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$F_q[M_2]$, $F_q[GL_2]$ AND $F_q[SL_2]$ AS QUANTIZED HYPERALGEBRAS

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Abstract. Within the quantum function algebra $F_q[SL_2]$, we study the subset $\mathcal{F}_q[SL_2]$ — introduced in [Ga1] — of all elements of $F_q[SL_2]$ which are $\mathbb{Z}\big[q,q^{-1}\big]$–valued when paired with $\mathcal{U}_q(\mathfrak{sl}_2)$, the unrestricted $\mathbb{Z}\big[q,q^{-1}\big]$–integral form of $U_q(\mathfrak{sl}_2)$ introduced by De Concini, Kac and Procesi. In particular we yield a presentation of it by generators and relations, and a nice $\mathbb{Z}\big[q,q^{-1}\big]$–spanning set (of PBW type). Moreover, we give a direct proof that $\mathcal{F}_q[SL_2]$ is a Hopf subalgebra of $F_q[SL_2]$, and that $\mathcal{F}_q[SL_2]\Big|_{q=1} \cong U_{\mathbb{Z}}(\mathfrak{sl}_2{}^*)$. We describe explicitly its specializations at roots of 1, say ε, and the associated quantum Frobenius (epi)morphism (also introduced in [Ga1]) from $\mathcal{F}_\varepsilon[SL_2]$ to $\mathcal{F}_1[SL_2] \cong U_{\mathbb{Z}}(\mathfrak{sl}_2{}^*)$. The same analysis is done for $\mathcal{F}_q[GL_2]$, with similar results, and also (as a key, intermediate step) for $\mathcal{F}_q[M_2]$.

Introduction

Let G be a semisimple, connected, simply connected affine algebraic group over $\mathbb{C}$, and $\mathfrak{g}$ its tangent Lie algebra. Let $U_q(\mathfrak{g})$ be the Drinfeld-Jimbo quantum group over $\mathfrak{g}$, defined over the field $\mathbb{Q}(q)$, where q is an indeterminate. There exist two integral forms of $U_q(\mathfrak{g})$ over $\mathbb{Z}\big[q,q^{-1}\big]$, the restricted one, say $\mathfrak{U}_q(\mathfrak{g})$, and the unrestricted one, say $\mathcal{U}_q(\mathfrak{g})$ — see [CP] and references therein. Both of them bear so called “quantum Frobenius morphisms”, namely Hopf algebra morphisms linking their specialisations at 1 with their specialisations at roots of 1. In particular, $\mathfrak{U}_q(\mathfrak{g})$ for $q \to 1$ specializes to $U_{\mathbb{Z}}(\mathfrak{g})$, the Kostant $\mathbb{Z}$–form of $U(\mathfrak{g})$; so $\mathfrak{g}$ becomes a Lie bialgebra, and G a Poisson group. Also, $\mathcal{U}_q(\mathfrak{g})$ for $q \to 1$ specializes to $F_{\mathbb{Z}}[G^*]$, a $\mathbb{Z}$–form of the function algebra on a Poisson group G^* dual to G .

Dually, one constructs a Hopf algebra $F_q[G]$ of matrix coefficients of $U_q(\mathfrak{g})$. It has two $\mathbb{Z}\big[q,q^{-1}\big]$–forms, say $\mathfrak{F}_q[G]$ and $\mathcal{F}_q[G]$, defined to be the subset of $F_q[G]$ of all $\mathbb{Z}\big[q,q^{-1}\big]$–valued functions on $\mathfrak{U}_q(\mathfrak{g})$, respectively on $\mathcal{U}_q(\mathfrak{g})$. At $q = 1$, $\mathfrak{F}_q[G]$ specializes to $F_{\mathbb{Z}}[G]$, while $\mathcal{F}_q[G]$ specializes to $U_{\mathbb{Z}}\big(\mathfrak{g}^*\big)$, a Kostant-like $\mathbb{Z}$–form of $U\big(\mathfrak{g}^*\big)$ — cf. [Ga1] for details.

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Moreover, both $\mathfrak{F}_q[G]$ and $\mathcal{F}_q[G]$ bear quantum Frobenius morphisms (relating their specialisations at 1 with those at roots of 1), which are dual to those of $\mathfrak{U}_q(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})$.

The aim of this article is to describe $\mathcal{F}_q[G]$, its specializations at roots of 1 and its quantum Frobenius morphisms when $G = SL_2$. Moreover, as the construction of these objects makes sense for $G = GL_2$ and $G = M_2 := Mat_2$ as well, we find similar results for them.

By [Ga1], $\mathcal{F}_q[M_2]$ should resemble $\mathfrak{U}_q(\mathfrak{gl}_2)$. Indeed, this is the case: $\mathcal{F}_q[M_2]$ is generated by quantum divided powers and quantum binomial coefficients, a PBW-like theorem hold for $\mathcal{F}_q[M_2]$, and the quantum Frobenius morphisms are given by an "ℓ–th root operation", if ℓ is the order of the root of unity. Similar (weaker) results hold for $\mathcal{F}_q[GL_2]$ and $\mathcal{F}_q[SL_2]$.

The general case of $M_n := Mat_n$, GL_n and SL_n is studied in [GR2], exploiting the same key ideas already developed here and the present results for $n = 2$.

Warning: an expanded, more detailed version of this paper is available on line, cf. [GR1]; the quotations in [GR2] about the present work refer in fact to [GR1].

DEDICATORY

This work grew out of a cooperation supported by an official agreement between the Department of Mathematics of the University of Rome "Tor Vergata" and the Faculty of Mathematics of the University of Belgrade in the period 2000–2003. Such agreement was the outcome of a common wish of peaceful, fruitful partnership, as an answer to the military aggression of NATO countries to the Federal Republic of Yugoslavia, which started in springs of 1999. This paper is dedicated to the memory of all victims of that war.

§ 1 Geometrical background and q–numbers

1.1 Poisson structures on linear groups. Let $\mathfrak{g} := \mathfrak{gl}_2(\mathbb{Q})$, with its basis given by the elementary matrices $e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $g_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $g_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $\mathfrak{g}$ has a natural structure of Lie algebra, and a Lie cobracket is defined on it by $\delta(e) = h \otimes e - e \otimes h$, $\delta(g_k) = 0$ (for $k = 1, 2$), $\delta(f) = h \otimes f - f \otimes h$, where $h := g_1 - g_2$; this makes $\mathfrak{g}$ into a *Lie bialgebra*. It follows that $U(\mathfrak{g})$ is naturally a co-Poisson Hopf algebra, whose co-Poisson bracket is the extension of the Lie cobracket of $\mathfrak{g}$. Finally, Kostant's $\mathbb{Z}$–integral form of $U(\mathfrak{g})$ — called also *hyperalgebra* in literature — is the unital $\mathbb{Z}$–subalgebra $U_{\mathbb{Z}}(\mathfrak{g})$ of $U(\mathfrak{g})$ generated by the "divided powers" $f^{(n)}$, $e^{(n)}$ and the binomial coefficients $\binom{g_k}{n}$ (for $k = 1, 2$, and $n \in \mathbb{N}$), where we use notation $x^{(n)} := x^n \big/ n!$ and $\binom{t}{n} := \frac{t(t-1)\cdots(t-n+1)}{n!}$. Again, this is a co-Poisson Hopf $\mathbb{Z}$–algebra; it is free as a $\mathbb{Z}$–module, with PBW-like $\mathbb{Z}$–basis the set of ordered monomials $\left\{ e^{(\eta)} \binom{g_1}{\gamma_1} \binom{g_2}{\gamma_2} f^{(\varphi)} \,\middle|\, \eta, \gamma_1, \gamma_2, \varphi \in \mathbb{N} \right\}$; see e.g. [Hu], Ch. VII.

A similar description holds for $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{Q})$, taking h instead of g_1 and g_2. The Kostant's $\mathbb{Z}$–form $U_{\mathbb{Z}}(\mathfrak{sl}_2)$ of $U(\mathfrak{sl}_2(\mathbb{Q}))$ is generated as above but for replacing the g_k's with h. Then $U_{\mathbb{Z}}(\mathfrak{sl}_2)$ is a co-Poisson Hopf subalgebra of $U_{\mathbb{Z}}(\mathfrak{gl}_2)$, free as a $\mathbb{Z}$–module with PBW $\mathbb{Z}$–basis as above but with h instead of the g_k's. Finally, $\mathfrak{sl}_2(\mathbb{Q})$ is a Lie sub-bialgebra of $\mathfrak{gl}_2(\mathbb{Q})$, and the embedding $\mathfrak{sl}_2 \longrightarrow \mathfrak{gl}_2$ is a section of the natural Lie bialgebra epimorphism $\mathfrak{gl}_2 \twoheadrightarrow \mathfrak{sl}_2$.

As $\mathfrak{gl}_2(\mathbb{Q})$ is a Lie bialgebra, by general theory $G := GL_2(\mathbb{Q})$ is then a *Poisson* group. Explicitly, the algebra $F[G]$ of regular functions on G is the unital associative commutative $\mathbb{Q}$–algebra with generators $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ and D^{-1}, where $D := det \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$ is the determinant.

The group structure on G yields on $F[G]$ the natural Hopf structure given by matrix product, while the Poisson structure is given by

$$\{\bar{a},\bar{b}\} = \bar{b}\,\bar{a}\,, \quad \{\bar{a},\bar{c}\} = \bar{c}\,\bar{a}\,, \quad \{\bar{b},\bar{c}\} = 0\,, \quad \{\bar{d},\bar{b}\} = -\bar{b}\,\bar{d}\,, \quad \{\bar{d},\bar{c}\} = -\bar{c}\,\bar{d}\,, \quad \{\bar{a},\bar{d}\} = 2\,\bar{b}\,\bar{c}\,.$$

We shall consider also the Poisson group-scheme $G_{\mathbb{Z}}$ associated to GL_2, for which a like analysis applies: in particular, its function algebra $F[G_{\mathbb{Z}}]$ is a Poisson Hopf $\mathbb{Z}$–algebra with the same presentation as $F[G]$ but over the ring $\mathbb{Z}$.

Similar constructions hold for $SL_2(\mathbb{Q})$ and the associated group-scheme (just set $D=1$).

Finally, the subalgebra of $F\big[(GL_2)_{\mathbb{Z}}\big]$ generated by $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ is a Poisson subbialgebra of $F\big[(GL_2)_{\mathbb{Z}}\big]$: indeed, it is the algebra $F\big[(M_2)_{\mathbb{Z}}\big]$ of regular functions of the $\mathbb{Z}$-scheme associated to the Poisson algebraic monoid M_2 of all (2×2)–matrices.

1.2 Dual Lie bialgebras and dual Poisson groups. By general theory, if $\mathfrak{g} := \mathfrak{gl}_2(\mathbb{Q})$ bears a Lie bialgebra structure then the dual space $\mathfrak{g}^*$ is a Lie bialgebra on its own. Let $\{e^*, g_1^*, g_2^*, f^*\}$ be the dual basis to the basis of elementary matrices for $\mathfrak{g}$, and let $\mathrm{e} := f^*\big/2$, $\mathrm{g}_1 := g_1^*$, $\mathrm{g}_2 := g_2^*$, $\mathrm{f} := e^*\big/2$; then $\{\mathrm{e}, \mathrm{g}_1, \mathrm{g}_2, \mathrm{f}\}$ is a basis of $\mathfrak{g}^*$. The Lie bracket of $\mathfrak{g}^*$ is given by $[\mathrm{g}_1, \mathrm{g}_2] = 0$, $[\mathrm{g}_1, \mathrm{f}] = +\mathrm{f}$, $[\mathrm{g}_2, \mathrm{f}] = -\mathrm{f}$, $[\mathrm{g}_1, \mathrm{e}] = +\mathrm{e}$, $[\mathrm{g}_2, \mathrm{e}] = -\mathrm{e}$, $[\mathrm{f}, \mathrm{e}] = 0$, and its Lie cobracket by $\delta(\mathrm{f}) = (\mathrm{g}_1 - \mathrm{g}_2)\wedge \mathrm{f}$, $\delta(\mathrm{g}_1) = 4\cdot \mathrm{f}\wedge \mathrm{e}$, $\delta(\mathrm{g}_2) = 4\cdot \mathrm{e}\wedge \mathrm{f}$, $\delta(\mathrm{e}) = \mathrm{e}\wedge(\mathrm{g}_1 - \mathrm{g}_2)$, where $x\wedge y := x\otimes y - y\otimes x$. These formulæ also provide a presentation of $U(\mathfrak{g}^*)$ as a co-Poisson Hopf algebra. Finally, we can define the Kostant's $\mathbb{Z}$–integral form, or hyperalgebra, $U_{\mathbb{Z}}(\mathfrak{g}^*)$ of $U(\mathfrak{g}^*)$ as the unital $\mathbb{Z}$–subalgebra generated by the divided powers $\mathrm{f}^{(n)}$, $\mathrm{e}^{(n)}$ and binomial coefficients $\binom{\mathrm{g}_k}{n}$ (for all $n\in\mathbb{N}$ and all $k=1,2$). This again is a co-Poisson Hopf $\mathbb{Z}$–algebra, free as a $\mathbb{Z}$–module, with PBW-like $\mathbb{Z}$–basis the set of ordered monomials $\Big\{ \mathrm{e}^{(\eta)} \binom{\mathrm{g}_1}{n_1} \binom{\mathrm{g}_2}{n_2} \mathrm{f}^{(\varphi)} \,\Big|\, \eta, n_1, n_2, \varphi\in\mathbb{N} \Big\}$.

A like description holds for $\mathfrak{sl}_2(\mathbb{Q})^*$: indeed, one has $\mathfrak{sl}_2(\mathbb{Q})^* = \mathfrak{gl}_2(\mathbb{Q})^*\Big/\big(\mathrm{g}_1 + \mathrm{g}_2\big)$, dually to $\mathfrak{sl}_2(\mathbb{Q}) \longrightarrow \mathfrak{gl}_2(\mathbb{Q})$, hence one simply has to set $\mathrm{h} := \mathrm{g}_1 \equiv -\mathrm{g}_2$ in the presentation above. All formulæ involving h follow from $\mathrm{h} \cong +\mathrm{g}_1 \cong -\mathrm{g}_2 \mod \big(\mathrm{g}_1 + \mathrm{g}_2\big)$. In particular $U_{\mathbb{Z}}\big(\mathfrak{sl}_2^*\big)$ is the $\mathbb{Z}$–subalgebra of $U\big(\mathfrak{sl}_2(\mathbb{Q})^*\big)$ generated by divided powers and binomial coefficients as above but taking h instead of the g_k's. Then $U_{\mathbb{Z}}\big(\mathfrak{sl}_2^*\big)$ is a co-Poisson Hopf $\mathbb{Z}$–subalgebra of $U_{\mathbb{Z}}\big(\mathfrak{gl}_2^*\big)$, with PBW $\mathbb{Z}$–basis as above but with h instead of the g_k's.

If $\mathfrak{g} = \mathfrak{gl}_2$, a simply connected algebraic Poisson group with tangent Lie bialgebra $\mathfrak{g}^*$ is the subgroup ${}_sG^*$ of $G\times G$ made of all pairs $\big(L, U\big)\in G\times G$ such that L is lower triangular, U is upper triangular, and their diagonals are inverse to each other. This is a Poisson subgroup of $G\times G$; its centre is $Z := \big\{ \big(zI, z^{-1}I\big) \,\big|\, z\in\mathbb{Q}\setminus\{0\} \big\}$, hence the associated adjoint group is ${}_aG^* := {}_sG^*\big/Z$. The same construction defines Poisson group-schemes ${}_sG_{\mathbb{Z}}^*$ and ${}_aG_{\mathbb{Z}}^*$. If $\mathfrak{g} = \mathfrak{sl}_2$ the construction of dual Poisson group-schemes ${}_sG_{\mathbb{Z}}^*$ and ${}_aG_{\mathbb{Z}}^*$ is entirely similar, just taking $G := SL_2$ instead of GL_2 in the previous recipe.

1.3 q–numbers, q–divided powers and q–binomial coefficients. Let q be an indeterminate. For all $s,n\in\mathbb{N}$, let $(n)_q := \frac{q^n-1}{q-1}$ $(\in\mathbb{Z}[q])$, $(n)_q! := \prod_{r=1}^n (r)_q$, $\binom{n}{s}_q := \frac{(n)_q!}{(s)_q!(n-s)_q!}$ $(\in\mathbb{Z}[q])$, and $[n]_q := \frac{q^n-q^{-n}}{q-q^{-1}}$ $\big(\in\mathbb{Z}\big[q,q^{-1}\big]\big)$, $[n]_q! := \prod_{r=1}^n [r]_q$, $\begin{bmatrix} n \\ s \end{bmatrix}_q := \frac{[n]_q!}{[s]_q![n-s]_q!}$ $\big(\in\mathbb{Z}\big[q,q^{-1}\big]\big)$. Furthermore, we set $\binom{-n}{s}_q := (-1)^s q^{-ns-\binom{s}{2}} \frac{(n-1+s)_q!}{(s)_q!}$ $(\in\mathbb{Z}[q])$ for all $n,s\in\mathbb{N}$, and $(2k)_q!! := \prod_{r=1}^k (2r)_q$, $(2k-1)_q!! := \prod_{r=1}^k (2r-1)_q$, for all $k\in\mathbb{N}_+$.

If A is any $\mathbb{Q}(q)$–algebra, q–divided powers and q–binomial coefficients are: $X^{(n)} := X^n \Big/ [n]_q!$, $\binom{X\,;\,c}{n} := \prod_{s=1}^n \frac{q^{c+1-s}X-1}{q^s-1}$; also, $\left\{ {X\,;\,c \atop n\,,\,r} \right\} := \sum_{s=0}^r q^{\binom{s+1}{2}} \binom{r}{s}_q \cdot \binom{X\,;\,c+s}{n-r}$ (for every $X \in A$, $n, r \in \mathbb{N}$, $c \in \mathbb{Z}$). Furthermore, if $Z \in A$ is *invertible* we define also $\left[{Z\,;\,c \atop n} \right] := \prod_{s=1}^n \frac{q^{c+1-s}Z^{+1} - q^{s-1-c}Z^{-1}}{q^{+s}-q^{-s}}$ for every $n \in \mathbb{N}$ and $c \in \mathbb{Z}$.

For later use, we remark that the q–binomial coefficients in $X \in A$ satisfy the relations

$$\begin{gathered}
\prod_{s=1}^n \big(q^s-1\big) \binom{X\,;\,c}{n} = \prod_{s=1}^n \big(q^{1-s+c}X - 1\big) \\
\binom{X\,;\,c}{t}\binom{X\,;\,c-t}{s} = \binom{t+s}{t}_q \binom{X\,;\,c}{t+s}, \qquad \binom{X\,;\,c+1}{t} - q^t \binom{X\,;\,c}{t} = \binom{X\,;\,c}{t-1} \\
\binom{X\,;\,c}{m}\binom{X\,;\,s}{n} = \binom{X\,;\,s}{n}\binom{X\,;\,c}{m}, \qquad \binom{X\,;\,c}{t} = \sum_{p \geq 0}^{p \leq c,t} q^{(c-p)(t-p)} \binom{c}{p}_q \binom{X\,;\,0}{t-p} \\
\binom{X\,;\,c}{0} = 1, \quad \binom{X\,;\,-c}{t} = \sum_{p=0}^t (-1)^p q^{-t(c+p)+p(p+1)/2} \binom{p+c-1}{p}_q \binom{X\,;\,0}{t-p} \\
\binom{X\,;\,c+1}{t} - \binom{X\,;\,c}{t} = q^{c-t+1} \left(1 + (q-1)\binom{X\,;\,0}{1}\right) \binom{X\,;\,c}{t-1}
\end{gathered} \tag{1.1}$$

Similarly, the q–divided powers in $X \in A$ satisfy the relations

$$X^{(r)} X^{(s)} = \left[{r+s \atop s} \right]_q X^{(r+s)}, \qquad X^{(0)} = 1 \tag{1.2}$$

Finally, let $\ell \in \mathbb{N}_+$ be odd, set $\mathbb{Z}_\varepsilon := Z[q] \Big/ \big(\phi_\ell(q)\big)$ where $\phi_\ell(q)$ is the ℓ-th cyclotomic polynomial in q, and let $\varepsilon := \overline{q}$, a (formal) primitive ℓ-th root of 1 in $\mathbb{Z}_\varepsilon$. Similarly let $\mathbb{Q}_\varepsilon := \mathbb{Q}[q] \Big/ \big(\phi_\ell(q)\big)$, the field of quotients of $\mathbb{Z}_\varepsilon$. If M is a module over $\mathbb{Z}\big[q, q^{-1}\big]$ or $\mathbb{Q}\big[q, q^{-1}\big]$ we shall set $M_\varepsilon := M \Big/ \big(\phi_\ell(q)\big) M$, which is a module over $\mathbb{Z}_\varepsilon$ or over $\mathbb{Q}_\varepsilon$.

§ 2 Quantum groups

2.1 Quantum enveloping algebras $U_q(\mathfrak{gl}_2)$ and $U_q(\mathfrak{sl}_2)$, their integral forms and specializations. Let $U_q(\mathfrak{gl}_2)$ be the unital $\mathbb{Q}(q)$–algebra with generators F, G_1, G_2, G_1^{-1}, G_2^{-1}, E and relations

$$\begin{gathered}
G_i^{\pm 1} G_i^{\mp 1} = 1 = G_i^{\mp 1} G_i^{\pm 1} \ \big(i = 1,2\big), \quad G_1 G_2 = G_2 G_1, \quad EF - FE = \frac{G_1 G_2^{-1} - G_1^{-1} G_2}{q - q^{-1}} \\
G_1^{\pm 1} F = q^{\mp 1} F G_1^{\pm 1}, \ G_2^{\pm 1} F = q^{\pm 1} F G_2^{\pm 1}, \ G_1^{\pm 1} E = q^{\pm 1} E G_1^{\pm 1}, \ G_2^{\pm 1} E = q^{\mp 1} E G_2^{\pm 1}
\end{gathered}$$

Moreover, $U_q(\mathfrak{gl}_2)$ is also a Hopf algebra with $\Delta(F) = F \otimes G_1^{-1} G_2 + 1 \otimes F$, $\epsilon(F) = 0$, $S(F) = -F G_1 G_2^{-1}$, $\Delta\big(G_i^{\pm 1}\big) = G_i^{\pm 1} \otimes G_i^{\pm 1}$, $\epsilon\big(G_i^{\pm 1}\big) = 1$, $S\big(G_i^{\pm 1}\big) = G_i^{\mp 1}$ (for $i = 1, 2$), $\Delta(E) = E \otimes 1 + G_1 G_2^{-1} \otimes E$, $\epsilon(E) = 0$, $S(E) = -G_1^{-1} G_2 E$.

Now let $K^{\pm 1} := G_1^{\pm 1} G_2^{\mp 1}$. The unital $\mathbb{Q}(q)$–subalgebra of $U_q(\mathfrak{gl}_2)$ generated by F, $K^{\pm 1}$ and E is the well-known Drinfeld-Jimbo's quantum algebra $U_q(\mathfrak{sl}_2)$. From the presentation of $U_q(\mathfrak{gl}_2)$ one argues one of $U_q(\mathfrak{sl}_2)$ too, and also sees that the latter is a Hopf subalgebra of the former; indeed, it is also a quotient via $F \mapsto F$, $G_1 \mapsto K$, $G_2 \mapsto K^{-1}$, $E \mapsto E$.

The quantum version of the PBW theorem for $U_q(\mathfrak{gl}_2)$ claims that the set of ordered monomials $B^g := \big\{ E^\eta \, G_1^{\gamma_1} \, G_2^{\gamma_2} \, F^\varphi \,\big|\, \eta, \varphi \in \mathbb{N}, \gamma_1, \gamma_2 \in \mathbb{Z} \big\}$ is a $\mathbb{Q}(q)$–basis of $U_q(\mathfrak{gl}_2)$. Similarly, the set $B^s := \big\{ E^\eta \, K^\kappa \, F^\varphi \,\big|\, \eta, \varphi \in \mathbb{N}, \kappa \in \mathbb{Z} \big\}$ is a $\mathbb{Q}(q)$–basis of $U_q(\mathfrak{sl}_2)$.

As to integral forms, let $\mathfrak{U}_q(\mathfrak{gl}_2)$ be the unital $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $U_q(\mathfrak{gl}_2)$ generated by $F^{(m)}$, $G_1^{\pm 1}$, $\Big(\!{G_1\,;\,c \atop m}\!\Big)$, $G_2^{\pm 1}$, $\Big(\!{G_2\,;\,c \atop m}\!\Big)$, $E^{(m)}$, for all $m \in \mathbb{N}$ and $c \in \mathbb{Z}$. This $\mathfrak{U}_q(\mathfrak{gl}_2)$ is a $\mathbb{Z}\big[q,q^{-1}\big]$–integral form of $U_q(\mathfrak{gl}_2)$ as a Hopf algebra, and specializes to $U_{\mathbb{Z}}(\mathfrak{gl}_2)$ for $q \mapsto 1$, that is $\mathfrak{U}_q(\mathfrak{gl}_2)\Big/(q-1)\,\mathfrak{U}_q(\mathfrak{gl}_2) \cong U_{\mathbb{Z}}(\mathfrak{gl}_2)$ as co-Poisson Hopf algebras; therefore we call $\mathfrak{U}_q(\mathfrak{gl}_2)$ a *quantum (*or *quantized) hyperalgebra*. Finally, for every root of 1, say ε, of odd order ℓ, a *quantum Frobenius morphism* $\mathfrak{Fr}_{\mathfrak{gl}_n}^{\mathbb{Z}} : \mathfrak{U}_\varepsilon(\mathfrak{gl}_2) \relbar\joinrel\twoheadrightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{gl}_n)$ exists (a Hopf algebra epimorphism): the left-hand side is $\mathfrak{U}_\varepsilon(\mathfrak{gl}_2) := \big(\mathfrak{U}_q(\mathfrak{gl}_2)\big)_\varepsilon$, and $\mathfrak{Fr}_{\mathfrak{gl}_n}^{\mathbb{Z}}$ is defined on generators by "dividing out by ℓ" the order of each quantum divided power and each quantum binomial coefficient, if this makes sense, and mapping to zero otherwise.

Similarly, one defines the integral form of $U_q(\mathfrak{sl}_2)$, say $\mathfrak{U}_q(\mathfrak{sl}_2)$, replacing the $G_k^{\pm 1}$'s by $K^{\pm 1}$, and the $\Big(\!{G_k\,;\,c \atop m}\!\Big)$'s by the $\Big(\!{K\,;\,c \atop m}\!\Big)$'s; totally similar results then hold (see [DL], [Ga1]).

As to *unrestricted* integral forms and their specializations, first set $\overline{X} := \big(q-q^{-1}\big)X$ as notation. We define $\mathcal{U}_q(\mathfrak{gl}_2)$ to be the unital $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $U_q(\mathfrak{gl}_2)$ generated by $\big\{\,\overline{F}\,,\, G_1^{\pm 1},\, G_2^{\pm 1},\, \overline{E}\,\big\}$. From the presentation of $U_q(\mathfrak{gl}_2)$ one argues a presentation for $\mathcal{U}_q(\mathfrak{gl}_2)$ as well, and then sees that the latter is a Hopf subalgebra of the former. Moreover, $\mathcal{U}_q(\mathfrak{gl}_2)$ is a free $\mathbb{Z}\big[q,q^{-1}\big]$–module with basis $\mathcal{B}^g := \Big\{\,\overline{F}^{\,\varphi} G_1^{\gamma_1} G_2^{\gamma_2} \overline{E}^{\,\eta} \,\Big|\, \varphi, \gamma_1, \gamma_2, \eta \in \mathbb{N}\Big\}$. Note that $\mathcal{U}_q(\mathfrak{gl}_2)$ is another $\mathbb{Z}\big[q,q^{-1}\big]$–integral form of $U_q(\mathfrak{gl}_2)$, as a Hopf algebra, in that it is a Hopf $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra such that $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{U}_q(\mathfrak{gl}_2) \cong U_q(\mathfrak{gl}_2)$.

Adapting results in [DP], [Ga1] and [Ga3–4], one has that $\mathcal{U}_q(\mathfrak{gl}_2)$ is a *quantization* of $F\big[({}_sGL_2{}^*)_{\mathbb{Z}}\big]$, i.e. $\mathcal{U}_1(\mathfrak{gl}_2) := \big(\mathcal{U}_q(\mathfrak{gl}_2)\big)_1 \cong F\big[({}_sGL_2{}^*)_{\mathbb{Z}}\big]$ as Poisson Hopf algebras, where on left-hand side we consider the standard Poisson structure inherited from $\mathcal{U}_q(\mathfrak{gl}_2)$. Finally, let ℓ and ε be as in § 1.3. Set $\mathcal{U}_\varepsilon(\mathfrak{gl}_2) := \big(\mathcal{U}_q(\mathfrak{gl}_2)\big)_\varepsilon$: then there is a Hopf algebra monomorphism $\mathcal{F}r_{\mathfrak{gl}_2}^{\mathbb{Z}} : F\big[({}_sGL_2{}^*)_{\mathbb{Z}}\big] \cong \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{U}_1(\mathfrak{gl}_2) \lhook\joinrel\longrightarrow \mathcal{U}_\varepsilon(\mathfrak{gl}_2)$ given by $\overline{F}\,\big|_{q=1} \mapsto \overline{F}^{\,\ell}\Big|_{q=\varepsilon}$, $G_k^{\pm 1}\big|_{q=1} \mapsto G_k^{\pm \ell}\big|_{q=\varepsilon}$, $\overline{E}\,\big|_{q=1} \mapsto \overline{E}^{\,\ell}\Big|_{q=\varepsilon}$ ($k=1,2$). This is the *quantum Frobenius morphism* for ${}_sGL_2{}^*$.

Again, the same constructions can be done with $\mathfrak{sl}_2$ too. One defines the unrestricted $\mathbb{Z}\big[q,q^{-1}\big]$–integral form $\mathcal{U}_q(\mathfrak{sl}_2)$ of $U_q(\mathfrak{sl}_2)$, simply following the recipe above but replacing the $G_k^{\pm 1}$'s with $K^{\pm 1}$. Then similar results to those for $\mathcal{U}_q(\mathfrak{gl}_2)$ hold for $U_q(\mathfrak{sl}_2)$ as well, e.g. $\mathcal{U}_q(\mathfrak{sl}_2)$ is a *quantization* of $F\big[({}_aSL_2{}^*)_{\mathbb{Z}}\big]$, that is $\mathcal{U}_1(\mathfrak{sl}_2) := \big(\mathcal{U}_q(\mathfrak{sl}_2)\big)_1 \cong F\big[({}_aSL_2{}^*)_{\mathbb{Z}}\big]$ as Poisson Hopf algebras (like above). See [Ga1] and references therein for further details.

The embedding $U_q(\mathfrak{sl}_2) \lhook\joinrel\longrightarrow U_q(\mathfrak{gl}_2)$ restricts to Hopf embeddings $\mathfrak{U}_q(\mathfrak{sl}_2) \lhook\joinrel\longrightarrow \mathfrak{U}_q(\mathfrak{gl}_2)$ and $\mathcal{U}_q(\mathfrak{sl}_2) \lhook\joinrel\longrightarrow \mathcal{U}_q(\mathfrak{gl}_2)$. The specializations of the latter ones at $q=\varepsilon$ and at $q=1$ are compatible (in the obvious sense) with the quantum Frobenius morphisms.

2.2 Quantum function algebras $F_q[M_2]$, $F_q[GL_2]$ and $F_q[SL_2]$, their integral forms and specializations. Let $F_q[M_2]$ be the well-known quantum function algebra over M_2 introduced by Manin. Namely, $F_q[M_2]$ is the unital associative $\mathbb{Q}(q)$–algebra with generators a, b, c, d and relations

$$ab = q\,ba\,, \quad ac = q\,ca\,, \quad bd = q\,db\,, \quad cd = q\,dc\,, \quad bc = cb\,, \quad ad - da = \big(q-q^{-1}\big)\,bc\,.$$

This is also a $\mathbb{Q}(q)$–bialgebra (yet not a Hopf algebra), with coalgebra structure given by

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c\,, \quad \epsilon(a) = 1\,, \qquad \Delta(b) = a \otimes b + b \otimes d\,, \quad \epsilon(b) = 0 \\ \Delta(c) &= c \otimes a + d \otimes c\,, \quad \epsilon(c) = 0\,, \qquad \Delta(d) = c \otimes b + d \otimes d\,, \quad \epsilon(d) = 1\,. \end{aligned}$$

In particular, the quantum determinant $D_q := ad - q\,bc \in F_q[M_2]$ is central and group-like in the bialgebra $F_q[M_2]$. Finally, it follows from definitions that $F_q[M_2]$ admits as $\mathbb{Q}(q)$–basis the set of ordered monomials $B_{M_2} := \Big\{ b^\beta a^\alpha d^\delta c^\kappa \,\Big|\, \beta, \alpha, \delta, \kappa \in \mathbb{N} \Big\}$.

We define $\mathfrak{F}_q[M_2]$ to be the unital $\mathbb{Z}\big[q, q^{-1}\big]$–subalgebra of $F_q[M_2]$ generated by a, b, c and d; this in fact is a *sub-bialgebra*, and admits the same presentation as $F_q[M_2]$ but over $\mathbb{Z}\big[q, q^{-1}\big]$. It follows that B_{M_2} is also a $\mathbb{Z}\big[q, q^{-1}\big]$–basis of $\mathfrak{F}_q[M_2]$, hence $\mathfrak{F}_q[M_2]$ is a $\mathbb{Z}\big[q, q^{-1}\big]$–*integral form of* $F_q[M_2]$. From its presentation one sees that $\mathfrak{F}_q[M_2]$ is a *quantization* of $F\big[(M_2)_{\mathbb{Z}}\big]$, i.e. $\mathfrak{F}_1[M_2] := \big(\mathfrak{F}_q[M_2]\big)_1 \cong F\big[(M_2)_{\mathbb{Z}}\big]$ as Poisson bialgebras (where on left-hand side we consider the standard Poisson structure inherited from $\mathfrak{F}_q[M_2]$); using notation of § 1.1, the isomorphism is given by $x\big|_{q=1} \cong \bar{x}$ for all $x \in \{a, b, c, d\}$.

Let $F_q[GL_2] := \big(F_q[M_2]\big)\big[D_q^{-1}\big]$, the extension of $F_q[M_2]$ by a formal inverse to D_q. Then $F_q[GL_2]$ is the unital associative $\mathbb{Q}(q)$–algebra with generators a, b, c, d and D_q^{-1}, and relations like for $F_q[M_2]$ plus those saying that D_q^{-1} is central and inverse to D_q. This $F_q[GL_2]$ is a Hopf algebra, whose coproduct and counit on the generators is given by the formulæ in § 2.2 and by those saying that D_q^{-1} is group-like plus $S(a) = d\,D_q^{-1}$, $S(b) = -q^{-1} b\,D_q^{-1}$, $S(c) = -q^{+1} c\,D_q^{-1}$, $S(d) = a\,D_q^{-1}$ and $S\big(D_q^{-1}\big) = ab - q\,bc \equiv D_q$.

It follows that $F_q[GL_2]$ is $\mathbb{Q}(q)$–spanned by $B_{GL_2} := \Big\{ b^\beta a^\alpha d^\delta c^\kappa D_q^{-n} \,\Big|\, \beta, \alpha, \delta, \kappa, n \in \mathbb{N} \Big\}$.

Let $\mathfrak{F}_q[GL_2]$ be the unital $\mathbb{Z}\big[q, q^{-1}\big]$–subalgebra of $F_q[GL_2]$ generated by a, b, c, d and D_q^{-1} (note that $D_q \in \mathfrak{F}_q[M_2]$). This in fact is a Hopf $\mathbb{Z}\big[q, q^{-1}\big]$–subalgebra, and admits the same presentation as $F_q[GL_2]$ but over $\mathbb{Z}\big[q, q^{-1}\big]$. Then B_{GL_2} is also a $\mathbb{Z}\big[q, q^{-1}\big]$–spanning set of $\mathfrak{F}_q[GL_2]$, hence $\mathfrak{F}_q[GL_2]$ is a $\mathbb{Z}\big[q, q^{-1}\big]$–*integral form of* $U_q(\mathfrak{gl}_2)$. Also, $\mathfrak{F}_q[GL_2]$ is a *quantization* of $F\big[(GL_2)_{\mathbb{Z}}\big]$ as a Poisson Hopf algebra, with $D_q^{\pm 1}\big|_{q=1} \cong D^{\pm 1}$.

Let $F_q[SL_2]$ be the quotient $F_q[SL_2] := F_q[GL_2]\big/\big(D_q - 1\big) \cong F_q[M_2]\big/\big(D_q - 1\big)$ where $\big(D_q - 1\big)$ is the two-sided ideal of $F_q[GL_2]$ or of $F_q[M_2]$ generated by the central element $D_q - 1$. This is a Hopf ideal of $F_q[GL_2]$, so $F_q[SL_2]$ is a Hopf algebra too: it admits the like presentation as $F_q[M_2]$ or $F_q[GL_2]$ but with the additional relation $D_q - 1 = 0$. Moreover, $F_q[SL_2]$ has the Hopf structure given as for $F_q[GL_2]$ but setting $D_q^{-1} = 1$. It also follows that $F_q[SL_2]$ admits $B_{SL_2} := \Big\{ b^\beta a^\alpha d^\delta c^\kappa \,\Big|\, \beta, \alpha, \delta, \kappa \in \mathbb{N}\,,\, 0 \in \{\alpha, \delta\} \Big\}$ as PBW-like basis over $\mathbb{Q}(q)$. The definition of the integral form $\mathfrak{F}_q[SL_2]$, as well as its properties, are exactly like those of $\mathfrak{F}_q[GL_2]$, up to switching "$\mathfrak{gl}$" with "$\mathfrak{sl}$" and "GL" with SL".

Another description of $\mathfrak{F}_q[M_2]$, $\mathfrak{F}_q[GL_2]$ and $\mathfrak{F}_q[SL_2]$ is possible. Indeed, using a characterization as algebra of matrix coefficients, $F_q[M_2]$ naturally embeds into $U_q(\mathfrak{gl}_2)^*$. In particular, there is a perfect (= non-degenerate) Hopf pairing between $F_q[M_2]$ and $U_q(\mathfrak{gl}_2)$, which we denote by $\langle\ ,\ \rangle : F_q[M_2] \times U_q(\mathfrak{gl}_2) \longrightarrow \mathbb{Q}(q)$ (see e.g. [No] for details). Then $\mathfrak{F}_q[M_2] = \Big\{ f \in F_q[M_2] \,\Big|\, \big\langle f, \mathfrak{U}_q(\mathfrak{gl}_2) \big\rangle \subseteq \mathbb{Z}\big[q, q^{-1}\big] \Big\}$. This leads us to define (cf. [Ga1])

$$\mathcal{F}_q[M_2] \;:=\; \Big\{ f \in F_q[M_2] \,\Big|\, \big\langle f, \mathcal{U}_q(\mathfrak{gl}_2) \big\rangle \subseteq \mathbb{Z}\big[q, q^{-1}\big] \Big\}\ .$$

The arguments in [Ga1], *mutatis mutandis*, prove also that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{F}_q[M_2]$ is a $\mathbb{Q}\big[q, q^{-1}\big]$–integral form of $F_q[M_2]$. Moreover, the analysis therein together with [Ga3], § 7.10, proves that $\mathbb{Q} \cdot \mathcal{F}_q[M_2]$ is a *quantization* of $U\big(\mathfrak{gl}_2^*\big)$, i.e. $\big(\mathbb{Q} \otimes \mathcal{F}_q[M_2]\big)_1 \cong U\big(\mathfrak{gl}_2^*\big)$ as co-Poisson bialgebras (taking on left-hand side the co-Poisson structure inherited from $\mathbb{Q} \otimes \mathcal{F}_q[M_2]$).

The perfect Hopf pairing between $F_q[M_2]$ and $U_q(\mathfrak{gl}_2)$ uniquely extends to a similar pairing

$\langle\ ,\ \rangle : F_q[GL_2] \times U_q(\mathfrak{gl}_2) \longrightarrow \mathbb{Q}(q)$, and $\mathfrak{F}_q[GL_2] = \Big\{ f \in F_q[GL_2] \,\Big|\, \big\langle f, \mathfrak{U}_q(\mathfrak{gl}_2) \big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big] \Big\}$. This gives the idea (like in [Ga1]) to define

$$\mathcal{F}_q[GL_2] \;:=\; \Big\{ f \in F_q[G] \,\Big|\, \big\langle f, \mathcal{U}_q(\mathfrak{g}) \big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big] \Big\} \;.$$

Again, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{F}_q[GL_2]$ is a $\mathbb{Q}\big[q,q^{-1}\big]$–integral form of $F_q[GL_2]$, and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{F}_q[GL_2]$ is a *quantization* of $U\big(\mathfrak{gl}_2^*\big)$, i.e. $\big(\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{F}_q[GL_2]\big)_1 \cong U\big(\mathfrak{gl}_2^*\big)$ as co-Poisson Hopf algebras.

The pairing between $F_q[M_2]$ (or $F_q[GL_2]$) and $U_q(\mathfrak{gl}_2)$ induces a perfect pairing between $F_q[SL_2]$ and $U_q(\mathfrak{sl}_2)$, giving $\mathfrak{F}_q[SL_2] = \Big\{ f \in F_q[SL_2] \,\Big|\, \big\langle f, \mathfrak{U}_q(\mathfrak{sl}_2) \big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big] \Big\}$. Then

$$\mathcal{F}_q[SL_2] \;:=\; \Big\{ f \in F_q[SL_2] \,\Big|\, \big\langle f, \mathcal{U}_q(\mathfrak{sl}_2) \big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big] \Big\}$$

and similar results to those for $\mathcal{F}_q[GL_2]$ hold for $\mathcal{F}_q[SL_2]$ too — see [Ga1].

Finally, let $\ell \in \mathbb{N}_+$ be odd, and let ε be a (formal) primitive ℓ-th root of 1 as in § 1.3. Set $\mathcal{F}_\varepsilon[M_2] := \big(\mathcal{F}_q[M_2]\big)_\varepsilon$: then again [Ga1] and [Ga3] show there is an epimorphism

$$\mathcal{F}r_{M_2}^{\mathbb{Q}} : \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}_\varepsilon} \mathcal{F}_\varepsilon[M_2] \relbar\joinrel\twoheadrightarrow \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}\big(\mathfrak{gl}_2^*\big) \;=\; \mathbb{Q}_\varepsilon \otimes_{\mathbb{Q}} U\big(\mathfrak{gl}_2^*\big) \tag{2.1}$$

of bialgebras, which we call *quantum Frobenius morphism* for $\mathfrak{gl}_2^*$. Similarly, there exist

$$\mathcal{F}r_{GL_2}^{\mathbb{Q}} : \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}_\varepsilon} \mathcal{F}_\varepsilon[GL_2] \relbar\joinrel\twoheadrightarrow \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{F}_1[GL_2] \;\cong\; \mathbb{Q}_\varepsilon \otimes_{\mathbb{Q}} U\big(\mathfrak{gl}_2^*\big) \tag{2.2}$$

a Hopf algebra epimorphism extending $\mathcal{F}r_{M_2}^{\mathbb{Q}}$, the *quantum Frobenius morphism* for $\mathfrak{gl}_2{}^*$, and a Hopf algebra epimorphism, uniquely induced by $\mathcal{F}r_{M_2}^{\mathbb{Z}}$ or $\mathcal{F}r_{GL_2}^{\mathbb{Z}}$,

$$\mathcal{F}r_{SL_2}^{\mathbb{Q}} : \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}_\varepsilon} \mathcal{F}_\varepsilon[SL_2] \relbar\joinrel\twoheadrightarrow \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}\big(\mathfrak{sl}_2^*\big) \;=\; \mathbb{Q}_\varepsilon \otimes_{\mathbb{Q}} U\big(\mathfrak{sl}_2^*\big) \tag{2.3}$$

where $\mathcal{F}_\varepsilon[SL_2] := \big(\mathcal{F}_q[SL_2]\big)_\varepsilon$, which we call *quantum Frobenius morphism* for $\mathfrak{sl}_2^*$.

By construction a bialgebra and a Hopf algebra epimorphism $F_q[M_2] \relbar\joinrel\twoheadrightarrow F_q[SL_2]$ and $F_q[GL_2] \relbar\joinrel\twoheadrightarrow F_q[SL_2]$ exist, dual to $U_q(\mathfrak{sl}_2) \hookrightarrow U_q(\mathfrak{gl}_2)$, and similarly there are epimorphisms $\mathfrak{F}_q[M_2] \relbar\joinrel\twoheadrightarrow \mathfrak{F}_q[SL_2]$ and $\mathfrak{F}_q[GL_2] \relbar\joinrel\twoheadrightarrow \mathfrak{F}_q[SL_2]$ dual to $\mathfrak{U}_q(\mathfrak{sl}_2) \hookrightarrow \mathfrak{U}_q(\mathfrak{gl}_2)$.

2.3 Dual quantum enveloping algebras. The linear dual $U_q(\mathfrak{sl}_2)^*$ of $U_q(\mathfrak{sl}_2)$ can be seen again (cf. [Ga1]) as a quantum group on its own: indeed, we set $U_q\big(\mathfrak{sl}_2{}^*\big) := U_q(\mathfrak{sl}_2)^*$, a notation used because $U_q\big(\mathfrak{sl}_2{}^*\big)$ stands for the Lie bialgebra $\mathfrak{sl}_n^*$ just like $U_q(\mathfrak{sl}_2)$ stands for $\mathfrak{sl}_2$. Namely, $U_q\big(\mathfrak{sl}_2{}^*\big)$ is a topological Hopf $\mathbb{Q}(q)$–algebra, with two integral forms $\mathfrak{U}_q\big(\mathfrak{sl}_2{}^*\big)$ and $\mathcal{U}_q\big(\mathfrak{sl}_2{}^*\big)$ which play for $U_q\big(\mathfrak{sl}_2{}^*\big)$ the same rôle as $\mathfrak{U}_q(\mathfrak{sl}_2)$ and $\mathcal{U}_q(\mathfrak{sl}_2)$ for $U_q(\mathfrak{sl}_2)$.

The construction goes as follows. Let $\mathbf{H}_q^g$ be the unital associative $\mathbb{Q}(q)$–algebra with generators F, $\Lambda_1^{\pm 1}$, $\Lambda_2^{\pm 1}$, E and relations

$$EF = FE \,, \quad \Lambda_i^{-1}\Lambda_i = 1 = \Lambda_i\Lambda_i^{-1} \,, \quad \Lambda_i^{\pm 1}\Lambda_j^{\pm 1} = \Lambda_j^{\pm 1}\Lambda_i^{\pm 1} \,, \quad \Lambda_i^{\mp 1}\Lambda_j^{\pm 1} = \Lambda_j^{\pm 1}\Lambda_i^{\mp 1} \quad \forall\, i,j$$
$$\Lambda_1^{\pm 1}E = q^{\pm 1}E\Lambda_1^{\pm 1} \,, \quad \Lambda_1^{\pm 1}F = q^{\pm 1}F\Lambda_1^{\pm 1} \,, \quad \Lambda_2^{\pm 1}E = q^{\mp 1}E\Lambda_2^{\pm 1} \,, \quad \Lambda_2^{\pm 1}F = q^{\mp 1}F\Lambda_2^{\pm 1} \;.$$

Let also $\mathbf{H}_q^s$ be the obtained from $\mathbf{H}_q^g$ adding the relation $\Lambda_1\Lambda_2 = 1$.

The set of PBW-like ordered monomials $B_*^g := \big\{ E^\eta \Lambda_1^{\lambda_1} \Lambda_2^{\lambda_2} F^\varphi \,\big|\, \eta, \lambda_1, \lambda_2, \varphi \in \mathbb{N} \big\}$ is a $\mathbb{Q}(q)$–basis for $\mathbf{H}_q^g$; similarly $B_*^s := \big\{ E^\eta \Lambda_1^{\lambda_1} F^\varphi \,\big|\, \eta, \varphi \in \mathbb{N}, \lambda_1 \in \mathbb{Z} \big\}$ is a $\mathbb{Q}(q)$–basis for $\mathbf{H}_q^s$.

One defines $U_q\big(\mathfrak{sl}_2{}^*\big)$ as a suitable completion of $\mathbf{H}_q^s$, so that $U_q\big(\mathfrak{sl}_2{}^*\big)$ is a topological $\mathbb{Q}(q)$–algebra topologically generated by $\mathbf{H}_q^s$, and B_*^s is a $\mathbb{Q}(q)$–basis of $U_q\big(\mathfrak{sl}_2{}^*\big)$ in topological sense. Then $U_q\big(\mathfrak{sl}_2{}^*\big)$ is also a topological Hopf $\mathbb{Q}(q)$–algebra (see [Ga1]). The same construction makes sense with $\mathbf{H}_q^g$ instead of $\mathbf{H}_q^s$ and yields the definition of $U_q\big(\mathfrak{gl}_2{}^*\big)$, a topological Hopf algebra with B_*^g as (topological) $\mathbb{Q}(q)$–basis. Then by construction $U_q\big(\mathfrak{sl}_2^{\,*}\big)$ is a quotient of $U_q\big(\mathfrak{gl}_2^{\,*}\big)$, as a topological Hopf algebra, via $U_q\big(\mathfrak{sl}_2^{\,*}\big)_q \cong U_q\big(\mathfrak{gl}_2^{\,*}\big)\Big/(\Lambda_1\Lambda_2-1)$.

The restricted integral form $\mathfrak{U}_q\big(\mathfrak{sl}_2^{\,*}\big)$ of $U_q\big(\mathfrak{sl}_2^{\,*}\big)$ is, by definition, a dense Hopf $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of the subset of linear functionals in $U_q\big(\mathfrak{sl}_2^{\,*}\big)$ which are $\mathbb{Z}\big[q,q^{-1}\big]$–valued onto $\mathcal{U}_q\big(\mathfrak{sl}_2^{\,*}\big)$. In order to describe it, set $L^{\pm 1} := \Lambda_1^{\pm 1}$, and let $\mathfrak{H}_q^s$ be the $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $\mathbf{H}_q^s$ generated by all the $F^{(m)}$'s, $E^{(m)}$'s, $L^{\pm 1}$ and $\binom{L\,;\,c}{n}$'s (for $m\in\mathbb{N}$, $c\in\mathbb{Z}$): then $\mathfrak{B}_*^s := \Big\{ E^{(\eta)} \binom{L\,;\,0}{l} L^{-Ent(l/2)} F^{(\varphi)} \;\Big|\; \eta, l, \varphi \in \mathbb{N} \Big\}$ is a $\mathbb{Z}\big[q,q^{-1}\big]$–basis of $\mathfrak{H}_q^s$, while $\mathfrak{U}_q\big(\mathfrak{sl}_2^{\,*}\big)$ is the topological closure of $\mathfrak{H}_q^s$, and $\mathfrak{B}_*^s$ is a topological $\mathbb{Z}\big[q,q^{-1}\big]$–basis of $\mathfrak{U}_q\big(\mathfrak{sl}_2^{\,*}\big)$.

Similarly, for the integral form $\mathfrak{U}_q\big(\mathfrak{gl}_2^{\,*}\big)$ of $U_q\big(\mathfrak{gl}_2^{\,*}\big)$, take the Λ_h's ($h=1,2$) instead of L and $\mathfrak{B}_*^g := \Big\{ E^{(\eta)} \binom{\Lambda_1;\,0}{\lambda_1} \Lambda_1^{-Ent(\lambda_1/2)} \binom{\Lambda_2;\,0}{\lambda_2} \Lambda_2^{-Ent(\lambda_2/2)} F^{(\varphi)} \;\Big|\; \eta, \lambda_1, \lambda_2, \varphi \in \mathbb{N} \Big\}$ instead of $\mathfrak{B}_*^s$. By construction $\mathfrak{H}_q^s$ is a quotient algebra of $\mathfrak{H}_q^g$ — restricting $\mathbf{H}_q^g\Big/(\Lambda_1\Lambda_2-1) \cong \mathbf{H}_q^s$ to $\mathfrak{H}_q^g$ — so $\mathfrak{U}_q\big(\mathfrak{sl}_2^{\,*}\big)$ is a quotient of $\mathfrak{U}_q\big(\mathfrak{gl}_2^{\,*}\big)$, as a topological Hopf algebra.

We can describe $\mathfrak{H}_q^s$ rather explicitly: it is the unital associative $\mathbb{Z}\big[q,q^{-1}\big]$–algebra with generators $F^{(m)}$, $E^{(m)}$, $L^{\pm 1}$, $\binom{L\,;\,c}{m}$ — for $m\in\mathbb{N}$, $c\in\mathbb{Z}$ — and relations

$$\textit{relations } (1.1) \textit{ for } X = L\,, \qquad L\,L^{-1} = 1 = L^{-1}L\,, \qquad \textit{relations } (1.2) \textit{ for } X \in \big\{F\,,E\big\}$$

$$L^{\pm 1} F^{(m)} = q^{\pm m} F^{(m)} L^{\pm 1}\,, \qquad E^{(r)} F^{(s)} = F^{(s)} E^{(r)}\,, \qquad L^{\pm 1} E^{(m)} = q^{\pm m} E^{(m)} L^{\pm 1}$$

$$\binom{L\,;\,c}{t} E^{(m)} = E^{(m)} \binom{L\,;\,c{+}m}{t}\,, \qquad \binom{L\,;\,c}{t} F^{(m)} = F^{(m)} \binom{L\,;\,c{+}m}{t}\,.$$

Similarly, $\mathfrak{H}_q^g$ is the unital associative $\mathbb{Z}\big[q,q^{-1}\big]$–algebra with generators $F^{(m)}$, $E^{(m)}$, $\Lambda_k^{\pm 1}$, $\binom{\Lambda_k\,;\,c}{m}$ (for $m\in\mathbb{N}$, $c\in\mathbb{Z}$, $k\in\{1,2\}$) and relations

$$\Lambda_k\Lambda_k^{-1} = 1 = \Lambda_k^{-1}\Lambda_k\,, \qquad \binom{\Lambda_h\,;\,c}{m}\binom{\Lambda_k\,;\,s}{n} = \binom{\Lambda_k\,;\,s}{n}\binom{\Lambda_h\,;\,c}{m}$$

$$\textit{relations } (1.1) \textit{ for all } X \in \big\{\Lambda_1, \Lambda_2\big\}\,, \qquad \textit{relations } (1.2) \textit{ for all } X \in \big\{F\,,E\big\}$$

$$E^{(r)} F^{(s)} = F^{(s)} E^{(r)}\,, \qquad \Lambda_k^{\pm 1} Y^{(m)} = q^{\pm(\delta_{k,1}-\delta_{k,2})m} Y^{(m)} \Lambda_k^{\pm 1} \qquad \forall\; Y \in \big\{F\,,E\big\}$$

$$\binom{\Lambda_k\,;\,c}{t} Y^{(m)} = Y^{(m)} \binom{\Lambda_k\,;\,c+(\delta_{k,1}-\delta_{k,2})\,m}{t} \qquad \forall\; Y \in \big\{F\,,E\big\}$$

In this paper we do not need the Hopf structure of $\mathfrak{U}_q\big(\mathfrak{sl}_2^{\,*}\big)$ and $\mathfrak{U}_q\big(\mathfrak{sl}_2^{\,*}\big)$ (cf. [Ga1]).

$\mathfrak{U}_q\big(\mathfrak{sl}_2^{\,*}\big)$ is a *quantization* of $U_{\mathbb{Z}}\big(\mathfrak{sl}_2^*\big)$, for $\mathfrak{U}_1\big(\mathfrak{sl}_2^{\,*}\big) := \big(\mathfrak{U}_q\big(\mathfrak{sl}_2^{\,*}\big)\big)_1 \cong U_{\mathbb{Z}}\big(\mathfrak{sl}_2^*\big)$ as co-Poisson Hopf algebras, with on left-hand side the co-Poisson structure inherited from $\mathfrak{U}_q\big(\mathfrak{sl}_2^{\,*}\big)$. In terms of generators (notation of § 1.2) this reads $F^{(m)}\big|_{q=1} \cong \mathrm{f}^{(m)}$, $\binom{L\,;\,0}{m}\Big|_{q=1} \cong \binom{\mathrm{h}}{m}$, $L^{\pm 1}\big|_{q=1} \cong 1$, $E^{(m)}\big|_{q=1} \cong \mathrm{e}^{(m)}$ for $m\in\mathbb{N}$. Similarly $\mathfrak{U}_1\big(\mathfrak{gl}_2^{\,*}\big) \cong U_{\mathbb{Z}}\big(\mathfrak{gl}_2^*\big)$, with $F^{(m)}\big|_{q=1} \cong \mathrm{f}^{(m)}$, $\binom{\Lambda_k\,;\,0}{m}\Big|_{q=1} \cong \binom{\mathrm{g}_k}{m}$, $\Lambda_k^{\pm 1}\big|_{q=1} \cong 1$, $E^{(m)}\big|_{q=1} \cong \mathrm{e}^{(m)}$ for $m\in\mathbb{N}$ and $k\in\{1,2\}$.

Finally, let ℓ and ε be as in § 1.3. Set $\mathfrak{U}_\varepsilon\big(\mathfrak{sl}_2^{\,*}\big) := \big(\mathfrak{U}_q\big(\mathfrak{sl}_2^{\,*}\big)\big)_\varepsilon$ and $\mathfrak{H}_\varepsilon^s := \big(\mathfrak{H}^s\big)_\varepsilon$. Then (cf. [Ga1], § 7.7) the embedding $\mathfrak{H}_\varepsilon^s \longhookrightarrow \mathfrak{U}_\varepsilon\big(\mathfrak{sl}_2^{\,*}\big)$ is an isomorphism, thus $\mathfrak{U}_\varepsilon\big(\mathfrak{sl}_2^{\,*}\big) = \mathfrak{H}_\varepsilon^s$. Similarly (with like notation) $\mathfrak{U}_\varepsilon\big(\mathfrak{gl}_2^{\,*}\big) = \mathfrak{H}_\varepsilon^g$. Also, there are Hopf algebra epimorphisms

$$\mathfrak{Fr}_{\mathfrak{sl}_2^*}^{\mathbb{Z}} : \mathfrak{U}_\varepsilon(\mathfrak{sl}_2) = \mathfrak{H}_\varepsilon^s \longrightarrow\!\!\!\!\rightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathfrak{H}_1^s = \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathfrak{U}_1\big(\mathfrak{sl}_2^{\,*}\big) \cong \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}\big(\mathfrak{sl}_2^*\big) \tag{2.4}$$

$$\mathfrak{Fr}_{\mathfrak{gl}_2^*}^{\mathbb{Z}} : \mathfrak{U}_\varepsilon(\mathfrak{gl}_2) = \mathfrak{H}_\varepsilon^g \longrightarrow\!\!\!\!\rightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathfrak{H}_1^g = \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathfrak{U}_1\big(\mathfrak{gl}_2^{\,*}\big) \cong \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}\big(\mathfrak{gl}_2^*\big) \tag{2.5}$$

defined by $X^{(s)}\big|_{q=\varepsilon} \mapsto \mathrm{x}^{(s/\ell)}$, $\binom{Y\,;\,0}{s}\Big|_{q=\varepsilon} \mapsto \binom{\mathrm{k}}{s/\ell}$ if $\ell \big| s$, $X^{(s)}\big|_{q=\varepsilon} \mapsto 0$, $\binom{Y\,;\,0}{s}\Big|_{q=\varepsilon} \mapsto 0$ if $\ell \not| s$, and $Y^{\pm 1}\Big|_{q=1} \mapsto 1$, with $(X,\mathrm{x}) \in \big\{(F,\mathrm{f}),(E,\mathrm{e})\big\}$, and $(Y,\mathrm{k}) = (L,\mathrm{h})$ in the $\mathfrak{sl}_n$ case, $(Y,\mathrm{k}) \in \big\{(\Lambda_i,\mathrm{g}_i)\big\}_{i=1,2}$ for $\mathfrak{gl}_n$. These are *quantum Frobenius morphism* for $\mathfrak{sl}_n^{\,*}$ and $\mathfrak{gl}_n^{\,*}$.

The above epimorphism $\pi_q : \mathfrak{U}_q\big(\mathfrak{gl}_2^{\,*}\big) \twoheadrightarrow \mathfrak{U}_q\big(\mathfrak{sl}_2^{\,*}\big)$ of topological Hopf $\mathbb{Z}\big[q,q^{-1}\big]$–algebras is compatible with the quantum Frobenius morphisms, in the obvious sense.

To finish with, the natural evaluation pairing $\langle\ ,\ \rangle : U_q(\mathfrak{g}^*) \times U_q(\mathfrak{g}) \longrightarrow \mathbb{Q}(q)$ (for $\mathfrak{g} \in \{\mathfrak{sl}_2,\mathfrak{gl}_2\}$) is uniquely determined by its values on PBW bases: we have them via

$$\left\langle E^{(\eta)} \binom{L\,;\,0}{l} L^{-Ent(l/2)} F^{(\varphi)} \,,\, \overline{F}^{\,f} K^{\kappa}\, \overline{E}^{\,e} \right\rangle \;=\; (-1)^{\eta}\, \delta_{e,\varphi}\, \delta_{f,\eta} \binom{\kappa}{l}_{\!q} q^{-\kappa Ent(l/2)} \tag{2.6}$$

$$\begin{aligned} \left\langle E^{(\eta)} \binom{\Lambda_1\,;\,0}{\lambda_1} \Lambda_1^{-Ent(\lambda_1/2)} \binom{\Lambda_2\,;\,0}{\lambda_2} \Lambda_2^{-Ent(\lambda_2/2)} F^{(\varphi)} \,,\, \overline{F}^{\,f} G_1^{\gamma_1} G_2^{\gamma_2}\, \overline{E}^{\,e} \right\rangle \;=\; & \\ =\; (-1)^{\eta}\, \delta_{e,\varphi}\, \delta_{f,\eta} \binom{\gamma_1}{\lambda_1}_{\!q} \binom{\gamma_2}{\lambda_2}_{\!q} q^{-\gamma_1 Ent(\lambda_1/2) - \gamma_2 Ent(\lambda_2/2)} & \end{aligned} \tag{2.7}$$

2.4 Embedding quantum function algebras into quantum enveloping algebras. Let $G \in \big\{SL_2, GL_2\big\}$ and $\mathfrak{g} := \mathit{Lie}(G)$. By definition $F_q[G]$ embeds into $U_q(\mathfrak{g}^*) := U_q(\mathfrak{g})^*$, via a monomorphism $\xi : F_q[G] \hookrightarrow U_q(\mathfrak{g}^*)$ of topological Hopf $\mathbb{Q}(q)$–algebras. Moreover $\mathcal{F}_q[G] = \xi^{-1}\big(\mathfrak{U}_q(\mathfrak{g}^*)\big)$, so ξ restricts to a monomorphism $\widehat{\xi} : \mathcal{F}_q[G] \hookrightarrow \mathfrak{U}_q(\mathfrak{g}^*)$ too, and similarly $\widetilde{\xi} : \mathfrak{F}_q[G] \hookrightarrow \mathcal{U}_q(\mathfrak{g}^*)$. These verify $\xi\big(F_q[G]\big) \subseteq \mathbf{H}_q^x$ and $\widehat{\xi}\big(\mathcal{F}_q[G]\big) \subseteq \mathfrak{H}_q^x$ (with $x \in \{s,g\}$, according to the type of G) so $F_q[G] = \xi^{-1}(\mathbf{H}_x)$ and $\mathcal{F}_q[G] = \widehat{\xi}^{-1}(\mathfrak{H}_x)$. Furthermore, $\widehat{\xi}$ is compatible with specializations and quantum Frobenius morphisms, that is $\Big(\mathrm{id}_{\mathbb{Q}_\varepsilon} \otimes_{\mathbb{Z}} \widehat{\xi}\,\big|_{q=1}\Big) \circ \mathcal{F}r_G^{\mathbb{Q}} = \Big(\mathrm{id}_{\mathbb{Q}_\varepsilon} \otimes_{\mathbb{Z}_\varepsilon} \mathfrak{Fr}_{\mathfrak{g}^*}^{\mathbb{Z}}\Big) \circ \Big(\mathrm{id}_{\mathbb{Q}_\varepsilon} \otimes_{\mathbb{Z}_\varepsilon} \widehat{\xi}\,\big|_{q=\varepsilon}\Big)$. As $F_q[M_2]$ embeds into $F_q[GL_2]$, restricting $\xi : F_q[GL_2] \hookrightarrow U_q\big(\mathfrak{gl}_2^{\,*}\big)$ yields an embedding $\xi : F_q[M_2] \hookrightarrow U_q\big(\mathfrak{gl}_2^{\,*}\big)$, and similarly we have an embedding $\widehat{\xi} : \mathcal{F}_q[M_2] \hookrightarrow \mathfrak{U}_q\big(\mathfrak{gl}_2^{\,*}\big)$, which factors through $\mathcal{F}_q[GL_2]$ (and similarly $\widetilde{\xi} : \mathfrak{F}_q[M_2] \hookrightarrow \mathcal{U}_q\big(\mathfrak{gl}_2^{\,*}\big)$, which factors through $\mathfrak{F}_q[GL_2]$).

In [Ga1], Appendix, embeddings ξ and $\widetilde{\xi}$ as above are described for SL_2, namely given by $\xi : a \mapsto L - \overline{F} L^{-1} \overline{E}$, $b \mapsto -\overline{F} L^{-1}$, $c \mapsto +L^{-1}\overline{E}$, $d \mapsto L^{-1}$. Similarly (and exploiting the analysis in [Ga2], §§ 5.2/4), we find an analogous ξ for GL_2, namely $\xi : F_q[GL_2] \hookrightarrow U_q\big(\mathfrak{gl}_2^{\,*}\big)$, $a \mapsto \Lambda_1 - \overline{F}\Lambda_2\overline{E}$, $b \mapsto -\overline{F}\Lambda_2$, $c \mapsto +\Lambda_2\overline{E}$, $d \mapsto \Lambda_2$, $D_q^{-1} \mapsto \big(\Lambda_1\Lambda_2\big)^{-1}$. The same formulæ describe $\widetilde{\xi} : \mathfrak{F}_q[GL_2] \hookrightarrow \mathcal{U}_q\big(\mathfrak{gl}_2^{\,*}\big)$. Discarding $D_q^{-1} \mapsto \big(\Lambda_1\Lambda_2\big)^{-1}$ they describe also the embedding $\xi : F_q[M_2] \hookrightarrow U_q\big(\mathfrak{gl}_2^{\,*}\big)$, $a \mapsto \Lambda_1 - \overline{F}\Lambda_2\overline{E}$, $b \mapsto -\overline{F}\Lambda_2$, $c \mapsto +\Lambda_2\overline{E}$, $d \mapsto \Lambda_2$, obtained restricting $\xi : F_q[GL_2] \hookrightarrow U_q\big(\mathfrak{gl}_2^{\,*}\big)$, and its restriction $\widetilde{\xi} : \mathfrak{F}_q[M_2] \hookrightarrow \mathcal{U}_q\big(\mathfrak{gl}_2^{\,*}\big)$.

Finally, the various embeddings ξ and their restrictions to integral forms also are compatible — in the obvious sense — with the epimorphisms $U_q\big(\mathfrak{gl}_2^{\,*}\big) \twoheadrightarrow U_q\big(\mathfrak{sl}_2^{\,*}\big)$ and $F_q[GL_2] \twoheadrightarrow F_q[SL_2]$ or $F_q[M_2] \twoheadrightarrow F_q[SL_2]$ and their restrictions to integral forms.

§ 3 The structure of $\mathcal{F}_q[M_2]$, $\mathcal{F}_q[GL_2]$ and $\mathcal{F}_q[SL_2]$, their specializations and quantum Frobenius epimorphisms.

We need some more notation. First set $\mathbf{b} := \big(q - q^{-1}\big)^{-1} b$ and $\mathbf{c} := \big(q - q^{-1}\big)^{-1} c$. Then for all $n \in \mathbb{N}$, we set $\mathbf{b}^{(n)} := \mathbf{b}^n \big/ [n]_q!$, $\mathbf{c}^{(n)} := \mathbf{c}^n \big/ [n]_q!$ — like in § 1.2.

Theorem 3.1.

(a) $\mathcal{F}_q[M_2]$ *is a free* $\mathbb{Z}[q,q^{-1}]$*–module, with basis the set of ordered monomials*

$$\mathcal{B}_{M_2} = \left\{ \mathbf{b}^{(\beta)} \binom{a\,;0}{\alpha} \binom{d\,;0}{\delta} \mathbf{c}^{(\kappa)} \;\middle|\; \alpha,\beta,\kappa,\delta \in \mathbb{N} \right\}$$

(a PBW-like basis). Similarly, any other set obtained from $\mathcal{B}_{M_2}$ *via permutations of factors (of the monomials in* $\mathcal{B}_{M_2}$*) is a* $\mathbb{Z}[q,q^{-1}]$*–basis of* $\mathcal{F}_q[M_2]$ *as well.*

(b) $\mathcal{F}_q[GL_2]$ *is the* $\mathbb{Z}[q,q^{-1}]$*–span of the set of ordered monomials*

$$\mathcal{S}_{GL_2} = \left\{ \mathbf{b}^{(\beta)} \binom{a\,;0}{\alpha} \binom{d\,;0}{\delta} \mathbf{c}^{(\kappa)} D_q^{-\nu} \;\middle|\; \alpha,\beta,\kappa,\delta,\nu \in \mathbb{N} \right\} \quad .$$

Similarly, any other set obtained from $\mathcal{S}_{GL_2}$ *via permutations of factors (of the monomials in* $\mathcal{S}_{GL_2}$*) is a* $\mathbb{Z}[q,q^{-1}]$*–spanning set for* $\mathcal{F}_q[GL_2]$ *as well. Moreover, if* $f \in \mathcal{F}_q[GL_2]$ *then* f *can be expanded into a* $\mathbb{Z}[q,q^{-1}]$*–linear combination of elements of* $\mathcal{S}_{GL_2}$ *which all bear the same exponent* ν*; similarly for the other spanning sets mentioned above.*

(c) $\mathcal{F}_q[SL_2]$ *is the* $\mathbb{Z}[q,q^{-1}]$*–span of the set of ordered monomials*

$$\mathcal{S}_{SL_2} = \left\{ \mathbf{b}^{(\beta)} \binom{a\,;0}{\alpha} \binom{d\,;0}{\delta} \mathbf{c}^{(\kappa)} \;\middle|\; \alpha,\beta,\kappa,\delta \in \mathbb{N} \right\} \quad .$$

Similarly, any other set obtained from $\mathcal{S}_{SL_2}$ *via permutations of factors (of the monomials in* $\mathcal{S}_{SL_2}$*) is a* $\mathbb{Z}[q,q^{-1}]$*–spanning set for* $\mathcal{F}_q[SL_2]$ *as well.*

Proof. (a) For all $\alpha,\beta,\kappa,\delta \in \mathbb{N}$, let $\mathcal{M}_{\alpha,\beta,\kappa,\delta} := \mathbf{b}^{(\beta)} \binom{a\,;0}{\alpha} \binom{d\,;0}{\delta} \mathbf{c}^{(\kappa)}$. Due to the formulæ for $\xi \colon F_q[M_2] \hookrightarrow U_q(\mathfrak{gl}_2^{\,*})$ in § 2.4 and to Lemma 4.2*(b)* later on, one has

$$\xi\big(\mathcal{M}_{\alpha,\beta,\kappa,\delta}\big) = \big(-\overline{F}\Lambda_2\big)^{(\beta)} \cdot \binom{\Lambda_1 - \overline{F}\Lambda_2\overline{E}\,;0}{\alpha} \cdot \binom{\Lambda_2\,;0}{\delta} \cdot \big(+\Lambda_2\overline{E}\big)^{(\kappa)} = \textstyle\sum_{r=0}^{\alpha} (-1)^{\beta+r} \times$$
$$\times\, q^{\binom{\kappa}{2}-\binom{\beta}{2}-\binom{r}{2}-r(\alpha+1)} \big(q-q^{-1}\big)^r \, [r]_q! \begin{bmatrix}\beta+r\\ \beta\end{bmatrix}_q \begin{bmatrix}\kappa+r\\ \kappa\end{bmatrix}_q F^{(\beta+r)} \left\{ \begin{matrix}\Lambda_1\,;0\\ \alpha\,,\,r\end{matrix} \right\} \binom{\Lambda_2\,;0}{\delta} \Lambda_2^{\;\beta+r+\kappa} E^{(\kappa+r)}$$

so that $\xi\big(\mathcal{M}_{\alpha,\beta,\kappa,\delta}\big) \in \mathfrak{H}_q^g$, which implies, thanks to $\mathcal{F}_q[G] = \widehat{\xi}^{-1}(\mathfrak{H}_x)$ (see § 2.4),

$$\mathcal{M}_{\alpha,\beta,\kappa,\delta} \in \widehat{\xi}^{-1}\big(\mathfrak{H}_q^g\big) \bigcap F_q[M_2] = \mathcal{F}_q[M_2] \, . \tag{3.1}$$

This proves that $\mathcal{B}_{M_2} \subseteq \mathcal{F}_q[M_2]$, hence the $\mathbb{Z}[q,q^{-1}]$–span of $\mathcal{B}_{M_2}$ is contained in $\mathcal{F}_q[M_2]$.

Now pick $f \in \mathcal{F}_q[M_2]$. Clearly $\mathcal{B}_{M_2}$ is a $\mathbb{Q}(q)$–basis of $F_q[M_2]$, hence there is a unique expansion $f = \sum_{\alpha,\beta,\kappa,\delta\in\mathbb{N}} \chi_{\alpha,\beta,\kappa,\delta}\, \mathcal{M}_{\alpha,\beta,\kappa,\delta}$ with all coefficients $\chi_{\alpha,\beta,\kappa,\delta} \in \mathbb{Q}(q)$; we must show that these belong to $\mathbb{Z}[q,q^{-1}]$. Let β_0 and κ_0 in $\mathbb{N}$ be the least indices such that $\chi_{\alpha,\beta_0,\kappa_0,\delta} \neq 0$ for some $\alpha,\delta \in \mathbb{N}$. The previous description of $\xi\big(\mathcal{M}_{\alpha,\beta,\gamma,\delta}\big)$ and (2.7) yield

$$\Big\langle \mathcal{M}_{\alpha,\beta,\kappa,\delta}\, ,\, \overline{E}^{\,\eta} G_1^{\gamma_1} G_2^{\gamma_2}\, \overline{F}^{\,\varphi} \Big\rangle = 0 \qquad \text{if} \quad \eta < \beta \quad \text{or} \quad \varphi < \kappa$$
$$\Big\langle \mathcal{M}_{\alpha,\beta,\kappa,\delta}\, ,\, \overline{E}^{\,\beta} G_1^{\gamma_1} G_2^{\gamma_2}\, \overline{F}^{\,\kappa} \Big\rangle = (-1)^\beta\, q^{\binom{\kappa}{2}-\binom{\beta}{2}+(\beta+\kappa)\gamma_2} \binom{\gamma_1}{\alpha}_q \binom{\gamma_2}{\delta}_q$$

This gives $\Big\langle f\, ,\, \overline{E}^{\,\beta_0} G_1^{\gamma_1} G_2^{\gamma_2}\, \overline{F}^{\,\kappa_0} \Big\rangle = q^{\binom{\kappa_0}{2}-\binom{\beta_0}{2}+(\beta_0+\kappa_0)\gamma_2} \sum_{\alpha,\delta} \chi_{\alpha,\beta_0,\kappa_0,\delta} \binom{\gamma_1}{\alpha}_q \binom{\gamma_2}{\delta}_q$. By assumption the last term belongs to $\mathbb{Z}[q,q^{-1}]$, whence also

$$y_{\gamma_1,\gamma_2} := \sum_{\alpha,\delta} \chi_{\alpha,\beta_0,\kappa_0,\delta} \binom{\gamma_1}{\alpha}_q \binom{\gamma_2}{\delta}_q \in \mathbb{Z}[q,q^{-1}] \qquad \forall\; \gamma_1,\gamma_2 \in \mathbb{Z},\; \alpha,\delta \in \mathbb{N} \, . \tag{3.2}$$

Using as indices the pairs $(\alpha,\delta) \in \mathbb{N}^2$ and $(\gamma_1,\gamma_2) \in \mathbb{N}^2$, the set of identities (3.2) can be read as a change of variables from $\big\{\chi_{\alpha,\beta_0,\kappa_0,\delta}\big\}_{(\alpha,\delta)\in\mathbb{N}^2}$ to $\big\{y_{\gamma_1,\gamma_2}\big\}_{(\gamma_1,\gamma_2)\in\mathbb{N}^2}$; fixing in $\mathbb{N}^2$ any total order $\preceq$ such that $(m,n) \preceq (m',n')$ if $m \leq m$ or $m' \leq n'$, the (infinite size) matrix ruling this change of variables, i.e. $\Big(\binom{\gamma_1}{\alpha}_q \binom{\gamma_2}{\delta}_q \Big)_{(\gamma_1,\gamma_2),(\alpha,\delta)\in\mathbb{N}^2}$, has entries in $\mathbb{Z}\big[q,q^{-1}\big]$ and is lower triangular unipotent. Thus it is invertible and its inverse also is lower triangular unipotent with entries in $\mathbb{Z}\big[q,q^{-1}\big]$, so $\chi_{\alpha,\beta_0,\kappa_0,\delta} \in \mathbb{Z}\big[q,q^{-1}\big]$ for all $\alpha,\delta \in \mathbb{N}$.

The previous analysis gives $f' := f - \sum_{\alpha,\delta} \chi_{\alpha,\beta_0,\kappa_0,\delta}\, \mathcal{M}_{\alpha,\beta_0,\kappa_0,\delta} \in \mathcal{F}_q[M_2]$; moreover, by construction the expansion of f' as a $\mathbb{Q}(q)$–linear combination of elements of $\mathcal{B}_{M_2}$ has less non-trivial summands than f: then we can apply the same argument, and iterate till we find that all coefficients $\chi_{\alpha,\beta,\kappa,\delta}$ in the original expansion of f do belong to $\mathbb{Z}\big[q,q^{-1}\big]$.

Finally, the last observation about other bases is clear.

(b) By claim *(a)*, every monomial of type $\mathbf{b}^{(\beta)} \binom{a\,;0}{\alpha} \binom{d\,;0}{\delta} \mathbf{c}^{(\kappa)}$ is $\mathbb{Z}\big[q,q^{-1}\big]$–integer-valued on $\mathcal{U}_q(\mathfrak{gl}_2)$; on the other hand, the same is true for $D_q^{\,-\nu}$ ($\forall\ \nu \in \mathbb{N}$), because $D_q^{\,-\nu} \in \mathfrak{F}_q[GL_2]$ and $\mathfrak{F}_q[GL_2] \subseteq \mathcal{F}_q[GL_2]$ since $\mathfrak{U}_q(\mathfrak{gl}_2) \supseteq \mathcal{U}_q(\mathfrak{gl}_2)$. Finally,

$$\Big\langle \mathbf{b}^{(\beta)} \binom{a\,;0}{\alpha} \binom{d\,;0}{\delta} \mathbf{c}^{(\kappa)}\, D_q^{\,-\nu}, \mathcal{U}_q(\mathfrak{gl}_2) \Big\rangle = \Big\langle \mathbf{b}^{(\beta)} \binom{a\,;0}{\alpha} \binom{d\,;0}{\delta} \mathbf{c}^{(\kappa)} \otimes D_q^{\,-\nu}, \Delta\big(\mathcal{U}_q(\mathfrak{gl}_2)\big) \Big\rangle \subseteq$$
$$\subseteq \Big\langle \mathbf{b}^{(\beta)} \binom{a\,;0}{\alpha} \binom{d\,;0}{\delta} \mathbf{c}^{(\kappa)}, \mathcal{U}_q(\mathfrak{gl}_2) \Big\rangle \cdot \big\langle D_q^{\,-\nu}, \mathcal{U}_q(\mathfrak{gl}_2) \big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big]$$

so $\mathbf{b}^{(\beta)} \binom{a\,;0}{\alpha} \binom{d\,;0}{\delta} \mathbf{c}^{(\kappa)} D_q^{\,-\nu} \in \mathcal{F}_q[GL_2]$, and the $\mathbb{Z}\big[q,q^{-1}\big]$–span of $\mathcal{S}_{GL_2}$ sits inside $\mathcal{F}_q[GL_2]$.

Conversely, let $f \in \mathcal{F}_q[GL_2]$. Then there exists $N \in \mathbb{N}$ such that $f\, D_q^{\,N} \in F_q[M_2]$. In addition, $\big\langle f\, D_q^{\,N}, \mathcal{U}_q(\mathfrak{gl}_2) \big\rangle = \big\langle f \otimes D_q^{\,N}, \Delta\big(\mathcal{U}_q(\mathfrak{gl}_2)\big) \big\rangle \subseteq \langle f, \mathcal{U}_q(\mathfrak{gl}_2) \rangle \cdot \big\langle D_q^{\,N}, \mathcal{U}_q(\mathfrak{gl}_2) \big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big]$ because $f, D_q^{\,N} \in \mathcal{F}_q[GL_2]$. Thus $f\, D_q^{\,N} \in \mathcal{F}_q[M_2]$; then, by claim *(a)*, $f\, D_q^{\,N}$ belongs to the $\mathbb{Z}\big[q,q^{-1}\big]$–span of $\mathcal{B}_{M_2}$, whence the claim follows at once.

Finally, the last observation about other spanning sets is self-evident.

(c) The projection epimorphism $F_q[M_2] \overset{\pi}{\twoheadrightarrow} F_q[M_2] \big/ \big(D_q\big) \cong F_q[SL_2]$ (given by restriction from $U_q(\mathfrak{gl}_2)$ to $U_q(\mathfrak{sl}_2)$) maps $\mathcal{B}_{M_2}$ onto $\mathcal{S}_{SL_2}$: it follows directly from definitions that $\pi\big(\mathcal{F}_q[M_2]\big) \subseteq \mathcal{F}_q[SL_2]$, hence in particular (thanks to claim *(a)*) the $\mathbb{Z}\big[q,q^{-1}\big]$–span of $\mathcal{S}_{SL_2}$ is contained in $\mathcal{F}_q[SL_2]$. Conversely, let $f \in \mathcal{F}_q[SL_2]$. Since B_{SL_2} in § 2.2 is a $\mathbb{Q}(q)$–basis of $F_q[SL_2]$ it follows that any $f \in \mathcal{F}_q[SL_2]$ has a unique expansion

$$\begin{aligned} f \;&= \textstyle\sum_{\alpha,\beta,\kappa\in\mathbb{N}} \chi^{\alpha}_{\beta,\kappa} \cdot \mathbf{b}^{(\beta)} a^{\alpha} \mathbf{c}^{(\kappa)} + \sum_{\beta,\kappa,\delta\in\mathbb{N}} \eta^{\delta}_{\beta,\kappa} \cdot \mathbf{b}^{(\beta)} d^{\,\delta} \mathbf{c}^{(\kappa)} = \\ &= \textstyle\sum_{\beta,\kappa\in\mathbb{N}} \mathbf{b}^{(\beta)} \big(\sum_{\alpha\in\mathbb{N}} \chi^{\alpha}_{\beta,\kappa}\, a^{\alpha} + \varphi + \sum_{\delta\in\mathbb{N}} \eta^{\delta}_{\beta,\kappa}\, d^{\,\delta} \big) \mathbf{c}^{(\kappa)} \end{aligned} \tag{3.3}$$

for some $\chi^{\alpha}_{\beta,\kappa}, \eta^{\delta}_{\beta,\kappa}, \varphi \in \mathbb{Q}(q)$. Since $D_q := a\,d - q\,b\,c = 1$ in $\mathcal{F}_q[SL_2]$, we can rewrite (3.3) as $f = \sum\limits_{\beta,\kappa\in\mathbb{N}} \mathbf{b}^{(\beta)} \bigg(\sum\limits_{\alpha\in\mathbb{N}} \chi^{\alpha}_{\beta,\kappa}\, a^{\alpha} \cdot D_q^{\,\mu} + \varphi \cdot D_q^{\,\mu} + \sum\limits_{\delta\in\mathbb{N}} \eta^{\delta}_{\beta,\kappa}\, d^{\,\delta} \cdot D_q^{\,\mu-\delta} \bigg) \mathbf{c}^{(\kappa)}$, where $\mu := \max\big\{ \delta \in \mathbb{N} \,\big|\, \eta^{\delta}_{\beta,\kappa} \neq 0 \big\}$. Now consider the element of $F_q[M_2]$

$$f' = \textstyle\sum_{\beta,\kappa\in\mathbb{N}} \mathbf{b}^{(\beta)} \Big(\sum_{\alpha\in\mathbb{N}} \chi^{\alpha}_{\beta,\kappa}\, a^{\alpha} \cdot D_q^{\,\mu} + \varphi \cdot D_q^{\,\mu} + \sum_{\delta\in\mathbb{N}} \eta^{\delta}_{\beta,\kappa}\, d^{\,\delta} \cdot D_q^{\,\mu-\delta} \Big) \mathbf{c}^{(\kappa)}$$

By construction, $\pi(f') = f$; moreover, a straightforward check shows that

$$\Big\langle f' \,,\, \overline{E}^{\,\eta} K^{\kappa}\, G_2^{\,\gamma}\, \overline{F}^{\,\varphi} \Big\rangle = q^{\gamma\mu} \cdot \Big\langle f \,,\, \overline{E}^{\,\eta} K^{\kappa}\, \overline{F}^{\,\varphi} \Big\rangle \in \mathbb{Z}\big[q,q^{-1}\big]$$

(for all $\eta,\varphi \in \mathbb{N}$, $\kappa,\gamma \in \mathbb{Z}$) because $f \in \mathcal{F}_q[SL_2]$ by hypothesis. Since the monomials $\overline{E}^{\,\eta} K^{\kappa}\, G_2^{\,\gamma}\, \overline{F}^{\,\varphi}$ form a $\mathbb{Z}\big[q,q^{-1}\big]$–basis of $\mathcal{U}_q(\mathfrak{gl}_2)$ (by § 2.1, as $K := G_1 G_2^{\,-1}$), it follows that $f' \in \mathcal{F}_q[M_2]$. Then claim *(a)* and the fact that $\pi(f') = f$ imply that f lies in the $\mathbb{Z}\big[q,q^{-1}\big]$–span of $\mathcal{S}_{SL_2}$. Finally, the last observation about other bases is clear. $\square$

Theorem 3.2.

(a) $\mathcal{F}_q[M_2]$ *is the unital associative* $\mathbb{Z}\big[q,q^{-1}\big]$*–algebra with generators*

$$\binom{a\,;r}{n}\,,\quad \mathbf{b}^{(n)}\,,\quad \mathbf{c}^{(n)}\,,\quad \binom{d\,;s}{m} \qquad\qquad \forall\;\, m,n\in\mathbb{N},\; r,s\in\mathbb{Z}$$

and relations

$$\mathbf{b}^{(r)}\mathbf{c}^{(s)} \,=\, \mathbf{c}^{(s)}\mathbf{b}^{(r)}$$

relations (1.1) *for* $X\in\big\{a,d\big\}$, *relations* (1.2) *for* $X\in\{b,c\}$

$$\bigg[\binom{a\,;r}{n},\binom{d\,;s}{m}\bigg] \;=\; \sum_{j=1}^{n\wedge m} q^{j\,((r+s)-(n+m))+\binom{j}{2}}\big(q-q^{-1}\big)^j\,[j]_q!\begin{Bmatrix} d\,;s\\ m\,,j\end{Bmatrix}\,\mathbf{c}^{(j)}\,\mathbf{b}^{(j)}\begin{Bmatrix} a\,;r\\ n\,,j\end{Bmatrix}$$

$$\binom{a\,;r}{t}\mathbf{b}^{(n)} \;=\; \mathbf{b}^{(n)}\binom{a\,;r+n}{t}\quad ,\qquad \binom{a\,;r}{t}\mathbf{c}^{(n)} \;=\; \mathbf{c}^{(n)}\binom{a\,;r+n}{t}$$

$$\binom{d\,;s}{t}\mathbf{b}^{(n)} \;=\; \mathbf{b}^{(n)}\binom{d\,;s-n}{t}\quad ,\qquad \binom{d\,;s}{t}\mathbf{c}^{(n)} \;=\; \mathbf{c}^{(n)}\binom{d\,;s-n}{t}\,.$$

Moreover, $\mathcal{F}_q[M_2]$ *is a* $\mathbb{Z}\big[q,q^{-1}\big]$*–bialgebra, whose bialgebra structure is given by*

$$\Delta\bigg(\binom{a\,;r}{n}\bigg) \;=\; \sum_{k=0}^{n} q^{k(r-n)}\big(q-q^{-1}\big)^k\,[k]_q!\cdot\Big(\mathbf{b}^{(k)}\otimes 1\Big)\cdot\begin{Bmatrix} a\otimes a\,;\,r-k\\ n\,,\,k\end{Bmatrix}\cdot\Big(1\otimes\mathbf{c}^{(k)}\Big)$$

$$\Delta\Big(\mathbf{b}^{(n)}\Big) \;=\; \sum_{k=0}^{n} q^{-k(n-k)}\cdot a^k\,\mathbf{b}^{(n-k)}\otimes\mathbf{b}^{(k)}\,d^{n-k}$$

$$\Delta\Big(\mathbf{c}^{(n)}\Big) \;=\; \sum_{k=0}^{n} q^{-k(n-k)}\cdot\mathbf{c}^{(k)}\,d^{n-k}\,\otimes a^k\,\mathbf{c}^{(n-k)}$$

$$\Delta\bigg(\binom{d\,;s}{m}\bigg) \;=\; \sum_{k=0}^{m} q^{k(s-m)}\big(q-q^{-1}\big)^k\,[k]_q!\cdot\Big(1\otimes\mathbf{b}^{(k)}\Big)\cdot\begin{Bmatrix} d\otimes d\,;\,s-k\\ m\,,\,k\end{Bmatrix}\cdot\Big(\mathbf{c}^{(k)}\otimes 1\Big)$$

$$\epsilon\bigg(\binom{a\,;r}{n}\bigg)=\binom{r}{n}_q\,,\quad \epsilon\Big(\mathbf{b}^{(\ell)}\Big)=0\,,\quad \epsilon\Big(\mathbf{c}^{(\ell)}\Big)=0\,,\quad \epsilon\bigg(\binom{d\,;s}{m}\bigg)=\binom{s}{m}_q \qquad (\ell,m,n\geq 1)$$

(notation of § 1.2) where $a \,=\, 1+(q-1)\binom{a\,;0}{1}$, $d \,=\, 1+(q-1)\binom{d\,;0}{1}$, *and terms like* $\binom{x\otimes x\,;\sigma}{t}$ *(with* $x\in\{a,d\}$, $\sigma\in\mathbb{Z}$ *and* $t\in\mathbb{N}_+$ *) must be expanded following the rule*

$$\binom{x\otimes x\,;2\tau}{t} \;=\; \sum_{r+s=\nu} q^{-sr}\binom{x\,;\tau}{r}\otimes\binom{x\,;\tau}{s}x^r \;=\; \sum_{r+s=\nu} q^{-rs}x^s\binom{x\,;\tau}{r}\otimes\binom{x\,;\tau}{s}$$

$$\binom{x\otimes x\,;2\tau{+}1}{t}=\sum_{r+s=\nu} q^{-(1-s)r}\binom{x\,;\tau}{r}\otimes\binom{x\,;\tau{+}1}{s}x^r=\sum_{r+s=\nu} q^{-(1-r)s}x^s\binom{x\,;\tau{+}1}{r}\otimes\binom{x\,;\tau}{s}$$

according to whether σ *is even* ($=2\tau$) *or odd* ($=2\tau+1$), *and consequently for* $\begin{Bmatrix} x\otimes x\,;\sigma\\ t\,,\ell\end{Bmatrix}$.

In particular, $\mathcal{F}_q[M_2]$ *is a* $\mathbb{Z}\big[q,q^{-1}\big]$*–integral form (as a bialgebra) of* $F_q[M_2]$.

(b) $\mathcal{F}_q[GL_2]$ *is the unital associative* $\mathbb{Z}\big[q,q^{-1}\big]$*–algebra with generators*

$$\binom{a\,;r}{n}\,,\qquad \mathbf{b}^{(n)}\,,\qquad \mathbf{c}^{(n)}\,,\qquad \binom{d\,;r}{n}\,,\qquad D_q^{-1} \qquad\qquad \forall\;\, n\in\mathbb{N}\,,\, r\in\mathbb{Z}$$

and relations as in claim (a), plus the additional relations

$$D_q^{-1}\binom{x\,;r}{n} = \binom{x\,;r}{n} D_q^{-1}\,, \quad D_q^{-1}\,\mathbf{y}^{(n)} = \mathbf{y}^{(n)}\,D_q^{-1} \qquad \big(\mathbf{y}\in\{\mathbf{b},\mathbf{c}\},\ r\in\mathbb{Z}\,,\ n\in\mathbb{N}\big)$$

$$D_q^{-1} + (q-1)\binom{a\,;0}{1} D_q^{-1} + (q-1)\binom{d\,;0}{1} D_q^{-1} + \\ + (q-1)^2\binom{a\,;0}{1}\binom{d\,;0}{1} D_q^{-1} - q\left(q-q^{-1}\right)^2 \mathbf{b}^{(1)}\mathbf{c}^{(1)}\,D_q^{-1} = 1 \ .$$

Moreover, $\mathcal{F}_q[GL_2]$ is a Hopf $\mathbb{Z}\big[q,q^{-1}\big]$–algebra, whose Hopf algebra structure is given by the same formulæ as in claim (a) for Δ and ϵ plus the formulæ

$$S\left(\binom{a\,;r}{n}\right) = \binom{d\,D_q^{-1}\,;\,r}{n}\,, \qquad S\left(\mathbf{b}^{(n)}\right) = (-1)^n q^{-n}\,\mathbf{b}^{(n)}\,D_q^{-n}$$

$$S\left(\mathbf{c}^{(n)}\right) = (-1)^n q^{+n}\mathbf{c}^{(n)}\,D_q^{-n}\,, \qquad S\left(\binom{d\,;r}{n}\right) = \binom{a\,D_q^{-1}\,;\,r}{n}$$

$$\Delta\big(D_q^{-1}\big) = D_q^{-1}\otimes D_q^{-1}\,, \qquad \varepsilon\big(D_q^{-1}\big) = 1\,, \qquad S\big(D_q^{-1}\big) = D_q\ .$$

In particular, $\mathcal{F}_q[GL_2]$ is a $\mathbb{Z}\big[q,q^{-1}\big]$–integral form (as a Hopf algebra) of $F_q[GL_2]$.

(c) $\mathcal{F}_q[SL_2]$ is generated, as a unital associative $\mathbb{Z}\big[q,q^{-1}\big]$–algebra, by generators as in claim (a). These generators enjoy all relations in (a), plus some additional relations, springing out of the relation $D_q = 1$ in $F_q[SL_2]$. Moreover, $\mathcal{F}_q[SL_2]$ is a Hopf $\mathbb{Z}\big[q,q^{-1}\big]$–algebra, whose Hopf algebra structure is given as in (a) and (b), but setting $D_q = 1$.

In particular, $\mathcal{F}_q[SL_2]$ is a $\mathbb{Z}\big[q,q^{-1}\big]$–integral form (as a Hopf algebra) of $F_q[SL_2]$.

Proof. (a) Thanks to Theorem 3.1*(a)*, the set of elements considered in the statement does generate $\mathcal{F}_q[M_2]$. As for relations, the third line ones are those springing out of the relation $a\,d - d\,a = \left(q-q^{-1}\right)b\,c$ in $F_q[M_2]$; those in fourth and fifth line are the ones following from the relations $a\,b = q\,b\,a$, $a\,c = q\,d\,c$, $b\,d = q\,d\,b$ and $c\,d = q\,d\,c$; the first line ones follow from $b\,c = c\,b$, and those in second line are obvious. The sole non-trivial relations are the third line ones, which we shall now prove, by induction on m.

We set $A^r_{n,k} := \left\{ {a\,;\,r \atop n\,,\,k} \right\}$, $D^s_{m,h} := \left\{ {d\,;\,s \atop m\,,\,h} \right\}$. The basis of induction ($m=1$) follows from

$$\big[A^t_{n,k}\,,d\,\big] = q^{t+k-n}\big(q-q^{-1}\big)^2\,\mathbf{c}^{(1)}\,\mathbf{b}^{(1)}A^t_{n,k+1} \tag{3.4}$$

which in turn is easily proved by induction on k using the commutation formula

$$\left[\binom{a\,;\ell}{u}\,,\,d\right] = q^{\ell-u}\big(q-q^{-1}\big)^2\,\mathbf{c}^{(1)}\,\mathbf{b}^{(1)}A^\ell_{u,1} \tag{3.5}$$

that directly follows from definitions and the relation $a\,d - d\,a = \left(q-q^{-1}\right)b\,c$.

For the general case ($m>1$), using formulæ (3.4–5) we get

$$\left[\binom{a\,;r}{n},\binom{d\,;s}{m+1}\right] = \left[\binom{a\,;r}{n},\binom{d\,;s}{m}\right]\frac{d\,q^{s-m}-1}{q^{m+1}-1} + \binom{d\,;s}{m}\left[\binom{a\,;r}{n}\,,\,\frac{d\,q^{s-m}-1}{q^{m+1}-1}\right] =$$

$$= \left(q^{m+1}-1\right)^{-1}\left(\sum_{j=1}^{n\wedge m} q^{j((r+s)-(n+m))}\big(q-q^{-1}\big)^j q^{\binom{j}{2}}\,[j]_q!\,D^s_{m,j}\left(q^{s-k+2j}d-1\right)\mathbf{c}^{(j)}\,\mathbf{b}^{(j)}A^r_{n,j} + \right.$$

$$\left. + \textstyle\sum_{i=1}^{n\wedge m} q^{i\,((r+s)-(n+m))+\binom{i}{2}}\big(q-q^{-1}\big)^{i+1}\,[i+1]_q!\,q^{r+i+s}\big(q^i-q^{-i}\big)\,D^s_{m,i}\,\mathbf{c}^{(i+1)}\,\mathbf{b}^{(i+1)}A^r_{n,i+1}\right)$$

Comparing the previous result with the expected formula for $(m+1)$, we see that the latter holds if and only if the following identity holds

$$q^j\, D^s_{m,j}\, \big(q^{s-m+2j} d - 1\big) \,+\, \big(q^{2j}-1\big)\, D^s_{m,j-1} \;=\; \big(q^{m+1}-1\big) D^s_{m+1,j}$$

which is just a special case of Lemma 4.5*(a)* later on.

The previous analysis shows that the given relations do hold in $\mathcal{F}_q[M_2]$; in order to prove the claim, we must show that these generate the ideal of all possible relations. This amounts to show that the algebra enjoying only the given relations is in fact isomorphic to $\mathcal{F}_q[M_2]$. To this end, it is enough to prove the following. Let $\mathcal{B}'$ be any one of the PBW–like bases provided by Theorem 3.1*(a)*: then the given relations are enough to expand any product of the given generators as a $\mathbb{Z}\big[q,q^{-1}\big]$–linear combination of the monomials in $\mathcal{B}'$.

Now, if $\mathcal{B}' = \Big\{ \binom{d\,;\,0}{\delta}\, \mathbf{c}^{(\kappa)}\, \mathbf{b}^{(\beta)} \binom{a\,;\,0}{\alpha} \;\Big|\; \alpha,\beta,\kappa,\delta \in \mathbb{N} \Big\}$, then the given relations clearly allow to write any product of the generators as an element of the $\mathbb{Z}\big[q,q^{-1}\big]$–span of $\mathcal{B}'$.

As to the bialgebra structure, everything is just a matter of computation. Yet we can point out just one key detail: namely, by definition we have

$$\Delta\left(\binom{a\,;\,r}{n}\right) \;=\; \binom{\Delta(a)\,;\,r}{n} \;=\; \binom{a\otimes a + b\otimes c\,;\; r}{n} \;=\; \binom{a\otimes a + \big(q - q^{-1}\big)^2\, \mathbf{b}\otimes\mathbf{c}\;;\; r}{n}$$

and then one gets the formula in the claim via Lemma 4.2; similarly for $\Delta\left(\binom{d\,;\,s}{m}\right)$.

(b) The fact that $\mathcal{F}_q[GL_2]$ admits the given presentation is a direct consequence of claim *(a)* and of Theorem 3.1*(b)*, but for the additional relations in first line. The latter mean that ${D_q}^{-1}$ is central (because D_q is) while the second line relation is a reformulation of the relation $D_q\, {D_q}^{-1} = 1$. The statement on the Hopf structure also follows from claim *(a)* and Theorem 3.1*(b)* and from the formulæ for the antipode in $F_q[GL_2]$ (cf. § 2.2), but for the formulæ for ${D_q}^{-1}$ which follow from $\Delta(D_q) = D_q \otimes D_q$, $\varepsilon(D_q) = 1$, $S\big(D_q\big) = {D_q}^{-1}$.

Now, the formulæ for Δ and ϵ show that $\mathcal{F}_q[GL_2]$ is a $\mathbb{Z}\big[q,q^{-1}\big]$–subbialgebra of $F_q[GL_2]$. For the antipode, $\Big\langle S\big(\mathcal{F}_q[GL_2]\big)\,,\, \mathcal{U}_q(\mathfrak{gl}_2) \Big\rangle \;=\; \Big\langle \mathcal{F}_q[GL_2]\,,\, S\big(\mathcal{U}_q(\mathfrak{gl}_2)\big) \Big\rangle \;\subseteq\; \mathbb{Z}\big[q,q^{-1}\big]$, which gives $S\big(\mathcal{F}_q[GL_2]\big) \subseteq \mathcal{F}_q[GL_2]$. The claim follows.

(c) This follows again from claim *(a)* and Theorem 3.1. $\square$

Corollary 3.3. *For every* $X \in \{M, GL\}$, *let* $\big(D_q - 1\big)$ *be the two-sided ideal of* $F_q[X_2]$ *generated by* $\big(D_q - 1\big)$, *and let* $\mathcal{D}(X_2) := \big(D_q - 1\big) \cap \mathcal{F}_q[X_2]$, *a two-sided ideal of* $\mathcal{F}_q[X_2]$.

(a) The epimorphism $F_q[M_2] \overset{\pi}{\twoheadrightarrow} F_q[M_2]\Big/\big(D_q - 1\big) \;\cong\; F_q[SL_2]$ *restricts to an epimorphism* $\mathcal{F}_q[M_2] \overset{\pi}{\twoheadrightarrow} \mathcal{F}_q[M_2]\Big/\mathcal{D}(M_2) \;\cong\; \mathcal{F}_q[SL_2]$ *of* $\mathbb{Z}\big[q,q^{-1}\big]$*–bialgebras.*

(b) The epimorphism $F_q[GL_2] \overset{\pi}{\twoheadrightarrow} F_q[GL_2]\Big/\big(D_q - 1\big) \;\cong\; F_q[SL_2]$ *restricts to an epimorphism* $\mathcal{F}_q[GL_2] \overset{\pi}{\twoheadrightarrow} \mathcal{F}_q[GL_2]\Big/\mathcal{D}(GL_2) \;\cong\; \mathcal{F}_q[SL_2]$ *of Hopf* $\mathbb{Z}\big[q,q^{-1}\big]$*–algebras.* $\square$

3.4 Remarks: *(a)* Besides those given in Theorem 3.2, there are several other (equivalent) commutation relations between the $\binom{a\,;\,r}{n}$'s and the $\binom{d\,;\,s}{m}$'s — see [GR1], § 3.3, for details.

(b) One should compute an expression for $S\left(\binom{a\,;\,r}{n}\right) = \binom{d\,{D_q}^{-1}\,;\,r}{n}$ and $S\left(\binom{d\,;\,r}{n}\right) = \binom{a\,{D_q}^{-1}\,;\,r}{n}$ in terms of the generators in Theorem 3.2! In fact, this is a very tough task.

3.5 Relations in $\mathcal{F}_q[SL_2]$. By Theorem 3.2*(c)*, in $\mathcal{F}_q[SL_2]$ the generators $\binom{a\,;\,r}{n}$, $\mathbf{b}^{(n)}$, $\mathbf{c}^{(n)}$ and $\binom{d\,;\,r}{n}$ — for all $n \in \mathbb{N}$, $r \in \mathbb{Z}$ — enjoy the same relations as in Theorem 3.2 *plus some additional ones*, springing out of the relation $a\,d - q\,b\,c = 1$ in $F_q[SL_2]$.

A **first (set of) relation(s)** is the following (for all $\alpha, \delta \in \mathbb{N}$, $r, s \in \mathbb{Z}$):

$$(\alpha)_q\,(\delta)_q \cdot \binom{a\,;\,r}{\alpha}\binom{d\,;\,s}{\delta} + (\alpha)_q \cdot \binom{a\,;\,r}{\alpha}\binom{d\,;\,s}{\delta-1} + \\ + (\delta)_q \cdot \binom{a\,;\,r}{\alpha-1}\binom{d\,;\,s}{\delta} + (2-\alpha-\delta+r+s)_q \cdot \binom{a\,;\,r}{\alpha-1}\binom{d\,;\,s}{\delta-1} + \\ + q^{1-\alpha-\delta+r+s}\,(q-1)\,(2)_q^{\,2} \cdot \binom{a\,;\,r}{\alpha-1}\,\mathbf{b}^{(1)}\,\mathbf{c}^{(1)}\binom{d\,;\,s}{\delta-1} \; = \; 0$$

A **second (set of) relation(s)** is (for all $h, n \in \mathbb{N}$, $r, s \in \mathbb{Z}$):

$$\sum_{\substack{j=0 \\ (j,\ell)\neq(0,0)}}^{h}\sum_{\ell=0}^{n} q^{\binom{j}{2}+\binom{\ell}{2}}\,(q-1)^{j+\ell-1}\,(j)_q!\,(\ell)_q!\binom{h}{j}_q\binom{n}{\ell}_q \cdot \binom{a\,;\,r}{j}\binom{d\,;\,s}{\ell} \; = \; \big(n(r+s)\big)_q + \\ + q^{n(r+s)}\sum_{\substack{i=0 \\ (j,\ell)\neq(0,0)}}^{h-n}\sum_{t=0}^{n} q^{t^2+\binom{i}{2}}\,(q-1)^{i-1}\,\big(q-q^{-1}\big)^{2t}\,(i)_q!\,[t]_q!^2\binom{h-n}{i}_q\binom{n}{t}_{q^2} \cdot \binom{a\,;\,r}{i}\,\mathbf{b}^{(t)}\,\mathbf{c}^{(t)}$$

$$\sum_{\substack{j=0 \\ (j,\ell)\neq(0,0)}}^{h}\sum_{\ell=0}^{n} q^{\binom{j}{2}+\binom{\ell}{2}}\,(q-1)^{j+\ell-1}\,(j)_q!\,(\ell)_q!\binom{h}{j}_q\binom{n}{\ell}_q \cdot \binom{a\,;\,r}{j}\binom{d\,;\,s}{\ell} \; = \; \big(h(r+s)\big)_q + \\ + q^{h(r+s)}\sum_{\substack{i=0 \\ (i,t)\neq(0,0)}}^{n-h}\sum_{t=0}^{h} q^{t^2+\binom{i}{2}}\,(q-1)^{i-1}\,\big(q-q^{-1}\big)^{2t}\,(i)_q!\,[t]_q!^2\binom{n-h}{i}_q\binom{h}{t}_{q^2} \cdot \mathbf{b}^{(t)}\,\mathbf{c}^{(t)}\binom{d\,;\,s}{i}$$

where the first identity holds for all $h \geq n$ and the second for all $h \leq n$.

A **third (set of) relation(s)** is for all $n \in \mathbb{N}_+$,

$$\binom{a\,;\,0}{n} + \binom{d\,;\,0}{n} \; = \; \sum_{h=1}^{n}\widetilde{\alpha}_h^n\,\mathbf{b}^{(h)}\,\mathbf{c}^{(h)} + \sum_{\substack{h=1 \\ h\leq i}}^{n}\widetilde{\beta}_{h,i}^n\left(\binom{a\,;\,0}{h}\binom{d\,;\,0}{i} + \binom{a\,;\,0}{i}\binom{d\,;\,0}{h}\right)$$

with (for all $1 \leq h \leq i \leq n$)

$$\widetilde{\alpha}_h^n \; = \; q^{h^2+\binom{n-h+1}{2}}\,[h]_q!^2\,\frac{(q-q^{-1})^{2h}}{(n)_q!\,(q-1)^h}\,\binom{n}{2(n-h)}_q\big(2(n-h)-1\big)_q!!$$

$$\widetilde{\beta}_{h,i}^n \; = \; -\big(1+\delta_{h,i}\big)^{-1}\,q^{\binom{n-h-i+1}{2}}\,(q-1)^{h+i-n}\,\frac{(h)_q!}{(n-i)_q!}\binom{i}{n-h}_q$$

All these formulæ are proved in detail in [GR1].

Further byproducts of Theorem 3.2 concern the specializations of $\mathcal{F}_q[M_2]$, $\mathcal{F}_q[GL_2]$ and $\mathcal{F}_q[SL_2]$ at roots of unity, including the case $q = 1$, as follows:

Corollary 3.6.

(a) There exists a $\mathbb{Z}$–bialgebra isomorphism $\mathcal{F}_1[M_2] \cong U_{\mathbb{Z}}\big(\mathfrak{gl}_2^{\,*}\big)$ *given by*

$$\binom{a\,;0}{n}\bigg|_{q=1} \mapsto \binom{\mathrm{g}_1}{n} \;, \quad \mathbf{b}^{(n)}\Big|_{q=1} \mapsto \mathrm{f}^{(n)} \;, \quad \mathbf{c}^{(n)}\Big|_{q=1} \mapsto \mathrm{e}^{(n)} \;, \quad \binom{d\,;0}{m}\bigg|_{q=1} \mapsto \binom{\mathrm{g}_2}{m} \quad .$$

In particular $\mathcal{F}_1[M_2]$ *is a Hopf $\mathbb{Z}$–algebra, isomorphic to* $U_{\mathbb{Z}}\big(\mathfrak{gl}_2^*\big)$ *.*

(b) There exists a Hopf $\mathbb{Z}$–algebra isomorphism $\mathcal{F}_1[GL_2] \cong U_{\mathbb{Z}}\big(\mathfrak{gl}_2^*\big)$ *, which is uniquely determined by the formulæ in claim (a).*

(c) There exists a Hopf $\mathbb{Z}$–algebra isomorphism $\mathcal{F}_1[SL_2] \cong U_{\mathbb{Z}}\big(\mathfrak{sl}_2^*\big)$ *given by the same formulæ as in claim (a), where one must read* $\mathrm{g}_1 = \mathrm{h}$ *,* $\mathrm{g}_2 = -\mathrm{h}$ *.* □

Proof. At $q = 1$, Theorem 3.2*(a)* provides a presentation for $\mathcal{F}_1[M_2]$; a straightforward comparison then shows that the latter is the standard presentation of $U_{\mathbb{Z}}\big(\mathfrak{gl}_2^*\big)$, following the correspondence given in the claim (actually, in the first presentation one also has the specialization at $q = 1$ of the $\binom{x\,;r}{k}$'s — with $x \in \{a,d\}$ — but these are generated by the specializations of the $\binom{x\,;0}{\nu}$'s). This yields a $\mathbb{Z}$–algebra isomorphism: a moment's check shows that it is one of $\mathbb{Z}$–bialgebras too. This proves *(a)* and, with minimal changes, *(b)* and *(c)* too. □

Proposition 3.7. *Let ε be a root of unity, of odd order, and apply notation of § 1.3.*

(a) The specialization $\mathcal{F}_\varepsilon[M_2] \lhook\joinrel\longrightarrow \mathfrak{H}_\varepsilon^g$ *at $q = \varepsilon$ of the embedding* $\mathcal{F}_q[M_2] \lhook\joinrel\longrightarrow \mathfrak{H}_q^g$ *is a* $\mathbb{Z}_\varepsilon$*–algebra isomorphism.*

(b) The embedding $\mathcal{F}_\varepsilon[M_2] \lhook\joinrel\longrightarrow \mathcal{F}_\varepsilon[GL_2]$ *of $\mathbb{Z}_\varepsilon$–bialgebras is an isomorphism. In particular,* $\mathcal{F}_\varepsilon[M_2]$ *and* $\mathfrak{H}_\varepsilon^g$ *both are Hopf $\mathbb{Z}_\varepsilon$–algebras isomorphic to* $\mathcal{F}_\varepsilon[GL_2]$ *.*

(c) The specialization $\mathcal{F}_\varepsilon[SL_2]$ *is a Hopf $\mathbb{Z}_\varepsilon$–algebra, isomorphic to* $\mathfrak{H}_\varepsilon^s$ *via the specialization of the embedding* $\mathcal{F}_q[SL_2] \lhook\joinrel\longrightarrow \mathfrak{H}_q^s$ *.*

Proof. The embedding $\mathcal{F}_q[M_2] \lhook\joinrel\longrightarrow \mathcal{F}_q[GL_2]$ induces an embedding $\mathcal{F}_\varepsilon[M_2] \lhook\joinrel\longrightarrow \mathcal{F}_\varepsilon[GL_2]$. Then *(a)* will follow by proving that $D_\varepsilon := D_q \mod (q-\varepsilon)\,\mathcal{F}_q[M_2]$ is invertible in $\mathcal{F}_\varepsilon[M_2]$.

Let ℓ be the (multiplicative) order of ε. Lemma 4.3 gives $a^\ell = 1$ in $\mathcal{F}_\varepsilon[M_2]$, so a is invertible in $\mathcal{F}_\varepsilon[M_2]$ with $a^{-1} = a^{\ell-1} \in \mathcal{F}_\varepsilon[M_2]$, and also $\mathbf{b}^{\,\ell} = 0 \in \mathcal{F}_\varepsilon[M_2]$, so $b^\ell = \big(\varepsilon - \varepsilon^{-1}\big)^\ell\, \mathbf{b}^{\,\ell} = 0$ in $\mathcal{F}_\varepsilon[M_2]$. Similarly, $d^{-1} = d^{\,\ell-1} \in \mathcal{F}_\varepsilon[M_2]$ and $c^\ell = 0$. The power series expansion of $(1-x)^{-1}$ then gives

$$D_\varepsilon^{-1} \;=\; \big(1 - \varepsilon\, b\, d^{-1} a^{-1} c\big)^{-1} d^{-1} a^{-1} \;=\; \sum_{n=0}^{\ell-1} \varepsilon^n\, b^n \big(d^{\,\ell-1} a^{\ell-1}\big)^n c^n\, d^{\,\ell-1} a^{\ell-1} \in \mathcal{F}_\varepsilon[M_2] \quad .$$

As to the second part, note that the embedding $\xi : F_q[GL_2] \lhook\joinrel\longrightarrow \mathbf{H}_q^g$ extends to an identity $F_q[GL_2]\big[d^{-1}\big] = \mathbf{H}_q^g$: this comes from [DL], § 1.8 (adapted to the case of GL_2) or directly from the explicit description of ξ in § 2.4. This yields also $\mathfrak{H}_q^g = \mathcal{F}_q[GL_2]\big[d^{-1}\big]$. In fact, if $\eta \in \mathfrak{H}_q^g = \mathcal{F}_q[GL_2]\big[d^{-1}\big]$ then there is $n \in \mathbb{N}$ such that $\eta\, d^n \in \mathcal{F}_q[GL_2]$, and also $\big\langle \eta\, d^n, \mathcal{U}_q(\mathfrak{gl}_2)\big\rangle = \Big\langle \eta \otimes d^{\otimes n}, \Delta^{(n+1)}\big(\mathcal{U}_q(\mathfrak{gl}_2)\big)\Big\rangle \subseteq \big\langle \eta, \mathcal{U}_q(\mathfrak{gl}_2)\big\rangle \cdot \big\langle d, \mathcal{U}_q(\mathfrak{gl}_2)\big\rangle^n \subseteq \mathbb{Z}\big[q,q^{-1}\big]$ because $\eta, d \in \mathfrak{H}_q^g$. Thus $\eta\, d^n \in F_q[GL_2] \bigcap \mathfrak{H}_q^g = \mathcal{F}_q[GL_2]$, whence $\eta \in \mathcal{F}_q[GL_2]\big[d^{-1}\big]$; the outcome is $\mathfrak{H}_q^g \subseteq \mathcal{F}_q[GL_2]\big[d^{-1}\big]$, and the converse is clear. Now $\mathfrak{H}_\varepsilon^g = \mathcal{F}_\varepsilon[GL_2]\big[d^{-1}\big]$; but we found $d^{-1} \in \mathcal{F}_\varepsilon[M_2] = \mathcal{F}_\varepsilon[GL_2]$, so $\mathfrak{H}_\varepsilon^g = \mathcal{F}_\varepsilon[GL_2]\big[d^{-1}\big] = \mathcal{F}_\varepsilon[GL_2] = \mathcal{F}_\varepsilon[M_2]$.

The above proves claim *(a)* and *(b)*, noting that $\mathcal{F}_\varepsilon[GL_2]$ is clearly a Hopf $\mathbb{Z}_\varepsilon$–algebra. Similarly, *(c)* can be proved like *(a)*, or deduced from the latter. □

Theorem 3.8. *Let ε be a root of unity, of odd order ℓ.*

(a) The quantum Frobenius morphism (2.1) is defined over $\mathbb{Z}_\varepsilon$, i.e. it restricts to an epimorphism of $\mathbb{Z}_\varepsilon$–bialgebras $\mathcal{F}r_{M_2}^{\mathbb{Z}} : \mathcal{F}_\varepsilon[M_2] \twoheadrightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{F}_1[M_2] \cong \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{gl}_2^{\,*})$, coinciding, via Corollary 3.6 and Proposition 3.7, with (2.5), and given on generators by

$$\mathcal{F}r_{M_2}^{\mathbb{Z}} : \begin{cases} \left.\binom{a\,;\,0}{n}\right|_{q=\varepsilon} \mapsto \left.\binom{a\,;\,0}{n/\ell}\right|_{q=1} = \binom{\mathrm{g}_1}{n/\ell} \;\; \textit{if} \;\; \ell \,\big|\, n \,, & \left.\binom{a\,;\,0}{n}\right|_{q=\varepsilon} \mapsto 0 \;\; \textit{if} \;\; \ell \not\big|\, n \\ \left.\mathbf{b}^{(n)}\right|_{q=\varepsilon} \mapsto \left.\mathbf{b}^{(n/\ell)}\right|_{q=1} = \mathrm{f}^{(n/\ell)} \;\; \textit{if} \;\; \ell \,\big|\, n \,, & \left.\mathbf{b}^{(n)}\right|_{q=\varepsilon} \mapsto 0 \;\; \textit{if} \;\; \ell \not\big|\, n \\ \left.\mathbf{c}^{(n)}\right|_{q=\varepsilon} \mapsto \left.\mathbf{c}^{(n/\ell)}\right|_{q=1} = \mathrm{e}^{(n/\ell)} \;\; \textit{if} \;\; \ell \,\big|\, n \,, & \left.\mathbf{c}^{(n)}\right|_{q=\varepsilon} \mapsto 0 \;\; \textit{if} \;\; \ell \not\big|\, n \\ \left.\binom{d\,;\,0}{n}\right|_{q=\varepsilon} \mapsto \left.\binom{d\,;\,0}{n/\ell}\right|_{q=1} = \binom{\mathrm{g}_2}{n/\ell} \;\; \textit{if} \;\; \ell \,\big|\, n \,, & \left.\binom{d\,;\,0}{n}\right|_{q=\varepsilon} \mapsto 0 \;\; \textit{if} \;\; \ell \not\big|\, n \end{cases}$$

(b) The quantum Frobenius morphism (2.2) is defined over $\mathbb{Z}_\varepsilon$, i.e. it restricts to an epimorphism of $\mathbb{Z}_\varepsilon$–bialgebras $\mathcal{F}r_{GL_2}^{\mathbb{Z}} : \mathcal{F}_\varepsilon[GL_2] \twoheadrightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{F}_1[GL_2] \cong \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{gl}_2^{\,*})$ coinciding, via Corollary 3.6 and Proposition 3.7, with (2.5) and with $\mathcal{F}r_{M_2}^{\mathbb{Z}}$ in (a).

(c) The quantum Frobenius morphism (2.3) is defined over $\mathbb{Z}_\varepsilon$, i.e. it restricts to an epimorphism of $\mathbb{Z}_\varepsilon$–bialgebras $\mathcal{F}r_{SL_2}^{\mathbb{Z}} : \mathcal{F}_\varepsilon[SL_2] \twoheadrightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{F}_1[SL_2] \cong \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{sl}_2^{\,*})$ coinciding, via Corollary 3.6 and Proposition 3.7, with (2.4), and described by formulæ like in (a) with $\mathrm{g}_1 = +\mathrm{h}$ and $\mathrm{g}_2 = -\mathrm{h}$.

Proof. By definition (cf. [Ga1]) the morphism $\mathcal{F}r_{M_2}^{\mathbb{Q}} : \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}_\varepsilon} \mathcal{F}_\varepsilon[M_2] \twoheadrightarrow \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{gl}_2^{\,*})$ is the restriction (via $\widehat{\xi} : \mathcal{F}_q[M_2] \hookrightarrow \mathfrak{H}_q^g$ at $q = \varepsilon$ and $q = 1$) of the similar epimorphism $\mathfrak{Fr}_{\mathfrak{gl}_2^{\,*}}^{\mathbb{Q}} : \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}_\varepsilon} \mathfrak{H}_\varepsilon^g \twoheadrightarrow \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}} \mathfrak{H}_1^g$ obtained by scalar extension from (2.5). From this, direct computation (taking into account that $[\ell]_\varepsilon = 0$) gives, thanks to Lemma 4.2*(a-1)*,

$$\mathfrak{Fr}_{\mathfrak{gl}_2^{\,*}}^{\mathbb{Q}}\Big(\widehat{\xi}\,\Big|_{q=\varepsilon}\Big(\left.\binom{a\,;\,0}{n}\right|_{q=\varepsilon}\Big)\Big) = \mathfrak{Fr}_{\mathfrak{gl}_2^{\,*}}^{\mathbb{Z}}\Big(\left.\binom{\Lambda_1 - \overline{F}\Lambda_2\overline{E}\,;\,0}{n}\right|_{q=\varepsilon}\Big) =$$

$$= \sum_{k=0}^{\ell-1} \varepsilon^{-kn-\binom{k}{2}} \big(\varepsilon^{-1} - \varepsilon\big)^k [k]_\varepsilon!\, \mathfrak{Fr}_{\mathfrak{gl}_2^{\,*}}^{\mathbb{Z}}\big(F^{(k)}\big)\, \mathfrak{Fr}_{\mathfrak{gl}_2^{\,*}}^{\mathbb{Z}}\big(\Lambda_2^{\,k}\big)\, \mathfrak{Fr}_{\mathfrak{gl}_2^{\,*}}^{\mathbb{Z}}\big(E^{(k)}\big)\, \mathfrak{Fr}_{\mathfrak{gl}_2^{\,*}}^{\mathbb{Z}}\Big(\Big(\Big\{ {\Lambda_1\,;\,0 \atop n\,,\,k} \Big\}\Big) =$$

$$= \mathfrak{Fr}_{\mathfrak{gl}_2^{\,*}}^{\mathbb{Z}}\Big(\left.\Big\{ {\Lambda_1\,;\,0 \atop n\,,\,0} \Big\}\right|_{q=\varepsilon}\Big) = \mathfrak{Fr}_{\mathfrak{gl}_2^{\,*}}^{\mathbb{Z}}\Big(\left.\binom{\Lambda_1\,;\,0}{n}\right|_{q=\varepsilon}\Big) = \begin{cases} \left.\binom{\Lambda_1\,;\,0}{n/\ell}\right|_{q=1} = \binom{\mathrm{g}_1}{n/\ell} & \text{if} \;\; \ell \,|\, n \\ 0 & \text{if} \;\; \ell \not|\, n \end{cases}$$

by the very description of $\mathfrak{Fr}_{\mathfrak{gl}_2^{\,*}}^{\mathbb{Z}}$; on the other hand, we also have $\widehat{\xi}\,\Big|_{q=1}\Big(\left.\binom{a\,;\,0}{n/\ell}\right|_{q=1}\Big) = \left.\binom{\Lambda_1 - \overline{F}\Lambda_2\overline{E}\,;\,0}{n/\ell}\right|_{q=1} = \Big(\sum_{k=0}^{\ell-1} q^{-kn-\binom{k}{2}} \big(q^{-1} - q\big)^k [k]_q! \cdot F^{(k)} \Lambda_2^{\,k} E^{(k)} \Big\{ {\Lambda_1\,;\,0 \atop n/\ell\,,\,k} \Big\}\Big)\Big|_{q=1} = \left.\binom{\Lambda_1\,;\,0}{n/\ell}\right|_{q=1} = \binom{\mathrm{g}_1}{n/\ell}$ by Corollary 3.6*(a)* and Lemma 4.2*(a-1)* again. This together with $\widehat{\xi}\,\big|_{q=\varepsilon} \circ \big(\mathrm{id}_{\mathbb{Q}_\varepsilon} \otimes_{\mathbb{Z}_\varepsilon} \mathfrak{Fr}_{\mathfrak{g}^*}^{\mathbb{Z}}\big) = \big(\mathrm{id}_{\mathbb{Q}_\varepsilon} \otimes_{\mathbb{Z}_\varepsilon} \widetilde{\xi}\,\big|_{q=1}\big) \circ \mathcal{F}r_G^{\mathbb{Q}}$ (see § 2.4) give $\mathcal{F}r_{M_2}^{\mathbb{Z}}\Big(\left.\binom{a\,;\,0}{n}\right|_{q=\varepsilon}\Big) = \left.\binom{a\,;\,0}{n/\ell}\right|_{q=1} = \binom{\mathrm{g}_1}{n/\ell}$ if $\ell \big| n$, and otherwise $\mathcal{F}r_{M_2}^{\mathbb{Z}}\Big(\left.\binom{a\,;\,0}{n}\right|_{q=\varepsilon}\Big) = 0$, as claimed. Similarly,

$$\mathfrak{Fr}_{\mathfrak{gl}_2^{\,*}}^{\mathbb{Q}}\Big(\widehat{\xi}\,\Big|_{q=\varepsilon}\Big(\mathbf{b}^{(n)}\big|_{q=\varepsilon}\Big)\Big) = \mathfrak{Fr}_{\mathfrak{gl}_2^{\,*}}^{\mathbb{Z}}\Big(-F\big|_{q=\varepsilon}\Lambda_2\big|_{q=\varepsilon}\Big)^{(n)} =$$

$$= (-1)^n \varepsilon^{-\binom{n}{2}} \cdot \mathfrak{Fr}_{\mathfrak{gl}_2^{\,*}}^{\mathbb{Z}}\big(F^{(n)}\big)\, \mathfrak{Fr}_{\mathfrak{gl}_2^{\,*}}^{\mathbb{Z}}\big(\Lambda^n\big) = \begin{cases} (-1)^n \varepsilon^{-\binom{n}{2}} \cdot F^{(n/\ell)}\Big|_{q=1} = (-1)^n \varepsilon^{-\binom{n}{2}} \cdot \mathrm{f}^{(n/\ell)} \\ 0 \end{cases}$$

where the upper identity holds if $\ell \big| n$ and the lower line holds if not. On the other hand $\widehat{\xi}\,\Big|_{q=1}\Big(\mathbf{b}^{(n/\ell)}\big|_{q=1}\Big) = (-1)^{n/\ell}\varepsilon^{-\binom{n/\ell}{2}} \cdot F^{(n/\ell)}\Big|_{q=1} = (-1)^{n/\ell}\varepsilon^{-\binom{n/\ell}{2}} \cdot \mathrm{f}^{(n/\ell)}$, and a moment's check shows that $(-1)^n\varepsilon^{-\binom{n}{2}} = (-1)^{n/\ell}\varepsilon^{-\binom{n/\ell}{2}}$ whence we conclude. Similarly

$$\mathfrak{Fr}^{\mathbb{Q}}_{\mathfrak{gl}_2{}^*}\Big(\widehat{\xi}\,\Big|_{q=\varepsilon}\Big(\mathbf{c}^{(n)}\big|_{q=\varepsilon}\Big)\Big) \;=\; \begin{cases} \varepsilon^{\binom{n}{2}} \cdot E^{(n/\ell)}\Big|_{q=1} = \varepsilon^{\binom{n}{2}} \cdot \mathrm{e}^{(n/\ell)} \\ \quad 0 \end{cases}$$

where the upper identity holds if $\ell \big| n$ and the lower line if not, while $\widehat{\xi}\,\Big|_{q=1}\Big(\mathbf{c}^{(n/\ell)}\big|_{q=1}\Big) = \varepsilon^{\binom{n/\ell}{2}} \cdot \mathrm{e}^{(n/\ell)}$, and again one checks that $\varepsilon^{\binom{n}{2}} = \varepsilon^{\binom{n/\ell}{2}}$ so the claim for $\mathbf{c}^{(n)}$ follows. Finally,

$$\mathfrak{Fr}^{\mathbb{Q}}_{\mathfrak{gl}_2{}^*}\bigg(\widehat{\xi}\,\bigg|_{q=\varepsilon}\bigg(\binom{d\,;0}{n}\bigg|_{q=\varepsilon}\bigg)\bigg) \;=\; \mathfrak{Fr}^{\mathbb{Z}}_{\mathfrak{gl}_2{}^*}\bigg(\binom{\Lambda_2\,;0}{n}\bigg|_{q=\varepsilon}\bigg) \;=\; \begin{cases} \binom{\Lambda_2\,;0}{n/\ell}\Big|_{q=1} = \binom{\mathrm{g}_2}{n/\ell} & \text{if } \ \ell \,|\, n \\ \quad 0 & \text{if } \ \ell \not|\, n \end{cases}$$

while $\widehat{\xi}\,\Big|_{q=1}\Big(\binom{d\,;0}{n/\ell}\Big|_{q=1}\Big) = \binom{\Lambda_2\,;0}{n/\ell}\Big|_{q=1} = \binom{\mathrm{g}_2}{n/\ell}$ due to Corollary 3.6*(a)*, so $\mathcal{F}r^{\mathbb{Z}}_{M_2}\Big(\binom{d\,;0}{n}\Big|_{q=\varepsilon}\Big) = \binom{d\,;0}{n/\ell}\Big|_{q=1} = \binom{\mathrm{g}_2}{n/\ell}$ whenever $\ell \big| n$, and otherwise $\mathcal{F}r^{\mathbb{Z}}_{M_2}\Big(\binom{d\,;0}{n}\Big|_{q=\varepsilon}\Big) = 0$.

All this accounts for claim *(a)*. Claims *(b)* and *(c)* can be proved with the same arguments, or deduced from *(a)* in force of Proposition 3.7 and of Corollary 3.3. □

§ 4 Miscellaneous results on q–numbers and q–functions.

In several steps along the present work we need special, technical results about q–numbers and their combinatorics. We collect them in the present section, referring to [GR1] for proofs.

Lemma 4.1. *For all* $k \in \mathbb{N}$, *let* $\Pi_k := (q-1)^k (k)_q!$ *and* $(x\,;k) := \Pi_k \cdot \binom{x\,;0}{k}$. *Then*

$$(x\,;n) \;=\; \sum_{k=0}^n (-1)^{n-k}\, q^{-\binom{k}{2}} \binom{n}{k}_{q^{-1}} x^k \quad , \qquad x^n \;=\; \sum_{k=0}^n q^{\binom{k}{2}} \binom{n}{k}_q (x\,;\,k) \quad . \quad \square$$

Lemma 4.2. *Let* A *be any* $\mathbb{Q}(q)$*–algebra, and let* $x,y,z,w \in A$ *be such that* $x\,w = q^2\,w\,x$, $x\,y = q\,y\,x$, $x\,z = q\,z\,x$ *and* $y\,z = z\,y$. *Then for all* $n \in \mathbb{N}$ *and* $t \in \mathbb{Z}$ *we have*

(a-1) $$\binom{x+(q-q^{-1})^2\, w\,;\,t}{n} \;=\; \sum_{r=0}^n q^{r(t-n)} \big(q-q^{-1}\big)^r \cdot w^{(r)} \left\{ {x\,;\,t \atop n\,,\,r} \right\} \;=$$

(a-2) $$=\; \sum_{r=0}^n q^{r(t-n)} \big(q-q^{-1}\big)^r \cdot \left\{ {x\,;\,t-2\,r \atop n\,,\,r} \right\} w^{(r)}$$

(b) $$\binom{x+(q-q^{-1})^2\, y\,z\,;\,t}{n} \;=\; \sum_{r=0}^n q^{r(t-n)} \big(q-q^{-1}\big)^r\, [r]_q! \cdot y^{(r)} \left\{ {x\,;\,t-r \atop n\,,\,r} \right\} z^{(r)} \quad . \quad \square$$

Lemma 4.3. *Let Ω be any $\mathbb{Z}\big[q,q^{-1}\big]$–algebra, ε be a (formal) primitive ℓ-th root of 1, with $\ell \in \mathbb{N}_+$, and $\Omega_\varepsilon := \Omega \Big/ (q-\varepsilon)\,\Omega$ the specialization of Ω at $q=\varepsilon$. Then for each $x, y \in \Omega$*

$$\Omega \ni \binom{x\,;0}{\ell} \implies \big(x\big|_{q=\varepsilon}\big)^\ell = 1 \;\; in \;\; \Omega_\varepsilon \,, \qquad \Omega \ni y^{(\ell)} \implies \big(y\big|_{q=\varepsilon}\big)^\ell = 0 \;\; in \;\; \Omega_\varepsilon \,. \quad \square$$

Lemma 4.4. *For all $n \in \mathbb{N}$, the identity* $a^n\,d^n = \sum_{k=0}^{n} q^{k^2} \binom{n}{k}_{q^2} b^k\,c^k$ *holds in $F_q[SL_2]$.* $\square$

Lemma 4.5. *The following identities holds (notation of § 1.2):*

(a) $$q^j \left\{ {x\,;\,s \atop m\,,\,j} \right\} \big(q^{s-m+2j}\,x-1\big) + \big(q^{2j}-1\big) \left\{ {x\,;\,s \atop m\,,\,j-1} \right\} = \big(q^{m+1}-1\big) \left\{ {x\,;\,s \atop m+1\,,\,j} \right\}$$

(b) $$q^{\binom{s}{2}} \binom{n}{s}_q = \sum_{j=1}^{s} (-1)^{j-1}\, q^{\binom{s-j}{2}} \binom{n}{j}_q \binom{n-j}{s-j}_q$$

(c) $$\sum_{j=0}^{k} (-1)^j\, q^{\binom{j}{2}} \binom{n-j}{k-j}_{q^2} \binom{n}{j}_q = q^{k^2}\,(q-1)^k \binom{n}{2\,k}_q \big(2k-1\big)_q!! \qquad \forall \;\; k \leq n$$

Proof. To give an idea, we sketch the proof of *(c)* — the complete proof is in [GR1]. First we fix some more notation: for all $s,k,n \in \mathbb{N}$ set $\langle s \rangle_q := \frac{q^s+1}{q+1}$, $\langle s \rangle_q! := \prod_{r=1}^{s} \langle r \rangle_q$, $\left\langle {n \atop k} \right\rangle_q := \frac{\langle n \rangle_q!}{\langle k \rangle_q!\,\langle n-k \rangle_q!}$, $(x\,;n)_q := \prod_{s=1}^{n} \big(x\,q^{1-s}-1\big)$, $\langle x\,;n \rangle_q := \prod_{s=1}^{n} \big(x\,q^{1-s}+1\big)$. The following identities then are clear from definitions:

$$\binom{n}{k}_{q^2} = \binom{n}{k}_q \left\langle {n \atop k} \right\rangle_q \;,\quad \binom{n}{j}_q \binom{n-j}{k-j}_q = \binom{n}{k}_q \binom{k}{j}_q \;,\quad \binom{n}{k}_q = \frac{(q^n;k)_q}{(q^k;k)_q}$$

$$\textstyle\prod_n^- := \prod_{i=1}^{n} \big(q^j-1\big) = \big(q^n;n\big)_q \;,\quad \prod_n^+ := \prod_{i=1}^{n} \big(q^j+1\big) = \big\langle q^n;n \big\rangle_q \;,\quad \left\langle {n \atop k} \right\rangle_q = \frac{\langle q^n\,;k \rangle_q}{\langle q^k;k \rangle_q}$$

Using them, we transform a bit both sides of *(c)*. Namely, we get

$$\textstyle\sum_{j=0}^{k} (-1)^j\, q^{\binom{j}{2}} \binom{n-j}{k-j}_{q^2} \binom{n}{j}_q = \sum_{j=0}^{k} (-1)^j\, q^{\binom{j}{2}} \left\langle {n-j \atop k-j} \right\rangle_q \binom{n-j}{k-j}_q \binom{n}{j}_q =$$
$$\textstyle = \sum_{j=0}^{k} (-1)^j\, q^{\binom{j}{2}} \binom{n}{k}_q \binom{k}{j}_q \left\langle {n-j \atop k-j} \right\rangle_q = \binom{n}{k}_q \sum_{j=0}^{k} (-1)^j\, q^{\binom{j}{2}} \binom{k}{j}_q \left\langle {n-j \atop k-j} \right\rangle_q$$

for the l.h.s., and similarly for the r.h.s. we find

$$q^{k^2}\,(q-1)^k\, x \binom{n}{2\,k}_q \big(2k-1\big)_q!! = q^{k^2}\,(q-1)^k\, \frac{(n)_q\,(n-1)_q \cdots (n-2k+1)_q}{(2\,k)_q!!} = q^{k^2} \binom{n}{k}_q \binom{n-k}{k}_q \cdot \frac{\prod_k^-}{\prod_k^+}$$

In light of this, we must prove that $\sum_{j=0}^{k} (-1)^j\, q^{\binom{j}{2}} \binom{k}{j}_q \left\langle {n-j \atop k-j} \right\rangle_q = q^{k^2} \binom{n-k}{k}_q \cdot \frac{\prod_k^-}{\prod_k^+}$, or

$$\sum_{j=0}^{k} (-1)^{k-j}\, q^{\binom{k-j}{2}} \binom{k}{j}_q \big\langle q^{n-k+j}\,;j \big\rangle_q \big\langle q^k\,;k-j \big\rangle_q = q^{k^2} \big(q^{n-k}\,;k\big)_q \quad . \tag{4.1}$$

To this end, we shall prove a more general identity. Let

$$R_{k,t}^n := \textstyle\sum_{j=0}^{k} (-1)^{k-j}\, q^{\binom{k-j}{2}} \binom{k}{j}_q \big\langle q^{n-k+j}\,;t+j \big\rangle_q \big\langle q^{k+t}\,;k-j \big\rangle_q \quad ; \tag{4.2}$$

we shall prove that

$$R_{k,t}^n = q^{k(k+t)} \big\langle q^{n-k}\,;\,t \big\rangle_q \big(q^{n-k-t}\,;\,k\big)_q \quad . \tag{4.3}$$

Note that, by definition, (4.3) for $t = 0$ yields exactly (4.1), so the latter is just a special case of the former. The proof of (4.3) follows easily by induction on k from the following identity:

$$R^n_{(k+1),t} \;=\; -\,q^k \left(q^{k+t+1}+1\right) R^{n-1}_{k,t} \;+\; R^n_{k,t+1} \qquad (\forall\, k,t,n\,)\,. \tag{4.4}$$

Let us prove now the above identity. We have

$$\begin{aligned}
R^n_{(k+1),t} \;&=\; \textstyle\sum_{j=0}^{k+1} (-1)^{k+1-j}\, q^{\binom{k+1-j}{2}} \binom{k+1}{j}_q \left\langle q^{n-k-1+j}\,;\, t+j\,\right\rangle_q \left\langle q^{k+1+t}\,;\, k+1-j\,\right\rangle_q \;= \\
&= \sum_{j=0}^{k+1} (-1)^{k+1-j}\, q^{\binom{k+1-j}{2}} \left(\binom{k}{j-1}_q + q^j \binom{k}{j}_q\right) \left\langle q^{n-k-1+j}\,;\, t+j\,\right\rangle_q \left\langle q^{k+1+t}\,;\, k+1-j\,\right\rangle_q \;= \\
&= \;-\,q^k \textstyle\sum_{j=0}^{k} (-1)^{k-j}\, q^{\binom{k-j}{2}} \binom{k}{j}_q \left\langle q^{n-k-1+j}\,;\, t+j\,\right\rangle_q \left\langle q^{k+1+t}\,;\, k+1-j\,\right\rangle_q + \\
&\qquad + \textstyle\sum_{j=1}^{k+1} (-1)^{k+1-j}\, q^{\binom{k+1-j}{2}} \binom{k}{j-1}_q \left\langle q^{n-k-1+j}\,;\, t+j\,\right\rangle_q \left\langle q^{k+1+t}\,;\, k+1-j\,\right\rangle_q \;= \\
&= \;-\,q^k \left(q^{1+k+t}+1\right) \textstyle\sum_{j=0}^{k} (-1)^{k-j}\, q^{\binom{k-j}{2}} \binom{k}{j}_q \left\langle q^{(n-1)-k+j}\,;\, t+j\,\right\rangle_q \left\langle q^{k+t}\,;\, k-j\,\right\rangle_q + \\
&\qquad + \textstyle\sum_{j=0}^{k} (-1)^{k-j}\, q^{\binom{k-j}{2}} \binom{k}{j}_q \left\langle q^{n-k+j}\,;\, (t+1)+j\,\right\rangle_q \left\langle q^{k+(t+1)}\,;\, k-j\,\right\rangle_q \;= \\
&= \;-\,q^k \left(q^{k+t+1}+1\right) R^{n-1}_{k,t} \;+\; R^n_{k,t+1} \;, \qquad \text{q.e.d.}
\end{aligned}$$

Now the induction on k to prove (4.3) goes as follows. For all $n \in \mathbb{N}_+$ and all $t \in \mathbb{N}$, the right hand sides of (4.2) and (4.3) are equal, because in that case (4.4) gives

$$\begin{aligned}
&-\left(q^{n-1}+1\right)\cdots\left(q^{nt+1}+1\right)\left(q^{t+1}+1\right) + \left(q^n+1\right)\cdots\left(q^{n-t}+1\right) \;= \\
&= \left(q^{n-1}+1\right)\cdots\left(q^{n-t}+1\right)\left(q^n+1-q^{t+1}-1\right) \;=\; q^{t+1}\left(q^{n-1}+1\right)\cdots\left(q^{n-t}+1\right)\left(q^{n-t-1}-1\right)\,.
\end{aligned}$$

This sets the basis of induction. For the inductive step, assume the r.h.s.'s of (4.2) and (4.3) be equal for all $t \in \mathbb{N}$ and $k', n \in \mathbb{N}_+$ with $k' \leq k$, for some $k \in \mathbb{N}_+$, $k \leq n$. Then

$$\begin{aligned}
R^n_{(k+1),t} \;&=\; -\,q^k \left(q^{k+t+1}+1\right) R^{n-1}_{k,t} + R^n_{k,t+1} \;= \\
&= \;-\,q^k \left(q^{k+t+1}+1\right) q^{k(k+t)} \left\langle q^{n-1-k}\,; t\right\rangle_q \left(q^{n-1-k-t}\,; k\right)_q + \\
&\qquad + q^{k(k+(t+1))} \left\langle q^{n-k};\, t+1\right\rangle_q \left(q^{n-k-t-1}\,; k\right)_q \;= \\
&= -\,q^{k(k+t+1)} \left(\left(q^{k+t+1}+1\right) \left\langle q^{n-1-k}\,; t\right\rangle_q \left(q^{n-1-k-t}\,; k\right)_q + \left\langle q^{n-k}\,; t+1\right\rangle_q \left(q^{n-k-t-1}\,; k\right)_q\right) = \\
&= \;q^{k(k+(t+1))} \left\langle q^{n-1-k}\,; t\right\rangle_q \left(q^{n-1-k-t}\,; k\right)_q \left(q^{n-k}+1-1-q^{k+t+1}\right) \;= \\
&= \;q^{k^2+k\,(t+1)+k+t+1} \left\langle q^{n-1-k}\,; t\right\rangle_q \left(q^{n-1-k-t}\,; k\right)_q \left(q^{n-2(k+1)-t+1}-1\right) \;= \\
&= \;q^{(k+1)\,((k+1)+t)} \left\langle q^{n-(k+1)}\,; t\right\rangle_q \left(q^{n-(k+1)-t}\,; k+1\right)_q
\end{aligned}$$

which gives exactly (4.3) with $k+1$ instead of k. □

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