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# Stein–Malliavin approximations for nonlinear functionals of random eigenfunctions on $\mathbb{S}^d$

Domenico Marinucci *, Maurizia Rossi

*Department of Mathematics, University of Rome Tor Vergata, Via della Ricerca Scientifica, 00133 Roma, Italy*

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ABSTRACT

We investigate Stein–Malliavin approximations for nonlinear functionals of geometric interest for random eigenfunctions on the unit $d$-dimensional sphere $\mathbb{S}^d$, $d \geq 2$. All our results are established in the high energy limit, i.e. as the corresponding eigenvalues diverge. In particular, we prove a quantitative Central Limit Theorem for the excursion volume of Gaussian eigenfunctions; this goal is achieved by means of several results of independent interest, concerning the asymptotic analysis for the variance of moments of Gaussian eigenfunctions, the rates of convergence in various probability metrics for Hermite subordinated processes, and quantitative Central Limit Theorems for arbitrary polynomials of finite order or general, square-integrable, nonlinear transforms. Some related issues were already considered in the literature for the 2-dimensional case $\mathbb{S}^2$; our results are new or improve the existing bounds even in these special circumstances. Proofs are based on the asymptotic analysis of moments of all order for Gegenbauer polynomials, and make extensive use of the recent literature on so-called fourth-moment theorems by Nourdin and Peccati.



* Corresponding author.
*E-mail addresses:* marinucc@mat.uniroma2.it (D. Marinucci), rossim@mat.uniroma2.it (M. Rossi).

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## 1. Introduction

The characterization of the asymptotic behavior in the high energy limit, i.e. as the corresponding eigenvalues diverge, for geometric functionals of random eigenfunctions $f$ on a compact manifold $\mathcal{M}$ is a topic which has recently drawn considerable attention. For instance, several papers have focused on the investigation of nodal sets, i.e. $f^{-1}(0)$, and nodal domains (connected components of the complement $\mathcal{M} \setminus f^{-1}(0)$) in Gaussian setups, see e.g. [35,13,14,7,6].

In particular, much effort has been devoted to the case of the 2-dimensional unit sphere $\mathbb{S}^2$: indeed, the asymptotic behavior of nodal lines and nodal domains for spherical Gaussian eigenfunctions (random spherical harmonics) has been studied in [26,37,38], whereas the area of excursion sets has been considered in [21,22]. Of course, boundary length and excursion area are just two instances of the so-called intrinsic volumes, or Lipschitz–Killing curvatures [1]. In the case of $\mathbb{S}^2$, the family of Lipschitz–Killing curvatures is completed by the Euler–Poincaré characteristic, which has been investigated in [10,11,20]. Most of these papers have considered the computation of asymptotic expected values and variances in the high energy limit; to the best of our knowledge, Central Limit Theorem (CLT) results have only been established for the so-called Defect (i.e. the difference of the measure of the positive and negative regions) in [22] and for the excursion area in [23].

This literature has been largely motivated by applications from Mathematical Physics. In particular, according to Berry's Universality conjecture [5], Gaussian monochromatic waves (similarly to e.g. random spherical harmonics) could model deterministic eigenfunctions on a "generic" manifold with or without boundary; this heuristic has strongly motivated the analysis of nodal sets of the former. On the other hand, it is also well-known that random eigenfunctions are the Fourier components of square integrable isotropic fields on manifolds. Spherical random fields are customarily used to model several data sets in astrophysics and cosmology; the analysis of polynomial transforms or geometric functionals of spherical random eigenfunctions has hence become a major statistical tool for these disciplines. For instance, analytic expectations on the values of geometric functionals can be used for testing the goodness of fit of theoretical models vs observational data (e.g., on Cosmic Microwave Background radiation, see [16,24] or the monograph [19]).

A CLT by itself can often provide little guidance to the actual distribution of random functionals, as it is only an asymptotic result with no information on the speed of convergence to the limiting distribution. More refined results indeed aim at the investigation of the asymptotic behavior for various probability metrics, such as Kolmogorov, Total Variation and Wasserstein distances (to be defined below). In this respect, a major development in the last few years has been provided by the so-called *fourth-moments literature*, which is summarized in the recent monograph [28]. In short, in this rapidly expanding literature it has been shown how to establish sharp bounds on probability distances between multiple stochastic integrals and the standard Gaussian distribution, by means of the analysis of the fourth-moments/fourth cumulants alone. Such results are currently being generalized in several directions, including Poisson processes, free probability, random matrices, Markov subordinators and information theory (see e.g. [3, 15,27,29,31]).

The main purpose of the present paper is to exploit these results to establish *quantitative* Central Limit Theorems for the excursion volume of Gaussian eigenfunctions on $\mathbb{S}^d$, $d \geq 2$. We note that there are already results in the literature giving rates of convergence in CLTs for value distributions of eigenfunctions of the spherical Laplacian, see in particular [25], which investigates however the complementary situation to the one considered here, i.e. the limit for eigenfunctions of fixed degree $\ell$ and increasing dimension $d$.

To achieve our goal, we will provide a number of intermediate results of independent interest, namely the asymptotic analysis for the variance of moments of Gaussian eigenfunctions, the rates of convergence for various probability metrics for so-called Hermite subordinated processes, the analysis of arbitrary polynomials of finite order and square integrable nonlinear transforms. All these results could be useful to attack other problems, for instance quantitative Central Limit Theorems for intrinsic volumes/Lipschitz–Killing curvatures of arbitrary order. A more precise statement of our results and a short outline of the proof is given in Section 2.

## 2. Main result and outline of the proof

### 2.1. Main result

Let us start by fixing some notation. For any two positive sequences $a_n$, $b_n$, we shall write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and $a_n \ll b_n$ or $a_n = O(b_n)$ if the sequence $\frac{a_n}{b_n}$ is bounded; moreover $a_n = o(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. Also, we write as usual $dx$ for the Lebesgue measure on the unit $d$-dimensional sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, so that $\int_{\mathbb{S}^d} dx = \mu_d$ where $\mu_d := \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}$. The triple $(\Omega, \mathscr{F}, \mathbb{P})$ shall denote a probability space and $\mathbb{E}$ shall stand for the expectation w.r.t. $\mathbb{P}$; convergence (resp. equality) in law shall be denoted by $\xrightarrow{\mathcal{L}}$ (resp. $\overset{\mathcal{L}}{=}$) and finally, as usual, $\mathcal{N}(\mu, \sigma^2)$ shall stand for a Gaussian random variable with mean $\mu$ and variance $\sigma^2$.

Now let $\Delta_{\mathbb{S}^d}$ $(d \geq 2)$ denote the usual spherical Laplacian operator on $\mathbb{S}^d$ and $(Y_{\ell,m;d})_{\ell,m}$ the orthonormal system of (real-valued) spherical harmonics, i.e. for $\ell \in \mathbb{N}$ the set of eigenfunctions

$$\Delta_{\mathbb{S}^d} Y_{\ell,m;d} = -\ell(\ell + d - 1) Y_{\ell,m;d}, \quad m = 1, 2, \ldots, n_{\ell;d}.$$

As well-known, the spherical harmonics $(Y_{\ell,m;d})_{m=1}^{n_{\ell;d}}$ represent a family of linearly independent homogeneous polynomials of degree $\ell$ in $d+1$ variables restricted to $\mathbb{S}^d$ of size

$$n_{\ell;d} := \frac{2\ell + d - 1}{\ell} \binom{\ell + d - 2}{\ell - 1} \sim \frac{2}{(d-1)!} \ell^{d-1}, \quad \text{as } \ell \to +\infty,$$

see e.g. [2] for further details. It is then customary to construct, for $\ell \in \mathbb{N}$, the random eigenfunction $T_\ell$ on $\mathbb{S}^d$ by taking

$$T_\ell(x) := \sum_{m=1}^{n_{\ell;d}} a_{\ell,m} Y_{\ell,m;d}(x), \quad x \in \mathbb{S}^d, \tag{2.1}$$

with the coefficients $(a_{\ell,m})_{m=1}^{n_{\ell;d}}$ Gaussian i.i.d. random variables, satisfying the relation

$$\mathbb{E}[a_{\ell,m} a_{\ell,m'}] = \frac{\mu_d}{n_{\ell;d}} \delta_m^{m'},$$

where $\delta_a^b$ denotes the Kronecker delta function and $\mu_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}$ the hypersurface volume of $\mathbb{S}^d$, as we wrote before.

It is then readily checked that $(T_\ell)_{\ell\in\mathbb{N}}$ represents a sequence of isotropic, zero-mean Gaussian random fields on $\mathbb{S}^d$, that is, for every fixed $\ell$ we have a collection of random variables $(T_\ell(x))_{x\in\mathbb{S}^d}$ indexed by the points of $\mathbb{S}^d$, such that the map

$$T_\ell : \Omega \times \mathbb{S}^d \to \mathbb{R}; \qquad (\omega, x) \mapsto T_\ell(\omega, x)$$

is $\mathscr{F} \otimes \mathscr{B}(\mathbb{S}^d)$-measurable, where $\mathscr{B}(\mathbb{S}^d)$ denotes the Borel $\sigma$-field of $\mathbb{S}^d$. Isotropy of $T_\ell$ simply means that the probability laws of the two random fields $T_\ell(\cdot)$ and $T_\ell^g(\cdot) := T_\ell(g\cdot)$ are equal (in the sense of finite dimensional distributions) for every $g \in SO(d+1)$.

It is also well-known that every Gaussian and isotropic random field $T$ on $\mathbb{S}^d$ satisfies in the $L^2(\Omega \times \mathbb{S}^d)$-sense the spectral representation (see [17,19] and also [1,4])

$$T(x) = \sum_{\ell=1}^{\infty} c_\ell T_\ell(x), \quad x \in \mathbb{S}^d,$$

where $\mathbb{E}\left[T^2\right] = \sum_{\ell=1}^\infty c_\ell^2 < \infty$; hence the spherical Gaussian eigenfunctions $(T_\ell)_{\ell\in\mathbb{N}}$ can be viewed as the Fourier components of the field $T$ (note that w.l.o.g. we are implicitly assuming that $T$ is centered). Equivalently these random eigenfunctions (2.1) could be defined by their covariance function, which equals

$$\mathbb{E}[T_\ell(x)T_\ell(y)] = G_{\ell;d}(\cos d(x,y)), \quad x, y \in \mathbb{S}^d. \tag{2.2}$$

Here and in the sequel, $d(x,y)$ is the spherical distance between $x, y \in \mathbb{S}^d$, and $G_{\ell;d} : [-1,1] \to \mathbb{R}$ is the $\ell$-th normalized Gegenbauer polynomial, i.e.

$$G_{\ell;d} \equiv \frac{P_\ell^{(\frac{d}{2}-1,\frac{d}{2}-1)}}{\binom{\ell+\frac{d}{2}-1}{\ell}},$$

where $P_\ell^{(\alpha,\beta)}$ are the Jacobi polynomials; throughout the paper therefore $G_{\ell;d}(1) = 1$. As a special case, for $d = 2$, it equals $G_{\ell;2} \equiv P_\ell$, the degree-$\ell$ Legendre polynomial. Remark that the Jacobi polynomials $P_\ell^{(\alpha,\beta)}$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function $w(t) = (1-t)^\alpha(1+t)^\beta$ and satisfy $P_\ell^{(\alpha,\beta)}(1) = \binom{\ell+\alpha}{\ell}$, see e.g. [36] for more details.

In order to formulate our main result, let us introduce the usual Wasserstein distance $d_W$ between random variables $Z$, $N$:

$$d_W(Z, N) = \sup_{h\in \mathrm{Lip}(1)} |\mathbb{E}[h(Z)] - \mathbb{E}[h(N)]| , \tag{2.3}$$

where Lip(1) is the set of Lipschitz functions whose Lipschitz constant equals 1.

Recall now the notion of the excursion volume of $T_\ell$, which for any fixed $z \in \mathbb{R}$ can be defined as

$$S_\ell(z) := S_\ell(\mathbb{I}(\cdot > z)) = \int_{\mathbb{S}^d} \mathbb{I}(T_\ell(x) > z)\, dx, \tag{2.4}$$

where $\mathbb{I}(\cdot > z)$ denotes the indicator function of the interval $(z,\infty)$; note that $\mathbb{E}[S_\ell(z)] = \mu_d(1 - \Phi(z))$, where $\Phi(z)$ is the cdf of the standard Gaussian law. The variance of this excursion volume will be shown below to have the following asymptotic behavior (as $\ell \to +\infty$)

$$\mathrm{Var}[S_\ell(z)] = \frac{z^2\phi(z)^2}{2} \frac{\mu_d^2}{n_{\ell,d}} + O(\ell^{-d}), \tag{2.5}$$

where $\phi$ denotes the standard Gaussian density and, as above, $\mu_d$ is the volume of the spherical hypersurface and $n_{\ell,d} \sim \frac{2}{(d-1)!}\ell^{d-1}$. Note that the variance is of order smaller than $\ell^{-(d-1)}$ if and only if $z = 0$. An analogous cancellation has been earlier noted for other geometric functionals of excursion sets of random eigenfunctions at zero level, see for instance [5,38] for the length of nodal lines, and [9] for the variance of extrema, in both cases in dimension 2. We leave for future research a more general investigation on this issue.

The main result of this paper is then as follows.

**Theorem 2.1.** *The excursion volume $S_\ell(z)$ in (2.4) of Gaussian eigenfunctions $T_\ell$ on $\mathbb{S}^d$, $d \geq 2$, satisfies a quantitative CLT as $\ell \to +\infty$, with rate of convergence in the Wasserstein distance given by, for $z \neq 0$*

$$d_W\left(\frac{S_\ell(z) - \mu_d(1 - \Phi(z))}{\sqrt{\mathrm{Var}[S_\ell(z)]}}, \mathcal{N}(0, 1)\right) = O\left(\ell^{-1/2}\right).$$

An outline of the main steps and auxiliary results to prove this theorem is given in the following subsection.

### 2.2. Outline of the proof

The first tool to investigate quantitative CLTs for the excursion volume of Gaussian eigenfunctions on $\mathbb{S}^d$ (compare, for $d = 2$, [23]) is to study the asymptotic behavior, as $\ell \to \infty$, of the random variables $h_{\ell;q,d}$ defined for $\ell = 1, 2, \dots$ and $q = 0, 1, \dots$ as

$$h_{\ell;q,d} = \int_{\mathbb{S}^d} H_q(T_\ell(x))\, dx, \tag{2.6}$$

where $H_q$ represent the family of Hermite polynomials [28,32]. The latter are defined as usual by $H_0 \equiv 1$ and for $q \geq 1$

$$H_q(t) = (-1)^q \mathrm{e}^{\frac{t^2}{2}} \frac{d^q}{dt^q} \mathrm{e}^{-\frac{t^2}{2}}, \quad t \in \mathbb{R}. \tag{2.7}$$

The rationale to investigate these sequences is the fact that the excursion volume, and more generally any square integrable transform of $(T_\ell)_\ell$, admits the Wiener–Ito chaos decomposition (see Section 3 and for more details e.g. [28, §2.2]), i.e. a series expansion in the $L^2(\Omega)$-sense of the form

$$S_\ell(z) = \sum_{q=0}^{+\infty} \frac{J_q(z)}{q!} h_{\ell;q,d}, \tag{2.8}$$

where $J_0(z) = 1 - \Phi(z)$ and for $q \geq 1$, $J_q(z) = (-1)^{q+1}\phi^{(q-1)}(z)$, $\Phi$ and $\phi$ denoting again respectively the cdf and the density of the standard Gaussian law.

The main idea in our argument will then be to establish first a CLT for each of the summands in the series, and then to deduce from this a CLT for the excursion volume. The starting point will then be the analysis of the asymptotic variances for $h_{\ell;q,d}$, as $\ell \to +\infty$.

To this aim, note first that, for all $d$

$$h_{\ell;0,d} = \mu_d, \qquad h_{\ell;1,d} = 0$$

a.s., and therefore it is enough to restrict our discussion to $q \geq 2$. Moreover $\mathbb{E}[h_{\ell;q,d}] = 0$ and

$$\operatorname{Var}[h_{\ell;q,d}] = q!\mu_d\mu_{d-1} \int_0^\pi G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta \tag{2.9}$$

(see Appendix A for more details). Gegenbauer polynomials satisfy the symmetry relationships

$$G_{\ell;d}(t) = (-1)^\ell G_{\ell;d}(-t),$$

whence the r.h.s. integral in (2.9) vanishes identically when both $\ell$ and $q$ are odd; therefore in these cases $h_{\ell;q,d} = 0$ a.s. For the remaining cases we have

$$\mathrm{Var}[h_{\ell;q,d}] = 2q!\mu_d\mu_{d-1}\int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^q(\sin\vartheta)^{d-1}\,d\vartheta. \tag{2.10}$$

We have hence the following asymptotic result for these variances, whose proof is given in Appendix A.

**Proposition 2.2.** *As $\ell\to\infty$, for $q, d\geq 3$,*

$$\int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^q(\sin\vartheta)^{d-1}\,d\vartheta = \frac{c_{q;d}}{\ell^d}(1+o_{q;d}(1)). \tag{2.11}$$

*The constants $c_{q;d}$ are given by the formula*

$$c_{q;d} = \left(2^{\frac{d}{2}-1}\left(\frac{d}{2}-1\right)!\right)^q \int_0^{+\infty} J_{\frac{d}{2}-1}(\psi)^q\psi^{-q(\frac{d}{2}-1)+d-1}d\psi, \tag{2.12}$$

*where $J_{\frac{d}{2}-1}$ is the Bessel function of order $\frac{d}{2}-1$. The r.h.s. integral in (2.12) is absolutely convergent for any pair $(d,q)\neq(3,3)$ and conditionally convergent for $d=q=3$.*

It is well known that for $d\geq 2$, the second moment of the Gegenbauer polynomials is given by

$$\int_0^{\pi} G_{\ell;d}(\cos\vartheta)^2(\sin\vartheta)^{d-1}\,d\vartheta = \frac{\mu_d}{\mu_{d-1}\,n_{\ell;d}}, \tag{2.13}$$

whence

$$\mathrm{Var}[h_{\ell;2,d}] = 2\frac{\mu_d^2}{n_{\ell;d}} \sim 4\mu_d\mu_{d-1}\frac{c_{2;d}}{\ell^{d-1}}, \quad \text{as } \ell\to+\infty, \tag{2.14}$$

where $c_{2;d} := \frac{(d-1)!\mu_d}{4\mu_{d-1}}$. For $d=2$ and every $q$, the asymptotic behavior of these integrals was resolved in [22]. In particular, it was shown that for $q=3$ or $q\geq 5$

$$\mathrm{Var}[h_{\ell;q,2}] = (4\pi)^2 q!\int_0^{\frac{\pi}{2}} P_\ell(\cos\vartheta)^q\sin\vartheta\,d\vartheta = (4\pi)^2q!\frac{c_{q;2}}{\ell^2}(1+o_q(1)), \tag{2.15}$$

where

$$c_{q;2} = \int_0^{+\infty} J_0(\psi)^q \psi \, d\psi, \tag{2.16}$$

$J_0$ being the Bessel function of order 0 and the above integral being absolutely convergent for $q \geq 5$ and conditionally convergent for $q = 3$. On the other hand, for $q = 4$, as $\ell \to \infty$,

$$\mathrm{Var}[h_{\ell;4,2}] \sim 24^2 \frac{\log \ell}{\ell^2}. \tag{2.17}$$

Clearly for any $d, q \geq 2$, the constants $c_{q;d}$ are nonnegative and it is obvious that $c_{q;d} > 0$ for all even $q$. We conjecture that this strict inequality holds for every $(d, q)$, but leave this issue as an open question for future research; also, in view of the previous discussion on the symmetry properties of Gegenbauer polynomials, to simplify the discussion in the sequel we restrict ourselves to even multipoles $\ell$.

As argued earlier, the following step is to establish quantitative CLTs for $h_{\ell;q,d}$ (see Section 4) in various probability metrics (2.3), (2.18) and (2.19). Here the crucial point to stress is that the Gaussian eigenfunctions $(T_\ell)_\ell$ can be always expressed as isonormal Gaussian processes, i.e., as stochastic integrals with respect to a suitably defined Gaussian white noise measure on $\mathbb{S}^d$. As a consequence, the random sequences $h_{\ell;q,d}$ can themselves be represented as multiple Wiener–Ito integrals, and therefore fall inside the domain of quantitative CLTs by means of the Nourdin–Peccati approach. It is thus sufficient to investigate the so-called circular components of their normalized fourth-order cumulants (Proposition 3.1 below) to establish the following Proposition 2.3.

We first need to introduce some more notation. Denote the usual Kolmogorov $d_K$, Total Variation $d_{TV}$ probability distances between the random variables $Z$, $N$

$$d_K(Z, N) = \sup_{z \in \mathbb{R}} \left| \mathbb{P}(Z \leq z) - \mathbb{P}(N \leq z) \right|, \tag{2.18}$$

$$d_{TV}(Z, N) = \sup_{A \in \mathscr{B}(\mathbb{R})} \left| \mathbb{P}(Z \in A) - \mathbb{P}(N \in A) \right|, \tag{2.19}$$

where $\mathscr{B}(\mathbb{R})$ denotes the Borel $\sigma$-field of $\mathbb{R}$. Recall the definition (2.3) of the Wasserstein distance $d_W$.

**Proposition 2.3.** *For all $d, q \geq 2$, $d_{\mathcal{D}} = d_{TV}, d_W, d_K$ we have*

$$d_{\mathcal{D}}\left( \frac{h_{2\ell;q,d}}{\sqrt{\mathrm{Var}[h_{2\ell;q,d}]}}, \mathcal{N}(0,1) \right) = O(R(\ell;q,d)),$$

*where for $d = 2$*

$$R(\ell;q,2) = \begin{cases} \ell^{-1/2}, & q = 2,3, \\ (\log \ell)^{-1}, & q = 4, \\ (\log \ell)\ell^{-1/4}, & q = 5,6, \\ \ell^{-1/4}, & q \geq 7; \end{cases} \tag{2.20}$$

*and for* $d \geq 3$

$$R(\ell; q, d) = \begin{cases} \ell^{-\left(\frac{d-1}{2}\right)}, & q = 2, \\ \ell^{-\left(\frac{d-5}{4}\right)}, & q = 3, \\ \ell^{-\left(\frac{d-3}{4}\right)}, & q = 4, \\ \ell^{-\left(\frac{d-1}{4}\right)}, & q \geq 5. \end{cases} \tag{2.21}$$

The following corollary is hence immediate.

**Corollary 2.4.** *For all* $q$ *such that* $(d, q) \neq (3, 3), (3, 4), (4, 3), (5, 3)$ *and* $c_{q;d} > 0$, $d \geq 2$,

$$\frac{h_{2\ell;q,d}}{\sqrt{\mathrm{Var}[h_{2\ell;q,d}]}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1), \quad \text{as } \ell \to +\infty. \tag{2.22}$$

**Remark 2.5.** For $d = 2$, the CLT in (2.22) was already provided by [23]; nevertheless Theorem 2.3 improves the existing bounds on the speed of convergence to the asymptotic Gaussian distribution. More precisely, for $d = 2$, $q = 2, 3, 4$ the same rate of convergence as in (2.20) was given in their Proposition 3.4; however for arbitrary $q$ the total variation rate was only shown to satisfy (up to logarithmic terms) $d_{TV} = O(\ell^{-\delta_q})$, where $\delta_4 = \frac{1}{10}$, $\delta_5 = \frac{1}{7}$, and $\delta_q = \frac{q-6}{4q-6} < \frac{1}{4}$ for $q \geq 7$.

**Remark 2.6.** The cases not included in Corollary 2.4 correspond to the pairs where $q = 4$ and $d = 3$, or $q = 3$ and $d = 3, 4, 5$; in these circumstances the bounds we establish on fourth-order cumulants in Proposition 4.3 are not sufficient to ensure that the CLT holds. Most probably, these four special cases can be dealt with ad hoc arguments based on the explicit evaluations of multiple integrals of spherical harmonics by means of so-called Clebsch–Gordan coefficients, following the steps of Lemma 3.3 in [23], see also [18,19]. Such computations, however, seem of limited interest for the present paper, and we therefore omit the investigation of these special cases for brevity's sake.

As briefly anticipated earlier in this subsection, the random variables $h_{\ell;q,d}$ defined in (2.6) are the basic building blocks for the analysis of any square integrable nonlinear transforms of Gaussian eigenfunctions on $\mathbb{S}^d$. Indeed, let us first consider generic polynomial functionals of the form

$$Z_\ell = \sum_{q=0}^{Q} b_q \int_{\mathbb{S}^d} T_\ell(x)^q \, dx, \quad Q \in \mathbb{N}, \; b_q \in \mathbb{R}, \tag{2.23}$$

which include, for instance, the so-called polyspectra (see e.g. [19, p. 148]) of isotropic random fields defined on $\mathbb{S}^d$. Note

$$Z_\ell = \sum_{q=0}^{Q} \beta_q h_{2\ell;q,d} \tag{2.24}$$

for some $\beta_q \in \mathbb{R}$. It is easy to establish CLTs for generic polynomials (2.24) from convergence results on $h_{2\ell;q,d}$, see e.g. [33]. It is more difficult to investigate the speed of convergence in the CLT in terms of the probability metrics we introduced earlier; indeed, in Section 5 we establish the following.

**Proposition 2.7.** *As $\ell \to \infty$,*

$$d_{\mathcal{D}}\left(\frac{Z_\ell - \mathbb{E}[Z_\ell]}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O(R(Z_\ell; d)),$$

*where $d_{\mathcal{D}} = d_{TV}, d_W, d_K$ and for $d \geq 2$*

$$R(Z_\ell; d) = \begin{cases} \ell^{-\left(\frac{d-1}{2}\right)} & \text{if } \beta_2 \neq 0, \\ \max_{q=3,\dots,Q:\ \beta_q, c_{q;d} \neq 0} R(\ell; q, d) & \text{if } \beta_2 = 0. \end{cases}$$

The previous results can be summarized as follows: for polynomials of *Hermite rank* 2, i.e. such that $\beta_2 \neq 0$ (more details later on the notion of Hermite rank) the asymptotic behavior of $Z_\ell$ is dominated by the single term $h_{\ell;2,d}$, whose variance is of order $\ell^{-(d-1)}$ rather than $\ell^{-d}$ as for the other terms. On the other hand, when $\beta_2 = 0$, the convergence rate to the asymptotic Gaussian distribution for a generic polynomial is the slowest among the rates for the Hermite components into which $Z_\ell$ can be decomposed, i.e. the terms $\beta_q h_{2\ell;q,d}$ in (2.24).

The fact that the bound for generic polynomials is of the same order as for the Hermite case (and not slower) is indeed rather unexpected; it can be shown to be due to the cancellation of some cross-product terms, which are dominant in the general Nourdin–Peccati framework, while they vanish for spherical eigenfunctions of arbitrary dimension (see (5.2) and Remark 5.1). An inspection of our proof will reveal that this result is a by-product of the orthogonality of eigenfunctions corresponding to different eigenvalues; it is plausible that similar ideas may be exploited in many related circumstances, for instance random eigenfunction on generic compact manifolds.

Proposition 2.7 shows that the asymptotic behavior of arbitrary polynomials of Hermite rank 2 is of particularly simple nature. Our result below will show that this feature holds in much greater generality, at least as far as the Wasserstein distance $d_W$ is concerned. Indeed, we shall consider the case of functionals of the form

$$S_\ell(M) = \int_{\mathbb{S}^d} M(T_\ell(x))\, dx, \tag{2.25}$$

where $M : \mathbb{R} \to \mathbb{R}$ is any square integrable, measurable nonlinear function. As briefly anticipated above, it is well known that for such transforms the following chaos expansion holds (in the $L^2(\Omega)$-sense)

$$M(T_\ell) = \sum_{q=0}^{\infty} \frac{J_q(M)}{q!} H_q(T_\ell), \quad \mathbb{E}[M(T_\ell)^2] < \infty, \quad J_q(M) := \mathbb{E}[M(T_\ell)H_q(T_\ell)]. \tag{2.26}$$

Therefore the asymptotic analysis, as $\ell \to \infty$, of $S_\ell(M)$ in (2.25) directly follows from the Gaussian approximation for $h_{\ell;q,d}$ and their polynomial transforms $Z_\ell$. More precisely, in Section 6 we prove the following result.

**Proposition 2.8.** *Let $Z \sim \mathcal{N}(0,1)$. For functions $M$ in (2.25) such that $\mathbb{E}[M(Z)H_2(Z)] = J_2(M) \neq 0$, we have*

$$d_W\left(\frac{S_{2\ell}(M) - \mathbb{E}[S_{2\ell}(M)]}{\sqrt{\mathrm{Var}[S_{2\ell}(M)]}}, \mathcal{N}(0,1)\right) = O(\ell^{-1/2}), \quad \text{as } \ell \to \infty, \tag{2.27}$$

*in particular*

$$\frac{S_{2\ell}(M) - \mathbb{E}[S_{2\ell}(M)]}{\sqrt{\mathrm{Var}[S_{2\ell}(M)]}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1). \tag{2.28}$$

Proposition 2.8 provides a Breuer–Major like result on nonlinear functionals, in the high-frequency limit (compare for instance [30]). While the CLT in (2.28) is somewhat expected, the square-root speed of convergence (2.27) to the limiting distribution may be considered quite remarkable; it is mainly due to some specific features in the chaos expansion of Gaussian eigenfunctions, which is dominated by a single term at $q = 2$. Note that the function $M$ need not be smooth in any meaningful sense; indeed, as we explained in Section 2.1, our main motivating rationale here is the analysis of the asymptotic behavior of the excursion volume in (2.4) $S_\ell(z) = S_\ell(M)$, where $M(\cdot) = M_z(\cdot) = \mathbb{I}(\cdot > z)$ is again the indicator function of the interval $(z, +\infty)$. An application of Proposition 2.8 (compare (2.8) to (2.26)) provides a quantitative CLT for $S_\ell(z)$, $z \neq 0$, thus completing the proof of our main result.

The plan of this paper is as follows: we collect in Section 3 some background results on isonormal processes and quantitative CLT for their nonlinear transforms; in Section 4 we establish the quantitative CLT for the sequences $h_{\ell;q,d,}$, while Section 5 extends these results to generic finite-order polynomials. The results for general nonlinear transforms and excursion volumes are given in Section 6; most technical proofs and (hard) estimates, including in particular evaluations of asymptotic variances, are collected in Appendix A.

## 3. Background

In a number of recent papers summarized in the monograph [28], a beautiful connection has been established between Malliavin calculus and the so-called Stein method to prove Berry–Esseen bounds and quantitative CLTs on functionals of Gaussian subordinated random fields. In this section, we first briefly review some notation and the main results in this area, which we shall deeply exploit in the sequel of the paper.

### *3.1. Stein–Malliavin normal approximations*

Let us consider the measure space $(X, \mathcal{X}, \mu)$, where $X$ is a Polish space, $\mathcal{X}$ is the $\sigma$-field on $X$ and $\mu$ is a positive, $\sigma$-finite and non-atomic measure on $(X, \mathcal{X})$. Denote $H = L^2(X, \mathcal{X}, \mu)$ the real (separable) Hilbert space of square integrable functions on $X$ w.r.t. $\mu$, with inner product $\langle f, g\rangle_H = \int_X f(x)g(x)\,d\mu(x)$. Let us recall the construction of an isonormal Gaussian field on $H$. First consider a Gaussian white noise on $X$, i.e. a centered Gaussian family $W$

$$W = \{W(A):\ A \in \mathcal{X},\ \mu(A) < +\infty\}$$

such that for $A, B \in \mathcal{X}$ of finite measure, we have

$$\mathbb{E}[W(A)W(B)] = \int_X \mathbb{I}(A \cap B)\,d\mu.$$

We define a Gaussian random field $T$ on $H$ as follows. For each $f \in H$, let

$$T(f) = \int_X f(x)\,dW(x), \tag{3.1}$$

i.e. the Wiener–Ito integral of $f$ with respect to $W$. The random field $T$ is the isonormal Gaussian field on $H$; indeed

$$\mathrm{Cov}(T(f), T(g)) = \langle f, g\rangle_H.$$

Let us recall now the notion of Wiener chaoses. Define the space of constants $C_0 := \mathbb{R} \subseteq L^2(\Omega)$, and for $q \ge 1$, let $C_q$ be the closure in $L^2(\Omega)$ of the linear subspace generated by random variables of the form

$$H_q(T(f)), \quad f \in H, \quad \|f\|_H = 1,$$

where $H_q$ is the $q$-th Hermite polynomial (2.7). $C_q$ is called the $q$-th Wiener chaos. The following, well-known property will be useful in the sequel: let $Z_1, Z_2 \sim \mathcal{N}(0, 1)$ be jointly Gaussian; then, for all $q_1, q_2 \ge 0$

$$\mathbb{E}[H_{q_1}(Z_1)H_{q_2}(Z_2)] = q_1!\mathbb{E}[Z_1 Z_2]^{q_1}\,\delta_{q_2}^{q_1}. \tag{3.2}$$

Moreover the following chaotic Wiener–Ito expansion holds:

$$L^2(\Omega) = \bigoplus_{q=0}^{+\infty} C_q,$$

the above sum being orthogonal from (3.2). Equivalently, each random variable $F \in L^2(\Omega)$ admits a unique decomposition in the $L^2(\Omega)$-sense of the form

$$F = \sum_{q=0}^{\infty} J_q(F), \tag{3.3}$$

where $J_q : L^2(\Omega) \to C_q$ is the orthogonal projection operator. Remark that $J_0(F) = \mathbb{E}[F]$.

We denote by $H^{\otimes q}$ and $H^{\odot q}$ the $q$-th tensor product and the $q$-th symmetric tensor product of $H$ respectively. In particular $H^{\otimes q} = L^2(X^q, \mathcal{X}^q, \mu^q)$ and $H^{\odot q} = L^2_s(X^q, \mathcal{X}^q, \mu^q)$ where by $L^2_s$ we mean the square integrable and symmetric functions. Note that for $(x_1, x_2, \ldots, x_q) \in X^q$ and $f \in H$, we have

$$f^{\otimes q}(x_1, x_2, \ldots, x_q) = f(x_1)f(x_2)\ldots f(x_q).$$

Now for $q \geq 1$ define the map $I_q$ as

$$I_q(f^{\otimes q}) := H_q(T(f)), \quad f \in H, \tag{3.4}$$

which can be extended to a linear isometry between $H^{\odot q}$ equipped with the modified norm $\sqrt{q!}\|\cdot\|_{H^{\odot q}}$ and the $q$-th Wiener chaos $C_q$. Moreover for $q = 0$, set $I_0(c) = c \in \mathbb{R}$. Under the new notation the equality (3.3) becomes

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{3.5}$$

where $f_0 = \mathbb{E}[F]$ and for $q \geq 1$, the kernels $f_q \in H^{\odot q}$ are uniquely determined.

In our case, it is well known that for $h \in H^{\odot q}$, $I_q(h)$ coincides with the multiple Wiener–Ito integral of $h$ with respect to the Gaussian measure $W$, i.e.

$$I_q(h) = \int_{X^q} h(x_1, x_2, \ldots, x_q)\, dW(x_1)dW(x_2)\ldots dW(x_q) \tag{3.6}$$

and, in words, $F$ in (3.5) can be seen as a series of (multiple) stochastic integrals.

For every $p, q \geq 1$, $f \in H^{\otimes p}, g \in H^{\otimes q}$ and $r = 1, 2, \ldots, p \wedge q$, the so-called *contraction* of $f$ and $g$ of order $r$ is the element $f \otimes_r g \in H^{\otimes p+q-2r}$ defined as

$$
\begin{aligned}
&(f \otimes_r g)(x_1, \dots, x_{p+q-2r}) \\
&\quad = \int_{X^r} f(x_1, \dots, x_{p-r}, y_1, \dots, y_r) g(x_{p-r+1}, \dots, x_{p+q-2r}, y_1, \dots, y_r)\, d\mu(y_1) \dots d\mu(y_r).
\end{aligned}
\tag{3.7}
$$

For $p = q = r$, we have $f \otimes_r g = \langle f, g\rangle_{H^{\otimes r}}$ and for $r = 0$, $f \otimes_0 g = f \otimes g$. Denote by $f \widetilde{\otimes}_r g$ the canonical symmetrization of $f \otimes_r g$. The following multiplication formula is well-known; for $p, q = 1, 2, \dots$, $f \in H^{\odot p}$, $g \in H^{\odot q}$, we have

$$
I_p(f) I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g).
$$

We now briefly recall some basic Malliavin calculus formulas for this setting. For $q, r \ge 1$, the $r$-th Malliavin derivative of a random variable $F = I_q(f) \in C_q$ where $f \in H^{\odot q}$, can be identified as the element $D^r F : \Omega \to H^{\odot r}$ given by

$$
D^r F = \frac{q!}{(q-r)!} I_{q-r}(f), \tag{3.8}
$$

for $r \le q$, and $D^r F = 0$ for $r > q$. So that, the $r$-th Malliavin derivative of the random variable $F$ in (3.5) could be written as

$$
D^r F = \sum_{q=r}^{+\infty} \frac{q!}{(q-r)!} I_{q-r}(f_q).
$$

For simplicity of notation, we shall write $D$ instead of $D^1$. We say that $F$ as in (3.5) belongs to $\mathbb{D}^{r,q}$ if

$$
\|F\|_{\mathbb{D}^{r,q}} := \left(\mathbb{E}[|F|^q] + \dots + \mathbb{E}[\|D^r F\|^q_{H^{\odot r}}]\right)^{\frac{1}{q}} < +\infty;
$$

it is easy to check that $F \in \mathbb{D}^{1,2}$ if and only if

$$
\mathbb{E}[\|DF\|^2_H] = \sum_{q=1}^{\infty} q \|J_q(F)\|^2_{L^2(\Omega)} < +\infty.
$$

We need to introduce also the generator of the Ornstein–Uhlenbeck semigroup, defined as

$$
L = -\sum_{q=0}^{\infty} q J_q,
$$

where $J_q$ is the orthogonal projection operator on $C_q$, as in (3.3). The domain of $L$ is $\mathbb{D}^{2,2}$, equivalently the space of Gaussian subordinated random variables $F$ such that

$$\sum_{q=1}^{+\infty} q^2 \|J_q(F)\|^2_{L^2(\Omega)} < +\infty.$$

The pseudo-inverse operator of $L$ is defined as

$$L^{-1} = -\sum_{q=1}^{\infty} \frac{1}{q} J_q$$

and satisfies for each $F \in L^2(\Omega)$

$$LL^{-1}F = F - \mathbb{E}[F].$$

The connection between stochastic calculus and probability metrics is summarized in the following celebrated result (see e.g. [28, Theorem 5.1.3]), which will provide the basis for most of our results to follow.

**Proposition 3.1.** *Let $F \in \mathbb{D}^{1,2}$ such that $\mathbb{E}[F] = 0$, $\mathbb{E}[F^2] = \sigma^2 < +\infty$. Then we have*

$$d_W(F, \mathcal{N}(0,1)) \le \sqrt{\frac{2}{\sigma^2 \pi}} \mathbb{E}[|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H|].$$

*Also, assuming in addition that $F$ has a density*

$$d_{TV}(F, \mathcal{N}(0,1)) \le \frac{2}{\sigma^2} \mathbb{E}[|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H|],$$

$$d_K(F, \mathcal{N}(0,1)) \le \frac{1}{\sigma^2} \mathbb{E}[|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H|].$$

*Moreover if $F \in \mathbb{D}^{1,4}$, we have also*

$$\mathbb{E}[|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H|] \le \sqrt{\mathrm{Var}[\langle DF, -DL^{-1}F\rangle_H]}.$$

Furthermore, in the special case where $F = I_q(f)$ for $f \in H^{\odot q}$, then from [28, Theorem 5.2.6]

$$\mathbb{E}[|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H|] \le \sqrt{\frac{1}{q^2} \sum_{r=1}^{q-1} r^2 r!^2 \binom{q}{r}^4 (2q-2r)! \|f \widetilde{\otimes}_r f\|^2_{H^{\otimes 2q-2r}}}. \tag{3.9}$$

Note that in (3.9) we can replace $\|f \widetilde{\otimes}_r f\|^2_{H^{\otimes 2q-2r}}$ with the norm of the unsymmetryzed contraction $\|f \otimes_r f\|^2_{H^{\otimes 2q-2r}}$ for the upper bound, because $\|f \widetilde{\otimes}_r f\|^2_{H^{\otimes 2q-2r}} \le \|f \otimes_r f\|^2_{H^{\otimes 2q-2r}}$ by the triangular inequality.

### 3.2. Polynomial transforms in Wiener chaoses

As mentioned earlier in Section 2, we shall be concerned first with random variables $h_{\ell;q,d}$, $\ell \geq 1$, $q, d \geq 2$

$$h_{\ell;q,d} = \int_{\mathbb{S}^d} H_q(T_\ell(x))\, dx,$$

and their (finite) linear combinations

$$Z_\ell = \sum_{q=2}^{Q} \beta_q h_{\ell;q,d}, \quad \beta_q \in \mathbb{R},\ Q \in \mathbb{N}. \tag{3.10}$$

Our first objective is to represent (3.10) as a (finite) sum of (multiple) stochastic integrals as in (3.5), in order to apply the results recalled in Section 3.1. More explicitly, we shall first provide the isonormal representation (3.1) on $L^2(\mathbb{S}^d)$ for the Gaussian random eigenfunctions $T_\ell$, $\ell \geq 1$, i.e., we shall show that the following identity in law holds:

$$T_\ell(x) \stackrel{\mathcal{L}}{=} \int_{\mathbb{S}^d} \sqrt{\frac{n_{\ell;d}}{\mu_d}} G_{\ell;d}(\cos d(x,y))\, dW(y), \quad x \in \mathbb{S}^d,$$

where $W$ is a Gaussian white noise on $\mathbb{S}^d$. To compare with (3.1), $T_\ell(x) = T(f_x)$, where $T$ is the isonormal Gaussian field on $L^2(\mathbb{S}^d)$ and $f_x(\cdot) := \sqrt{\frac{n_{\ell;d}}{\mu_d}} G_{\ell;d}(\cos d(x,\cdot))$. Moreover we have immediately that

$$\mathbb{E}\left[\int_{\mathbb{S}^d} \sqrt{\frac{n_{\ell;d}}{\mu_d}} G_{\ell;d}(\cos d(x,y))\, dW(y)\right] = 0,$$

and by the reproducing formula for Gegenbauer polynomials [36]

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{S}^d} \sqrt{\frac{n_{\ell;d}}{\mu_d}} G_{\ell;d}(\cos d(x_1,y_1))\, dW(y_1) \int_{\mathbb{S}^d} \sqrt{\frac{n_{\ell;d}}{\mu_d}} G_{\ell;d}(\cos d(x_2,y_2))\, dW(y_2)\right] \\
&= \frac{n_{\ell;d}}{\mu_d} \int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1,y)) G_{\ell;d}(\cos d(x_2,y)) dy = G_{\ell;d}(\cos d(x_1,x_2)).
\end{aligned}$$

Note that by (3.4), we also have

$$\begin{aligned}
H_q(T_\ell(x)) &= I_q(f_x^{\otimes q}) \\
&= \int_{(\mathbb{S}^d)^q} \left(\frac{n_{\ell;d}}{\mu_d}\right)^{q/2} G_{\ell;d}(\cos d(x,y_1)) \dots G_{\ell;d}(\cos d(x,y_q))\, dW(y_1) \dots dW(y_q),
\end{aligned}$$

so that

$$h_{\ell;q,d} \stackrel{\mathcal{L}}{=} \int\limits_{(\mathbb{S}^d)^q} g_{\ell;q}(y_1,\dots,y_q)\, dW(y_1)\dots dW(y_q),$$

where

$$g_{\ell;q}(y_1,\dots,y_q) := \int\limits_{\mathbb{S}^d} \left(\frac{n_{\ell;d}}{\mu_d}\right)^{q/2} G_{\ell;d}(\cos d(x,y_1))\dots G_{\ell;d}(\cos d(x,y_q))\, dx. \tag{3.11}$$

Thus we just established that $h_{\ell;q,d} \stackrel{\mathcal{L}}{=} I_q(g_{\ell;q})$ and therefore

$$Z_\ell \stackrel{\mathcal{L}}{=} \sum_{q=2}^{Q} I_q(\beta_q g_{\ell;q}), \tag{3.12}$$

as required. It should be noted that for such random variables $Z_\ell$, the conditions of Proposition 3.1 are trivially satisfied.

## 4. The quantitative Central Limit Theorem for $h_{\ell;q,d}$

In this section we prove Proposition 2.3 with the help of Proposition 3.1 and (3.9) in particular. The identifications of Section 3.2 lead to some very explicit expressions for the contractions (3.7), as detailed in the following result.

For $\ell \geq 1$, $q \geq 2$, let $g_{\ell;q}$ be defined as in (3.11).

**Lemma 4.1.** *For all $q_1, q_2 \geq 2$, $r = 1,\dots,q_1 \wedge q_2 - 1$, we have the identities*

$$\|g_{\ell;q_1} \otimes_r g_{\ell;q_2}\|^2_{H^{\otimes n}} = \int\limits_{(\mathbb{S}^d)^4} G^r_{\ell;d}(\cos d(x_1,x_2)) G^{q_1\wedge q_2 - r}_{\ell;d}(\cos d(x_2,x_3)) \times G^r_{\ell;d}(\cos d(x_3,x_4)) G^{q_1\wedge q_2 - r}_{\ell;d}(\cos d(x_1,x_4))\, d\underline{x},$$

*where we set $d\underline{x} := dx_1 dx_2 dx_3 dx_4$ and $n := q_1 + q_2 - 2r$.*

**Proof.** Assume w.l.o.g. $q_1 \leq q_2$ and set for simplicity of notation $d\underline{t} := dt_1 \dots dt_r$. The contraction (3.7) here takes the form

$$(g_{\ell;q_1} \otimes_r g_{\ell;q_2})(y_1,\dots,y_n) = \int\limits_{(\mathbb{S}^d)^r} g_{\ell;q_1}(y_1,\dots,y_{q_1-r},t_1,\dots,t_r) g_{\ell;q_2}(y_{q_1-r+1},\dots,y_n,t_1,\dots,t_r)\, d\underline{t}$$

$$= \int_{(\mathbb{S}^d)^r} \int_{\mathbb{S}^d} \left(\frac{n_{\ell;d}}{\mu_d}\right)^{q_1/2} G_{\ell;d}(\cos d(x_1, y_1)) \dots G_{\ell;d}(\cos d(x_1, t_r))\, dx_1$$
$$\times \int_{\mathbb{S}^d} \left(\frac{n_{\ell;d}}{\mu_d}\right)^{q_2/2} G_{\ell;d}(\cos d(x_2, y_{q_1-r+1})) \dots G_{\ell;d}(\cos d(x_2, t_r))\, dx_2\, d\underline{t}$$
$$= \int_{(\mathbb{S}^d)^2} \left(\frac{n_{\ell;d}}{\mu_d}\right)^{n/2} G_{\ell;d}(\cos d(x_1, y_1)) \dots G_{\ell;d}(\cos d(x_1, y_{q_1-r}))$$
$$\times G_{\ell;d}(\cos d(x_2, y_{q_1-r+1})) \dots G_{\ell;d}(\cos d(x_2, y_n)) G^r_{\ell;d}(\cos d(x_1, x_2))\, dx_1 dx_2,$$

where in the last equality we have repeatedly used the reproducing property of Gegenbauer polynomials [36]. Now set $d\underline{y} := dy_1 \dots dy_n$. It follows at once that

$$\|g_{\ell;q_1} \otimes_r g_{\ell;q_2}\|^2_{H^{\otimes n}}$$
$$= \int_{(\mathbb{S}^d)^n} (g_{\ell;q_1} \otimes_r g_{\ell;q_2})^2 (y_1, \dots, y_n)\, d\underline{y}$$
$$= \int_{(\mathbb{S}^d)^n} \int_{(\mathbb{S}^d)^2} \left(\frac{n_{\ell;d}}{\mu_d}\right)^n G_{\ell;d}(\cos d(x_1, y_1)) \dots G_{\ell;d}(\cos d(x_2, y_n)) G^r_{\ell;d}(\cos d(x_1, x_2)) dx_1 dx_2$$
$$\times \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_4, y_1)) \dots G_{\ell;d}(\cos d(x_3, y_n)) G^r_{\ell;d}(\cos d(x_3, x_4))\, dx_3 dx_4\, d\underline{y}$$
$$= \int_{(\mathbb{S}^d)^4} G^r_{\ell;d}(\cos d(x_1, x_2)) G^{q_1-r}_{\ell;d}(\cos d(x_2, x_3)) G^r_{\ell;d}(\cos d(x_3, x_4)) G^{q_1-r}_{\ell;d}(\cos d(x_1, x_4))\, d\underline{x},$$

as claimed. □

We need now to introduce some further notation, i.e. for $q \geq 2$ and $r = 1, \dots, q-1$

$$\mathcal{K}_\ell(q; r) := \int_{(\mathbb{S}^d)^4} G^r_{\ell;d}(\cos d(x_1, x_2)) G^{q-r}_{\ell;d}(\cos d(x_2, x_3))$$
$$\times G^r_{\ell;d}(\cos d(x_3, x_4)) G^{q-r}_{\ell;d}(\cos d(x_1, x_4))\, dx_1 dx_2 dx_3 dx_4,$$

Lemma 4.1 asserts that

$$\mathcal{K}_\ell(q; r) = \|g_{\ell;q} \otimes_r g_{\ell;q}\|^2_{H^{\otimes 2q-2r}}; \tag{4.1}$$

it is immediate to check that

$$\mathcal{K}_\ell(q; r) = \mathcal{K}_\ell(q; q-r). \tag{4.2}$$

In the following two propositions we bound each term of the form $\mathcal{K}(q;r)$ (from (4.2) it is enough to consider $r = 1, \dots, \left[\frac{q}{2}\right]$). As noted in Section 2.2, these bounds improve the existing literature even for the case $d = 2$, from which we start our analysis.

For $d = 2$, as previously recalled, Gegenbauer polynomials become standard Legendre polynomials $P_\ell$, for which it is well-known that (see (2.13))

$$\int_{\mathbb{S}^2} P_\ell(\cos d(x_1,x_2))^2 \, dx_1 = O\left(\frac{1}{\ell}\right); \tag{4.3}$$

also, from [23, Lemma 3.2] we have that

$$\int_{\mathbb{S}^2} P_\ell(\cos d(x_1,x_2))^4 \, dx_1 = O\left(\frac{\log \ell}{\ell^2}\right). \tag{4.4}$$

Finally, it is trivial to show that

$$\int_{\mathbb{S}^2} |P_\ell(\cos d(x_1,x_2))| \, dx_1 \le \sqrt{\int_{\mathbb{S}^2} P_\ell(\cos d(x_1,x_2))^2 \, dx_1} = O\left(\frac{1}{\sqrt{\ell}}\right) \tag{4.5}$$

and

$$\int_{\mathbb{S}^2} |P_\ell(\cos d(x_2,x_3))|^3 \, dx_2 \le \sqrt{\int_{\mathbb{S}^2} P_\ell(\cos d(x_2,x_3))^2 \, dx_2}\sqrt{\int_{\mathbb{S}^2} P_\ell(\cos d(x_1,x_2))^4 \, dx_1}$$
$$= O\left(\sqrt{\frac{\log \ell}{\ell^3}}\right). \tag{4.6}$$

These results will be the main tools to establish the upper bounds which are collected in the following proposition, whose proof is deferred to Appendix A.

**Proposition 4.2.** *For all $r = 1, 2, \dots, q-1$, we have*

$$\mathcal{K}_\ell(q;r) = O\left(\frac{1}{\ell^5}\right) \quad \text{for } q = 3, \tag{4.7}$$

$$\mathcal{K}_\ell(q;r) = O\left(\frac{1}{\ell^4}\right) \quad \text{for } q = 4, \tag{4.8}$$

$$\mathcal{K}_\ell(q;r) = O\left(\frac{\log \ell}{\ell^{9/2}}\right) \quad \text{for } q = 5, 6 \tag{4.9}$$

*and*

$$\mathcal{K}_\ell(q;1) = \mathcal{K}_\ell(q;q-1) = O\left(\frac{1}{\ell^{9/2}}\right),$$
$$\mathcal{K}_\ell(q;r) = O\left(\frac{1}{\ell^5}\right), \quad r = 2, \dots, q-2, \ \text{for } q \ge 7. \tag{4.10}$$

We can now move to the higher-dimensional case, as follows. Let us start with the bounds for all order moments of Gegenbauer polynomials. From (2.13)

$$\int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1,x_2))^2 dx_1 = O\left(\frac{1}{\ell^{d-1}}\right); \tag{4.11}$$

also, from Proposition 2.2, we have that if $q = 2p$, $p = 2, 3, 4, \dots,$

$$\int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1,x_2))^q dx_1 = O\left(\frac{1}{\ell^{d}}\right). \tag{4.12}$$

Finally, it is trivial to show that

$$\int_{\mathbb{S}^d} |G_{\ell;d}(\cos d(x_2,x_3))|\, dx_2 \le \sqrt{\int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_2,x_3))^2 dx_2} = O\left(\frac{1}{\sqrt{\ell^{d-1}}}\right), \tag{4.13}$$

$$\begin{aligned}&\int_{\mathbb{S}^d} |G_{\ell;d}(\cos d(x_2,x_3))|^3\, dx_2 \\ &\le \sqrt{\int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_2,x_3))^2 dx_2}\sqrt{\int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1,x_2))^4 dx_1} = O\left(\frac{1}{\ell^{d-\frac{1}{2}}}\right)\end{aligned} \tag{4.14}$$

and for $q \ge 5$ odd,

$$\begin{aligned}&\int_{\mathbb{S}^d} |G_{\ell;d}(\cos d(x_2,x_3))|^q\, dx_2 \\ &\le \sqrt{\int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_2,x_3))^4 dx_2}\sqrt{\int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1,x_2))^{2(q-2)} dx_1} = O\left(\frac{1}{\ell^{d}}\right).\end{aligned} \tag{4.15}$$

Analogously to the 2-dimensional case, we can exploit the previous results to obtain the following bounds, whose proof is again collected in Appendix A.

**Proposition 4.3.** *For all* $r = 1, 2, \dots, q-1$,

$$\mathcal{K}_\ell(q;r) = O\left(\frac{1}{\ell^{2d+\frac{d-5}{2}}}\right) \quad \textit{for } q = 3, \tag{4.16}$$

$$\mathcal{K}_\ell(q;r) = O\left(\frac{1}{\ell^{2d+\frac{d-3}{2}}}\right) \quad \textit{for } q = 4, \tag{4.17}$$

*and*

$$\mathcal{K}_\ell(q;1) = \mathcal{K}_\ell(q;q-1) = O\left(\frac{1}{\ell^{2d+\frac{d-1}{2}}}\right),$$

$$\mathcal{K}_\ell(q;r) = O\left(\frac{1}{\ell^{3d-1}}\right), \quad r = 2, \dots, q-2, \ \textit{for } q \ge 5. \tag{4.18}$$

Exploiting the results in this section and the variance evaluation in Proposition 2.2 of Appendix A, we have the proof of our first quantitative CLT.

**Proof of Proposition 2.3.** By Parseval's identity, the case $q = 2$ can be treated as a sum of independent random variables and the proof follows from standard Berry–Esseen arguments, as in Lemma 8.3 of [19] for the case $d = 2$. For $q \geq 3$, from Proposition 3.1 and (3.9), for $d_{\mathcal{D}} = d_K, d_{TV}, d_W$

$$d_{\mathcal{D}}\left(\frac{h_{\ell;q}}{\sqrt{\mathrm{Var}[h_{\ell;q,d}]}}, \mathcal{N}(0,1)\right) = O\left(\sup_r \sqrt{\frac{\mathcal{K}_\ell(q;r)}{\mathrm{Var}[h_{\ell;q,d}]^2}}\right). \tag{4.19}$$

The proof is thus an immediate consequence of the previous equality and the results in Proposition 2.2, Proposition 4.2 and Proposition 4.3. □

## 5. General polynomials

In this section, we show how the previous results can be extended to establish quantitative CLTs, for the case of general, nonHermite polynomials. To this aim, we need to introduce some more notation, namely (for $Z_\ell$ defined as in (3.10))

$$\mathcal{K}(Z_\ell; d) := \max_{q:\beta_q \neq 0} \max_{r=1,\dots,q-1} \mathcal{K}_\ell(q;r),$$

and as in Proposition 2.7

$$R(Z_\ell; d) = \begin{cases} \frac{1}{\ell^{\frac{d-1}{2}}}, & \text{for } \beta_2 \neq 0, \\ \max_{q=3,\dots,Q:\ \beta_q, c_{q;d} \neq 0} R(\ell; q, d), & \text{for } \beta_2 = 0. \end{cases}$$

In words, $\mathcal{K}(Z_\ell; d)$ is the largest contraction term among those emerging from the analysis of the different Hermite components, and $R(Z_\ell; d)$ is the slowest convergence rate of the same components. The next result is stating that these are the only quantities to look at when considering the general case.

**Proof of Theorem 2.7.** We apply Proposition 3.1. In our case $H = L^2(\mathbb{S}^d)$ and

$$\begin{aligned}\mathrm{Var}[\langle DZ_\ell, -DL^{-1}Z_\ell\rangle_H] &= \mathrm{Var}\left[\langle \sum_{q_1=2}^{Q} \beta_{q_1} Dh_{\ell;q_1,d}, -\sum_{q_{21}=2}^{Q} \beta_{q_2} DL^{-1} h_{\ell;q_2,d}\rangle_H\right] \\ &= \mathrm{Var}\left[\sum_{q_1=2}^{Q}\sum_{q_1=2}^{Q} \beta_{q_1}\beta_{q_2}\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H\right].\end{aligned}$$

From Section 3 recall that for $q_1 \neq q_2$

$$E[\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H] = 0,$$

whence we write

$$\begin{aligned}
&\mathrm{Var}\left[\sum_{q_1=2}^{Q}\sum_{q_2=2}^{Q}\beta_{q_1}\beta_{q_2}\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H\right]\\
&\quad=\sum_{q_1=2}^{Q}\sum_{q_2=2}^{Q}\beta_{q_1}^2\beta_{q_2}^2\,\mathrm{Cov}\left(\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_1,d}\rangle_H, \langle Dh_{\ell;q_2,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H\right)\\
&\qquad+\sum_{q_1=2}^{Q}\sum_{q_2\neq q_1}^{Q}\sum_{q_3=2}^{Q}\sum_{q_4\neq q_3}^{Q}\beta_{q_1}\beta_{q_2}\beta_{q_3}\beta_{q_4}\\
&\qquad\times\mathrm{Cov}\left(\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H, \langle Dh_{\ell;q_3,d}, -DL^{-1}h_{\ell;q_4,d}\rangle_H\right).
\end{aligned}$$

Now of course we have

$$\begin{aligned}
&\mathrm{Cov}\left(\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_1,d}\rangle_H, \langle Dh_{\ell;q_2,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H\right)\\
&\quad\leq\left(\mathrm{Var}\left[\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_1,d}\rangle_H\right]\mathrm{Var}\left[\langle Dh_{\ell;q_2,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H\right]\right)^{1/2},\\
&\mathrm{Cov}\left(\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H, \langle Dh_{\ell;q_3,d}, -DL^{-1}h_{\ell;q_4,d}\rangle_H\right)\\
&\quad\leq\left(\mathrm{Var}\left[\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H\right]\mathrm{Var}\left[\langle Dh_{\ell;q_3,d}, -DL^{-1}h_{\ell;q_4,d}\rangle_H\right]\right)^{1/2}.
\end{aligned}$$

Applying [28, Lemma 6.2.1] it is immediate to show that

$$\begin{aligned}
&\mathrm{Var}\left[\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_1,d}\rangle_H\right]\\
&\quad\leq q_1^2\sum_{r=1}^{q_1-1}((r-1)!)^2\binom{q_1-1}{r-1}^4(2q_1-2r)!\,\|g_{\ell;q_1}\otimes_r g_{\ell;q_1}\|^2_{H^{\otimes 2q_1-2r}}\\
&\quad= q_1^2\sum_{r=1}^{q_1-1}((r-1)!)^2\binom{q_1-1}{r-1}^4(2q_1-2r)!\mathcal{K}_\ell(q_1;r).
\end{aligned}$$

Also, for $q_1 < q_2$

$$\begin{aligned}
&\mathrm{Var}\left[\langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d}\rangle_H\right]\\
&\quad= q_1^2\sum_{r=1}^{q_1}((r-1)!)^2\binom{q_1-1}{r-1}^2\binom{q_2-1}{r-1}^2(q_1+q_2-2r)!\,\left\|g_{\ell;q_1}\widetilde{\otimes}_r g_{\ell;q_2}\right\|^2_{H^{\otimes(q_1+q_2-2r)}}\\
&\quad= q_1^2((q_1-1)!)^2\binom{q_2-1}{q_1-1}^2(2q_1-2r)!\,\left\|g_{\ell;q_1}\widetilde{\otimes}_{q_1} g_{\ell;q_2}\right\|^2_{H^{\otimes(q_2-q_1)}}\\
&\qquad+q_1^2\sum_{r=1}^{q_1-1}((r-1)!)^2\binom{q_1-1}{r-1}^2\binom{q_2-1}{r-1}^2(q_1+q_2-2r)!\,\left\|g_{\ell;q_1}\widetilde{\otimes}_r g_{\ell;q_2}\right\|^2_{H^{\otimes(q_1+q_2-2r)}}\\
&\quad=: A+B.
\end{aligned}$$

Let us focus on the first summand $A$, which includes terms that, from Lemma 4.1, take the form

$$\begin{aligned}
&\left\|g_{\ell;q_1}\,\widetilde{\otimes}_{q_1}\,g_{\ell;q_2}\right\|^2_{H^{\otimes(q_2-q_1)}}\\
&\quad\le \left\|g_{\ell;q_1}\otimes_{q_1} g_{\ell;q_2}\right\|^2_{H^{\otimes(q_2-q_1)}}\\
&\quad= \int\limits_{(\mathbb{S}^d)^{q_2-q_1}}\ \int\limits_{(\mathbb{S}^d)^2}\left(\frac{n_{\ell;d}}{\mu_d}\right)^{q_2-q_1} G_{\ell;d}(\cos d(x_2,y_1))\dots G_{\ell;d}(\cos d(x_2,y_{q_2-q_1}))\\
&\qquad\times G_{\ell;d}(\cos d(x_1,x_2))^{q_1}\,dx_1dx_2\\
&\qquad\times \int\limits_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_3,y_1))\dots G_{\ell;d}(\cos d(x_3,y_{q_2-q_1}))G_{\ell;d}(\cos d(x_3,x_4))^{q_1}\,dx_3dx_4\,d\underline{y}\\
&\quad=: I,
\end{aligned}$$

where for the sake of simplicity we have set $d\underline{y} := dy_1\dots dy_{q_2-q_1}$. Applying $q_2-q_1$ times the reproducing formula for Gegenbauer polynomials [36] we get

$$I = \int\limits_{(\mathbb{S}^d)^4} G_{\ell;d}(\cos d(x_1,x_2))^{q_1} G_{\ell;d}(\cos d(x_2,x_3))^{q_2-q_1} G_{\ell;d}(\cos d(x_3,x_4))^{q_1}\,d\underline{x}. \tag{5.1}$$

In graphical terms, these contractions correspond to the diagrams such that all $q_1$ edges corresponding to vertex 1 are linked to vertex 2, vertex 2 and 3 are connected by $q_2-q_1$ edges, vertex 3 and 4 by $q_1$ edges, and no edges exist between 1 and 4, i.e. the diagram has no proper loop.

Now immediately we write

$$\begin{aligned}
(5.1) &= \int\limits_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1,x_2))^{q_1}\,dx_1 \int\limits_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_3,x_4))^{q_1}\,dx_4\\
&\qquad\times \int\limits_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_2,x_3))^{q_2-q_1}\,dx_2dx_3\\
&= \frac{1}{(q_1!)^2}\operatorname{Var}[h_{\ell;q_1,d}]^2 \int\limits_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_2,x_3))^{q_2-q_1}\,dx_2dx_3.
\end{aligned}$$

Moreover we have

$$\int\limits_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_2,x_3))^{q_2-q_1}\,dx_2dx_3 = 0, \quad \text{if } q_2-q_1=1 \tag{5.2}$$

and from (2.13) if $q_2-q_1\ge 2$

$$\int_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_2,x_3))^{q_2-q_1}\,dx_2dx_3 \le \mu_d \int_{\mathbb{S}^d} G_{\ell;d}(\cos d(x,y))^2\,dx = O\left(\frac{1}{\ell^{d-1}}\right).$$

It follows that

$$\|g_{\ell;q_1} \otimes_{q_1} g_{\ell;q_2}\|^2_{H^{\otimes(q_2-q_1)}} = O\left(\mathrm{Var}[h_{\ell;q_1,d}]^2 \frac{1}{\ell^{d-1}}\right) \tag{5.3}$$

always. For the second term, still from [28, Lemma 6.2.1] we have

$$\begin{aligned} B &\le \frac{q_1^2}{2}\sum_{r=1}^{q_1-1}((r-1)!)^2\binom{q_1-1}{r-1}^2\binom{q_2-1}{r-1}^2(q_1+q_2-2r)! \\ &\quad\times\left(\|g_{\ell;q_1}\otimes_{q_1-r} g_{\ell;q_1}\|^2_{H^{\otimes 2r}} + \|g_{\ell;q_2}\otimes_{q_2-r} g_{\ell;q_2}\|^2_{H^{\otimes 2r}}\right) \\ &= \frac{q_1^2}{2}\sum_{r=1}^{q_1-1}((r-1)!)^2\binom{q_1-1}{r-1}^2\binom{q_2-1}{r-1}^2(q_1+q_2-2r)! \\ &\quad\times(\mathcal{K}_\ell(q_1;r)+\mathcal{K}_\ell(q_2;r)), \end{aligned} \tag{5.4}$$

where the last step follows from Lemma 4.1.

Let us first investigate the case $d = 2$. From (2.14), (2.15) and (2.17) it is immediate that

$$\mathrm{Var}[Z_\ell] = \sum_{q=2}^{Q}\beta_q^2\,\mathrm{Var}[h_{\ell;q,2}] = \begin{cases} O(\ell^{-1}), & \text{for } \beta_2 \ne 0, \\ O(\ell^{-2}\log\ell), & \text{for } \beta_2 = 0,\ \beta_4 \ne 0, \\ O(\ell^{-2}), & \text{otherwise.} \end{cases} \tag{5.5}$$

Hence we have that for $\beta_2 \ne 0$

$$d_{TV}\left(\frac{Z_\ell - EZ_\ell}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O\left(\frac{\sqrt{\mathcal{K}_\ell(2;r)}}{\mathrm{Var}[Z_\ell]}\right) = O\left(\ell^{-1/2}\right);$$

for $\beta_2 = 0$, $\beta_4 \ne 0$,

$$d_{TV}\left(\frac{Z_\ell - EZ_\ell}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O\left(\frac{\sqrt{\mathcal{K}_\ell(4;r)}}{\mathrm{Var}[Z_\ell]}\right) = O\left(\frac{1}{\log\ell}\right)$$

and for $\beta_2 = \beta_4 = 0$, $\beta_5 \ne 0$ and $c_5 > 0$

$$d_{TV}\left(\frac{Z_\ell - EZ_\ell}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O\left(\frac{\sqrt{\mathcal{K}_\ell(5;r)}}{\mathrm{Var}[Z_\ell]}\right) = O\left(\frac{\log\ell}{\ell^{1/4}}\right),$$

and analogously we deal with the remaining cases, so that we obtain the claimed result for $d = 2$.

For $d \geq 3$ from (2.13) and Proposition 2.2, it holds

$$\mathrm{Var}[Z_\ell] = \sum_{q=2}^{Q} \beta_q^2 \,\mathrm{Var}[h_{\ell;q,d}] = \begin{cases} O(\ell^{-(d-1)}), & \text{for } \beta_2 \neq 0, \\ O(\ell^{-d}), & \text{otherwise.} \end{cases}$$

Hence we have for $\beta_2 \neq 0$

$$d_{TV}\left(\frac{Z_\ell - EZ_\ell}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O\left(\frac{\sqrt{\mathcal{K}_\ell(2;r)}}{\mathrm{Var}[Z_\ell]}\right) = O\left(\frac{1}{\ell^{\frac{d-1}{2}}}\right).$$

Likewise for $\beta_2 = 0$, $\beta_3, c_{3;d} \neq 0$,

$$d_{TV}\left(\frac{Z_\ell - EZ_\ell}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O\left(\frac{\sqrt{\mathcal{K}_\ell(3;r)}}{\mathrm{Var}[Z_\ell]}\right) = O\left(\frac{1}{\ell^{\frac{d-5}{4}}}\right)$$

and for $\beta_2 = \beta_3 = 0$, $\beta_4 \neq 0$

$$d_{TV}\left(\frac{Z_\ell - EZ_\ell}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O\left(\frac{\sqrt{\mathcal{K}_\ell(4;r)}}{\mathrm{Var}[Z_\ell]}\right) = O\left(\frac{1}{\ell^{\frac{d-3}{2}}}\right).$$

Finally if $\beta_2 = \beta_3 = \beta_4 = 0$, $\beta_q, c_{q;d} \neq 0$ for some $q$, then

$$d_{TV}\left(\frac{Z_\ell - EZ_\ell}{\sqrt{\mathrm{Var}[Z_\ell]}}, \mathcal{N}(0,1)\right) = O\left(\frac{\sqrt{\mathcal{K}_\ell(q;r)}}{\mathrm{Var}[Z_\ell]}\right) = O\left(\sqrt{\frac{\ell^{2d}}{\ell^{2d+\frac{d}{2}-\frac{1}{2}}}}\right) = O\left(\frac{1}{\ell^{\frac{d-1}{4}}}\right). \qquad \square$$

**Remark 5.1.** To compare our result in these specific circumstances with the general bound obtained by Nourdin and Peccati, we note that for (5.1), these authors are exploiting the inequality

$$\|g_{\ell;q_1} \otimes_{q_1} g_{\ell;q_2}\|^2_{H^{\otimes(q_2-q_1)}} \leq \|g_{\ell;q_1}\|^2_{H^{\otimes q_1}} \|g_{\ell;q_2} \otimes_{q_2-q_1} g_{\ell;q_2}\|_{H^{\otimes 2q_1}},$$

see [28, Lemma 6.2.1]. In the special framework we consider here (i.e., orthogonal eigenfunctions), this provides, however, a less efficient bound than (5.3): indeed from (5.1), repeating the same argument as in Lemma 4.1, one obtains

$$\begin{aligned} &\|g_{\ell;q_1} \otimes_{q_1} g_{\ell;q_2}\|^2_{H^{\otimes(q_2-q_1)}} \\ &= \int_{(\mathbb{S}^d)^4} G_{\ell;d}(\cos d(x_1,x_2))^{q_1} G_{\ell;d}(\cos d(x_2,x_3))^{q_2-q_1} G_{\ell;d}(\cos d(x_3,x_4))^{q_1}\, d\underline{x} \\ &\leq \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_1,x_2))^{q_1}\, dx_1 dx_2 \end{aligned}$$

$$\times \sqrt{\int\limits_{(\mathbb{S}^d)^4} G_{\ell;d}(\cos d(x_1,x_2))^{q_1} G_{\ell;d}(\cos d(x_2,x_3))^{q_2-q_1} G_{\ell;d}(\cos d(x_3,x_4))^{q_1} G_{\ell;d}(\cos d(x_1,x_4))^{q_2-q_1}\, d\underline{x}}$$

$$= O\left(\mathrm{Var}[h_{\ell;q_1,d}]\sqrt{\mathcal{K}_\ell(q_2,q_1)}\right),$$

yielding a bound of order

$$O\left(\sqrt{\frac{\mathrm{Var}[h_{\ell;q_1,d}]\sqrt{\mathcal{K}_\ell(q_2,q_1)}}{\mathrm{Var}[h_{\ell;q_1,d}]^2}}\right) = O\left(\frac{\sqrt[4]{\mathcal{K}_\ell(q_2,q_1)}}{\sqrt{\mathrm{Var}[h_{\ell;q_1,d}]}}\right) \tag{5.6}$$

rather than

$$O\left(\sqrt{\frac{\mathcal{K}_\ell(q_2,q_1)}{\mathrm{Var}[h_{\ell;q_1,d}]^2}}\right); \tag{5.7}$$

for instance, for $d = 2$ note that (5.6) is typically $= O(\ell \times \ell^{-9/8}) = O(\ell^{-1/8})$, while we have established for (5.7) bounds of order $O(\ell^{-1/4})$.

**Remark 5.2.** Clearly the fact that $\|g_{\ell;q_1} \otimes_{q_1} g_{\ell;q_2}\|^2_{H^{\otimes(q_2-q_1)}} = 0$ for $q_2 = q_1 + 1$ entails that the contraction $g_{\ell;q_1} \otimes_{q_1} g_{\ell;q_2}$ is identically null. Indeed repeating the same argument as in Lemma 4.1

$$\begin{aligned} &g_{\ell;q_1} \otimes_{q_1} g_{\ell;q_1+1} \\ &= \int\limits_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_1,y)) G_{\ell;d}(\cos d(x_1,x_2))^{q_1}\, dx_1 dx_2 \\ &= \int\limits_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1,y)) \left( \int\limits_{\mathbb{S}^d} G_{\ell;d}(\cos d(x_1,x_2))^{q_1}\, dx_2 \right) dx_1 = 0, \end{aligned}$$

as expected, because the inner integral in the last equation does not depend on $x_1$ by rotational invariance.

## 6. General nonlinear functionals and excursion sets

The techniques and results developed in Sections 4, 5 are restricted to finite-order polynomials. In the special case of the Wasserstein distance, we shall show below how they can indeed be extended to general nonlinear functionals of the form (2.25)

$$S_\ell(M) = \int\limits_{\mathbb{S}^d} M(T_\ell(x))dx;$$

here $M : \mathbb{R} \to \mathbb{R}$ is a measurable function such that $\mathbb{E}[M(T_\ell)^2] < \infty$, as in Section 2.2, and $J_2(M) \neq 0$, where we recall that $J_q(M) := \mathbb{E}[M(T_\ell)H_q(T_\ell)]$.

**Remark 6.1.** Without loss of generality, the first two coefficients $J_0(M)$, $J_1(M)$ can always be taken to be zero in the present framework. Indeed, $J_0(M) := \mathbb{E}[M(T_\ell)] = 0$, assuming we work with centered variables and moreover as we noted earlier $h_{\ell;1,d} = \int_{\mathbb{S}^d} T_\ell(x)\,dx = 0$.

**Proof of Proposition 2.8.** As in [21], from (2.26) we write the expansion

$$S_\ell(M) = \int_{\mathbb{S}^d} \sum_{q=2}^{\infty} \frac{J_q(M)H_q(T_\ell(x))}{q!} dx.$$

Precisely, we write for $d = 2$

$$S_\ell(M) = \frac{J_2(M)}{2} h_{\ell;2,2} + \frac{J_3(M)}{3!} h_{\ell;3,2} + \frac{J_4(M)}{4!} h_{\ell;4,2} + \int_{\mathbb{S}^2} \sum_{q=5}^{\infty} \frac{J_q(M)H_q(T_\ell(x))}{q!} dx, \tag{6.1}$$

whereas for $d \geq 3$

$$S_\ell(M) = \frac{J_2(M)}{2} h_{\ell;2,d} + \int_{\mathbb{S}^d} \sum_{q=3}^{\infty} \frac{J_q(M)H_q(T_\ell(x))}{q!} dx. \tag{6.2}$$

Let us first investigate the case $d = 2$. Set for the sake of simplicity

$$S_\ell(M;1) := \frac{J_2(M)}{2} h_{\ell;2,2} + \frac{J_3(M)}{3!} h_{\ell;3,2} + \frac{J_4(M)}{4!} h_{\ell;4,2},$$
$$S_\ell(M;2) := \int_{\mathbb{S}^2} \sum_{q=5}^{\infty} \frac{J_q(M)H_q(T_\ell(x))}{q!} dx.$$

Hence from (6.1) and the triangular inequality

$$\begin{aligned}
& d_W\left(\frac{S_\ell(M)}{\sqrt{\mathrm{Var}[S_\ell(M)]}}, \mathcal{N}(0,1)\right) \\
& \leq d_W\left(\frac{S_\ell(M)}{\sqrt{\mathrm{Var}[S_\ell(M)]}}, \frac{S_\ell(M;1)}{\sqrt{\mathrm{Var}[S_\ell(M)]}}\right) \\
& \quad + d_W\left(\frac{S_\ell(M;1)}{\sqrt{\mathrm{Var}[S_\ell(M)]}}, \mathcal{N}\left(0, \frac{\mathrm{Var}[S_\ell(M;1)]}{\mathrm{Var}[S_\ell(M)]}\right)\right) \\
& \quad + d_W\left(\mathcal{N}\left(0, \frac{\mathrm{Var}[S_\ell(M;1)]}{\mathrm{Var}[S_\ell(M)]}\right), \mathcal{N}(0,1)\right)
\end{aligned}$$

$$\leq \frac{1}{\sqrt{\mathrm{Var}[S_\ell(M)]}}\mathbb{E}\left[\left(\int_{\mathbb{S}^2}\sum_{q=5}^{\infty}\frac{J_q(M)H_q(T_\ell(x))}{q!}dx\right)^2\right]^{1/2}$$
$$+ d_W\left(\frac{S_\ell(M;1)}{\sqrt{\mathrm{Var}[S_\ell(M)]}}, \mathcal{N}(0, \frac{\mathrm{Var}[S_\ell(M;1)]}{\mathrm{Var}[S_\ell(M)]})\right)$$
$$+ d_W\left(\mathcal{N}(0, \frac{\mathrm{Var}[S_\ell(M;1)]}{\mathrm{Var}[S_\ell(M)]}), \mathcal{N}(0,1)\right).$$

Let us bound the first term of the previous summation. Of course

$$\mathrm{Var}[S_\ell(M)] = \mathrm{Var}[S_\ell(M;1)] + \mathrm{Var}[S_\ell(M;2)];$$

now we have (see [21])

$$\mathrm{Var}[S_\ell(M;1)] = \frac{J_2^2(M)}{2^2}\mathrm{Var}[h_{\ell;2,2}] + \frac{J_3^2(M)}{6^2}\mathrm{Var}[h_{\ell;3,2}] + \frac{J_4^2(M)}{(4!)^2}\mathrm{Var}[h_{\ell;4,2}]$$

and moreover

$$\mathrm{Var}[S_\ell(M;2)] = \mathbb{E}\left[\left(\int_{\mathbb{S}^2}\sum_{q=5}^{\infty}\frac{J_q(M)H_q(T_\ell(x))}{q!}dx\right)^2\right] = \sum_{q=5}^{\infty}\frac{J_q^2(M)}{(q!)^2}\mathrm{Var}[h_{\ell;q,2}]$$
$$\ll \frac{1}{\ell^2}\sum_{q=5}^{\infty}\frac{J_q^2(M)}{q!} \ll \frac{1}{\ell^2},$$

where the last bound follows from (2.15) and (2.16). Therefore recalling also (2.14) and (2.17)

$$\frac{1}{\mathrm{Var}[S_\ell(M)]}\mathbb{E}\left[\left(\int_{\mathbb{S}^2}\sum_{q=5}^{\infty}\frac{J_q(M)H_q(T_\ell(x))}{q!}dx\right)^2\right] \ll \frac{1}{\ell}.$$

On the other hand, from Proposition 2.7

$$d_W\left(\frac{S_\ell(M;1)}{\sqrt{\mathrm{Var}[S_\ell(M)]}}, \mathcal{N}\left(0, \frac{\mathrm{Var}[S_\ell(M;1)]}{\mathrm{Var}[S_\ell(M)]}\right)\right) = O\left(\frac{1}{\sqrt{\ell}}\right)$$

and finally, using Proposition 3.6.1 in [28],

$$d_W\left(\mathcal{N}\left(0, \frac{\mathrm{Var}[S_\ell(M;1)]}{\mathrm{Var}[S_\ell(M)]}\right), \mathcal{N}(0,1)\right) \leq \sqrt{\frac{2}{\pi}}\left|\frac{\mathrm{Var}[S_\ell(M;1)]}{\mathrm{Var}[S_\ell(M)]} - 1\right| = O\left(\frac{1}{\ell}\right),$$

so that the proof for $d = 2$ is completed.

The proof in the general case $d \geq 3$ is indeed analogous, just setting

$$S_\ell(M;1) := \frac{J_2(M)}{2} h_{\ell;2,d},$$

$$S_\ell(M;2) := \int_{\mathbb{S}^2} \sum_{q=3}^{\infty} \frac{J_q(M) H_q(T_\ell(x))}{q!} dx$$

and recalling from (2.13) that $\mathrm{Var}[h_{\ell;2,d}] = O(\frac{1}{\ell^{d-1}})$ whereas for $q \geq 3$, $\mathrm{Var}[h_{\ell;q,d}] = O(\frac{1}{\ell^d})$ from Proposition 2.2. □

We are now in the position to establish our main result, concerning the volume of the excursion sets, which we recall for any fixed $z \in \mathbb{R}$ is given by

$$S_\ell(z) := S_\ell(\mathbb{I}(\cdot > z)) = \int_{\mathbb{S}^d} \mathbb{I}(T_\ell(x) > z) dx.$$

Again, $\mathbb{E}[S_\ell(z)] = \mu_d(1 - \Phi(z))$, where $\Phi(z)$ is the cdf of the standard Gaussian law, and in this case we have $M = M_z := \mathbb{I}(\cdot > z)$, $J_2(M_z) = z\phi(z)$, $\phi$ denoting the standard Gaussian density. The proof of Theorem 2.1 is then just an immediate consequence of Proposition 2.8.

**Remark 6.2.** It should be noted that the rate obtained here is much sharper than the one provided by [34] for the Euclidean case with $d = 2$. The asymptotic setting we consider is rather different from his, in that we consider the case of spherical eigenfunction with diverging eigenvalues, whereas he focuses on functionals evaluated on increasing domains $[0, T]^d$ for $T \to \infty$. However the contrast in the converging rates is not due to these different settings, indeed [8] establish rates of convergence analogous to those by [34] for spherical random fields with more rapidly decaying covariance structure than the one we are considering here. The main point to notice is that the slow decay of Gegenbauer polynomials entails some form of long range dependent behavior on random spherical harmonics; in this sense, hence, our results may be closer in spirit to the work by [12] on empirical processes for long range dependent stationary processes on $\mathbb{R}$.

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## Appendix A

### A.1. On the variance of $h_{\ell;q,d}$

In this appendix we study the variance of $h_{\ell;q,d}$ defined in (2.6). By (3.2) and the definition of Gaussian random eigenfunctions (2.2), it follows that (2.9) hold at once:

$$\begin{aligned}\mathrm{Var}[h_{\ell;q,d}] &= \mathbb{E}\left[\left(\int_{\mathbb{S}^d} H_q(T_\ell(x))\,dx\right)^2\right] = \int_{(\mathbb{S}^d)^2} \mathbb{E}[H_q(T_\ell(x_1))H_q(T_\ell(x_2))]\,dx_1dx_2 \\ &= q!\int_{(\mathbb{S}^d)^2} \mathbb{E}[T_\ell(x_1)T_\ell(x_2)]^q\,dx_1dx_2 = q!\int_{(\mathbb{S}^d)^2} G_{\ell;d}(\cos d(x_1,x_2))^q\,dx_1dx_2 \\ &= q!\mu_d\mu_{d-1}\int_0^\pi G_{\ell;d}(\cos\vartheta)^q(\sin\vartheta)^{d-1}\,d\vartheta.\end{aligned}$$

Now we prove Proposition 2.2, inspired by the proof of [22, Lemma 5.2].

**Proof of Proposition 2.2.** By the Hilb asymptotic formula for Jacobi polynomials (see [36, Theorem 8.21.12]), we have uniformly for $\ell \geq 1$, $\vartheta \in [0, \frac{\pi}{2}]$

$$(\sin\vartheta)^{\frac{d}{2}-1}G_{\ell;d}(\cos\vartheta) = \frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{\ell}}\left(a_{\ell,d}\left(\frac{\vartheta}{\sin\vartheta}\right)^{\frac{1}{2}}J_{\frac{d}{2}-1}(L\vartheta)+\delta(\vartheta)\right),$$

where $L = \ell + \frac{d-1}{2}$,

$$a_{\ell,d} = \frac{\Gamma(\ell+\frac{d}{2})}{(\ell+\frac{d-1}{2})^{\frac{d}{2}-1}\ell!} \sim 1 \quad \text{as } \ell\to\infty, \tag{A.1}$$

and the remainder is

$$\delta(\vartheta) \ll \begin{cases} \sqrt{\vartheta}\,\ell^{-\frac{3}{2}} & \ell^{-1} < \vartheta < \frac{\pi}{2}, \\ \vartheta^{(\frac{d}{2}-1)+2}\ell^{\frac{d}{2}-1} & 0 < \vartheta < \ell^{-1}. \end{cases}$$

Therefore, in light of (A.1) and $\vartheta \to \frac{\vartheta}{\sin\vartheta}$ being bounded,

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^q(\sin\vartheta)^{d-1}d\vartheta \\ &\quad= \left(\frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{\ell}}\right)^q a^q_{\ell,d}\int_0^{\frac{\pi}{2}}(\sin\vartheta)^{-q(\frac{d}{2}-1)}\left(\frac{\vartheta}{\sin\vartheta}\right)^{\frac{q}{2}}J^q_{\frac{d}{2}-1}(L\vartheta)(\sin\vartheta)^{d-1}d\vartheta\end{aligned}$$

$$+O\left(\frac{1}{\ell^{q(\frac{d}{2}-1)}}\int_0^{\frac{\pi}{2}}(\sin\vartheta)^{-q(\frac{d}{2}-1)}|J_{\frac{d}{2}-1}(L\vartheta)|^{q-1}\delta(\vartheta)(\sin\vartheta)^{d-1}d\vartheta\right), \tag{A.2}$$

where we used

$$\binom{\ell+\frac{d}{2}-1}{\ell}\ll\frac{1}{\ell^{\frac{d}{2}-1}}$$

(note that we readily neglected the smaller terms, corresponding to higher powers of $\delta(\vartheta)$). We rewrite (A.2) as

$$\int_0^{\frac{\pi}{2}}G_{\ell;d}(\cos\vartheta)^q(\sin\vartheta)^{d-1}d\vartheta=N+E, \tag{A.3}$$

where

$$\begin{aligned}N&=N(d,q;\ell)\\&:=\left(\frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{\ell}}\right)^q a_{\ell,d}^q\int_0^{\frac{\pi}{2}}(\sin\vartheta)^{-q(\frac{d}{2}-1)}\left(\frac{\vartheta}{\sin\vartheta}\right)^{\frac{q}{2}}J_{\frac{d}{2}-1}(L\vartheta)^q(\sin\vartheta)^{d-1}d\vartheta\end{aligned} \tag{A.4}$$

and

$$E=E(d,q;\ell)\ll\frac{1}{\ell^{q(\frac{d}{2}-1)}}\int_0^{\frac{\pi}{2}}(\sin\vartheta)^{-q(\frac{d}{2}-1)}|J_{\frac{d}{2}-1}(L\vartheta)|^{q-1}\delta(\vartheta)(\sin\vartheta)^{d-1}d\vartheta. \tag{A.5}$$

To bound the error term $E$ we split the range of the integration in (A.5) and write

$$\begin{aligned}E&\ll\frac{1}{\ell^{q(\frac{d}{2}-1)}}\int_0^{\frac{1}{\ell}}(\sin\vartheta)^{-q(\frac{d}{2}-1)}|J_{\frac{d}{2}-1}(L\vartheta)|^{q-1}\vartheta^{(\frac{d}{2}-1)+2}\ell^{\frac{d}{2}-1}(\sin\vartheta)^{d-1}\,d\vartheta\\&\quad+\frac{1}{\ell^{q(\frac{d}{2}-1)}}\int_{\frac{1}{\ell}}^{\frac{\pi}{2}}(\sin\vartheta)^{-q(\frac{d}{2}-1)}|J_{\frac{d}{2}-1}(L\vartheta)|^{q-1}\sqrt{\vartheta}\,\ell^{-\frac{3}{2}}(\sin\vartheta)^{d-1}\,d\vartheta.\end{aligned} \tag{A.6}$$

For the first integral in (A.6) recall that $J_{\frac{d}{2}-1}(z)\sim z^{\frac{d}{2}-1}$ as $z\to 0$, so that as $\ell\to\infty$,

$$\begin{aligned}&\frac{1}{\ell^{(q-1)(\frac{d}{2}-1)}}\int_0^{\frac{1}{\ell}}\left(\frac{\vartheta}{\sin\vartheta}\right)^{q(\frac{d}{2}-1)-d+1}|J_{\frac{d}{2}-1}(L\vartheta)|^{q-1}\vartheta^{-(q-1)(\frac{d}{2}-1)+d+1}\,d\vartheta\\&\ll\int_0^{\frac{1}{\ell}}\vartheta^{d+1}\,d\vartheta=\frac{1}{\ell^{d+2}},\end{aligned} \tag{A.7}$$

which is enough for our purposes. Furthermore, since for $z$ big $|J_{\frac{d}{2}-1}(z)| = O(z^{-\frac{1}{2}})$ (and keeping in mind that $L$ is of the same order of magnitude as $\ell$), we may bound the second integral in (A.6) as

$$\begin{aligned} &\ll \frac{1}{\ell^{q(\frac{d}{2}-1)+\frac{3}{2}}} \int_{\frac{1}{\ell}}^{\frac{\pi}{2}} \left(\frac{\vartheta}{\sin\vartheta}\right)^{q(\frac{d}{2}-1)-d+1} |J_{\frac{d}{2}-1}(L\vartheta)|^{q-1} \vartheta^{-q(\frac{d}{2}-1)+d-\frac{1}{2}}\, d\vartheta \\ &\ll \frac{1}{\ell^{q(\frac{d}{2}-1)+\frac{3}{2}}} \int_{\frac{1}{\ell}}^{\frac{\pi}{2}} (\ell\vartheta)^{-\frac{q-1}{2}} \vartheta^{-q(\frac{d}{2}-1)+d-\frac{1}{2}}\, d\vartheta = \frac{1}{\ell^{q(\frac{d}{2}-\frac{1}{2})+2}} \int_{\frac{1}{\ell}}^{\frac{\pi}{2}} \vartheta^{-q(\frac{d}{2}-\frac{1}{2})+d}\, d\vartheta \\ &\ll \frac{1}{\ell^{(d+2)\wedge(q(\frac{d}{2}-\frac{1}{2})+1)}} = o(\ell^{-d}), \end{aligned} \tag{A.8}$$

where the last equality in (A.8) holds for $q \geq 3$. From (A.7) (bounding the first integral in (A.6)) and (A.8) (bounding the second integral in (A.6)) we finally find that the error term in (A.3) is

$$E = o(\ell^{-d}) \tag{A.9}$$

for $q \geq 3$, admissible for our purposes.

Therefore, substituting (A.9) into (A.3) we have

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta \\ &\quad = \left(\frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{\ell}}\right)^q a_{\ell,d}^q \int_0^{\frac{\pi}{2}} (\sin\vartheta)^{-q(\frac{d}{2}-1)} \left(\frac{\vartheta}{\sin\vartheta}\right)^{\frac{q}{2}} J_{\frac{d}{2}-1}(L\vartheta)^q (\sin\vartheta)^{d-1} d\vartheta + o(\ell^{-d}) \\ &\quad = \left(\frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{\ell}}\right)^q a_{\ell,d}^q \frac{1}{L} \int_0^{L\frac{\pi}{2}} (\sin\psi/L)^{-q(\frac{d}{2}-1)} \left(\frac{\psi/L}{\sin\psi/L}\right)^{\frac{q}{2}} J_{\frac{d}{2}-1}(\psi)^q (\sin\psi/L)^{d-1}\, d\psi \\ &\qquad + o(\ell^{-d}), \end{aligned} \tag{A.10}$$

where in the last equality we transformed $\psi/L = \vartheta$; it then remains to evaluate the first term in (A.10), which we denote by

$$N_L := \left(\frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{\ell}}\right)^q a_{\ell,d}^q \frac{1}{L} \int_0^{L\frac{\pi}{2}} (\sin\psi/L)^{-q(\frac{d}{2}-1)} \left(\frac{\psi/L}{\sin\psi/L}\right)^{\frac{q}{2}} J_{\frac{d}{2}-1}(\psi)^q (\sin\psi/L)^{d-1}\, d\psi.$$

Now recall that as $\ell \to \infty$

$$\binom{\ell+\frac{d}{2}-1}{\ell} \sim \frac{\ell^{\frac{d}{2}-1}}{(\frac{d}{2}-1)!};$$

moreover (A.1) holds, therefore we find of course that as $L \to \infty$

$$N_L \sim \frac{(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!)^q}{L^{q(\frac{d}{2}-1)+1}} \int_0^{L\frac{\pi}{2}} (\sin\psi/L)^{-q(\frac{d}{2}-1)} \left(\frac{\psi/L}{\sin\psi/L}\right)^{\frac{q}{2}} J_{\frac{d}{2}-1}(\psi)^q (\sin\psi/L)^{d-1}\, d\psi. \tag{A.11}$$

In order to finish the proof of Proposition 2.2, it is enough to check that, as $L \to \infty$

$$L^d \frac{(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!)^q}{L^{q(\frac{d}{2}-1)+1}} \int_0^{L\frac{\pi}{2}} (\sin\psi/L)^{-q(\frac{d}{2}-1)} \left(\frac{\psi/L}{\sin\psi/L}\right)^{\frac{q}{2}} J_{\frac{d}{2}-1}(\psi)^q (\sin\psi/L)^{d-1}\, d\psi \to c_{q;d},$$

actually from (A.10) and (A.11), we have

$$\begin{aligned}
&\lim_{\ell\to+\infty} \ell^d \int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta \\
&\quad = \lim_{L\to+\infty} L^d \frac{(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!)^q}{L^{q(\frac{d}{2}-1)+1}} \int_0^{L\frac{\pi}{2}} (\sin\psi/L)^{-q(\frac{d}{2}-1)} \\
&\qquad \times \left(\frac{\psi/L}{\sin\psi/L}\right)^{\frac{q}{2}} J_{\frac{d}{2}-1}(\psi)^q (\sin\psi/L)^{d-1}\, d\psi.
\end{aligned}$$

Now we write

$$\frac{\psi/L}{\sin\psi/L} = 1 + O\left(\psi^2/L^2\right),$$

so that

$$\begin{aligned}
&L^d \frac{\left(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\right)^q}{L^{q(\frac{d}{2}-1)+1}} \int_0^{L\frac{\pi}{2}} (\sin\psi/L)^{-q(\frac{d}{2}-1)} \left(\frac{\psi/L}{\sin\psi/L}\right)^{\frac{q}{2}} J_{\frac{d}{2}-1}(\psi)^q (\sin\psi/L)^{d-1}\, d\psi \\
&= \left(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\right)^q \int_0^{L\frac{\pi}{2}} \left(\frac{\psi/L}{\sin\psi/L}\right)^{q(\frac{d}{2}-\frac{1}{2})-d+1} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1}\, d\psi \\
&= \left(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\right)^q \int_0^{L\frac{\pi}{2}} \left(1 + O\left(\psi^2/L^2\right)\right)^{q(\frac{d}{2}-\frac{1}{2})-d+1} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1}\, d\psi
\end{aligned}$$

$$= \left(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\right)^q \int_0^{L\frac{\pi}{2}} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1}\, d\psi$$
$$+ O\left(\frac{1}{L^2}\int_0^{L\frac{\pi}{2}} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d+1}\, d\psi\right).$$

Note that as $L \to +\infty$, the first term of the previous summation converges to $c_{q;d}$ defined in (2.12), i.e.

$$\left(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\right)^q \int_0^{L\frac{\pi}{2}} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1}\, d\psi \to c_{q;d}. \tag{A.12}$$

It remains to bound the remainder

$$\frac{1}{L^2}\int_0^{L\frac{\pi}{2}} |J_{\frac{d}{2}-1}(\psi)|^q \psi^{-q(\frac{d}{2}-1)+d+1}\, d\psi = O(1) + \frac{1}{L^2}\int_1^{L\frac{\pi}{2}} |J_{\frac{d}{2}-1}(\psi)|^q \psi^{-q(\frac{d}{2}-1)+d+1}\, d\psi.$$

Now for the second term on the r.h.s.

$$\int_1^{L\frac{\pi}{2}} |J^q_{\frac{d}{2}-1}(\psi)| \psi^{-q(\frac{d}{2}-1)+d+1}\, d\psi \ll \int_1^{L\frac{\pi}{2}} \psi^{-q(\frac{d}{2}-\frac{1}{2})+d+1}\, d\psi$$
$$= O(1 + L^{-q(\frac{d}{2}-\frac{1}{2})+d+2}).$$

Therefore we obtain

$$\left(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\right)^q \int_0^{L\frac{\pi}{2}} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1}\, d\psi$$
$$+ O\left(\frac{1}{L^2}\int_0^{L\frac{\pi}{2}} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d+1}\, d\psi\right)$$
$$= \left(2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\right)^q \int_0^{L\frac{\pi}{2}} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1}\, d\psi + O(L^{-2} + L^{-q(\frac{d}{2}-\frac{1}{2})+d}),$$

so that we have just checked the statement of the present proposition for $q > \frac{2d}{d-1}$. This is indeed enough for each $q \geq 3$ when $d \geq 4$.

It remains to investigate separately just the case $d = q = 3$. Recall that for $d = 3$ we have an explicit formula for the Bessel function of order $\frac{d}{2} - 1$ [36], that is

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin(z),$$

and hence the integral in (2.12) is indeed convergent for $q = d = 3$ by integration by parts.

We have hence to study the convergence of the following integral

$$\frac{8}{\pi^{\frac{3}{2}}} \int_0^{L\frac{\pi}{2}} \left( \frac{\psi/L}{\sin \psi/L} \right) \frac{\sin^3 \psi}{\psi} \, d\psi.$$

To this aim, let us consider a large parameter $K \gg 1$ and divide the integration range into $[0, K]$ and $[K, \frac{\pi}{2}]$; the main contribution comes from the first term, whence we have to prove that the latter vanishes. Note that

$$\int_K^{L\frac{\pi}{2}} \left( \frac{\psi/L}{\sin \psi/L} \right) \frac{\sin^3 \psi}{\psi} \, d\psi \ll \frac{1}{K}, \tag{A.13}$$

where we use integration by part with the bounded function $I(T) = \int_0^T \sin^3 z \, dz$. On $[0, K]$, we write

$$\begin{aligned} \frac{8}{\pi^{\frac{3}{2}}} \int_0^{K} \left( \frac{\psi/L}{\sin \psi/L} \right) \frac{\sin^3 \psi}{\psi} \, d\psi &= \frac{8}{\pi^{\frac{3}{2}}} \int_0^{K} \frac{\sin^3 \psi}{\psi} \, d\psi + O\left( \frac{1}{L^2} \int_0^{K} \psi \sin^3 \psi \, d\psi \right) \\ &= \frac{8}{\pi^{\frac{3}{2}}} \int_0^{K} \frac{\sin^3 \psi}{\psi} \, d\psi + O\left( \frac{K^2}{L^2} \right). \end{aligned}$$

Consolidating the latter with (A.13) we find that

$$\frac{8}{\pi^{\frac{3}{2}}} \int_0^{L\frac{\pi}{2}} \left( \frac{\psi/L}{\sin \psi/L} \right) \frac{\sin^3 \psi}{\psi} \, d\psi = \frac{8}{\pi^{\frac{3}{2}}} \int_0^{K} \frac{\sin^3 \psi}{\psi} \, d\psi + O\left( \frac{1}{K} + \frac{K^2}{L^2} \right).$$

Now as $K \to +\infty$,

$$\frac{8}{\pi^{\frac{3}{2}}} \int_0^{K} \frac{\sin^3 \psi}{\psi} \, d\psi \to c_{3;3};$$

to conclude the proof, it is then enough to choose $K = K(L) \to \infty$ sufficiently slowly, i.e. $K = \sqrt{L}$. □

### A.2. Proofs of Propositions 4.2 and 4.3

**Proof of Proposition 4.2.** The bounds (4.7), (4.8) are known and indeed the corresponding integrals can be evaluated explicitly in terms of Wigner's 3j and 6j coefficients, see [18,19,23]. The bounds in (4.9), (4.10) derive from a simple improvement in the proof of Proposition 2.2 in [23], which can be obtained when focusing only on a subset of the terms (the circulant ones) considered in that reference. In the proof to follow, we exploit repeatedly (4.3), (4.4), (4.5) and (4.6).

Let us start investigating the case $q = 5$:

$$\begin{aligned}
\mathcal{K}_\ell(5;1) &= \int_{(\mathbb{S}^2)^4} |P_\ell(\cos d(x_1,x_2))|^4 \, |P_\ell(\cos d(x_2,x_3))| \\
&\quad \times |P_\ell(\cos d(x_3,x_4))|^4 \, |P_\ell(\cos d(x_4,x_1))| \, dx_1 dx_2 dx_3 dx_4 \\
&\le \int_{(\mathbb{S}^2)^4} |P_\ell(\cos d(x_1,x_2))|^4 \, |P_\ell(\cos d(x_2,x_3))| \, |P_\ell(\cos d(x_3,x_4))|^4 \, dx_1 dx_2 dx_3 dx_4 \\
&\le \int_{(\mathbb{S}^2)^3} |P_\ell(\cos d(x_1,x_2))|^4 \, |P_\ell(\cos d(x_2,x_3))| \\
&\quad \times \left\{ \int_{\mathbb{S}^2} |P_\ell(\cos d(x_3,x_4))|^4 \, dx_4 \right\} dx_1 dx_2 dx_3 \\
&\le O\left(\frac{\log \ell}{\ell^2}\right) \times \int_{\mathbb{S}^2\times\mathbb{S}^2} |P_\ell(\cos d(x_1,x_2))|^4 \left\{ \int_{\mathbb{S}^2} |P_\ell(\cos d(x_2,x_3))| \, dx_3 \right\} dx_1 dx_2 \\
&\le O\left(\frac{\log \ell}{\ell^2}\right) \times O\left(\frac{1}{\sqrt{\ell}}\right) \times \int_{\mathbb{S}^2\times\mathbb{S}^2} |P_\ell(\cos d(x_1,x_2))|^4 \, dx_1 dx_2 \\
&\le O\left(\frac{\log \ell}{\ell^2}\right) \times O\left(\frac{1}{\sqrt{\ell}}\right) \times O\left(\frac{\log \ell}{\ell^2}\right) = O\left(\frac{\log^2 \ell}{\ell^{9/2}}\right);
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_\ell(5;2) &= \int_{(\mathbb{S}^2)^4} |P_\ell(\cos d(x_1,x_2))|^3 \, |P_\ell(\cos d(x_2,x_3))|^2 \\
&\quad \times |P_\ell(\cos d(x_3,x_4))|^3 \, |P_\ell(\cos d(x_4,x_1))|^2 \, dx_1 dx_2 dx_3 dx_4 \\
&\le \int_{(\mathbb{S}^2)^4} |P_\ell(\cos d(x_1,x_2))|^3 \, |P_\ell(\cos d(x_2,x_3))|^2 \, |P_\ell(\cos d(x_3,x_4))|^3 \, dx_1 dx_2 dx_3 dx_4 \\
&\le \int_{(\mathbb{S}^2)^3} |P_\ell(\cos d(x_1,x_2))|^3 \, |P_\ell(\cos d(x_2,x_3))|^2
\end{aligned}$$

$$\times \left\{ \int_{\mathbb{S}^2} |P_\ell(\cos d(x_3,x_4))|^3 \, dx_4 \right\} dx_1 dx_2 dx_3$$

$$\leq O\left(\sqrt{\frac{\log \ell}{\ell^3}}\right) \times \int_{\mathbb{S}^2\times\mathbb{S}^2} |P_\ell(\cos d(x_1,x_2))|^3$$

$$\times \left\{ \int_{\mathbb{S}^2} |P_\ell(\cos d(x_2,x_3))|^2 \, dx_3 \right\} dx_1 dx_2$$

$$\leq O\left(\sqrt{\frac{\log \ell}{\ell^3}}\right) \times O\left(\frac{1}{\ell}\right) \times \int_{\mathbb{S}^2\times\mathbb{S}^2} |P_\ell(\cos d(x_1,x_2))|^3 \, dx_1 dx_2$$

$$\leq O\left(\sqrt{\frac{\log \ell}{\ell^3}}\right) \times O\left(\frac{1}{\ell}\right) \times O\left(\sqrt{\frac{\log \ell}{\ell^3}}\right) = O\left(\frac{\log \ell}{\ell^4}\right).$$

For $q = 6$ and $r = 1$ we simply note that $\mathcal{K}_\ell(6;1) \leq \mathcal{K}_\ell(5;1)$, actually

$$\mathcal{K}_\ell(6;1) = \int_{(\mathbb{S}^2)^4} |P_\ell(\cos d(x_1,x_2))|^5 \, |P_\ell(\cos d(x_2,x_3))|$$

$$\times |P_\ell(\cos d(x_3,x_4))|^5 \, |P_\ell(\cos d(x_4,x_1))| \, dx_1 dx_2 dx_3 dx_4$$

$$\leq \int_{(\mathbb{S}^2)^4} |P_\ell(\cos d(x_1,x_2))|^4 \, |P_\ell(\cos d(x_2,x_3))|$$

$$\times |P_\ell(\cos d(x_3,x_4))|^4 \, |P_\ell(\cos d(x_4,x_1))| \, dx_1 dx_2 dx_3 dx_4$$

$$= \mathcal{K}_\ell(5;1) = O\left(\frac{\log^2 \ell}{\ell^{9/2}}\right).$$

Then we find with analogous computations as for $q = 5$ that

$$\mathcal{K}_\ell(6;2) = \int_{(\mathbb{S}^2)^4} |P_\ell(\cos d(x_1,x_2))|^4 \, |P_\ell(\cos d(x_2,x_3))|^2$$

$$\times |P_\ell(\cos d(x_3,x_4))|^4 \, |P_\ell(\cos d(x_4,x_1))|^2 \, dx_1 dx_2 dx_3 dx_4$$

$$\leq \int_{(\mathbb{S}^2)^4} |P_\ell(\cos d(x_1,x_2))|^4 \, |P_\ell(\cos d(x_2,x_3))|^2$$

$$\times |P_\ell(\cos d(x_3,x_4))|^4 \, |P_\ell(\cos d(x_4,x_1))|^2 \, dx_1 dx_2 dx_3 dx_4$$

$$\leq \int_{\mathbb{S}^2\times\mathbb{S}^2} |P_\ell(\cos d(x_1,x_2))|^4 \, dx_1 \left\{ \int_{\mathbb{S}^2} |P_\ell(\cos d(x_2,x_3))|^2 \, dx_2 \right\}$$

$$\times \left\{ \int_{\mathbb{S}^2} |P_\ell(\cos d(x_3, x_4))|^4 \, dx_4 \right\} dx_3$$
$$= O\left(\frac{\log \ell}{\ell^2}\right) \times O\left(\frac{1}{\ell}\right) \times O\left(\frac{\log \ell}{\ell^2}\right) = O\left(\frac{\log^2 \ell}{\ell^5}\right)$$

and likewise

$$\mathcal{K}_\ell(6;3) = \int_{(\mathbb{S}^2)^4} |P_\ell(\cos d(x_1, x_2))|^3 \, |P_\ell(\cos d(x_2, x_3))|^3$$
$$\times |P_\ell(\cos d(x_3, x_4))|^3 \, |P_\ell(\cos d(x_4, x_1))|^3 \, dx_1 dx_2 dx_3 dx_4$$
$$\leq \int_{(\mathbb{S}^2)^4} |P_\ell(\cos d(x_1, x_2))|^3 \, |P_\ell(\cos d(x_2, x_3))|^3 \, |P_\ell(\cos d(x_3, x_4))|^3 \, dx_1 dx_2 dx_3 dx_4$$
$$= O\left(\frac{\sqrt{\log \ell}}{\ell^{3/2}}\right) \times O\left(\frac{\sqrt{\log \ell}}{\ell^{3/2}}\right) \times O\left(\frac{\sqrt{\log \ell}}{\ell^{3/2}}\right) = O\left(\frac{\log^{3/2} \ell}{\ell^{9/2}}\right).$$

Finally for $q = 7$

$$\mathcal{K}_\ell(7;1) = \int_{\mathbb{S}^2 \times \cdots \times S^2} |P_\ell(\cos d(x_1, x_2))|^6 \, |P_\ell(\cos d(x_2, x_3))|$$
$$\times |P_\ell(\cos d(x_3, x_4))|^6 \, |P_\ell(\cos d(x_4, x_1))| \, dx_1 dx_2 dx_3 dx_4$$
$$\leq \int_{\mathbb{S}^2 \times S^2} |P_\ell(\cos d(x_1, x_2))|^6 \, dx_1 \left\{ \int_{\mathbb{S}^2} |P_\ell(\cos d(x_2, x_3))| \, dx_3 \right\}$$
$$\times \left\{ \int_{\mathbb{S}^2} |P_\ell(\cos d(x_3, x_4))|^6 \, dx_4 \right\} dx_2$$
$$= O\left(\frac{1}{\ell^2}\right) \times O\left(\frac{1}{\ell^{1/2}}\right) \times O\left(\frac{1}{\ell^2}\right) = O\left(\frac{1}{\ell^{9/2}}\right)$$

and repeating the same argument we obtain

$$\mathcal{K}_\ell(7;2) = O\left(\frac{1}{\ell^5}\right) \quad \text{and} \quad \mathcal{K}_\ell(7;3) = O\left(\frac{\log^{9/2} \ell}{\ell^{11/2}}\right).$$

From (4.2), we have indeed computed the bounds for $\mathcal{K}_\ell(q;r)$, $q = 1, \ldots, 7$ and $r = 1, \ldots, q-1$.

To conclude the proof we note that, for $q > 7$

$$\max_{r=1,\ldots,q-1} \mathcal{K}_\ell(q;r) = \max_{r=1,\ldots,[\frac{q}{2}]} \mathcal{K}_\ell(q;r) \leq \max_{r=1,\ldots,3} \mathcal{K}_\ell(6;r) = O\left(\frac{1}{\ell^{9/2}}\right).$$

Moreover in particular

$$\max_{r=2,\dots,[\frac{q}{2}]} \mathcal{K}_\ell(q;r) \le \mathcal{K}_\ell(7;2) \vee \mathcal{K}_\ell(7;3) = O\left(\frac{1}{\ell^5}\right),$$

so that the dominant terms are of the form $\mathcal{K}_\ell(q;1)$. □

**Proof of Proposition 4.3.** The proof relies on the same argument of the proof of Proposition 4.2, therefore we shall omit some calculations. In what follows we exploit repeatedly the inequalities (4.12), (4.13), (4.14) and (4.15).

For $q = 3$ we immediately have

$$\begin{aligned}
\mathcal{K}_\ell(3;1) &= \int_{(\mathbb{S}^d)^4} |G_{\ell;d}(\cos d(x_1,x_2))|^2 \, |G_{\ell;d}(\cos d(x_2,x_3))| \\
&\quad \times |G_{\ell;d}(\cos d(x_3,x_4))|^2 \, |G_{\ell;d}(\cos d(x_4,x_1))| \, dx_1 dx_2 dx_3 dx_4 \\
&\le \int_{(\mathbb{S}^d)^4} |G_{\ell;d}(\cos d(x_1,x_2))|^2 \, |G_{\ell;d}(\cos d(x_2,x_3))| \\
&\quad \times |G_{\ell;d}(\cos d(x_3,x_4))|^2 \, dx_1 dx_2 dx_3 dx_4 \\
&= O\left(\frac{1}{\ell^{d-1}}\right) \times O\left(\frac{1}{\sqrt{\ell^{d-1}}}\right) \times O\left(\frac{1}{\ell^{d-1}}\right) = O\left(\frac{1}{\ell^{2d+\frac{d}{2}-\frac{5}{2}}}\right).
\end{aligned}$$

Likewise for $q = 4$

$$\begin{aligned}
\mathcal{K}_\ell(4;1) &= \int_{(\mathbb{S}^d)^4} |G_{\ell;d}(\cos d(x_1,x_2))|^3 \, |G_{\ell;d}(\cos d(x_2,x_3))| \\
&\quad \times |G_{\ell;d}(\cos d(x_3,x_4))|^3 \, |G_{\ell;d}(\cos d(x_4,x_1))| \, dx_1 dx_2 dx_3 dx_4 \\
&\le \int_{(\mathbb{S}^d)^4} |G_{\ell;d}(\cos d(x_1,x_2))|^3 \, |G_{\ell;d}(\cos d(x_2,x_3))| \\
&\quad \times |G_{\ell;d}(\cos d(x_3,x_4))|^3 \; dx_1 dx_2 dx_3 dx_4 \\
&= O\left(\frac{1}{\ell^{d-\frac{1}{2}}}\right) \times O\left(\frac{1}{\ell^{\frac{d}{2}-\frac{1}{2}}}\right) \times O\left(\frac{1}{\ell^{d-\frac{1}{2}}}\right) = O\left(\frac{1}{\ell^{2d+\frac{d}{2}-\frac{3}{2}}}\right)
\end{aligned}$$

and moreover

$$\begin{aligned}
\mathcal{K}_\ell(4;2) &= \int_{(\mathbb{S}^d)^4} |G_{\ell;d}(\cos d(x_1,x_2))|^2 \, |G_{\ell;d}(\cos d(x_2,x_3))|^2 \\
&\quad \times |G_{\ell;d}(\cos d(x_3,x_4))|^2 \, |G_{\ell;d}(\cos d(x_4,x_1))|^2 \, dx_1 dx_2 dx_3 dx_4
\end{aligned}$$

$$\leq \int_{(\mathbb{S}^d)^4} |G_{\ell;d}(\cos d(x_1,x_2))|^2 \, |G_{\ell;d}(\cos d(x_2,x_3))|^2$$

$$\times \, |G_{\ell;d}(\cos d(x_3,x_4))|^2 \; dx_1 dx_2 dx_3 dx_4$$

$$= O\left(\frac{1}{\ell^{d-1}}\right) \times O\left(\frac{1}{\ell^{d-1}}\right) \times O\left(\frac{1}{\ell^{d-1}}\right) = O\left(\frac{1}{\ell^{3d-3}}\right).$$

Similarly, for $q = 5$ we get the bounds

$$\mathcal{K}_\ell(5;1) = \int_{\mathbb{S}^d \times \cdots \times S^d} |G_{\ell;d}(\cos d(x_1,x_2))|^4 \, |G_{\ell;d}(\cos d(x_2,x_3))|$$

$$\times \, |G_{\ell;d}(\cos d(x_3,x_4))|^4 \, |G_{\ell;d}(\cos d(x_4,x_1))| \, dx_1 dx_2 dx_3 dx_4$$

$$\leq \int_{\mathbb{S}^d \times \cdots \times S^d} |G_{\ell;d}(\cos d(x_1,x_2))|^4 \, |G_{\ell;d}(\cos d(x_2,x_3))|$$

$$\times \, |G_{\ell;d}(\cos d(x_3,x_4))|^4 \, dx_1 dx_2 dx_3 dx_4$$

$$= O\left(\frac{1}{\ell^{d}}\right) \times O\left(\frac{1}{\ell^{\frac{d}{2}-\frac{1}{2}}}\right) \times O\left(\frac{1}{\ell^{d}}\right) = O\left(\frac{1}{\ell^{2d+\frac{d}{2}-\frac{1}{2}}}\right)$$

and

$$\mathcal{K}_\ell(5;2) = O\left(\frac{1}{\ell^{3d-2}}\right).$$

It is immediate to check that

$$\mathcal{K}_\ell(6;1) = \mathcal{K}_\ell(7;1) = O\left(\frac{1}{\ell^{2d+\frac{d}{2}-\frac{1}{2}}}\right), \qquad \mathcal{K}_\ell(6;2) = \mathcal{K}_\ell(7;2) = O\left(\frac{1}{\ell^{2d+d-1}}\right),$$

whereas

$$\mathcal{K}_\ell(6;3) = O\left(\frac{1}{\ell^{2d+d-\frac{3}{2}}}\right) \quad \text{and} \quad \mathcal{K}_\ell(7;3) = O\left(\frac{1}{\ell^{2d+d-\frac{1}{2}}}\right).$$

The remaining terms are indeed bounded thanks to (4.2).

In order to finish the proof, it is enough to note, as for that for $q > 7$

$$\max_{r=1,\dots,q-1} \mathcal{K}_\ell(q;r) = \max_{r=1,\dots,[\frac{q}{2}]} \mathcal{K}_\ell(q;r) \leq \max_{r=1,\dots,3} \mathcal{K}_\ell(6;r) = O\left(\frac{1}{\ell^{2d+\frac{d}{2}-\frac{1}{2}}}\right). \tag{A.14}$$

In particular we have

$$\max_{r=2,\dots,[\frac{q}{2}]} \mathcal{K}_\ell(q;r) \leq \mathcal{K}_\ell(7;2) \vee \mathcal{K}_\ell(7;3) = O\left(\frac{1}{\ell^{3d-1}}\right), \tag{A.15}$$

so that the dominant terms are again of the form $\mathcal{K}_\ell(q;1)$. □

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