol C for denoting a positive constant independent on n . The value of C is allowed to vary from line to line (and also in the same formula).

Fix $\varepsilon \in (0, 1)$ and for every n consider the new sequence of functions

$$w_n := \begin{cases} |\mathbf{A}_n| & \text{if } |\mathbf{A}_n| \geq \varepsilon, \\ |\mathbf{A}_n|^3 \varepsilon^{-2} & \text{if } |\mathbf{A}_n| \leq \varepsilon. \end{cases}$$

It is immediate that

$$|w_n|^2 \leq \varepsilon^{-4} |\mathbf{A}_n|^6, \quad |w_n|^2 \leq |\mathbf{A}_n|^2, \quad (27)$$

$$|\nabla w_n|^2 \leq 9|\nabla |\mathbf{A}_n||^2 \leq 9|\nabla \mathbf{A}_n|^2.$$

In particular $w_n \in H^1(\mathbb{R}^3, \mathbb{R})$ and, using (9),

$$\|w_n\|_{H^1(\mathbb{R}^3)}^2 \leq \varepsilon^{-4} \int_{\mathbb{R}^3} |\mathbf{A}_n|^6 dx + 9 \int_{\mathbb{R}^3} |\nabla \mathbf{A}_n|^2 dx \leq C\varepsilon^{-4}. \quad (28)$$

The object is to prove that

$$w_n \rightarrow 0 \text{ in } L^s(\mathbb{R}^3, \mathbb{R}) \quad \forall 2 < s < 6. \quad (29)$$

Indeed first assume $s \geq \frac{10}{3}$. By using Hölder's inequality, since $s = 2\frac{6-s}{4} + 6\frac{s-2}{4}$, for every $x \in \mathbb{R}^3$ we get

$$\begin{aligned} \int_{B(x,R)} |w_n|^s dx &\leq \left(\int_{B(x,R)} |w_n|^2 dx \right)^{\frac{6-s}{4}} \left(\int_{B(x,R)} |w_n|^6 dx \right)^{\frac{s-2}{4}} \\ &\leq C \left(\int_{B(x,R)} |w_n|^2 dx \right)^{\frac{6-s}{4}} \left(\int_{B(x,R)} (|w_n|^2 + |\nabla w_n|^2) dx \right)^{3\frac{s-2}{4}}, \end{aligned}$$

where C is independent on x and n . Since $3\frac{s-2}{4} \geq 1$, then

$$\int_{B(x,R)} |w_n|^s dx \leq C \|w_n\|_{L^2(B(x,R))}^{\frac{6-s}{2}} \|w_n\|_{H^1(\mathbb{R}^3)}^{3\frac{s-2}{2}-2} \int_{B(x,R)} (|w_n|^2 + |\nabla w_n|^2) dx. \quad (30)$$

Choosing a family of balls $\{B(x,R)\}$ whose union covers \mathbb{R}^3 such that each point in \mathbb{R}^3 is contained in at most k such balls, summing (30) over this family and using (27) and (28) we deduce

$$\begin{aligned} \int_{\mathbb{R}^3} |w_n|^s dx &\leq Ck \sup_{x \in \mathbb{R}^3} \left(\int_{B(x,R)} |w_n|^2 dx \right)^{\frac{6-s}{4}} \|w_n\|_{H^1(\mathbb{R}^3)}^{3\frac{s-2}{2}} \\ &\leq Ck \varepsilon^{-3(s-2)} \sup_{x \in \mathbb{R}^3} \left(\int_{B(x,R)} |\mathbf{A}_n|^2 dx \right)^{\frac{6-s}{4}} \rightarrow 0 \end{aligned}$$

by the hypotheses of the lemma. If $2 < s < \frac{10}{3}$, then $s = 2\frac{10-3s}{4} + \frac{10}{3}\frac{3(s-2)}{4}$ and from Hölder's inequality and from (28)

$$\begin{aligned} \int_{\mathbb{R}^3} |w_n|^s dx &\leq \left(\int_{\mathbb{R}^3} |w_n|^2 dx \right)^{\frac{10-3s}{4}} \left(\int_{\mathbb{R}^3} |w_n|^{10/3} dx \right)^{\frac{3(s-2)}{4}} \\ &\leq C \varepsilon^{3s-10} \left(\int_{\mathbb{R}^3} |w_n|^{10/3} dx \right)^{\frac{3(s-2)}{4}}; \end{aligned}$$

since by the case already established $\int_{\mathbb{R}^3} |w_n|^{10/3} dx \rightarrow 0$, we obtain

$$\int_{\mathbb{R}^3} |w_n|^s dx \rightarrow 0.$$

Then (29) holds. Hence using assumption (W3) we conclude

$$\begin{aligned} \int_{\mathbb{R}^3} |W(|\mathbf{A}_n|^2)| dx &\leq C \int_{\{|\mathbf{A}_n| \geq 1\}} |\mathbf{A}_n|^p dx + C \int_{\{|\mathbf{A}_n| \leq 1\}} |\mathbf{A}_n|^q dx \\ &\leq C \int_{\{|\mathbf{A}_n| \geq \varepsilon\}} |\mathbf{A}_n|^p dx + C \int_{\{|\mathbf{A}_n| \leq \varepsilon\}} |\mathbf{A}_n|^q dx \\ &\leq C \int_{\{|\mathbf{A}_n| \geq \varepsilon\}} |w_n|^p dx + C \varepsilon^{q-6} \int_{\{|\mathbf{A}_n| \leq \varepsilon\}} |\mathbf{A}_n|^6 dx \\ &\leq C \|w_n\|_{L^p(\mathbb{R}^3)}^p + C \varepsilon^{q-6} \|\mathbf{A}_n\|_{L^6(\mathbb{R}^3)}^6 \end{aligned}$$

by which, since $\|w_n\|_{L^p(\mathbb{R}^3)} \rightarrow 0$, $\limsup_n \int_{\mathbb{R}^3} |W(|\mathbf{A}_n|^2)| dx \leq C \varepsilon^{q-6}$. By the arbitrariness of ε we get the conclusion.

4 Proof of the Main Theorem

According to Lemma 1 a natural method to solve (7) would be to look for critical points of $\mathcal{E}_{|\mathcal{D}_{\mathcal{F}}^1}$. Anyway, rather than working directly on $\mathcal{E}_{|\mathcal{D}_{\mathcal{F}}^1}$, first we will consider a constrained minimization method. Then set

$$\Sigma := \left\{ \mathbf{A} \in \mathcal{D}_{\mathcal{F}}^1 \mid \int_{\mathbb{R}^3} W(|\mathbf{A}|^2) dx = 1 \right\}. \quad (31)$$

Observe that Σ is not empty. Indeed, as in [2], by (W2) for $R > 1$ define

$$v_R(x) = \begin{cases} \sqrt{\xi} & \text{if } |x| \leq R, \\ \sqrt{\xi}(R+1-|x|) & \text{if } |x| \in [R, R+1], \\ 0 & \text{if } |x| \geq R+1, \end{cases} \quad A_R(x) = v_R(x)\eta(r),$$

where $\eta \in C^\infty(\mathbb{R}, \mathbb{R})$ is such that $\eta = 0$ for $r \leq \frac{1}{2}$ and $\eta = 1$ for $r \geq 1$. Thus $\mathbf{A}(x) = \frac{A_R(x)}{r}(-x_2, x_1, 0) \in \mathcal{D}_{\mathcal{F}}^1$ and it is easily checked that

$$\begin{aligned} & \int_{\mathbb{R}^3} W(|\mathbf{A}|^2) dx \\ & \geq W(\xi) \text{meas}(B_R \cap \{r \geq 1\}) - \text{meas}(B_{R+1} \setminus B_R) \max_{s \in [0, \xi]} |W(s)| \\ & \quad - \text{meas}(B_R \cap \{r < 1\}) \max_{s \in [0, \xi]} |W(s)| \geq CR^3 - C'R^2 - C''R \end{aligned}$$

for some constants $C, C', C'' > 0$. For $R > 0$ large enough this shows that $\int_{\mathbb{R}^3} W(|\mathbf{A}|^2) dx > 0$; then, for a suitable rescaling parameter $\sigma > 0$, we have that the function $\mathbf{A}(\sigma x)$ belongs to Σ . Now consider the following constrained minimization problem:

$$\min \left\{ \int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 dx \mid \mathbf{A} \in \Sigma \right\}. \quad (32)$$

We will see in the last part of the paper that the solutions of the problem (32) are, modulo rescaling, critical points of the functional (14). In particular, analyzing the behaviour of the minimizing sequences for (32) we obtain the following result.

Proposition 2 *There exists a minimizing sequence of (32) which weakly converges to a function $\mathbf{A} \in \mathcal{D}_{\mathcal{F}}^1 \setminus \{0\}$.*

Proof. Let $(\mathbf{A}_n)_n$ be a minimizing sequence of (32), namely

$$\begin{aligned} & \int_{\mathbb{R}^3} |\nabla \mathbf{A}_n|^2 dx \rightarrow \inf_{\mathbf{A} \in \Sigma} \int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 dx, \\ & \int_{\mathbb{R}^3} W(|\mathbf{A}_n|^2) dx = 1 \quad \forall n. \end{aligned} \quad (33)$$

We claim that for every $R > 0$

$$\liminf_{n \rightarrow +\infty} \sup_{z \in \mathbb{R}^3} \int_{B(z, R)} |\mathbf{A}_n|^2 dx > 0. \quad (34)$$

Otherwise, there should exist $\tilde{R} > 0$ such that, up to a subsequence,

$$\lim_{n \rightarrow +\infty} \sup_{z \in \mathbb{R}^3} \int_{B(z, \tilde{R})} |\mathbf{A}_n|^2 dx \rightarrow 0,$$

and then, by Lemma 2,

$$\int_{\mathbb{R}^3} W(|\mathbf{A}_n|^2) dx \rightarrow 0$$

that contradicts (33). From (34) we deduce the existence of $R > 0$, $\varepsilon > 0$ and a sequence $(z_n)_n = ((z_{n,1}, z_{n,2}, z_{n,3}))_n$ in \mathbb{R}^3 such that

$$\int_{B(z_n, R)} |\mathbf{A}_n|^2 dx \geq \varepsilon, \quad \forall n.$$

From Hölder's inequality we get

$$\int_{B(z_n, R)} |\mathbf{A}_n|^6 dx \geq \delta, \quad \forall n$$

for a suitable $\delta > 0$. We claim that (z_n) is bounded in the directions x_1 and x_2 . Indeed the cylindrical symmetry of $|\mathbf{A}_n|$ implies that $\int_{B(z, R)} |\mathbf{A}_n|^6 dx \geq \delta$ for every $z = (z_1, z_2, z_{n,3})$ such that, using the notation (12), $r_z = \sqrt{z_1^2 + z_2^2} = \sqrt{z_{n,1}^2 + z_{n,2}^2} = r_{z_n}$. Geometric arguments assure that the number of disjoint balls of the kind $B(z, R)$ with $r_z = r_{z_n}$ grows as r_{z_n} grows. Then the boundedness of $(\mathbf{A}_n)_n$ in the L^6 -norm put an upper bound to the sequence $(r_{z_n})_n$. If we relabel $(\mathbf{A}_n)_n$ the sequence obtained making the translation in the x_3 -direction, i.e. $\mathbf{A}_n(\cdot + z_{n,3}\ell_3)$ (being $\ell_3 = (0, 0, 1)$), we obtain a new minimizing sequence in $\mathcal{D}_{\mathcal{F}}^1$ satisfying, possibly increasing the radius R ,

$$\int_{B(0, R)} |\mathbf{A}_n|^2 dx \geq \varepsilon \quad \forall n \in \mathbb{N}. \quad (35)$$

Since $(\mathbf{A}_n)_n$ is bounded, certainly there exists $\mathbf{A} \in \mathcal{D}_{\mathcal{F}}^1$ such that, up to a subsequence,

$$\mathbf{A}_n \rightharpoonup \mathbf{A} \quad \text{in } \mathcal{D}_{\mathcal{F}}^1.$$

On the other hand by (10) we have $\mathbf{A}_n \rightharpoonup \mathbf{A}$ in $H^1(B(0, R), \mathbb{R}^3)$ and, by (35) and the compact embedding $H^1(B(0, R), \mathbb{R}^3) \hookrightarrow L^2(B(0, R), \mathbb{R}^3)$, we deduce that $\mathbf{A} \neq 0$.

Now we are ready for the following

Proof of Theorem 1 According to Proposition 2 let $\mathbf{A} \in \mathcal{D}_{\mathcal{F}}^1 \setminus \{0\}$ be the weak limit of a minimizing sequence $(\mathbf{A}_n)_n$ for (32). First we introduce the following notation:

$$\forall \mathbf{C} \in \mathcal{D}_{\mathcal{F}}^1, \forall \mu > 0 : \mathbf{C}_\mu(x) = \mathbf{C}(\mu x).$$

Consider $\mathbf{B} \in \mathcal{D}_{\mathcal{F}}^1$, \mathbf{B} compactly supported. For every $n \in \mathbb{N}$ and $t > 0$ set

$$\mu_{n,t} := \left(\int_{\mathbb{R}^3} W(|\mathbf{A}_n + t\mathbf{B}|^2) dx \right)^{\frac{1}{3}},$$

so that $(\mathbf{A}_n + t\mathbf{B})_{\mu_n, t} \in \Sigma$. Since $(\mathbf{A}_n)_n$ is a minimizing sequence, for every $t > 0$ we deduce

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \frac{\int_{\mathbb{R}^3} |\nabla(\mathbf{A}_n + t\mathbf{B})_{\mu_n, t}|^2 dx - \int_{\mathbb{R}^3} |\nabla \mathbf{A}_n|^2 dx}{t} \\ = \frac{\liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |\nabla(\mathbf{A}_n + t\mathbf{B})_{\mu_n, t}|^2 dx - \theta}{t} \geq 0 \end{aligned} \quad (36)$$

where $\theta = \inf_{\mathbf{A} \in \Sigma} \int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 dx$. Consider two sequences $t_n, \varepsilon_n > 0$, $t_n \rightarrow 0^+$, $\varepsilon_n \searrow 0^+$; then for every $n \geq 1$ by (36) there exists \mathbf{A}_{k_n} such that

$$\frac{\int_{\mathbb{R}^3} |\nabla(\mathbf{A}_{k_n} + t_n \mathbf{B})_{\mu_{k_n, t_n}}|^2 dx - \int_{\mathbb{R}^3} |\nabla \mathbf{A}_{k_n}|^2 dx}{t_n} \geq -\varepsilon_n.$$

As a consequence we can extract a subsequence $(\mathbf{A}_{k_n})_n$ from $(\mathbf{A}_n)_n$, which we relabel $(\mathbf{A}_n)_n$, such that, setting $\mu_n = \mu_{k_n, t_n}$,

$$\liminf_{n \rightarrow +\infty} \frac{\int_{\mathbb{R}^3} |\nabla(\mathbf{A}_n + t_n \mathbf{B})_{\mu_n}|^2 dx - \int_{\mathbb{R}^3} |\nabla \mathbf{A}_n|^2 dx}{t_n} \geq 0. \quad (37)$$

On the other hand for every n we have

$$\begin{aligned} & \frac{\int_{\mathbb{R}^3} |\nabla(\mathbf{A}_n + t_n \mathbf{B})_{\mu_n}|^2 dx - \int_{\mathbb{R}^3} |\nabla \mathbf{A}_n|^2 dx}{t_n} \\ &= \frac{\int_{\mathbb{R}^3} \frac{1}{\mu_n} |\nabla(\mathbf{A}_n + t_n \mathbf{B})|^2 dx - \int_{\mathbb{R}^3} |\nabla \mathbf{A}_n|^2 dx}{t_n} \\ &= \frac{1}{\mu_n} \left[\frac{(1 - \mu_n)}{t_n} \int_{\mathbb{R}^3} |\nabla \mathbf{A}_n|^2 dx + t_n \int_{\mathbb{R}^3} |\nabla \mathbf{B}|^2 dx + 2 \int_{\mathbb{R}^3} (\nabla \mathbf{A}_n | \nabla \mathbf{B}) dx \right]. \end{aligned} \quad (38)$$

Since $\mathbf{A}_n \in \Sigma$, we have

$$\begin{aligned} \frac{1 - \mu_n}{t_n} &= \frac{1 - \mu_n^3}{t_n(1 + \mu_n + \mu_n^2)} \\ &= \frac{\int_{\mathbb{R}^3} W(|\mathbf{A}_n|^2) dx - \int_{\mathbb{R}^3} W(|\mathbf{A}_n + t_n \mathbf{B}|^2) dx}{t_n(1 + \mu_n + \mu_n^2)} \\ &= - \frac{2 \int_{\mathbb{R}^3} W'(|\mathbf{C}_n|^2) \mathbf{C}_n \cdot \mathbf{B} dx}{(1 + \mu_n + \mu_n^2)}, \end{aligned} \quad (39)$$

where $\mathbf{C}_n = \mathbf{A}_n + s_n \mathbf{B}$ for a suitable $s_n \in (0, t_n)$. Hence $\mathbf{C}_n \rightharpoonup \mathbf{A}$ in $\mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3)$; setting $U = \{x \in \mathbb{R}^3 \mid \mathbf{B}(x) \neq 0\}$, then U is bounded and, by

(10), $\mathbf{C}_n \rightharpoonup \mathbf{A}$ in $H^1(U, \mathbb{R}^3)$; consequently $\mathbf{C}_n \rightarrow \mathbf{A}$ in $L^p(U, \mathbb{R}^3)$. According to assumption (W3) $|W'(s^2)| \leq c|s|^{p-2}$ for every $s \in \mathbb{R}$; the continuity of the Nemytski operators leads to $W'(|\mathbf{C}_n|^2)\mathbf{C}_n \rightarrow W'(|\mathbf{A}|^2)\mathbf{A}$ in $L^{p/(p-1)}(U, \mathbb{R}^3)$, by which we deduce

$$\int_{\mathbb{R}^3} W'(|\mathbf{C}_n|^2)\mathbf{C}_n \cdot \mathbf{B} \, dx \rightarrow \int_{\mathbb{R}^3} W'(|\mathbf{A}|^2)\mathbf{A} \cdot \mathbf{B} \, dx. \quad (40)$$

By (39) and (40), since $t_n \rightarrow 0^+$, we have

$$\lim_{n \rightarrow +\infty} \mu_n = 1 \quad (41)$$

and then, again from (39),

$$\lim_{n \rightarrow +\infty} \frac{1 - \mu_n}{t_n} = -\frac{2}{3} \int_{\mathbb{R}^3} W'(|\mathbf{A}|^2)\mathbf{A} \cdot \mathbf{B} \, dx. \quad (42)$$

Inserting (41)-(42) in (38) and using (37) we deduce that

$$\int_{\mathbb{R}^3} (\nabla \mathbf{A} | \nabla \mathbf{B}) \, dx - \frac{\theta}{3} \int_{\mathbb{R}^3} W'(|\mathbf{A}|^2)\mathbf{A} \cdot \mathbf{B} \, dx \geq 0.$$

Replacing \mathbf{B} by $-\mathbf{B}$ and repeating the same arguments as before, we have that also the opposite inequality holds, and then

$$\int_{\mathbb{R}^3} (\nabla \mathbf{A} | \nabla \mathbf{B}) \, dx - \frac{\theta}{3} \int_{\mathbb{R}^3} W'(|\mathbf{A}|^2)\mathbf{A} \cdot \mathbf{B} \, dx = 0. \quad (43)$$

In general, if $\mathbf{B} \in \mathcal{D}_{\mathcal{F}}^1$, we can select $f \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$ such that $f = f(|x|)$, $0 \leq f \leq 1$, $f = 1$ in $B(0, 1)$, $f = 0$ in $\mathbb{R}^3 \setminus B(0, 2)$, $|\nabla f| \leq 2$ and set $f_n(x) = f(\frac{x}{n})$. Then it is easy to prove that $f_n \mathbf{B} \in \mathcal{D}_{\mathcal{F}}^1$ too, $f_n \mathbf{B}$ is compactly supported and $f_n \mathbf{B} \rightarrow \mathbf{B}$ in $\mathcal{D}_{\mathcal{F}}^1$. Hence by density we obtain that (43) holds for every $\mathbf{B} \in \mathcal{D}_{\mathcal{F}}^1$. The constrained minimization method has caused a Lagrange multiplier θ to appear in (43). We remark that $\theta > 0$, otherwise by (43), taking $\mathbf{B} = \mathbf{A}$, we would have $\int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 \, dx = 0$, i.e. $\mathbf{A} = 0$ which is impossible. The Lagrange multiplier θ can be removed by a rescaling argument: set

$$\bar{\mathbf{A}}(x) = \mathbf{A} \left(\sqrt{\frac{3}{\theta}} x \right) \in \mathcal{D}_{\mathcal{F}}^1.$$

By (43) it is easy to verify that

$$\int_{\mathbb{R}^3} (\nabla \bar{\mathbf{A}} | \nabla \mathbf{B}) \, dx = \int_{\mathbb{R}^3} W'(|\bar{\mathbf{A}}|^2) \bar{\mathbf{A}} \cdot \mathbf{B} \, dx \quad \forall \mathbf{B} \in \mathcal{D}_{\mathcal{F}}^1,$$

which means that $\bar{\mathbf{A}}$ is a critical point of $\mathcal{E}|_{\mathcal{D}_{\mathcal{F}}^1}$ and, consequently, by Lemma 1, a weak solution of (7).

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