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Linear and non-linear effects in structure formation

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To my father and my mother

Abstract

The subject matter of this thesis is the formation of large-scale structure in the universe. Most of the study has dealt with the non-linear evolution of cosmological fluctuations, focusing on the scalar sector of perturbation theory.

The period of transition between the radiation era and the matter era has been largely examined, extending the already known linear results to a non-standard matter model and to a non-linear analysis. The obtained second order solutions for the matter fluctuations variables have been used to find the skewness of the density and velocity distributions, an important statistical estimator measuring the level of non-Gaussianity of a distribution.

In the contest of cosmological perturbations a complete Post-Newtonian (1PN) treatment is presented with the aim of obtain a set of equations suitable in particular for the intermediate scales. The final result agrees with both the non linear Newtonian theory of small scales and the linear general relativistic theory of large scales. Analyzing the limit cases of our approach to 1PN cosmology, we have clarified the link between the Newtonian theory of gravity and General Relativity.

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The chapters 5, 6 and 7 are the themes of two articles in preparation, that will be shortly submitted:

- **"How the universe got its skewness"** M. Bruni, I. Milillo, K.Koyama
- **"Post-Newtonian Cosmology"** I. Milillo, D.Bertacca, M. Bruni

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Outline

This thesis deals with one of the main issues of modern cosmology: the theory of large scale structure in the universe, describing the clustering of matter in galaxies and clusters of galaxies.

Starting from the first studies of Jeans in 1928[1], this topic has been theoretically dealt with the perturbative approach, succeeding in reproducing the numerous experimental data, from the statistical data of density and velocity fields of large scale structure (LSS) [55], to the observed cosmic-microwave-background (CMB) anisotropies [59]. The usefulness of perturbation theory in interpreting such experimental results is based on the fact that in the gravitational instability scenario density fluctuations become small enough at large scales that a perturbative approach suffices to understand their evolution. Therefore, if at very large scale we can use the first-order relativistic perturbation theory (and most of the cosmological perturbation results are within this range), at small scale (up to about 20 Mpc) we necessary need the non-linear theory. On the other hand the imminent large amount of data from the next-generation large-scale galaxy surveys and the completion of Planck Surveyor satellites will give more accurate informations concerning small and intermediate angular scales. It follows that the non linearity in studying cosmological structure formation is becoming essential.

In this thesis I have analyzed, in a non-linear approach, two different aspects of cosmological perturbation theory: the evolution of a cold dark matter component (CDM) in the matter-radiation era and the application of the Post-Newtonian theory to large scale structure formation.

The epoch at which the energy density in matter equals that in radiation has a special significance for the generation of large scale structure and the development of CMB anisotropies, because the perturbations grow at different rates in the two different eras. This reflects on the power spectrum of the matter density perturbation that is the main link between theory and observation [75][76][29].

In the first part of the thesis I analyzed this period of the evolution of the universe using the second order perturbation theory. Under the assumption of considering the modes which enter the horizon during the radiation era, and so small scale perturbations, we shall regard that the radiation density contrast is negligible with respect to the CDM perturbation, so that the radiation is seen

[1] Actually the idea of gravitational instability goes back to Newton (letter to Bentley,1692).

as a smooth background component. At linear order this leads to the known Meszaros effect for which the matter perturbation remains frozen (constant) until the equidensity, and then it grows linearly in the scale factor as the matter era period. The importance of a second-order analysis of this effect is that such constant mode could signal a failure of the linearization to capture the essence of the evolution (this is what typically happens for dynamic systems with a constant marginally stable first-order solution), so that it is interesting to see whether second order terms could be relevant in the CDM evolution.

Another extremely important aspect of the non linear evolution of cosmological perturbations is the consequent arising of non-Gaussian signatures on the statistical distribution of the density field. Since the statistical approach is the only way to test cosmological theories, every theoretical investigations which could explicitly influence the most common used statistical estimators have a relevant importance. Moreover the research of non-Gaussianity has a further importance because it is connected with different types of primordial theory. In fact the simplest inflationary scenarios predict Gaussian initial fluctuations, but there are other models of inflation or models where structure is seeded by topological defects that generate non-Gaussian fluctuations. Besides these primordial sources of non-Gaussianity there are secondary-type sources arising from many different physical processes, among which there is the non-linearity.

Thanks to our second-order analysis of the matter-radiation equidensity period we were able to find the simpler statistical estimator of non-Gaussianity evidence, the skewness. It describes the degree of asymmetry of a distribution. In our case we found the time-dependent skewness of the density and the velocity field of the CDM component, for Gaussian type initial conditions. We recall that in the matter dominated era the solution for the normalized density skewness already exists , it is a constant value [2]. We have generalized such result at the previous period of the universe, the late radiation epoch approaching the matter-radiation equivalence. The final result is that the constant matter era solution is reached very early: looking at our time-dependent function of the normalized skewness, we can see that at the decoupling era, i.e. when the CMB features were imprinted, a large amount of skewness due to non-linearity is already present. Considering that the Planck satellite[3] and its successors may be sensitive to the non-Gaussianity of the cosmological perturbations at the level of second- or higher-order perturbation theory, this is an important result.

In the second part of the thesis I have applied the Post-Newtonian approach[4] to cosmological perturbation theory with the aim of giving a set of equations that are supposed to be valid over a large range of distances, from the large scales down to scales which experience slight departures from linearity.
This requires a careful analysis of the proper perturbation theory to use, i.e. the General Relativistic or the Newtonian theory.

As it is known, at scales much smaller than the horizon scale, the General

[2]Peebles found $S_3 = 34/7$ [77]

[3]http://planck.esa.int

[4]In chapter 6 we will give the appropriate bibliography, now we just cite the textbook [72]
.

Relativistic effects can be neglected and the Newtonian theory of gravitation perfectly suffices to describe the clustering process, mainly if we consider a collisionless matter component in a slow motion regime. It is clear that the smaller is the scales we deal with, the more appropriate is the Newtonian approximation, but also larger is the departure from linearity. At the opposite side instead, if we consider scales closer to the horizon and super-horizon scales where also the causality concept loses sense, we do have to use the General Relativistic theory, with all the difficulties that this implies. First of all there is the gauge problem, connected with the invariance of General Relativity under general coordinate transformations (diffeomorhisms in the language of differential geometry), leading to a not unique definition of perturbation. Fortunately at this scales the linear theory is appropriate and the relativistic equations give a simple result: the constantness of the gauge-invariant primordial potential (the Bardeen potential) that at small scales coincides with the Newtonian gravitational potential.

More difficult is to consider the intermediate scales, where the relativistic effects can not be ignored and the perturbations have a slight non-linear character. In this case the relativistic theory can be performed up to second or third order. Wide effort has been given in this sense also using a gauge-invariant formalism [7][69][19]. Such works maintain the general-relativistic spirit ignoring the fact that the physics is approaching a Newtonian-like form. Moreover their perturbative analysis do not make distinction between geometrical and matter perturbative order.

In my work I tried to obtain a set of perturbation equations which not only reproduce the linear general relativistic theory at very large scales and the non-linear Newtonian laws at small scales, but are suitable for studying the non-linear evolution of geometric and matter perturbations in a almost-Newtonian regime, treating the relativistic terms as corrections. Using the Post Newtonian formalism (1PN), I have approximated the equations of General Relativity in order to clearly show the Newtonian part and the first relativistic corrections, keeping the full non-linearity of the equations. This naturally implies that, from a perturbative point of view, the geometrical potentials and the matter density have a different perturbative order. In practice this means that the final equations can be applied to weak relativistic geometrical and kinematic effects (not so small to be ignored) and highly non linear density cases.

The advantage of such treatment is that it enables one to treat a large variety of scales and phenomena within the same computational technique, removing the highly relativistic terms that the relativistic theory retain only starting from the second-order.
We must however recall that, since its Newtonian point of view, our approach 1PN can not describe the purely general relativistic phenomena as the gravitational waves.

The thesis is organized as follows. Chapter 1 is a brief introduction to the smooth universe, here the basic equations of cosmology are given focusing on the main aspects connected with cosmological perturbation theory. Chapter 2 introduces the gravitational instability in a Newtonian contest illustrating the two approximations that we will use in the rest of the thesis, the Meszaros and the Zel'dovich approximation. In Chapter 3 we summarize the main concepts of the General Relativistic theory of cosmological perturbations, both in the metric and covariant approach. An application of the second approach with the 1+3 gauge-invariant formalism is given in Chapter 4; here we consider a perfect fluid with a not negligible velocity dispersion parametrized by the barotropic parameter $\alpha = \rho/p$. We show the density contrast behavior in the matter dominated era and in the matter-radiation equidensity period, using in this second case the Meszaros approximation. The purpose of this chapter is studying in a qualitatively way how a small pressure component can influence the gravitational collapse, generalizing the Meszaros effect to a fluid with pressure.

In Chapter 5 the second-order analysis of perturbations of a CDM component is performed in the period around the matter-radiation equidensity finding the matter density contrast up to this order. The result is used in order to compute the skewness of the density and the velocity field for which the time-dependent evolutions are plotted as functions of the scale factor. We show how the matter era limit is reached starting from the late radiation era. As a further application to the Meszaros effect we use the Zel'dovich approximation to solve the non-linear perturbative equations for the times around matter-radiation equidensity and afterwards obtaining the generalization of the known Zel'dovich results, only valid in the matter era.

In Chapter 6 the Post-Newtonian approach to cosmological perturbations is presented in the Poisson gauge; with a clarification of the Newtonian limit, we obtain a set of equations enabling to reproduce the results of the standard full relativistic linear theory at large scales and the full non-linear Newtonian theory at small scale; moreover we have the additional terms responsible for the first relativistic corrections, these are the most significant non-linear terms that are not negligible at intermediate scales. Through a changing of variables the equations are written as evolution and constraint equations, consequently some variables have not a dynamic character like the gravitational potential in the Newtonian (0PN) theory.

In Chapter 7 the relativistic covariant approach to cosmological perturbation theory is analyzed within the Post-Newtonian scheme. The Newtonian terms and the first relativistic corrections of the velocity covariant gradient tensor are found in order to obtain the 1PN Raychauduri equation. Further applications at this approach are the current object of investigations, we give a short summary of them.

Chapter 1

Introduction

Even if some important aspects of large-scale structure formation can be simply explained within the context of Newtonian theory, Cosmology is essentially a general relativistic topic so that the main basic notions of the Einstein theory are implied or briefly mentioned in this first chapter.
For the following basic concepts we referred to the textbooks [62] and [32].

1.1 Basic Cosmology

1.1.1 Background dynamics

The basic assumption of Cosmology is the homogeneity and isotropy of the universe at large scales (bigger than $300h^{-1}$Mpc[1]). This *cosmological principle* implies that the metric describing the universe in the General Relativistic theory, is the Friedmann-Robertson-Wallker metric (FRW):

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta + sin^2\theta d\varphi^2) \right], \tag{1.1}$$

where $a(t)$ is the scalar factor, t is the cosmic time and K is the spatial curvature. This latter describes the geometry of the space-time and can assume the values $K = -1, 0, 1$ respectively for an open, flat or closed universe. In all our work we will consider a flat universe so that in Cartesian coordinates the metric can be written as

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j. \tag{1.2}$$

[1] h is the uncertainty parameter of the Hubble constant: $H_0 = 100h$ Km·s^{-1}Mpc^{-1}. Observations suggests that it lies between 0.5 and 0.8 [51].

with δ_{ij} the Kronecker delta.
For further use, let us introduce the *conformal time* defined by

$$d\tau = \frac{dt}{a(t)}. \tag{1.3}$$

Starting point for studying the cosmological evolution of the universe are the Einstein's equations of General Relativity:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}, \tag{1.4}$$

where $G_{\mu\nu}, R_{\mu\nu}, R$ are respectively the Einstein tensor, the Ricci tensor and the Ricci scalar of the metric, Λ is the cosmological constant[2] and $T_{\mu\nu}$ is the energy-momentum tensor of the matter.
In our isotropic and homogeneous model the energy-momentum tensor is given by the following perfect fluid expression:

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu + p g_{\mu\nu}, \tag{1.5}$$

with ρ and p the energy density and pressure of the fluid and u_μ the 4-velocity of comoving observers.
The Einstein's equations (1.4), with the FRW metric (1.2) and defining the Hubble rate $H = \dot{a}/a$, are the so called Friedmann equations:

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \tag{1.6}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}, \tag{1.7}$$

here the dot denotes derivative with respect to time.
From the Bianchi identity

$$T^\mu{}_{\nu;\mu} = 0, \tag{1.8}$$

where the semicolon denotes the covariant derivative with respect to the coordinates x^μ, we obtain the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{1.9}$$

Equation (1.6) is often written as

$$\Omega + \Omega_\Lambda = 1, \tag{1.10}$$

with $\Omega = \rho/\rho_c$ and $\Omega_\Lambda = \Lambda/3H^2$, where the critical density is $\rho_c = 3H^2/(8\pi G)$.
The matter component is characterized once we give an equation of state relating the energy density and the pressure. Supposing to have a barotropic fluid so that

$$p = \alpha\rho, \tag{1.11}$$

[2]See [21] for a review.

with α a positive constant, the continuity equation (1.9) and the Friedmann equation (1.6) give

$$\rho \propto a^{-3(1+\alpha)}, \qquad a \propto t^{2/(3+3\alpha)}, \tag{1.12}$$

where temporal behavior of the scale factor is valid for $K = \Lambda = 0$.
Three cases are remarkable: when the universe is dominated by a pressureless fluid, generally called dust ($\alpha = 0$), by the radiation ($\alpha = 1/3$) or by the cosmological constant (Λ is equivalent to a fluid with $\alpha = -1$). We obtain

$$\rho_d \propto a^{-3} \qquad a \propto t^{\frac{2}{3}}, \tag{1.13a}$$

$$\rho_r \propto a^{-4} \qquad a \propto t^{\frac{1}{2}}, \tag{1.13b}$$

$$\rho_\Lambda = \text{const} \qquad a \propto e^{Ht}, \quad H = \text{const}. \tag{1.13c}$$

The current model of universe (ΛCDM), provided with many experimental tests, includes these three components: a matter component, formed mainly by a dust fluid (or cold dark matter, CDM), the cosmological constant and a relativistic component of radiation and neutrinos. The values of each density parameter today is found to be

$$\Omega_{tot,0} = \Omega_{mat,0} + \Omega_{\Lambda,0} + \Omega_{rel,0} \approx 1, \tag{1.14a}$$

$$\Omega_{mat,0} = \Omega_{CDM,0} + \Omega_{bar,0} \approx 0.3, \tag{1.14b}$$

$$\Omega_{\Lambda,0} \approx 0.7, \tag{1.14c}$$

$$\Omega_{rel,0} = \Omega_{r,0} + \Omega_{\nu,0} \approx 10^{-5}, \tag{1.14d}$$

where 0 denotes the quantity evaluated today. $\Omega_{bar,0}$ is the baryons density; from nucleosynthesis theory we obtain $\Omega_{bar,0} \approx 0.045$ but the observation data give a 35% smaller value, suggesting a baryonic dark matter presence. Note that we have allowed the presence of a neutrino density (Ω_ν) in the relativistic component.
The salient features of such model are
-the flatness of the universe,
-the large amount of dark matter, mainly made of non-baryonic matter,
-the presence of the cosmological constant, leading to the acceleration of the universe (1.13c).
With all the components the Friedmann equation (1.6), using (1.13), can be written as

$$\dot{a}^2 = a_o^2 H_0^2 \left[\Omega_{mat,0} \left(\frac{a_0}{a} \right) + \Omega_{rad,0} \left(\frac{a_0}{a} \right)^2 + \Omega_{\Lambda,0} \left(\frac{a_0}{a} \right)^{-2} \right]; \tag{1.15}$$

the different transition periods of the universe are given by the values of the density parameters (1.14).
Defining the redshift z

$$z = \frac{a_0}{a} - 1, \tag{1.16}$$

we have that at the equidensity epoch $\Omega_{rad} = \Omega_{mat}$ and for (1.13a) and (1.13b) the redshift satisfies

$$z_{eq} + 1 = \frac{\Omega_{mat,0}}{\Omega_{rad,0}}. \tag{1.17}$$

From the current data [51] $z_{eq} \approx 3200$. For $z >> z_{eq}$ the universe is radiation dominated and just the second term in the right hand part of (1.15) is relevant; for $z << z_{eq}$ the matter is the only considerable component until the cosmological constant becomes important and the universe starts to accelerate. This happens at $z \approx 0.2$.

1.1.2 Inflation

Inflation is an extensive topic, dealing with the early epoch of the universe. It constitutes a wide field of research, with a large number of different inflationary models.
Here we just present the original simpler idea of inflation as an accelerated expansion phase. In fact, even if in this thesis we will not consider any inflationary subject, we can not ignore such topic because it provides the initial conditions for structure formation and generally for cosmological perturbation theory.

The inflation theory was primarily introduced as an explanation of the two main problems of the standard cosmological model, the flatness problem and the horizon problem [48]. The former concerns the impressive fine tuning of the initial condition for Ω_{tot}; in fact its present value $\Omega_{tot,0} \approx 1$ implies that at big bang epoch it should be equal to 1 with a precision of 10^{-60}. This exact flatness is theoretically unsatisfactory.
The horizon problem is due to the fact that causally unconnected regions in the sky, have the same temperature, i.e. the CMB results isotropic over scales larger than the horizon scale at decoupling ($90h^{-1}$Mpc, corresponding to an angle $\theta \approx 1.5°$).
Both this problems are explained assuming an early epoch in which the scale factor of the universe accelerates rapidly. For a FRW universe this means

$$\ddot{a} = -\frac{4}{3}\pi G(\rho + p) > 0 \rightarrow p < -\frac{\rho}{3}. \tag{1.18}$$

This condition can be obtained introducing a scalar field φ minimally coupled to gravity. In the FRW metric, the Klein-Gordon equation for φ is

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0, \tag{1.19}$$

where V is the potential of the scalar field and the prime is the derivative with respect to φ. The energy density and the pressure of the scalar field are

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V, \tag{1.20}$$

$$p = \frac{1}{2}\dot{\varphi}^2 - V. \tag{1.21}$$

The condition (1.19) is equivalent to suppose that the potential V is quite flat, so that the scalar field slowly rolls over it, that is

$$\ddot{\varphi} << V. \tag{1.22}$$

Practically the inflation condition is implemented by the following *slow-roll parameters*

$$\epsilon \equiv \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2, \tag{1.23a}$$

$$\eta \equiv \frac{1}{3}\frac{V''}{H^2}; \tag{1.23b}$$

they have to satisfy $\epsilon << 1$, $\eta << 1$.
In this scenario all is homogeneous and isotropic, the universe during inflation being composed by a uniform scalar field and a uniform background metric. What really happens is that the fields fluctuate quantum mechanically creating the primordial perturbation responsible for most of the inhomogeneities and anisotropies in the universe.

1.1.3 Power spectrum of scalar perturbations

In addition to explain the horizon and the flatness problem, inflation is also a mechanism for generating primordial perturbations. They were produced as almost perfectly Gaussian quantistic fluctuations in the very early period before inflation started, when the relevant scales were causally connected. Then these scales are whisked outside the horizon by the acceleration period of inflation, and reenter much later, serving as initial condition for scalar perturbation theory. This initial condition is given through the power spectrum.
The power spectrum is defined as the variance in the Fourier space (in fact, as we will see, it is convenient to describe perturbations in terms of Fourier modes); for the field ϕ describing the gravitational potential we define the power spectrum $P_\phi(\mathbf{k})$ with

$$\left\langle \phi(\mathbf{k})\phi^*(\mathbf{k}')\right\rangle = (2\pi)^3 P_\phi(\mathbf{k})\delta^3(\mathbf{k}-\mathbf{k}'). \tag{1.24}$$

The procedure to find this statistical function uses a conserved quantity for super-horizon perturbations, the curvature variable ζ. It is a linear combination of ϕ and $\delta\varphi$ and allows to find ϕ at some time t^* post inflation as a function of $\delta\varphi$ evaluated at the horizon crossing time ($aH = k$). The relation is

$$\phi|_{\text{post inflation}} = \frac{2}{3}aH\frac{\delta\varphi}{\dot{\varphi}_b}\Big|_{\text{horizon crossing}}, \tag{1.25}$$

where the index b refers to background.
Using (1.24), (1.25) and (1.23a) we can obtain the power spectrum at the initial

time t^* before the modes of interest reenter the horizon but well after inflation started:

$$P_\phi = \frac{4}{9}\left(\frac{aH}{\dot{\varphi}_b}\right)^2 P_{\delta\varphi}\bigg|_{aH=k}. \tag{1.26}$$

As we said before, $P_{\delta\varphi}$ has a quantum-mechanics origin. In order to quantize the scalar field we rewrite the field equation (the perturbed version of eq. (1.19)) so that it looks like the simple harmonic oscillator equation. In this way the ordinary Hisemberg type quantization is easily performed, giving

$$P_{\delta\varphi} = \frac{H^2}{2k^3}. \tag{1.27}$$

We recall that this result is not exact; some term has been neglected thanks to the slow-roll condition.
The final expression for the initial power spectrum, putting (1.27) in (1.26) and writing in terms of the slow-roll parameters, is

$$P_\phi = \frac{8\pi G}{9k^3}\frac{H^2}{\epsilon}\bigg|_{aH=k}. \tag{1.28}$$

The parameter ϵ weakly depends on k so that in first approximation $k^3 P_\phi(k)$ is constant (in k); such a spectrum is called *scale-invariant* or *scale-free*.
Here we just gave a brief summary of the main results, the computations requires the perturbation theory which we will treat in Chapter 3 (in particular the ϕ variable here introduced corresponds to the scalar field of the Newtonian gauge in the negligible anisotropic stress case).
An analogous study for tensor perturbation leads again to a Gaussian type of perturbations, with an almost scale-invariant spectrum but, with respect to the scalar result, it is multiplied by an extra factor ϵ.
Instead vector type perturbations do not derive from any simple inflationary model.
In conclusion inflation predicts an almost scale-invariant spectrum of scalar and tensorial perturbations, the latter smaller than the former.

1.2 Overview of Structure Formation. Gravitational instability

Once the primordial perturbations are created at the initial time t^*, structure formation derives from gravitational instability: the seed of density inhomogeneity from quantum fluctuations grows for the influence of its gravitational force; the presence of pressure contrasts this growth, so that the balance between pressure forces and gravitational force governs the temporal behavior of density contrast. This scenario is complicated by the fact that the universe is

expanding, the expansion rate inhibits the enhancing of perturbations and introduces a characteristic length scale, the Hubble scale, which depends on the model and indicates the transition between small and large scales.
Considering the scale of the perturbations is essential in order to use the correct theory of structure formation; at small scales the Newtonian theory suffices to describe the physical processes but at large scales the general relativistic effects become relevant so that the full Einstein theory is needed.
In the Newtonian contest the first studies on density perturbations from gravitational instability where performed by Jeans in 1902. He considered a gravitating fluid in a static universe; at once, also in this simple case, the qualitative results are important and meaningful: a small density perturbation, described by a density contrast variable δ, at first order in the variations, follows this second degree differential equation:

$$\frac{\partial^2 \delta}{\partial t^2} = c_s^2 \nabla^2 \delta + 4\pi G \rho_b \delta, \tag{1.29}$$

where ρ_b is the background density and c_s is the sound velocity in the fluid defined as

$$c_s = \sqrt{\frac{\delta P}{\delta \rho}}. \tag{1.30}$$

For the presence of the Laplacian term in eq. (1.29), it is convenient to work in the Fourier space where all the variables are characterized by the wave number k. The linearity of the equation assures that we do not have convolution terms, so that the equation is easily solvable; two different regimes are found, relating the fluid proprieties with the k mode of the perturbation: an harmonic regime when $k > k_J = \sqrt{4\pi G \rho_b / c_s^2}$ and a growing exponential regime for $k < k_J$. The critic wavenumber k_J defines a critical length, the Jeans length

$$\lambda_J = \sqrt{\frac{\pi c_s^2}{G \rho_b}}; \tag{1.31}$$

it gives the limit size of the perturbation, under this value the pressure forces rule over gravity preventing the overdensity growth. Instead, for scales larger than λ_J density perturbation grows exponentially under the gravitational collapse. This analysis, neglecting the expansion and the curvature of the universe, can be valid only locally in a small region of the universe and so is not good for a cosmological contest. Adding the expansion rate has the effect of changing the growing progress from exponential type to power law type (in the flat case) and the harmonic solution in a damped oscillation. The Jeans condition remains the same apart for a scale factor contribution, so that for the gravitational collapse the wavenumber has to satisfy $k < ak_J$.
We can understand, also with this qualitative analysis, how much is important the presence of the sound speed c_s, and then the pressure: the smaller its value is, the smaller is λ_J and so the bigger is the part of the universe whose density

contrast is undergoing a gravitational collapse.
A fundamental role in structure formation is given by the dust component, a non collisional fluid that interacts only gravitationally. It constitutes an approximation for the cold dark matter in which the velocity dispersion is neglected and, according with the ΛCDM model, as we have seen, it is believed to dominate the other components of the universe for its last period, until the cosmological constant becomes relevant. In such period, as we will see in chapter 3, the linear theory of cosmological perturbation predicts that the density contrast grows linearly with the scale factor. This process is essential because it creates the potential wells that favorite the growth of density contrast of the baryons, the components of visible structures in our universe.

Chapter 2

Newtonian Cosmological Perturbation Theory

Introduction

In this chapter we review the Newtonian theory of gravitational instability , focusing on the main aspects that will be fundamental for the entire thesis work. The validity of such treatment in structure formation is clearly restricted at small scales .
After the usual presentation of the basic equations in comoving coordinates, we introduce, in the first section, the kinematic quantities related with the velocity gradient. In terms of these variables the evolution equation for the tidal force is obtained [58], emphasizing the non local aspect of the Newtonian theory.
The subsequent sections deal with the most important approximated ways of resolution: starting from the simple linear treatment in the matter era, we analyze the matter-radiation equidensity epoch illustrating the so called Meszaros approximation [71]; then we consider the non linear case introducing the famous Zel'dovich approximation [100][83].

2.1 Full non linear Newtonian Theory

2.1.1 Field and conservation equations

We consider the universe filled by a homogeneous fluid, and scales well within the horizon. The expansion of the universe implies that a good system of reference is that of the *comoving observers*, the observers that measure always zero momentum density. They are moving with the expansion of the universe, including the effect of its inhomogeneities.
The physical position of a comoving observer in Cartesian space coordinates is given by

$$\mathbf{r}(t) = a(t)\mathbf{x}(t), \tag{2.1}$$

where $a(t)$ is the scale factor of the expansion and the time dependence of the *comoving position* $\mathbf{x}$ is due to the perturbation (otherwise, in the unperturbed universe, it is constant). Taking the derivative with respect to the time, we have the total velocity, in which we can identify the purely homogeneous expansion part and the component due to the perturbation

$$\mathbf{V}(\mathbf{x},t) = \frac{d\mathbf{r}(t)}{dt} = H(t)\mathbf{r} + \frac{d\mathbf{x}}{dt}a(t) = H(t)\mathbf{r} + \mathbf{v}(\mathbf{x},t). \tag{2.2}$$

The *peculiar velocity* is defined as

$$\mathbf{v}(\mathbf{x},t) = \frac{d\mathbf{x}}{dt}a(t), \tag{2.3}$$

and $H(t) = a^{-1}da/dt$ is the Hubble parameter.
Differentiating (2.2) we obtain the acceleration of the comoving observer

$$\frac{d\mathbf{V}}{dt} = \frac{d^2a}{dt}\mathbf{x} + \frac{d\mathbf{v}}{dt} + H\mathbf{v} = \frac{d^2a}{dt^2}\mathbf{x} + \mathbf{g}, \tag{2.4}$$

where the first term is the acceleration of the local uniform-expansion observer and the second term is the acceleration of the comoving observer relative to this observer, called the *peculiar acceleration*.

Now we start studying the evolution of the perturbations.
The fluid element is characterized by a mass density $\rho(\mathbf{x},t)$ and a pressure $p(\mathbf{x},t)$, position $\mathbf{r}$, and velocity $\mathbf{V}$, for which (2.1) and (2.2) hold.
The Newtonian laws governing the dynamics are:

-the **continuity equation** , describing the mass conservation:

$$\frac{\partial \rho}{\partial t} + \mathbf{V}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{V} = 0; \tag{2.5}$$

-the **Euler equation** , the equation of motion for the fluid element:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V} = -\frac{1}{\rho}\nabla p - \nabla\Psi, \tag{2.6}$$

where Ψ is the gravitational potential;
-the **Poisson equation** , the field equation for Newtonian gravity:

$$\nabla^2\Psi = 4\pi G\rho. \tag{2.7}$$

These generic equations become the cosmological system we need once we have factorized out the uniform expansion effect; using (2.1) and (2.2) and writing the components for the gradient operator in the form

$$\nabla \to \frac{\partial}{\partial r^i} = \frac{1}{a}\frac{\partial}{\partial x^i} \to \frac{1}{a}\nabla_x, \tag{2.8}$$

we end up with the following system:

$$\frac{\partial \rho}{\partial t} + 3\rho H + \frac{1}{a}\nabla_x \cdot (\rho \mathbf{v}) = 0, \tag{2.9a}$$

$$\frac{\partial \mathbf{v}}{\partial t} + H\mathbf{v} + (\dot{H} + H^2)\mathbf{r} + \frac{\mathbf{v}}{a} \cdot \nabla_x \mathbf{v} = -\frac{1}{a\rho}\nabla_x p - \frac{1}{a}\nabla_x \Psi, \tag{2.9b}$$

$$\nabla_x^2 \Psi = 4\pi G \rho a^2. \tag{2.9c}$$

In the unperturbed universe the velocity is zero and the system above gives the Friedmann equation (1.6)

$$\dot{H} + H^2 = -\frac{4\pi G}{3}\rho_b, \tag{2.10}$$

$$H^2 = \frac{8\pi G}{3}\rho_b, \tag{2.11}$$

with the index b for background.
Now we specify the perturbation writing

$$\rho(\mathbf{x}, t) = \rho_b(t) + \delta\rho(\mathbf{x}, t) = \rho_b(1 + \delta(\mathbf{x}, t)), \tag{2.12a}$$

$$p(\mathbf{x}, t) = p(t) + \delta p(\mathbf{x}, t), \tag{2.12b}$$

$$\Psi(\mathbf{x}, t) = \Psi_b + \phi(\mathbf{x}, t), \tag{2.12c}$$

with ϕ the *peculiar gravitational potential* .
The exact perturbations equations, obtained from (2.9) using (2.12) and subtracting the background equations, are:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla_x \cdot \mathbf{v} + \frac{1}{a}\nabla_x \cdot (\delta \mathbf{v}) = 0, \tag{2.13a}$$

$$\frac{\partial \mathbf{v}}{\partial t} + H\mathbf{v} + \frac{\mathbf{v}}{a} \cdot \nabla_x \mathbf{v} = -\frac{1}{a\rho}\nabla_x \delta p - \frac{1}{a}\nabla_x \phi, \tag{2.13b}$$

$$\nabla_x^2 \phi = 4\pi G a^2 \delta\rho. \tag{2.13c}$$

The system is non linear, due to the quadratic terms in the continuity and Euler equation.
In order to simplify the notation, we omit he index x in the gradient operator and introduce the convective derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla, \tag{2.14}$$

so that we have

$$\frac{d\delta}{dt} + \frac{(1+\delta)}{a}\nabla \cdot \mathbf{v} = 0, \tag{2.15a}$$
$$\frac{d\mathbf{v}}{dt} + H\mathbf{v} = -\frac{1}{a\rho}\nabla\delta p - \frac{1}{a}\nabla\phi, \tag{2.15b}$$
$$\nabla^2\phi = 4\pi G a^2 \delta\rho. \tag{2.15c}$$

We recall that this result is valid for a single fluid model, without cosmological constant Λ and in the flat background geometry. The multifluid case with Λ is easily obtained when we consider collisionless fluids. What we have to do is to add the continuity and Euler equations for every single component and consider, in the Poisson equation, the total density variable. Therefore for the ith fluid we have

$$\frac{d\delta_{(i)}}{dt} + \frac{(1+\delta_{(i)})}{a}\nabla \cdot \mathbf{v}_{(i)} = 0, \tag{2.16a}$$
$$\frac{d\mathbf{v}_{(i)}}{dt} + H\mathbf{v}_{(i)} = -\frac{1}{a\rho}\nabla\delta p_{(i)} - \frac{1}{a}\nabla\phi, \tag{2.16b}$$

and the gravitational potential is given by

$$\nabla^2\phi = 4\pi G a^2 \sum_i \delta_{(i)}\rho_{(i)}. \tag{2.16c}$$

Each species feels its own pressure gradient, but both the gravitational acceleration and the background Hubble parameter depend on the total density. At the background level all the density components are connected by the Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_{(i)b} + \frac{\Lambda}{3}. \tag{2.17}$$

This equation is obtained from (2.9) with $v = 0$ and considering a total density variable in the Poisson equation of the form $\rho_{tot} = \sum_i \rho_{(i)} + \Lambda/8\pi G$.

2.1.2 Velocity Gradient Tensor

The system (2.13) describes the fully non linear dynamics of the inhomogeneous fluid, until now no approximations have been done. Now, retaining the full non linearity, we want to characterize some propriety of the velocity field, in order to obtain a different form of the Euler equation. Considering the fluid flow and two neighboring points, their relative velocity is given by

$$d\mathbf{v} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right) \cdot d\mathbf{x}, \tag{2.18}$$

where $(\partial\mathbf{v}/\partial\mathbf{x})$ is the peculiar velocity gradient tensor. It is a measure of the steepness of velocity variation in the flow field as one moves from one location to another at a given instant in time. The velocity gradient tensor contains all the information on rotation and deformation of the material elements: splitting it in its symmetric and skew-symmetric part the deformation tensor and the vorticity tensor are defined:

$$\frac{\partial v_i}{\partial x^j} = v_{i,j} = \Theta_{ij} + \omega_{ij}. \tag{2.19}$$

The deformation tensor

$$\Theta_{ij} \equiv \frac{1}{2}(v_{i,j} + v_{j,i}) \tag{2.20}$$

is the rate of strain tensor, describing the rate at which neighboring material particles move with respect to each other independently of superimposed rigid rotations. This deformation can be a linear or a shear strain, the former producing changes in volume of the fluid element, the latter producing changes in angle between two line segments from the undeformed state to the deformed state. So we define the expansion scalar and the shear[1]:

$$\theta \equiv v_{i,i}, \tag{2.21}$$

$$\sigma_{ij} \equiv v_{(i,j)} - \frac{1}{3}\theta\delta_{ij}, \tag{2.22}$$

note that the linear strain is the first invariant form of the tensor Θ_{ij}, therefore it does not depend on the coordinate system.
The vorticity tensor

$$\omega_{ij} \equiv v_{[i,j]} \tag{2.23}$$

contains all the information about the local rates of rotation of material elements. If it is null (irrotational fluid), the velocity field is generated by a scalar potential:

$$\tag{2.24}$$

$$v_i \propto \varphi_{,i}. \tag{2.25}$$

The evolution equation for expansion, shear and vorticity are found by taking the gradient of Euler equation (2.15b) and taking the trace, the symmetric and

[1] Round and square brackets denote respectively symmetrization and antisymmetrization of the indices

the skew-symmetric part.
At this point it is convenient to use the scale factor $a(t)$ as a new time variable instead of t; in order to do this we introduce the new velocity defined as:

$$\mathbf{u} \equiv \frac{d\mathbf{x}}{da} = \frac{\mathbf{v}}{Ha^2} \tag{2.26}$$

and

$$u_{i,j} = \tilde{\theta}\delta_{ij} + \tilde{\sigma}_{ij} + \tilde{\omega}_{ij}. \tag{2.27}$$

In the single fluid case in presence of the cosmological constant, using (2.17), the Euler equation now is written as

$$\frac{d\mathbf{u}}{da} + \frac{\mathbf{u}}{a}\left(\frac{3}{2} + \frac{\Lambda}{2H^2}\right) + \frac{3}{2a}\nabla\tilde{\phi} = -\frac{\tilde{c}_s^2}{a}\nabla\delta\ , \tag{2.28}$$

with the rescaled potential and sound velocity

$$\tilde{\phi} = \frac{2}{3}H^{-2}a^{-3}\phi, \tag{2.29}$$

$$\tilde{c}_s^2 = H^{-2}a^{-3}c_s^2 \tag{2.30}$$

(note that the factor $H^{-2}a^3$ is constant in the Einstein-De Sitter[2] universe).
Now, from (2.28), the evolution equation for the gradient velocity tensor components (refering to $\mathbf{u}$) are

$$\frac{d\tilde{\theta}}{da} + \frac{1}{3}\tilde{\theta}^2 + 2\tilde{\sigma}^2 - 2\tilde{\omega}^2 + \frac{3}{2a}(B\tilde{\theta} + \nabla^2\tilde{\phi}) = -\frac{\tilde{c}_s^2}{a}\nabla^2\delta, \tag{2.31a}$$

$$\frac{d\tilde{\sigma}_{ij}}{da} + \frac{2}{3}\tilde{\theta}\tilde{\sigma}_{ij} + \tilde{\sigma}_{il}\tilde{\sigma}_{lj} + \tilde{\omega}_{il}\tilde{\omega}_{lj} - \frac{3}{2a}(B\tilde{\sigma}_{ij} + \tilde{E}_{ij}) = -\frac{\tilde{c}_s^2}{a}\left(\delta_{,ij} - \frac{1}{3}\nabla^2\delta\delta_{ij}\right), \tag{2.31b}$$

$$\frac{d\tilde{\omega}_{ij}}{da} + \frac{2}{3}\tilde{\theta}\tilde{\omega}_{ij} + 2\tilde{\sigma}_{[il}\tilde{\omega}_{lj]} + B\frac{3}{2a}\tilde{\omega}_{ij} = 0, \tag{2.31c}$$

where $B = \left(1 + \Lambda/3H^2\right)$ and

$$\tilde{\sigma}^2 = \frac{1}{2}\tilde{\sigma}_{ij}\tilde{\sigma}_{ij}, \qquad \tilde{\omega}^2 = \frac{1}{2}\tilde{\omega}_{ij}\tilde{\omega}_{ij}, \tag{2.32}$$

$$\tilde{E}_{ij} = \tilde{\phi}_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\tilde{\phi}. \tag{2.33}$$

[2]The Einstein-De Sitter universe is a flat model with a single pressureless fluid and without cosmological constant

Now we can use the set of equation (2.31) instead of the Euler equation in (2.15). Using also in (2.15a) and (2.15c) the scale factor a as the time variable and the new definitions (2.26) and (2.29), the system becomes

$$\frac{d\delta}{da} + (1+\delta)\tilde{\theta} = 0, \tag{2.34a}$$

$$\frac{d\tilde{\theta}}{da} + \frac{1}{3}\tilde{\theta}^2 + 2\tilde{\sigma}^2 - 2\tilde{\omega}^2 + \frac{3}{2a}(B\tilde{\theta} + \nabla^2\tilde{\phi}) = -\frac{\tilde{c}_s^2}{a}\nabla^2\delta, \tag{2.34b}$$

$$\frac{d\tilde{\sigma}_{ij}}{da} + \frac{2}{3}\tilde{\theta}\tilde{\sigma}_{ij} + \tilde{\sigma}_{il}\tilde{\sigma}_{lj} + \tilde{\omega}_{il}\tilde{\omega}_{lj} - \frac{3}{2a}(B\tilde{\sigma}_{ij} + \tilde{E}_{ij}) = -\frac{\tilde{c}_s^2}{a}\left(\delta_{,ij} - \frac{1}{3}\nabla^2\delta\delta_{ij}\right), \tag{2.34c}$$

$$\frac{d\tilde{\omega}_{ij}}{da} + \frac{2}{3}\tilde{\theta}\tilde{\omega}_{ij} + 2\tilde{\sigma}_{[il}\tilde{\omega}_{lj]} + B\frac{3}{2a}\tilde{\omega}_{ij} = 0, \tag{2.34d}$$

$$\nabla^2\tilde{\phi} = \frac{\delta}{a}\left(\frac{\rho_b}{\rho_b + 3\Lambda/8\pi G}\right). \tag{2.34e}$$

2.1.3 Tidal Force. Non locality of Newtonian theory

Now we consider an irrotational pressureless fluid; the zero pressure assumption is a good approximation if we consider that most of the mass in the large-scale structure formation period is in the form of cold dark matter (CDM); moreover eq.(2.34d) is homogeneous in $\tilde{\omega}_{ij}$ and so if we start with zero vorticity we remain with this propriety.
The final system for a pressureless fluid is

$$\frac{d\delta}{da} + (1+\delta)\tilde{\theta} = 0, \tag{2.35a}$$

$$\frac{d\tilde{\theta}}{da} + \frac{1}{3}\tilde{\theta}^2 + 2\tilde{\sigma}^2 + \frac{3}{2a}(B\tilde{\theta} + \nabla^2\tilde{\phi}) = 0, \tag{2.35b}$$

$$\frac{d\tilde{\sigma}_{ij}}{da} + \frac{2}{3}\tilde{\theta}\tilde{\sigma}_{ij} + \tilde{\sigma}_{il}\tilde{\sigma}_{lj} - \frac{3}{2a}(B\tilde{\sigma}_{ij} + \tilde{E}_{ij}) = 0, \tag{2.35c}$$

$$\nabla^2\tilde{\phi} = \frac{\delta}{a}C, \tag{2.35d}$$

with $C = \rho_b/(\rho_b + 3\Lambda/8\pi G)$. This system will be useful later.
Now we want to obtain an evolution equation for the gravitational potential, in particular for its tracefree gradient, the tidal force $\tilde{E}_{ij}$. We proceed writing the Euler equation (2.28) in terms of the rescaled velocity potential ($\tilde{\varphi}_{,i} = u_i$); we obtain

$$\frac{\partial\tilde{\varphi}}{\partial a} + \frac{1}{2}\nabla\tilde{\varphi}\cdot\nabla\tilde{\varphi} = -\frac{3}{2a}(B\tilde{\varphi} + \tilde{\phi}). \tag{2.36}$$

An evolution equation for the scalar $\tilde{\phi}$ can be found combining equations (2.35d) and (2.35a), we obtain

$$\nabla \cdot \left[\frac{\partial}{C\partial a} \nabla(a\tilde{\phi}) + (1 + \frac{a}{C}\nabla^2\tilde{\phi})\nabla\tilde{\varphi} \right] = 0. \tag{2.37}$$

The previous equation has the form $\nabla \cdot \mathbf{S}$, where $\mathbf{S}$ is the expression in the square brackets. For the Helmholtz theorem of the vector calculus we have

$$\mathbf{S}(\mathbf{x}) = \nabla \times \left[\frac{1}{4\pi} \int \frac{d^3x'}{|\mathbf{x} - \mathbf{x}'|} \nabla_{x'} \times \mathbf{S}(\mathbf{x}') \right], \tag{2.38}$$

so that substituting $\mathbf{S}$ we obtain

$$\frac{\partial}{C\partial a} \nabla(a\tilde{\phi}) + \nabla\tilde{\varphi} = \frac{a}{4\pi} \nabla\nabla \cdot \left[\int \frac{d^3x'}{|\mathbf{x} - \mathbf{x}'|} \nabla_{x'}\tilde{\varphi}\nabla^2_{x'}\tilde{\phi} \right]. \tag{2.39}$$

Now, taking the gradient of the last expression, we obtain the evolution equation for the tidal force [58][13]

$$\frac{d}{da}\tilde{E}_{ij} + \frac{1}{a}(\tilde{E}_{ij} + \tilde{\sigma}_{ij}) - \nabla\tilde{\varphi} \cdot \nabla\tilde{\phi}_{,ij} - \frac{1}{2}\nabla^2\tilde{\phi}\nabla^2\tilde{\varphi}\delta_{ij} = \frac{1}{4\pi}\nabla \cdot \left[\int \frac{d^3x'}{|\mathbf{x} - \mathbf{x}'|} \nabla_{x'}\tilde{\varphi}\nabla^2_{x'}\tilde{\phi} \right]_{,ij} \tag{2.40}$$

This equation shows that the Newtonian treatment gives a non local set of equations, in fact the integral term is not local. We can not obtain a set of solutions for all the variables, in a given point; we necessarily need to know the gravitational potential in other places as initial conditions.

2.1.4 Linear Newtonian Theory

As far as the perturbations remain small ($\delta << 1$), the linear approximation gives a good insight to gravitational instability. Keeping just the first order terms of the system (2.34), we obtain

$$\frac{d\delta}{da} + \tilde{\theta} = 0, \tag{2.41a}$$

$$\frac{d\tilde{\theta}}{da} + \frac{3}{2a}(B\tilde{\theta} + \nabla^2\tilde{\phi}) = -\frac{\tilde{c}_s^2}{a}\nabla^2\delta, \tag{2.41b}$$

$$\frac{d\tilde{\sigma}_{ij}}{da} - \frac{3}{2a}(B\tilde{\sigma}_{ij} + \tilde{E}_{ij}) = -\frac{\tilde{c}_s^2}{a}\left(\delta_{,ij} - \frac{1}{3}\nabla^2\delta\delta_{ij}\right), \tag{2.41c}$$

$$\nabla^2\tilde{\phi} = \frac{\delta}{a}C, \tag{2.41d}$$

now the total temporal derivative coincides with the partial derivative.
From the system above we obtain the following second order differential equation for δ:

$$\frac{d^2}{da^2}\delta + \frac{3B}{2a}\frac{d}{da}\delta - \frac{3C}{2a^2}\delta - \frac{\tilde{c}_s^2}{a}\nabla^2\delta = 0, \tag{2.42}$$

where B and $\tilde{c}_s^2$ depends on a and C is constant.
In the particular case of dust $c_s = 0$ and neglecting the cosmological constant ($B = C = 1$), the solution of (2.42) is

$$\delta_d = c_+ a + c_- a^{-3/2}, \tag{2.43}$$

with c_+ and c_- the two integration constants. Considering just the growing solution, from (2.41a) and (2.41d) we have that the peculiar velocity divergence is constant in time and equal to $-\nabla^2\tilde{\phi}$. Moreover the tidal force coincides with minus the shear:

$$\tilde{\phi} = \frac{2}{3}H_0^{-2}\phi(\mathbf{x}), \qquad \delta_d = \nabla^2\tilde{\phi}, \qquad \tilde{\theta} = -\nabla^2\tilde{\phi}, \qquad \tilde{E}_{ij} = -\tilde{\sigma}_{ij}, \tag{2.44}$$

where we used $a_0 = 1$.

If we include the fluid pressure, the term proportional to the sound velocity arises ,(2.42). Because of this spatial gradient, it is often convenient to write each quantity as a Fourier series,

$$\delta(\mathbf{x}, a) = \sum_{\mathbf{k}} \delta_{\mathbf{k}}(a) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.45}$$

Then, for a given $\mathbf{k}$, we can make the replacements

$$\nabla \to i\frac{\mathbf{k}}{a}; \qquad \nabla^2 \to -\left(\frac{k}{a}\right)^2. \tag{2.46}$$

Neglecting Λ, for the $\mathbf{k}$-mode we obtain

$$\frac{d^2}{da^2}\delta_k + \frac{3}{2a}\frac{d}{da}\delta_k + \frac{1}{a}\left(k^2\tilde{c}_s^2 - \frac{3}{2a}\right)\delta_k = 0, \tag{2.47}$$

so that the Jeans wave number is now defined as $k_J = (3/(2a\tilde{c}_s^2))^{1/2}$.

2.1.5 Meszaros Effect

Within the linear theory and considering a cold dark matter fluid without cosmological constant, an analytic solution to the perturbation equations can be found in the time of equality between radiation and matter. Being a multifluid case, the correct Newtonian approach asks to consider the full system of five

equations: the Euler and continuity equations for both matter and radiation and the Poisson equation coupling the two density components through gravity. The key point of this period is the fact that, even if the radiation density component is of the same order of the matter density, the density perturbation of radiation is much smaller then the matter density perturbation; this means that we can study the dust perturbation in the period around the equidensity neglecting the gravitational coupling given by Poisson, i.e. the peculiar gravitational potential is completely dominated by matter. In this framework the only contribution of radiation is present in the Hubble expansion scalar.
Such approximation, valid just for small scales, is reasonable if one notes that, differently to the radiation whose perturbation oscillates due to its pressure, the cold dark matter grows linearly in a and so even in the radiation era its perturbation start to be bigger than those of radiation.
Since we are interested in the transition phase between the radiation and the matter era, it is convenient to use as time variable the ratio

$$y = \frac{a}{a_{eq}} = \frac{\rho_d}{\rho_r}, \tag{2.48}$$

where a_{eq} is the scale factor in the radiation matter equality epoch; the second equivalence is due to the fact that the background densities scale as a^{-3} for dust and a^{-4} for radiation.
System (2.16), for the matter component (we omit the index d), neglecting $\delta_r \rho_r$ in (2.16c) and using (2.26) becomes:

$$\frac{d}{dy}\delta + (1+\delta)\nabla \cdot \mathbf{u} = 0, \tag{2.49a}$$

$$\frac{d}{dy}\mathbf{u} + \frac{3y+2}{2y(1+y)}\mathbf{u} = -\frac{3}{2}\frac{1}{(y+1)}\nabla\tilde{\phi}, \tag{2.49b}$$

$$\nabla^2\tilde{\phi} = \frac{\delta}{y}; \tag{2.49c}$$

now the peculiar gravitational potential is defined as

$$\tilde{\phi} = \phi\frac{a_{eq}^2}{4\pi G\rho_r^0}. \tag{2.50}$$

Linearizing and combining the equations we obtain the following differential equation for the density perturbation (Meszaros equation) [71][77],[97]

$$\frac{d^2\delta}{dy^2} + \frac{3y+2}{2y(y+1)}\frac{d\delta}{dy} - \frac{3}{2}\frac{1}{y(y+1)}\delta = 0, \tag{2.51}$$

with the general solution

$$\delta(y) = C_1\left(\frac{2}{3}+y\right) + C_2\left[\frac{3}{8}\left(\frac{2}{3}+y\right)\ln\frac{\sqrt{(1+y)}+1}{\sqrt{(1+y)}-1} - \frac{3}{4}\sqrt{1+y}\right]. \tag{2.52}$$

The second term (proportional to C_2) is the decaying mode that at late time ($y >> 1$) falls off as $y^{-\frac{3}{2}}$. The first term (proportional to C_1) is the growing solution scaling as $y \propto a$ at late time (matter era). For small values of y and so during the radiation epoch approaching the equality, the perturbation does not grow.
The reason of this behavior is the fact that the dominant energy of radiation drives the universe to expand so fast that the matter has no time to respond, and δ is frozen at a constant value. At late times, the radiation becomes negligible, and the growth increases smoothly to the Einstein-De Sitter behavior.
Let us note that eq.(2.51) does not depend on the wavenumber k of the perturbation so that the freeze of the density contrast occurs equally at all scales well within the horizon.

2.1.6 Zel'dovich Approximation

Until now we have used linear theory, it applies from the time the perturbation was very small up until it reaches order unity. Going beyond the linear regime requires the use of various techniques or approximations [80][6].
One of the most applied and successfully scheme is the Zel'dovich approximation [100][83]. It consists in using some first-order results as an ansatz for an approximation into the mildly non-linear regime.

Let us consider the system (2.34), for the irrotational dust, neglecting Λ :

$$\delta' + (1+\delta)\tilde{\theta} = 0, \tag{2.53a}$$
$$\tilde{\theta}' + \frac{1}{3}\tilde{\theta}^2 + 2\tilde{\sigma}^2 + \frac{3}{2a}(\tilde{\theta} + \nabla^2\tilde{\phi}) = 0, \tag{2.53b}$$
$$\tilde{\sigma}'_{ij} + \frac{2}{3}\tilde{\theta}\tilde{\sigma}_{ij} + \tilde{\sigma}_{il}\tilde{\sigma}_{lj} - \frac{3}{2a}(\tilde{\sigma}_{ij} + \tilde{E}_{ij}) = 0, \tag{2.53c}$$
$$\nabla^2\tilde{\phi} = \frac{\delta}{a}. \tag{2.53d}$$

(the prime denotes total derivative with respect to a).
Now we use the firs order result (see the last two expressions of (2.44)) for the velocity field: let us assume that also in the mildly non-linear regime the velocity potential is equal and opposite to the gravitational potential so that the shear tensor is equal to minus the tidal field. With these assumptions the system above becomes a local system (the tidal field disappears from the equations) whose solutions can explicitly be found in the shear eigenframe [68].
They are:

$$\delta^{(Zel)} = \frac{(\delta_I + 1)}{\prod_{i=1}^{3}(1+\lambda_i(a-a_I))} - 1, \qquad \tilde{\theta}^{(Zel)} = \sum_i \frac{\lambda_i}{(1+a\lambda_i)} \tag{2.54}$$

with λ_i the three eigenvalues of the initial tidal field [67] and a_I and δ_I the values at the initial time.
This solution is exact in the one-dimensional case [83].

Note that a negative eigenvalue corresponds to collapse along the associated shear eigendirection. In general one expects at least one of the λ_i to be negative and smaller than the other two, and on this basis the generic solution should to tend to a pancake (sheet-like structure) by contraction along one of the principal axes. The pancakes are so the first structures formed by gravitational clustering. Other structures like filaments and knots come from simultaneous contractions along two and three axes, respectively. In the case of a Gaussian random field the simultaneous vanishing of more than one eigeinvalues is quite unlikely. Thus, in this scenario, pancakes are the dominant features arising from the first stages of non-linear gravitational clustering.

The Zel'dovich approximation represents a significant step forward with respect to linear theory; it has been successfully applied, especially in the 90's, in the N-body simulations describing the large scale clustering in the distribution of galaxies.

Note: In order to distinguish from the relativistic variables, the Newtonian kinematic quantities used in this chapter should be written with an index N denoting the Newtonian character. In order to simplify the notation we have omitted such index.

Chapter 3

Relativistic Theory of Cosmological Perturbation

Introduction

In this chapter we review the General Relativistic theory of Cosmological Perturbation. This will be done considering two different formalisms, one is the usual metric approach in which all quantities are computed starting from the perturbed metric components and the geometry of space-time is the basic fundamental element [61][74][11]; the other formalism is the covariant approach [40][50][42][39][54][93] where the starting points are given by the kinematic quantities describing the fluid(s) elements that fill the universe: using the energy-momentum tensor and the Weyl tensor all the fundamental equations are written with respect to the fluid 4-velocity vector, obtaining a reference frame independent formalism.
Both the two approaches suffer the known problem of the gauge, that is the problem of mapping two space-time manifolds: the perturbed and the background world. Gauge-invariant quantities are defined in the two approaches following different schemes, and getting results that can be matched each other [41].
Here we take as known the standard metric approach to General Relativity [31][85] and give, in the first section, a summary of the covariant approach. After having explained the gauge problem in section 2.1, we present the main aspects of cosmological perturbation theory in both approaches (section 2.2 and 2.4); the application to the matter-radiation epoch is separately described, showing the connection with the Newtonian theory of chapter one.

3.1 Covariant 1+3 theory of General Relativity

Let us consider a 4-dimensional manifold endowed with a metric $g_{\mu\nu}$ with signature $(-+++)$.[1]
The fundamental starting point of the covariant description is the introduction of a timelike vector field u^μ that can be regarded as the 4-velocity of a family of observers. It allows a decomposition of space-time (1+3 threading) in a time direction given by u^μ and an observer's instantaneous rest space, given by the projector tensor

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu; \tag{3.1}$$

thanks to the propriety $u_\mu u^\mu = -1$ it is easy to see that it projects orthogonal to the 4-velocity. In the absence of rotation, the 4-velocity is hypersurface-orthogonal and $h_{\mu\nu}$ is the metric of the 3-dimensional spatial sections orthogonal to u_μ.
Using u^μ and $h^{\mu\nu}$ we can define the proper time derivative and the spatial derivative of every tensor $S^{\mu\nu..}{}_{\gamma\delta..}$ according to

$$\dot{S}^{\mu\nu..}{}_{\gamma\delta..} = S^{\mu\nu..}{}_{\gamma\delta..;\rho}u^\rho, \tag{3.2a}$$
$$D_\rho S^{\mu\nu..}{}_{\gamma\delta..} = h_\rho{}^\tau h_\alpha{}^\mu h_\beta{}^\nu h_\gamma{}^\xi h_\delta{}^\eta S^{\alpha\beta..}{}_{\xi\eta..;\tau}, \tag{3.2b}$$

respectively.
Kinematic quantities are defined by splitting the covariant derivative of u^μ in its trace, symmetric tracefree part, and skew-symmetric part:

$$u_{\mu;\nu} = D_\nu u_\mu - A_\mu u_\nu = \frac{1}{3}h_{\mu\nu}\Theta + \sigma_{\mu\nu} + \omega_{\mu\nu} - A_\mu u_\nu, \tag{3.3}$$

so that the volume expansion, the shear and the vorticity are given by

$$\Theta = D^\mu u_\mu \qquad \sigma_{\mu\nu} = D_{\langle\nu}u_{\mu\rangle} \qquad \omega_{\mu\nu} = D_{[\nu}u_{\mu]}. \tag{3.4}$$

The angle brackets denote orthogonal projections of vectors and the orthogonally projected symmetric trace-free part of tensors:

$$S^{<\mu>} = h^\mu{}_\nu S^\nu, \qquad S^{<\mu\nu>} = \left[h^{(\mu}{}_\delta h^{\nu)}{}_\rho - \frac{1}{3}h^{\mu\nu}h_{\delta\rho}\right]S^{\delta\rho}. \tag{3.5}$$

In addition the 4-acceleration vector A_μ is defined as

$$A_\mu = u_{\mu;\nu}u^\nu, \tag{3.6}$$

[1] Greek indices vary between 0 and 3 and refer to arbitrary coordinate; Latin indices run from 1 to 3 denoting spatial components. In this chapter we use geometrized units with $c = 1 = 8\pi G$.

it represents non gravitational forces and vanishes when matter moves under gravity alone.
Apart from the acceleration, these variables are the General Relativistic extension of the Newtonian definitions presented in section 1.2; they measure the relative motion of neighboring observers. The evolution equations for these quantities, the analog of (2.31), derive from the Ricci identities:

$$u_{\mu;\nu;\rho} - u_{\mu;\nu;\rho} = R_{\mu\nu\rho\gamma}u^{\gamma}. \tag{3.7}$$

Before dealing with this equation let us say somethings about the geometric and matter aspects of General Relativity.

The curvature Riemann tensor $R_{\mu\nu\rho\gamma}$ is the geometrical variable of space-time and represents the gravitational field. It can be split in its trace $R_{\mu\nu} = R^{\delta}{}_{\mu\delta\nu}$, i.e. the Ricci tensor, and its trace-free part, the Weyl tensor $C^{\mu\nu}{}_{\delta\rho}$:

$$R^{\mu\nu}{}_{\delta\rho} = C^{\mu\nu}{}_{\delta\rho} + g^{[\mu}{}_{[\delta}R^{\nu]}{}_{\rho]} - \frac{1}{3}Rg^{[\mu}{}_{[\delta}g^{\nu]}{}_{\rho]}. \tag{3.8}$$

The Weyl tensor represents the non-local part of the gravitational field, that is the part of the Riemann tensor that depends on the matter distribution at other points. Relatively to the fundamental observers it can be decomposed into its irreducible parts, the electric tensor $E_{\mu\nu}$ and the magnetic tensor $H_{\mu\nu}$. The electric part generalizes the tidal tensor of the Newtonian gravitational potential, instead $H_{\mu\nu}$ has no Newtonian counterpart. So we can write [42]

$$C_{\mu\nu}{}^{\delta\rho} = 4\left(u_{[\mu}u^{[\delta} + h_{[\mu}{}^{[\delta}\right)E_{\nu]}{}^{\rho]} + 2\varepsilon_{\mu\nu\gamma}u^{[\delta}H^{\rho]\gamma} + 2u_{[\mu}H_{\nu]\gamma}\varepsilon^{\delta\rho\gamma}, \tag{3.9}$$

with

$$E_{\mu\nu} = u^{\gamma}u^{\delta}C_{\mu\gamma\nu\delta}, \tag{3.10}$$

$$H_{\mu\nu} = \frac{1}{2}\eta_{\mu\delta}{}^{\gamma\rho}C_{\gamma\rho\nu\beta}u^{\delta}u^{\beta}, \tag{3.11}$$

where $\eta_{\mu\nu\delta\rho}$ and $\varepsilon_{\mu\delta\gamma}$ are the 4-D and 3-D permutation tensors.
The matter sector enters in the Ricci tensor through the Einstein equation

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{3.12}$$

$T_{\mu\nu}$ is the energy-momentum tensor of the matter relative to the fundamental observers, that for a generic imperfect fluid, has the form [1]

$$T_{\mu\nu} = \rho u_{\mu}u_{\nu} + ph_{\mu\nu} + 2q_{(\mu}u_{\nu)} + \pi_{\mu\nu}, \tag{3.13}$$

where $\rho = T_{\mu\nu}u^\mu u^\nu$ and $p = T_{\mu\nu}h^{\mu\nu}/3$ are the energy density and isotropic pressure, $q_\mu = -h_\mu{}^\nu T_{\nu\delta}u^\delta$ is the energy flux and $\pi_{\mu\nu} = T_{\langle\mu\nu\rangle}$ is the symmetric and trace-free anisotropic stress tensor.
From now on we consider a perfect fluid component; in this case there exist a unique comoving 4-velocity relative to which q_μ and $\pi_{\mu\nu}$ are zero. Therefore we have

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu}. \tag{3.14}$$

In order to fully characterize the fluid, an equation of state is needed giving p as a function of ρ and other thermodynamic variables as the entropy; in the barotropic case $p = p(\rho)$ and $c_s^2 = dp/d\rho$, in the dust case (CDM) $p = 0$.
Now let us come back to the Ricci identity related to u_μ, eq (3.7) (considering a perfect fluid we use the unique 4-velocity for which the (3.14) holds). Substituting in it the decompositions (3.3), (3.8), (3.9) and using (3.12) and (3.14), we obtain a set of dynamic and constraint equations. The formers are

$$\dot\Theta + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) + \frac{1}{2}(\rho + 3p) - \Lambda = D^\mu A_\mu + A^\mu A_\mu, \tag{3.15a}$$

$$\dot\sigma_{\langle\mu\nu\rangle} + \omega_\mu\omega_\nu + \sigma_{\rho\langle\mu}\sigma^\rho{}_{\rangle\nu} + \frac{2}{3}\Theta\sigma_{\mu\nu} + E_{\mu\nu} = D_{\langle\mu}A_{\nu\rangle} + A_{\langle\mu}A_{\nu\rangle}, \tag{3.15b}$$

$$\dot\omega_{\langle\mu\rangle} + \frac{2}{3}\Theta\omega_\mu - \sigma_{\mu\nu}\omega^\nu = -\frac{1}{2}\texttt{curl}A_\mu, \tag{3.15c}$$

with the curl defined as $\texttt{curl}A_\mu = \varepsilon_{\mu\nu\rho}D^\nu A^\rho$.
The vorticity vector ω_μ is defined as

$$\omega_\mu = \frac{1}{2}\varepsilon_{\mu\delta\gamma}\omega^{\delta\gamma}, \tag{3.16}$$

with $\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}/2$ and $\omega^2 = \omega_{\mu\nu}\omega^{\mu\nu}/2$ the shear and vorticity magnitude.
The equations above describe evolution of the kinematic variables; the equation concerning the scalar expansion (3.15a), is known as the Raychauduri equation [81][30].

Conservation equations are found projecting $T^{\mu\nu}{}_{;\nu}$ (twice contracted Bianchi identity) along and orthogonal to u^μ, getting respectively the energy density and momentum density conservation laws:

$$\dot\rho = -\Theta(\rho + p), \tag{3.17}$$

$$(\rho + p)A_\mu = -D_\mu p. \tag{3.18}$$

Finally the field equations can be given through the Bianchi identity (once contracted Bianchi identity) involving the Weyl tensor $C_{\mu\nu\gamma\delta}$:

$$2C^{\mu\nu\delta\gamma}{}_{;\gamma} = R^{\delta\mu;\nu} - R^{\delta\nu;\mu} + \frac{1}{6}(g^{\delta\nu}R^{;\mu} - g^{\gamma\mu}R^{;\nu}). \tag{3.19}$$

Decomposing the Weyl tensor in its electric and magnetic part, we obtain two propagation equations

$$\dot{E}_{<\mu\nu>} + \Theta E_{\mu\nu} - 3\sigma^{\delta}_{<\mu}E_{\nu>\delta} - \varepsilon_{\delta\rho<\mu}\left(2A^{\delta}H_{\nu>}^{\rho} - E_{\nu>}^{\rho}\omega^{\delta}\right) + \frac{1}{2}(\rho+p)\sigma_{\mu\nu} = \mathtt{curl}\ H_{\mu\nu}, \tag{3.20}$$

$$\dot{H}_{<\mu\nu>} + \Theta H_{\mu\nu} - 3\sigma^{\delta}_{<\mu}H_{\nu>\delta} + \varepsilon_{\delta\rho<\mu}\left(2A^{\delta}E_{\nu>}^{\rho} + H_{\nu>}^{\rho}\omega^{\delta}\right) = -\mathtt{curl}\ E_{\mu\nu}, \tag{3.21}$$

and two constraint equations

$$D^{\nu}E_{\mu\nu} = \frac{1}{3}D_{\mu}\rho - 3H_{\mu\nu}\omega^{\nu} + \varepsilon_{\mu\nu\delta}\sigma^{\nu}{}_{\gamma}H^{\delta\gamma}, \tag{3.22}$$

$$D^{\nu}E_{\mu\nu} = (\rho+p)\omega_{\mu} + 3E_{\mu\nu}\omega^{\nu} - \varepsilon_{\mu\nu\delta}\sigma^{\nu}{}_{\gamma}E^{\delta\gamma}. \tag{3.23}$$

We remark the key points of the covariant formalism: as we have seen, the Ricci identity gives the evolution equations for the kinematic variables; the field equations derive from the Bianchi identity after having substituted the Ricci tensor with the energy-momentum combination coming from the Einstein equations. This procedure, differently from the standard approach, allows to obtain a set of field equations with only first degree derivatives. The Einstein equations are here used just as algebraic relations connecting the Ricci tensor with the energy-momentum tensor.

3.2 Cosmological Perturbation Theory

Until now we considered a generic space-time, formulating the GR equations in the 1+3 covariant approach. Dealing with cosmology where the homogeneity and isotropy is assumed, the Friedmann-Robertson-Walker (FRW) metric correctly describes the universe on large scales, succeeding in reproducing the observational evidence (mainly the CMB).

It is clear from all the different objects we see in the universe, that this homogeneity and isotropy is just a good approximation but it has to break at some scale limit. If we take into account that the universe is a time dependent entity, this length limit connects with a temporal epoch of initial deviations so that structure formation theory is an evolution theory of FRW perturbation.

3.2.1 Gauge problem

The cosmological perturbative theory deals with two space-times: the real one (physical space-time) and the idealized FRW model. A gauge choice is a one-to-one correspondence (map) between this two manifolds. The arbitrariness of such a point-identification gives rise to not unique values of perturbative quantities, i.e. quantities defined in the physical space-time, this allowing the presence of unphysical *gauge modes* in the evolution of the perturbations.
The possibility of freely change the gauge choice, performing then what is called a *gauge transformation* , is the problem of the gauge [19][65][66].
One way to solve this is fixing the gauge, developing the machinery of perturbation theory and identifying the resulting gauge modes [11]. The alternative is to use gauge-invariant quantities. Analyzing how each variable changes under an infinitesimal coordinate transformation, to which the gauge transformation reduces, one can find a set of combinations that are gauge-invariant (Bardeen approach) [47] [5][57][56][73]. Unfortunately, most of these GI variables do not have a transparent geometrical meaning, since they are defined with respect to a given coordinate system.
In the contest of the covariant 1+3 formalism, gauge-invariant variables are easily defined, thanks to the following lemma (Stewart & Walker)[87][86]:

> If a quantity T vanishes in the background space-time, then it is gauge-invariant to all orders.

This can be easily seen considering that the effect of a gauge transformation induced by an infinitesimal vector field ξ on a tensor T in the perturbed universe is

$$T' = T + L_\xi T_0 \Rightarrow \delta T' = \delta T + L_\xi T_0, \qquad L_\xi T_0 = 0 \Rightarrow T = \delta T. \tag{3.24}$$

where L_ξ is the Lie derivative along ξ and T_0 is the background value.
Following this, the spatial gradients give naturally gauge-invariant quantities with respect to FRW metric.
Differently from the standard approach of Bardeen, this 1+3 GI variables are tensors defined in the real space-time, so that we can evaluate them in any coordinate system we like in that space-time.

3.2.2 Metric approach. Poisson gauge

A perturbed flat Friedmann-Robertson-Walker metric can be written in general as

$$ds^2 = a^2(\tau)\{-(1+2\phi)d\tau^2 + 2w_i d\tau dx^i + [(1-2\psi)\delta_{ij} + 2h_{ij}]dx^i dx^j\} = 0. \tag{3.25}$$

where h_{ij} is traceless and τ is conformal time.
The 10 degrees of freedom introduced with the new perturbation metric variables, can be decomposed in three sectors: scalars, vectors and tensors. The

scalar sector encloses, besides the two fields ψ and ϕ, the two scalar components respectively belonging from the longitudinal parts of vector w_i and tensor h_{ij}. We call them w and h. The vector sector encloses the transverse part of vector w_i ($w_i^\perp$), and the solenoidal part of h_{ij} (deriving from a vector that we call h_i). Finally the tensorial part is composed just by the transverse part of h_{ij} (h_{ij}^{TT}). For clarity we write the vectorial and tensorial decomposition with their proprieties:

$$w_i = w_{,i} + w_i^\perp, \tag{3.26a}$$
$$h_{ij} = D_{ij}h + h_{(i,j)} + h_{ij}^{TT}, \tag{3.26b}$$
$$w^{\perp i}{}_{,i} = 0, \qquad h^{TTi}{}_{j,i} = 0. \tag{3.26c}$$

D_{ij} is the 3-tensor defined as

$$D_{ij} = \nabla_i \nabla_j - \frac{1}{3}\delta_{ij}\nabla^2, \tag{3.27}$$

where the 3-vector covariant derivative operator ∇_i is associated with δ_{ij} (flat universe) so that definition (3.2.2) coincides with the elementary Cartesian notation.
This decomposition is useful because each sector represents a distinct physical phenomenon: the scalar sector is connected with ordinary Newtonian gravity, vectorial and tensorial parts represent respectively gravitomagnetism and gravitational radiation. Moreover, at linear order, these sectors are decoupled so that it is possible to study each type of perturbation neglecting the others. Using (3.26) the line element is written as

$$ds^2 = a^2(\tau)\{-(1+2\phi)d\tau^2 + 2(w_{,i} + w_i^\perp)d\tau dx^i + [(1-2\psi)\delta_{ij} + 2(D_{ij}h + h_{(i,j)} + h_{ij}^{TT})]dx^i dx^j\} = 0. \tag{3.28}$$

Now we recall the invariance under general coordinate transformations, this means that we can freely change our 4 coordinates through a gauge transformation. It follow that 4 of the independent components of $g_{\mu\nu}$ are not physical.
Choosing a gauge means fixing these 4 metric degrees of freedom. The most common gauges are the *syncronous gauge* which set to zero the two scalars ϕ, w and the vectorial component $w_i^\perp$, and the *Poisson gauge* where $w = h = h_i = 0$.

The first important results of cosmological perturbation theory concern the linear regime, which is a good approximation considering small perturbation from a FRW universe. The linearized equations to be considered are the conservation equations and the Einstein equations:

$$T^\mu{}_{\nu,\mu} = 0, \tag{3.29a}$$
$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{3.29b}$$

The energy-momentum tensor undergoes the effects of metric perturbations through the 4-velocity and its normalization condition $u^\mu g_{\mu\nu} u^\nu = -1$. At first order in the perturbations we have

$$T^0{}_0 = -\rho_b(1+\delta), \qquad T^i{}_0 = -(\rho_b + p_b)v^i,$$
$$T^0{}_i = (\rho_b + p_b)(v_i + w_i), \qquad T^i{}_j = p_b(1+\pi_L)\delta^i_j + \Pi^i_j, \tag{3.30a}$$

where v^i is the 3-velocity whose components are to be raised and lowered using δ^{ij} and δ_{ij}. It is a first order variable so that terms involving products of v^i and the metric perturbations are neglected. The anisotropic stress Π^i_j and the velocity can be decomposed as in (3.26):

$$\frac{v_i}{a} = v_{,i} + v_i^{\perp}, \tag{3.31a}$$
$$\Pi_{ij} = D_{ij}\Pi + \Pi_{(i,j)} + \Pi^{TT}_{ij} \tag{3.31b}$$

Using (3.30), together with decompositions (3.31) and (3.28), the linear equations for perturbations can be found from (3.29).
Here we report just the scalar sector relative to a CDM fluid. For further use we have chosen the Poisson gauge (without tensorial and vectorial modes it is also called longitudinal or conformal Newtonian gauge). The linearized equations are (in Fourier space)[2]

$$\phi = \psi, \tag{3.32a}$$
$$k^2\phi + 3\frac{a'}{a}\left(\phi' + \frac{a'}{a}\phi\right) = -\frac{1}{2}a^2\rho_b\delta, \tag{3.32b}$$
$$k^2\left(\phi' + \frac{a'}{a}\phi\right) = \frac{1}{2}a^2\rho_b v^i{}_{,i}, \tag{3.32c}$$
$$\delta' = -v^i{}_{,i} + 3\phi', \tag{3.32d}$$
$$v'^i{}_{,i} = -\frac{a'}{a}v^i{}_{,i} + k^2\phi, \tag{3.32e}$$

with the prime denoting the derivative with respect to conformal time τ.

3.2.3 Bardeen gauge-invariant variables

The metric and matter variables in (3.28) and (3.30) are gauge dependent. This leads to ambiguity in their physical meaning; for example also in an unperturbed

[2]See [64] for the general case. Note that here the definitions of ψ and ϕ are inverted

FRW universe where the density depends only on time, after a gauge transformation, it depends also on the new spatial coordinates. In order to find some gauge-invariant combinations of metric and matter variables, let us examine the effects of an infinitesimal coordinate transformation of the form

$$\hat{\tau} = \tau + \alpha(x,\tau), \qquad \hat{x}^i = x^i + \delta^{ij}\nabla_j\beta(x,\tau) + \epsilon^i(x,\tau). \tag{3.33}$$

We have split the spatial transformation into longitudinal and transverse parts ($\nabla \cdot \epsilon = 0$). The metric fields transform as [11]

$$\begin{aligned} \hat{\phi} &= \phi - \alpha' - \frac{a'}{a}\alpha, \qquad \hat{\psi} = \psi + \frac{1}{3}\nabla^2\beta + \frac{a'}{a}\alpha, \\ \hat{w}_i &= w_i + \nabla_i(\alpha - \beta') - \epsilon_i', \qquad \hat{h}_{ij} = h_{ij} - D_{ij}\beta - \nabla_{(i}\epsilon_{j)}. \end{aligned} \tag{3.34}$$

Analyzing the previous expressions, the following two combinations result to be gauge-invariant in the Fourier space (Bardeen potentials):

$$\Phi_A = \phi - \frac{a'}{a}(k^{-2}h' + k^{-1}w) - k^{-2}h'' - k^{-1}w' \tag{3.35}$$

$$\Phi_H = \psi - \frac{1}{3}h + \frac{a'}{a}(k^{-2}h' + k^{-1}w) \tag{3.36}$$

Note that in the Poisson gauge $h = w = 0$, so that from (3.35) and (3.36) the two scalars ϕ and ψ directly correspond to the two gauge-invariant potentials. Similarly, for the matter sector, we can find some gauge-invariant density and velocity perturbations [36].

$$\begin{aligned} D_g &= \delta + 3(1-w)\left(-\psi + \frac{1}{3}h\right), \\ V &= v - k^{-1}h'. \end{aligned} \tag{3.37a}$$

We can note that V corresponds at the velocity perturbation in the Poisson gauge, instead in this gauge we have

$$D_g = \delta^{(Poisson)} - 3\psi. \tag{3.38}$$

The Einstein's equations and the conservation equations can be finally written in terms of these gauge-invariant variables.
In the case of a dust component two kinds of solutions are found [35]:
i) super-horizon regime ($k\tau << 1$)

$$\Phi_A = \Phi_H = \Phi_0, \qquad D_g = -5\Phi_0, \qquad V = \frac{1}{3}\Phi_0 k\tau, \tag{3.39}$$

with Φ_0 an arbitrary constant.
ii) sub-horizon regime ($k\tau >> 1$)

$$\Phi_A = \Phi_H = \Phi_0, \qquad D_g = -\frac{1}{6}\Phi_0(k\tau)^2, \qquad V = \frac{1}{3}\Phi_0 k\tau. \tag{3.40}$$

Equation (3.40) for D_g tells us that, at small scales, the density fluctuations grow like the scale factor $\propto (k\tau)^2 \propto a$.

3.2.4 1+3 covariant gauge-invariant perturbation theory

In this section we give the main results of the covariant approach to perturbation theory, we focus on the scalar sector and consider the first order in the perturbations. For further purpose we consider the multifluid case [34].
Let us introduce, besides the 4-velocity vector of the fundamental observers u_μ introduced in section 3.1, the 4-velocity $u_\mu^{(i)}$ of the i-th fluid component. We have the following relation

$$u_\mu^{(i)} = \gamma^{(i)}\left(u_\mu + v_\mu^{(i)}\right), \tag{3.41}$$

where $\gamma^{(i)}$ is the Lorentz-boost factor and $v_\mu^{(i)}$ is the peculiar velocity of the i-th component relative to u_μ.
Now we want to define our gauge-invariant variables; according with the Stewart & Walker Lemma and remembering (3.4), the following quantities

$$\sigma_{\mu\nu}, \quad \omega_{\mu\nu}, \quad D_\mu\rho, \quad D_\mu\Theta, \quad D_\mu\rho^{(i)}, \quad D_\mu\Theta^{(i)}, \tag{3.42}$$

vanish in the FRW background and so they are gauge-invariant.
Although the above spatial gradients describe the fluids inhomogeneity, it was found that the physically relevant variables are the dimensionless comoving fractional spatial gradient of the energy density and the comoving spatial gradient of the expansion:

$$\Delta_\mu = a\frac{D_\mu\rho}{\rho}, \quad Z_\mu = aD_\mu\Theta, \quad \Delta_\mu^{(i)} = a\frac{D_\mu\rho^{(i)}}{\rho^{(i)}}, \quad Z_\mu^{(i)} = aD_\mu\Theta^{(i)}. \tag{3.43}$$

For non interacting perfect fluids the linear conservation equations of a single component is coupled with the equation for the total expansion rate. We have

$$\dot{\Delta}_\mu^{(i)} = -Z_\mu(1+c_s^{2(i)}) + \frac{1}{(1+w)}3(1+c_s^{2(i)})H(c_s^2\Delta_\mu + w\Sigma_\mu)$$
$$-a(1+c_s^2)D_\mu D^\nu v_\nu^{(i)}, \tag{3.44}$$
$$\dot{Z}_\mu = -2HZ_\mu - \frac{1}{2}\rho\Delta_\mu - \frac{c_s^2 D^2\Delta_\mu}{1+w} - \frac{wD^2\Sigma_\mu}{1+w} - 6ac_s^2 HD^\nu\omega_{\mu\nu}. \tag{3.45}$$

with $\Delta_\mu = (1/\rho)\sum_i \rho^{(i)}\Delta_\mu^{(i)}$, $\rho = \sum_i \rho^{(i)}$, $p = \sum_i p^{(i)}$, $w = (1/\rho)\sum_i \rho^{(i)} c_s^{2(i)}$ and $c_s^2 = [1/\rho(1+w)]\sum_i c_s^{2(i)}\rho^{(i)}(1+c_s^{2(i)})$.

The total effective entropy Σ_μ is not simply the sum of single entropies, in fact in the perfect fluids case it is given by

$$p\Sigma_\mu = \sum_i c_s^{2(i)}\rho^{(i)}\Delta_\mu^{(i)} - c_s^2\sum_i \rho^{(i)}\Delta_\mu^{(i)}. \tag{3.46}$$

In order to obtain a scalar set of equations, what we have to do is taking the comoving projected divergence of eq (3.44) and (3.45).
With the new variables

$$\begin{aligned} \Delta^{(i)} = aD^\mu\Delta_\mu^{(i)}, \quad \Sigma^{(i)} = aD^\mu\Sigma_\mu^{(i)}, \quad v^{(i)} = aD^\mu v_\mu^{(i)}, \\ \Delta = aD^\mu\Delta_\mu \quad \Sigma = aD^\mu\Sigma_\mu, \quad Z = aD^\mu Z_\mu, \end{aligned} \tag{3.47}$$

the final scalar set of equation is

$$\dot{\Delta}^{(i)} = -Z(1+c_s^{2(i)}) + \frac{1}{(1+w)}3(1+c_s^{2(i)})H(c_s^2\Delta + w\Sigma)$$

$$-a(1+c_s^2)D^2v^{(i)}, \tag{3.48a}$$

$$\dot{Z} = -2HZ - \frac{1}{2}\rho\Delta - \frac{c_s^2 D^2\Delta}{1+w} - \frac{wD^2\Sigma}{1+w}, \tag{3.48b}$$

$$\dot{v}^{(i)} = -(1+3c_s^{2(i)})Hv^{(i)} - \frac{c_s^{2(i)}\Delta^{(i)}}{a(1+c_s^{2(i)})} + \frac{c_s^2\Delta + w\Sigma}{a(1+w)}. \tag{3.48c}$$

Radiation-dust universe

Consider the universe filled with a CDM component and radiation. We neglect the radiation density perturbation, thinking to a homogeneous radiation distribution. This approximation is valid for small scales [71]. We have

$$\rho = \rho^{(r)} + \rho^{(d)}, \tag{3.49}$$

$$p = \frac{\rho^{(r)}}{3}, \tag{3.50}$$

$$c_s^2 = \frac{4\rho^{(r)}}{3(4\rho^{(r)} + 3\rho^{(d)})}, \tag{3.51}$$

$$3H^2 = \rho, \tag{3.52}$$

$$c_s^2\Delta + w\Sigma = 0. \tag{3.53}$$

so that the system (3.48) becomes

$$\dot{\Delta}^{(d)} = -Z + aD^2 v^{(d)}, \tag{3.54a}$$

$$\dot{Z} = -2HZ - \frac{1}{2}\rho\Delta, \tag{3.54b}$$

$$\dot{v}^{(d)} = -Hv^{(d)}. \tag{3.54c}$$

Taking the time derivative of (3.54a) (considering that time and spatial derivative do not commute) and changing the time variable with $y = a/a_{eq}$, we obtain

$$\Delta''^{(d)} + \frac{2+3y}{2y(1+y)}\Delta'^{(d)} + \frac{3}{2y(1+y)}\Delta^{(d)} = 0, \tag{3.55}$$

that is the same result of the Newtonian analysis.

Chapter 4

Generalized α-Fluid

Introduction

In the standard model of Cosmology, as we have seen, a fundamental role is given by the cold dark matter [15][33]. Since it does not interact with other particles, its density contrast can grow at all scales, having a null Jeans length. Such pressureless fluid model is just a good approximation that necessarily breaks down at astrophysical scales or every times that thermal motion can not be neglected.

Now we want to study the effects of a small pressure term in the perturbation dynamics, using the 1+3 covariant approach of chapter 3. In particular we focus on the period in which the universe passes from a radiation era to a matter domination, where now the matter equation of state little differs from that of the dust. Studies on models of fluid with pressure in connection with the dark matter and dark energy sector, can be found, for instance, in [52][3][4][78] and refs. therein.

First of all let us recall the background physics of a generic barotropic fluid with equation of state

$$p = \alpha\rho. \tag{4.1}$$

Solving the Friedmann equation in the homogeneous and isotropic metric we obtain (1.12):

$$\rho = \rho_{\alpha 0} a^{-3(1+\alpha)}, \tag{4.2}$$

so that we can easily recover the CDM and radiation case respectively for $\alpha = 0$ and $\alpha = 1/3$.

4.1 Single fluid

Let us consider a universe filled with a fluid with pressure given by (4.1) and the cosmological constant. In this case the linear theory of gauge invariant covariant perturbations gives from (3.48a) and (3.48b)[1]

$$\dot{\Delta} - 3H\alpha\Delta + (1+\alpha)Z = 0 \tag{4.3}$$
$$\dot{Z} + 2HZ + 4\pi G\rho\Delta + \frac{\alpha}{1+\alpha}D^2\Delta = 0 \tag{4.4}$$

This system can be rearranged in a second-order differential equation for Δ [17][5] :

$$\ddot{\Delta} + (2-3\alpha)H\dot{\Delta} - \left[\frac{1}{2}(1-\alpha)(1+3\alpha)8\pi G\rho + 2\alpha\Lambda\right]\Delta - \alpha D^2\Delta = 0. \tag{4.5}$$

We can discuss some qualitative features of this equation; it has the form of a wave equation with extra terms due to the expansion of the universe, gravity and the cosmological constant. The usual expansion damping term (the second term) is now α-dependent so that for $\alpha > 2/3$ it is negative so that it contributes to the growing of inhomogeneities rather than the damping. The matter term, the corresponding Newtonian term responsible for the gravitational aggregation, is no more always attractive; it vanishes for $\alpha = 1$ and $\alpha = -1/3$ and outside this limit it favors the inhomogeneities decreasing. If one considers the range $-1/3 < \alpha < 1$, i.e. the ordinary matter, this term always contributes to the growing of the density fluctuations, with a time dependent intensity given by the background behavior of ρ_b. Finally, for positive α, the cosmological constant term tends to cause gravitational aggregation and the laplacian term causes the smoothing of perturbations.
Changing the time variable from t to a and using the Friedmann equations (1.6) and (1.7), equation (4.5) becomes

$$\begin{aligned}\Delta'' &+ \left[2 - 3\alpha + \frac{\Omega_{\Lambda 0} - 1/2\Omega_{\alpha 0}a^{-3(1+\alpha)}(1+3\alpha)}{\Omega_{\Lambda 0} + \Omega_{\alpha 0}a^{-3(1+\alpha)}}\right]\frac{\Delta'}{a} \\ &- \left[\frac{(1+2\alpha-3\alpha^2)3/2\Omega_{\alpha 0}a^{-3(1+\alpha)} + 6\alpha\Omega_{\Lambda 0}}{\Omega_{\Lambda 0} + \Omega_{\alpha 0}a^{-3(1+\alpha)}}\right]\frac{\Delta}{a^2} \\ &- \frac{\alpha D^2\Delta}{a^2 H_0^2\left(\Omega_{\Lambda 0} + \Omega_{\alpha 0}a^{-3(1+\alpha)}\right)} = 0 \\ ,\end{aligned} \tag{4.6}$$

where $\Omega_{\alpha 0}$ is the density parameter of the α-fluid evaluated today.
Expanding each variables in spatial harmonics, in the Laplacian term we have

$$\frac{\alpha D^2\Delta}{a^2 H_0} = -\frac{\alpha k^2}{a^4 H_0^2}, \tag{4.7}$$

[1]Now we restore the $8\pi G$ factor retaining $c = 1$.

and the equation can be solved for varying α, the density parameters and the wave number k [5].
As for the Newtonian case we can find the Jeans length giving the wavelength limit over which the gravitational collapse occurs. Combining the matter and Λ term with the divergence term we have

$$\lambda_J = \frac{2\pi\alpha}{H_0 a\left[\frac{3}{2}(1-\alpha)(1+3\alpha)\Omega_{\alpha 0}a^{-3(1+\alpha)} + 6\alpha\Omega_{\Lambda 0}\right]^{1/2}}. \tag{4.8}$$

As expected the cosmological constant is relevant just at late time as we can see in this plot, showing the behavior of the Jeans length in the radiation case ($\alpha = 1/3$) as a function of the scale factor.

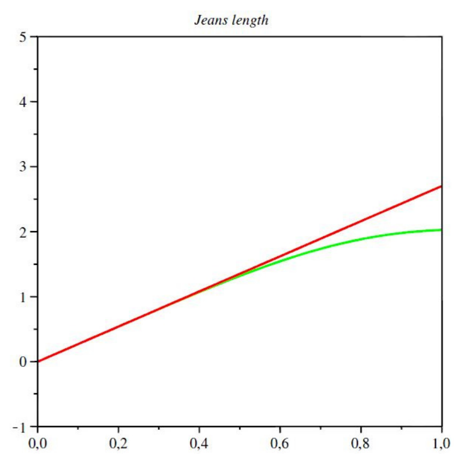

Figure 4.1: Jeans length as a function of a in the $\alpha = 1/3$ case, with and without cosmological constant (red line).

Now we show some qualitative aspects of the integration of eq.(4.6) through some plots obtained with the mathematical software Maple.
We are interested in small deviations from the dust case, so that, except for the first example, we have taken always values of α near to 0.

- Long-wavelength limit ($k = 0$), dust and radiation without the cosmological constant; in these two cases the analytic solutions are

$$\alpha = 0 \qquad \Delta_{d+} \propto a, \qquad \Delta_{d-} \propto a^{-3/2} \tag{4.9}$$

$$\alpha = \frac{1}{3} \qquad \Delta_{r+} \propto a^2, \qquad \Delta_{r-} \propto a^{-1} \tag{4.10}$$

- Long-wavelength limit ($k = 0$), α-fluid with and without the cosmological constant; as it should be the presence of Λ becomes relevant at late time.

We can see that the overall effect of the cosmological constant, in the near to zero α case, is the damping of the density growth. This is due to the expansion term proportional to $\dot{\Delta}$.

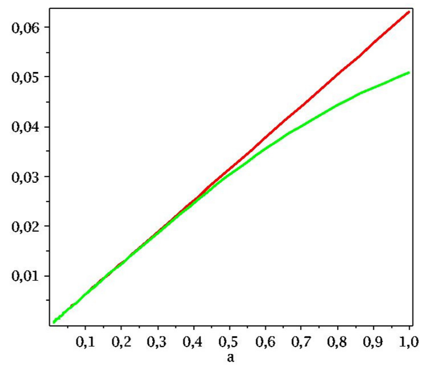

Figure 4.2: Δ for $\alpha = 0.002$ case, with and without cosmological constant (red line). k=0

- Long-wavelength limit ($k = 0$), α fluid with the cosmological constant; the higher is α, the higher is the growth of the density perturbation.

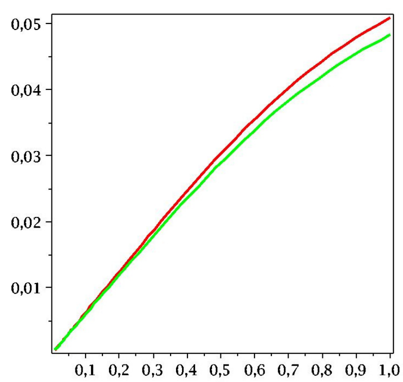

Figure 4.3: Dust and α-fluid (red line, $\alpha = 0.002$) with cosmological constant. k=0

Starting from the early time the small pressure term favors the gravitational aggregation at large scale.

- Short-wavelength limit case ($k \neq 0$) with different values of α and $\Lambda \neq 0$. Naturally the dust case does not depend on k. Switching on the barotropic index the coupling with the wavenumber stops the growing of fluctuations.

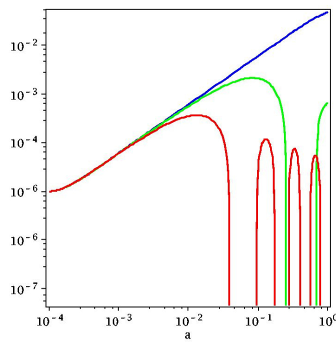

Figure 4.4: Δ for dust (blue), $\alpha = 0.001$ (green) and $\alpha = 0,007$ (red). $k = 10^{-2}$h·Mpc^{-1}

- Short-wavelength limit case, α-fluid with the cosmological constant for different values of the wave number. Increasing the wavenumber and so

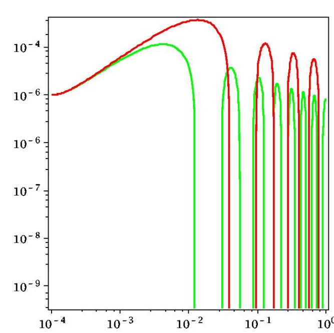

Figure 4.5: Δ for $k = 3 \cdot 10^{-2}$h·Mpc^{-1} (red line) and $k = 6 \cdot 10^{-2}$h·Mpc^{-1}(green line). $\alpha = 0.007$

looking at smaller scales, the damped oscillations start before with higher frequency and smaller amplitude.

- Short-wavelength limit case, α-fluid with and without the cosmological constant.
  The influence is seen at late time. Now the Jeans criterion can be applied: the Jeans length at late time is smaller in the $\Lambda \neq 0$, this reduces the oscillation regime with a consequent lowering of the frequency with respect to the $\Lambda = 0$ case.

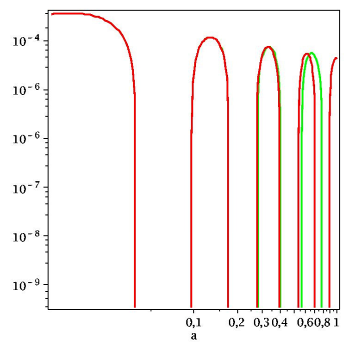

Figure 4.6: Δ with and without cosmological constant(green line). $\alpha = 0.007$

4.2 Multifluid case. Generalized Meszaros effect

Now we consider the universe filled with a mixture of radiation and α-fluid, generalizing the Meszaros approach and so neglecting the perturbation of the radiation density.
We will use system (3.48) with the approximation condition for the total density variable:

$$\Delta = \frac{\rho_\alpha}{\rho_\alpha + \rho_r}\Delta^{(\alpha)}, \tag{4.11}$$

the indices α and r denoting respectively the α-fluid and the radiation component. Having $w_\alpha = \alpha, w_r = 1/3$, the total sound velocity c_s^2, the total barotropic parameter w and the effective entropy are respectively

$$c_s^2 = \frac{3\alpha(1+\alpha)\rho_\alpha + \frac{4}{3}\rho_r}{3(1+\alpha)\rho_\alpha + 4\rho_r}, \tag{4.12a}$$

$$w = \frac{\alpha\rho_\alpha + \frac{1}{3}\rho_r}{\rho_r + \rho_d}, \tag{4.12b}$$

$$\Sigma = \frac{4(\alpha - \frac{1}{3})\rho_r\rho_\alpha}{\left[3(\alpha+1)\rho_\alpha + 4\rho_r\right]\left(\alpha\rho_\alpha + \frac{1}{3}\rho_r\right)}\Delta^{(\alpha)}; \tag{4.12c}$$

so that we obtain the following combination

$$c_s^2\Delta + w\Sigma = \frac{\alpha\rho_\alpha}{\rho_\alpha + \rho_r}\Delta^{(\alpha)}. \tag{4.13}$$

With these expressions, the set of scalar equations describing the perturbation dynamics of the α-fluid becomes

$$\dot{\Delta} = -(1+\alpha)Z - a(1+\alpha)D^2 v + \frac{9\alpha(1+\alpha)H\rho_\alpha}{4\rho_r + 3(1+\alpha)\rho_\alpha}\Delta \tag{4.14a}$$

$$\dot{Z} = -2HZ - 4\pi G\rho_\alpha\Delta - \frac{\alpha\rho_\alpha}{(\alpha+1)\rho_\alpha + \frac{4}{3}\rho_r}D^2\Delta \tag{4.14b}$$

$$\dot{v} = -(1-3\alpha)Hv - \frac{\alpha\Delta}{a}\left(\frac{1}{\alpha+1} - \frac{3\rho_\alpha}{4\rho_r - 3(1+\alpha)\rho_\alpha}\right) \tag{4.14c}$$

where the indices α in Δ and in v have been omitted.
From system (4.14), we can obtain a second-order differential equation for Δ. Considering the commutation rule

$$(D^2 v)^{\cdot} = D^2\dot{v} - 2HD^2 v, \tag{4.15}$$

we obtain

$$\ddot{\Delta} + (2H - B)\dot{\Delta} - \left[2HB + 4\pi G(1+\alpha)\rho_\alpha + \dot{B}\right]\Delta - \alpha D^2\Delta$$
$$+3\alpha(1+\alpha)aHD^2 v = 0, \tag{4.16}$$

with

$$B = \frac{9\alpha(1+\alpha)H\rho_\alpha}{4\rho_r + 3(1+\alpha)\rho_\alpha}. \tag{4.17}$$

We can see that neglecting the radiation density and putting $v = 0$ we recover the previous equation (4.5)of the single fluid model.
Now, as in the standard Meszaros equation, we introduce the new time variable

$$y = \frac{a}{a_{eq}}, \tag{4.18}$$

so that the following relation between the density components holds

$$\frac{\rho_r}{\rho_\alpha} = y^{-1+3\alpha}. \tag{4.19}$$

Using the Friedmann equation and the behavior of the radiation and the matter with respect to the scale factor, we obtain

$$\begin{aligned}
&\Delta'' - \frac{9(3\alpha^2+2\alpha-1)y^2 + 6(3\alpha^2+4\alpha-3)y^{3\alpha+1} - 8y^{6\alpha}}{2y^2\left(1+y^{3\alpha-1}\right)\left(3(\alpha+1)y+4y^{3\alpha}\right)}\Delta' \\
&+ \frac{27(\alpha+1)(3\alpha^3+\alpha^2-3\alpha-1)y^2 + 3(36\alpha^2-20\alpha-8)y^{3\alpha+1} + (36\alpha^2-12\alpha-8)y^{6\alpha}}{2\left(y^2\left(1+y^{3\alpha-1}\right)\left(3(\alpha+1)y+4y^{3\alpha}\right)\right)^2}\Delta \\
&+ \frac{\alpha a_{eq}^{3\alpha+1}}{\Omega_{\alpha 0}(y^{-3\alpha+1}+1)}\frac{k}{H_0^2}\Delta - \frac{3\alpha(\alpha+1)}{y\left(\Omega_{\alpha 0}a_{eq}^{-3\alpha+1}(y^{-3\alpha+1}+1)\right)^{1/2}}\frac{k^2}{H_0^2}v^2 = 0
\end{aligned} \tag{4.20}$$

In the previous expression we recognize the standard Meszaros equation (3.55) when we put $\alpha = 0$.
In the long-wavelength limit the last two terms are neglected and eq.(4.20) is the scale-invariant equation describing the density evolution.
In the general case we need equation (4.14c) for the velocity field, using y it becomes:

$$v' = -\frac{(1-3\alpha)}{y}v - \frac{\left(\frac{\Omega_{\alpha 0}}{\Omega_{r0}}\right)^{-\frac{1+3\alpha}{2(1-3\alpha)}}\alpha\Delta}{\left(y^{-3\alpha+1}+1\right)^{1/2}\Omega_{\alpha 0}^{1/2}H_0}\left(\frac{1}{\alpha+1} - \frac{3}{4y^{-1+3\alpha}+3(1+\alpha)}\right) \tag{4.21}$$

Now we show some plots of the density contrast for different values of the barotropic parameter. The solution has been found with the numerical implementation of equations (4.20) and (4.21).

- Long wavelengths, dust and α-fluid . In the large scale case the small pressure term enhances the growing of the density fluctuations.

  As we can seen in the first plot of Figure 4.2, if α is very small the difference with the pure CDM component is negligible; instead, for bigger values of α, the total enhancing effect is clearly visible in the second plot (logarithmic scales).

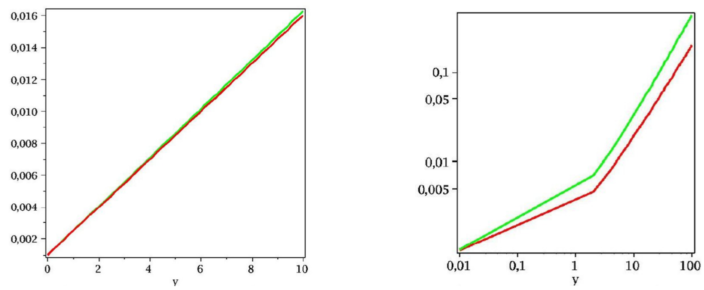

Figure 4.7: The first graph shows Δ for dust (red line) and $\alpha = 0.00015$ (green line). In the second graph Δ is plotted for $\alpha = 0.015$ (red line) and $\alpha = 0.055$ (green line).

- Short wavelengths.
  In the small scale case the pressure term couples with k and the density contrast starts to oscillate following the Jeans criterion.

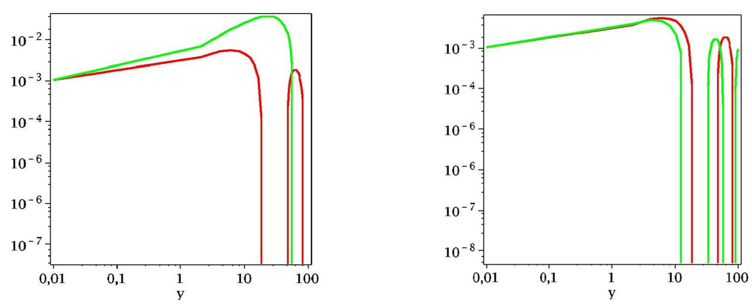

Figure 4.8: Δ in the small scale case. In the first graph the density contrast is shown varying the barotropic index ($\alpha = 0.015$ red line and $\alpha = 0.055$ green line). The second graph shows the variation with respect to the wavenumber ($k = 0.26$ h·Mpc^{-1} red line and $k = 0.31$ h·Mpc^{-1} green line. $\alpha = 0.007$)

Concluding this section we must remark that our analysis has been quite qualitative, we wanted just to see how the presence of a pressure term can modify the growth of the density contrast from the mathematical point of view. The general result agrees with the basic knowledge of the gravitational instability for which the pressure in the large wavenumber case does not allow the density contrast to grow. Both in the radiation-matter epoch and in the matter-dominated era the leading effect due to the barotropic parameter appears at the small wavelength limit. In fact, in the large scale limit, the necessarily small value of α (for a constraint limit see [52]) assures that the deviations from the dust behavior are negligible, this can be easily seen considering the large scale limit of eq.(4.6) and (4.20). It is only switching on the wavenumber k and so considering the small scale limit that the coupling with α becomes active and the solution is a damped oscillating regime in agreement with the Newtonian model.

We note that our results of the single α-fluid component agree with the work [78] where the CAMB code has been used. The multifluid case is instead a new result; we have generalized the Meszaros approximation including a small pressure term in the matter obtaining eq.(4.20). The plots show that for small values of α the damping effect is seen just at late time in the matter era; at the equivalence time in fact the density contrast is bigger for the α-fluid than for a pure CDM component, but in practice this difference is negligible.

Chapter 5

Generalizing Meszaros approximation

In this chapter we analyze the Newtonian cosmological perturbation approach of chapter 2, going behind the linear theory, so that we consider weakly non linear effects in the dynamic evolution of the matter density contrast. We focus again on the radiation-matter era, considering a mixture of radiation and cold dark matter, in the case of negligible radiation inhomogeneity [71].
In the first section a second order analysis is developed, finding the temporal behavior of the matter density contrast up to this order (see also [79] and [7]); this result enable us to start a statistical analysis directly connected with one of the topical issue of the modern Cosmology: the research of non-Gaussianity The simplest inflationary scenarios predict Gaussian initial fluctuations [49], but there are other models of inflation [10] or models where structure is seeded by topological defects [94][46] that generate non-Gaussian fluctuations. For this reason looking at possible evidences of non-Gaussianity from CMB anisotropies and from the statistics of large scale structure is fundamental in order to have insight for the early epoch of the universe. On the other hand there exist other sources of non-Gaussian features coming from different physical processes, so that it is important to separate these secondary anisotropies from those of primordial nature.
The non linearity of the gravitational instability evolution is one of these non-Gaussian sources, in fact the second order perturbative analysis allows to compute the first statistical estimator measuring the lowest-order deviation from the Gaussian distribution, the skewness[8][45][95].

In this chapter we compute the skewness of the density field and the velocity field relatively to the CDM component in the period around radiation-matter equidensity, starting from Gaussian initial conditions. We will find a time evolving skewness reaching the known limit of the matter dominated era [77].
This original result tell us how the universe (in its simpler model) got its statistical asymmetry, concluding that already at the decoupling period, i.e. when

the CMB features were imprinted, the matter density distribution presents the 86% of its maximum asymmetry level.
Since the CMB and LSS data will shortly improve dramatically with the Microwave Anisotropy Probe (MAP)[1] and Planck Surveyor satellites[2], and the Anglo-Australian 2-Degree Field (2dF) and Sloan Digital Sky Survey (SDSS), we will be able to make significant comparisons between the theoretical predictions and the experimental results. In this spirit our analysis is a first step towards a possible link between theory and observations.

In the second section of the chapter the non linear equidensity period is studied using the Zel'dovich approximation. We will show that such non-linear approximation is valid and easily generalized at the radiation-matter epoch.

5.1 Second order evolution

Let us consider the sub-horizon approximate system (2.49), describing the density and velocity evolution of a dust component in a radiation background. We now look at the second order perturbation in the density contrast.
Using the kinematic approach through the velocity gradient tensor, this system is written as

$$\delta' + \tilde{\theta} = -\tilde{\theta}\delta \tag{5.1a}$$

$$\tilde{\theta}' + \frac{3}{2(1+y)}\left[\frac{1}{3}\frac{(3y+2)}{y}\tilde{\theta} + \nabla^2\tilde{\phi}\right] = -\frac{1}{3}\tilde{\theta}^2 - 2\tilde{\sigma}^2 \tag{5.1b}$$

$$\tilde{\sigma}'_{ij} + \frac{3}{2(1+y)}\left[\frac{1}{3}\frac{(3y+2)}{y}\tilde{\sigma}_{ij} + \tilde{E}_{ij}\right] = -\frac{2}{3}\tilde{\theta}\tilde{\sigma}_{ij} - \tilde{\sigma}_{il}\tilde{\sigma}_{lj} + \frac{2}{3}\delta_{ij}\tilde{\sigma}^2 \tag{5.1c}$$

$$\tilde{\omega}'_{ij} + \frac{1}{2(1+y)}\frac{(3y+2)}{y}\tilde{\omega}_{ij} = -\frac{2}{3}\tilde{\theta}\tilde{\omega}_{ij} - 2\tilde{\sigma}_{[il}\tilde{\omega}_{lj]} \tag{5.1d}$$

$$\nabla^2\tilde{\phi} = \frac{\delta}{y} \tag{5.1e}$$

where now the prime denotes the total derivative with respect to y. This system is analogous to system (2.34) without pressure and cosmological constant terms. Now its validity refers not only to the matter era but also to the previous period and naturally, for big values of y, the two systems coincide.
Let us write every variables as the sum of a first order part and a second order part, for example in the case of the density contrast:

$$\delta = \varepsilon\delta_{(1)} + \frac{\varepsilon^2}{2}\delta_{(2)} \tag{5.2}$$

[1] http://map.gsfc.nasa.gov/
[2] http://astro.estec.esa.nl/SA-general/Projects/Planck/

with the perturbation parameter $\varepsilon << 1$. Keeping just the terms proportional to ε we obtain this first order set of equations

$$\delta'_{(1)} + \tilde{\theta}_{(1)} = 0 \tag{5.3a}$$

$$\tilde{\theta}'_{(1)} + \frac{3}{2(1+y)}\left[\frac{1}{3}\frac{(3y+2)}{y}\tilde{\theta}_{(1)} + \nabla^2\tilde{\phi}_{(1)}\right] = 0 \tag{5.3b}$$

$$\tilde{\sigma}'_{(1)ij} + \frac{3}{2(1+y)}\left[\frac{1}{3}\frac{(3y+2)}{y}\tilde{\sigma}_{(1)ij} + \tilde{E}_{(1)ij}\right] = 0 \tag{5.3c}$$

$$\tilde{\omega}'_{(1)ij} + \frac{1}{2(1+y)}\frac{(3y+2)}{y}\tilde{\omega}_{(1)ij} = 0 \tag{5.3d}$$

$$\nabla^2\tilde{\phi}_{(1)} = \frac{\delta_{(1)}}{y} \tag{5.3e}$$

As we already know, combining (5.3a), (5.3b) and (5.3e) we obtain the Meszaros equation, with the growing solution

$$\delta_{(1)} = C_1\left(\frac{2}{3} + y\right). \tag{5.4}$$

System (5.3) allow us to easily find the gravitational potential and the other variables connected with the velocity field:
-from (5.3e)

$$\tilde{\phi}_{(1)} = C_0\left(1 + \frac{2}{3}\frac{1}{y}\right), \qquad C_1 = \nabla^2 C_0; \tag{5.5}$$

-from (5.3a) and supposing zero vorticity ($\varphi_{(1)}$ is the velocity potential)

$$\theta_{(1)} = \nabla^2\varphi_{(1)} = -C_1 = -\nabla^2 C_0, \qquad \varphi_{(1)} = -C_0; \tag{5.6}$$

-from the constantness of the velocity and using eq.s (5.3b) and (5.3c)

$$\frac{3y+2}{3y}\tilde{\theta}_{(1)} = -\nabla^2\tilde{\phi}_{(1)}, \qquad E_{(1)ij} = -\left(1 + \frac{2}{3}\frac{1}{y}\right)\tilde{\sigma}_{(1)ij}, \tag{5.7}$$

so that, using (5.6)

$$\tilde{\sigma}_{(1)ij} = -D_{ij}C_0 \tag{5.8}$$

$$\tilde{E}_{(1)ij} = D_{ij}C_0\left(1 + \frac{2}{3}\frac{1}{y}\right), \tag{5.9}$$

with $D_{ij} = \partial_i \partial_j - 1/3 \nabla^2 \delta_{ij}$.
We use these results in the second order equations, obtained from system (5.1) keeping just terms proportional to ε^2.
As always happens in the perturbative schemes, the second order variables obey to inhomogeneous differential equations where the homogeneous part is made up of the same linear operator that defines the first order, in our case the system (5.3), while the source terms are quadratic in the first order variables. We obtain

$$\delta'_{(2)} + \tilde{\theta}_{(2)} = -2\tilde{\theta}_{(1)}\delta_{(1)} \tag{5.10a}$$

$$\tilde{\theta}'_{(2)} + \frac{3}{2(1+y)}\left[\frac{1}{3}\frac{(3y+2)}{y}\tilde{\theta}_{(2)} + \nabla^2\tilde{\phi}_{(2)}\right] = -\frac{2}{3}\tilde{\theta}^2_{(1)} - 4\tilde{\sigma}^2_{(1)} \tag{5.10b}$$

$$\tilde{\sigma}'_{(2)ij} + \frac{3}{2(1+y)}\left[\frac{1}{3}\frac{(3y+2)}{y}\tilde{\sigma}_{(2)ij} + \tilde{E}_{(2)ij}\right] = -\frac{4}{3}\tilde{\theta}_{(1)}\tilde{\sigma}_{(1)ij} - 2\tilde{\sigma}_{(1)il}\tilde{\sigma}_{(1)lj}$$
$$+\frac{4}{3}\delta_{ij}\tilde{\sigma}^2_{(1)} \tag{5.10c}$$

$$\tilde{\omega}'_{(2)ij} + \frac{1}{2(1+y)}\frac{(3y+2)}{y}\tilde{\omega}_{(2)ij} = -\frac{4}{3}\tilde{\theta}_{(1)}\tilde{\omega}_{(1)ij} - 4\tilde{\sigma}_{(1)[il}\tilde{\omega}_{(1)lj]} \tag{5.10d}$$

$$\nabla^2\tilde{\phi}_{(2)} = \frac{\delta_{(2)}}{y} \tag{5.10e}$$

The second order differential equation for the density contrast is

$$\delta''_{(2)} + \frac{3y+2}{2y(y+1)}\delta'_{(2)} - \frac{3}{2}\frac{1}{y(y+1)}\delta_{(2)} = -\frac{3y+2}{y(y+1)}\tilde{\theta}_{(1)}\delta_{(1)} + \frac{2}{3}\tilde{\theta}^2_{(1)} + 4\tilde{\sigma}^2_{(1)} - -(2\tilde{\theta}_{(1)}\delta_{(1)})' \tag{5.11}$$

We recall that here the prime denotes the total derivative with respect to y; if at linear order there is not difference between total and partial time derivative, at second order they differ because of the convective term proportional to the velocity field. This implies that the left side hand of (5.11) contains other second order terms. In order to make them explicit let us write the relation between total and partial derivative of δ at second order:

$$\left(\frac{d\delta}{dy}\right)_{(2)} = \frac{1}{2}\frac{\partial\delta_{(2)}}{\partial y} + \mathbf{u}_{(1)}\cdot\nabla\delta_{(1)}, \quad \left(\frac{d^2\delta}{dy^2}\right)_{(2)} = \frac{1}{2}\frac{\partial^2\delta_{(2)}}{\partial y^2} + 2\mathbf{u}_{(1)}\cdot\nabla\frac{\partial}{\partial y}\delta_{(1)}. \tag{5.12}$$

We are only interested in the extension to second order of the growing modes (4), hence we retain only these modes in our source, on the basis that other source terms made up of decaying modes would only give subdominant contributions to the full second order solution. Therefore from (5.11) and (5.12) we substitute

$$\delta_{(1)} = C_{0,ii}\left(\frac{2}{3}+y\right), \tag{5.13}$$

$$u_{(1)i} = -C_{0,i}, \tag{5.14}$$

getting

$$\delta''_{(2)} + \frac{3y+2}{2y(y+1)}\delta'_{(2)} - \frac{3}{2}\frac{1}{y(y+1)}\delta_{(2)} = \frac{7y^2+8y+\frac{4}{3}}{y(y+1)}C_{0,i}C_{0,jji} + \\ + \left[\frac{(3y+2)^2}{3y(y+1)}+2\right]C_{0,ii}^2 + 2C_{0,ij}C_{0,ij}. \tag{5.15}$$

The analytic particular solution is

$$\delta_{(2)}^{part} = C_{0,i}C_{0,jji}\left(2y^2+\frac{169}{70}y+\frac{227}{315}\right) + C_{0,ii}^2\left[\left(\frac{10}{7}y^2+\frac{744}{1225}y-\frac{3376}{11025}\right) + \frac{4}{35}\left(y+\frac{2}{3}\right)\ln y\right] + C_{0,ij}C_{0,ij}\left[\left(\frac{4}{7}y^2+\frac{4427}{2450}y+\frac{11321}{11025}\right) - \frac{4}{35}\left(y+\frac{2}{3}\right)\ln y\right]. \tag{5.16}$$

and combining with the homogeneous solution we choose initial conditions such that $\delta_{(2)}(y=0)=0$

$$\delta_{(2)} = C_{0,i}C_{0,jji}\left(2y^2+\frac{4}{3}y\right) + C_{0,ii}^2\left[\frac{10}{7}y^2+\frac{148}{105}y+\frac{8}{35}+\frac{8}{35}\left(y+\frac{2}{3}\right)\right. \\ \left.(\ln(\sqrt{(1+y)}+1)-\ln 2)-\frac{8}{35}\sqrt{1+y}\right] + C_{0,ij}C_{0,ij}\left[\frac{4}{7}y^2-\frac{8}{105}y-\frac{8}{35}\right. \\ \left.-\frac{8}{35}\left(y+\frac{2}{3}\right)(\ln(\sqrt{(1+y)}+1)-\ln 2)+\frac{8}{35}\sqrt{1+y}\right]. \tag{5.17}$$

Defining in accordance with Peebles [77]

$$\Delta_{,ii} = -4\pi\delta_{(1)} \qquad \Longrightarrow \qquad C_0 = -\frac{\Delta}{4\pi(y+\frac{2}{3})}, \tag{5.18}$$

we can now write[3]

$$\delta = \delta_{(1)} + \frac{1}{2}\delta_{(2)} = \delta_{(1)} - \frac{1}{4\pi}\delta_{(1),i}\Delta_{,i}h_f(y) + \delta_{(1)}^2 h_c(y) + \frac{1}{16\pi^2}\Delta_{,ij}\Delta_{,ij}h_d(y), \tag{5.19}$$

[3] Δ has not a physical meaning, it has been introduced for a practical purpose

where

$$h_f(y) = \frac{y}{\left(y+\frac{2}{3}\right)}, \tag{5.20a}$$

$$h_c(y) = \frac{\left(\frac{5}{7}y^2 + \frac{74}{105}y + \frac{4}{35}\right) + \frac{4}{35}(\ln(\sqrt{1+y}+1) - \ln 2)(y+\frac{2}{3}) - \frac{4}{35}\sqrt{1+y}}{(y+\frac{2}{3})^2}, \tag{5.20b}$$

$$h_d(y) = \frac{(\frac{2}{7}y^2 + \frac{4}{105}y - \frac{4}{35}) - \frac{4}{35}(\ln(\sqrt{1+y}+1) - \ln 2))(y+\frac{2}{3}) + \frac{4}{35}\sqrt{1+y}}{(y+\frac{2}{3})^2}. \tag{5.20c}$$

The solution (5.19) generalizes the known result of Peebles obtained in a pure Einstein-de Sitter model [77]. We have included the radiation contribute in the background so that we can consider its validity range starting from the late radiation era. Naturally the matter era behavior is recovered in the $y >> 1$ limit:

$$h_f(y) \quad \rightarrow \quad 1, \tag{5.21a}$$

$$h_c(y) \quad \rightarrow \quad \frac{5}{7}, \tag{5.21b}$$

$$h_d(y) \quad \rightarrow \quad \frac{2}{7}, \tag{5.21c}$$

and (5.19) coincides with the Peebles result.

5.2 Skewness

An important consequence of the second order analysis is that, departing from a Gaussian distribution of cosmic fields, as predicted by inflation, the non linearity inevitably leads to the growth of non-Gaussian features.
If a random Gaussian process is completely described by the two-point correlation function (power spectrum in Fourier modes), a non-Gaussian distribution is characterized by the other higher moments; the first interesting moment is the three point correlation function, or skewness, describing the asymmetry of the probability distribution.
Let us compute the first moments of the density field $< \delta^n >$.

For the conservation of the mass the $n = 1$ moment has to be zero also at second order, we can check this using the following relations [77])

$$< \delta_{(1),i}\Delta_{,i} >= - < \delta_1\Delta_{,ii} >= 4\pi < \delta_{(1)}^2 >= 4\pi\xi(0), \tag{5.22}$$

$$< \Delta_{,ij}\Delta_{,ij} >< \Delta_{,ii}\Delta_{jj} >= 16\pi^2\xi(0), \tag{5.23}$$

where we have defined the variance $\xi(0) =< \delta^2_{(1)} >$.
Averaging the solution (5.19) and using the previous expressions, we find the expected value $< \delta >= 0$.
The first non-trivial moment emerging from the second order terms is the skewness:

$$< \delta^3 >= 3 < \delta^2_{(1)} \frac{1}{2} \delta_{(2)} > . \tag{5.24}$$

This form is valid up to second order and we have used the fact that the first order distribution has zero skewness. From (5.19) we obtain:

$$< \delta^3 >= 3 \Big[h_c(y) < \delta^4_{(1)} > - \frac{1}{4\pi} h_f(y) < \delta^2_{(1)} \delta_{(1),i} \Delta_{,i} > + \frac{1}{16\pi^2} h_d(y) \\ < \delta^2_{(1)} \Delta_{,ij} \Delta_{,ij} > \Big] . \tag{5.25}$$

From the definitions of the moments, the proprieties of Gaussian distribution and analytic arrangements, we have [77]

$$< \delta^4_{(1)} >= 3\xi(0)^2, \tag{5.26a}$$
$$< \delta^2_{(1)} \delta_{(1),i} \Delta_{,i} >= 4\pi \xi(0)^2, \tag{5.26b}$$
$$< \delta^2_{(1)} \Delta_{,ij} \Delta_{,ij} >= \frac{80}{3} \pi^2 \xi(0)^2. \tag{5.26c}$$

Substituting in (5.25) we obtain the time dependent skewness:

$$< \delta^3 >= [9h_c(y) + 5h_d(y) - 3h_f(y)] \xi(0)^2 = \\ = \left[\frac{34}{7} y^2 + \frac{436}{105} y + \frac{16}{35} + \frac{16}{35} \left(\frac{2}{3} + y \right) (\ln(\sqrt{1+y} + 1) \\ - \ln(2)) - \frac{16}{35} \sqrt{1+y} \right] \left(y + \frac{2}{3} \right)^{-2} \xi(0)^2. \tag{5.27}$$

We can easily see that in the matter era limit

$$< \delta^3 >= \frac{34}{7} \xi(0)^2 \tag{5.28}$$

in agreement with the well known result [77][8][9].
The time dependence of the ratio

$$S_3 = \frac{< \delta^3 >}{< \delta^2 >^2} \tag{5.29}$$

in the matter-radiation era is a growing function of y going asymptotically to the constant value $S_3^\infty = 34/7$ (Figure 5.1).

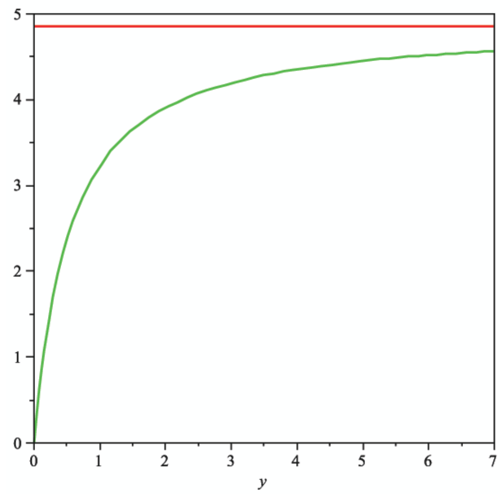

Figure 5.1: Skewness of δ. Comparison with the constant matter era result (red line)

The important conclusion of this analysis is that now we can infer how the universe approaches its constant value of skewness due to the gravitational clustering. From eq.(5.27) we note that at last scattering epoch, i.e. for $y \approx 3.3$ in a standard ΛCDM model with current parameter values [59][51], we get $S_3^{LS} = 4.2$. Given that the matter constant value is $S_3^\infty \approx 4.9$, our result implies that the skewness in the density perturbation distribution, representing the non-Gaussianity purely arising from the collapsing process, reaches its asymptotical value very early on in the matter dominated era, and it is already at 86% of that value at last scattering. This means that such signature from non-Gaussianity could be measurable by the coming CMB experiments.

In this contest the general way to deal with non-Gaussianity estimators is introducing the f_{NL} parameter. Dealing with primary anisotropies, in the real space it is defined through the Bardeen gravitational potential[60]:

$$\Phi = \Phi_L + f_{NL} \left(\Phi_L^2 - \langle \Phi_L^2 \rangle \right) , \tag{5.30}$$

with Φ_L the linear Gaussian part.
A further analysis will be to compute the analogous f_{NL} parameter in our model, also adding a primordial non-Gaussianity in the initial conditions (see [79] where a similar analysis has been done in Fourier space).

5.3 Moments of the expansion scalar

In analogy with the previous result we can compute the skewness of the expansion scalar, the velocity divergence $\tilde{\theta}$. Its perturbative expansion is

$$\tilde{\theta} = \varepsilon\tilde{\theta}_{(1)} + \frac{\varepsilon^2}{2}\tilde{\theta}_{(2)}, \tag{5.31}$$

where the first order is given by (5.6) and the second order satisfies equation (5.10a) so that:

$$\tilde{\theta}_{(2)} = -\delta'_{(2)} - 2\tilde{\theta}_{(1)}\delta_{(1)}. \tag{5.32}$$

Using (5.12), we write it in Eulerian coordinates:

$$\tilde{\theta}_{(2)} = -\frac{\partial\delta_{(2)}}{\partial y} - 2\tilde{\theta}_{(1)}\delta_{(1)} - 2\mathbf{u}_{(1)}\cdot\nabla\delta_{(1)}; \tag{5.33}$$

substituting (5.17) we obtain:

$$\begin{aligned}\tilde{\theta}_{(2)} = (C_{0,ii})^2 &\left(-\frac{6}{7}y - \frac{8}{105} - \frac{8}{35}[\ln(\sqrt{1+y}+1-\ln 2] - \frac{4}{35}\frac{y+\frac{2}{3}}{\sqrt{1+y}+1+y}\right.\\ &\left.+\frac{4}{35\sqrt{1+y}}\right) + C_{0,ij}C_{0,ij}\left(-\frac{8}{7}y + \frac{8}{105} + \frac{8}{35}[\ln(\sqrt{1+y}+1-\ln 2]\right.\\ &\left.+\frac{4}{35}\frac{y+\frac{2}{3}}{\sqrt{1+y}+1+y} - \frac{4}{35\sqrt{1+y}}\right) - 2C_{0,i}C_{0,jji}y,\end{aligned} \tag{5.34}$$

so that we can express the expansion up to second order as:

$$\tilde{\theta} = -C_{0,ii} + (C_{0,ii})^2\tilde{h}_c + C_{0,ij}C_{0,ij}\tilde{h}_d + C_{0,i}C_{0,jji}\tilde{h}_f, \tag{5.35}$$

with

$$\tilde{h}_c = -\frac{3}{7}y + \frac{4}{105} - \frac{4}{35}[\ln(\sqrt{1+y}+1-\ln 2] - \frac{2}{35}\frac{y+\frac{2}{3}}{\sqrt{1+y}+1+y} + \frac{2}{35\sqrt{1+y}}, \tag{5.36}$$

$$\tilde{h}_d = -\frac{4}{7}y - \frac{4}{105} + \frac{4}{35}[\ln(\sqrt{1+y}+1-\ln 2] + \frac{2}{35}\frac{y+\frac{2}{3}}{\sqrt{1+y}+1+y} - \frac{2}{35\sqrt{1+y}}, \tag{5.37}$$

$$\tilde{h}_f = -y. \tag{5.38}$$

In the matter era ($y >> 1$)

$$\begin{aligned}\tilde{h}_c &\rightarrow -\frac{3}{7}y, &(5.39)\\ \tilde{h}_d &\rightarrow -\frac{4}{7}y, &(5.40)\\ \tilde{h}_f &\rightarrow -y, &(5.41)\end{aligned}$$

and we have [9]

$$\tilde{\theta} = -C_{0,ii} - y\left[(C_{0,ii})^2\frac{3}{7} + C_{0,ij}C_{0,ij}\frac{2}{7} + C_{0,i}C_{0,jji}\right]. \tag{5.42}$$

Now we are ready to compute the first moments of the cosmic velocity divergence.
As in the previous paragraph we use the following relations, easily obtained under the only assumption of initial Gaussianity:

$$< C_{0,ii}C_{0,ii} >=< \tilde{\theta}_{(1)}\tilde{\theta}_{(1)} >= \xi_{\tilde{\theta}}(0), \tag{5.43a}$$

$$< C_{0,ij}C_{0,ij} >=< C_{0,ii}C_{0,ii} >= \xi_{\tilde{\theta}}(0), \tag{5.43b}$$

$$< C_{0,i}C_{0,jji} >= - < C_{0,ii}C_{0,ii} >= -\xi_{\tilde{\theta}}(0), \tag{5.43c}$$

$$< \tilde{\theta}^4_{(1)} >= 3\xi^2_{(\tilde{\theta})}(0), \tag{5.43d}$$

$$< (C_{0,ii})^2C_{0,i}C_{0,jji} >= -\frac{1}{3} < \tilde{\theta}^4_{(1)} >= -\xi^2_{(\tilde{\theta}}(0), \tag{5.43e}$$

$$< (C_{0,ii})^2C_{0,ij}C_{0,ij} >= \xi_{(\tilde{\theta})}(0) < C_{0,ij}C_{0,ij} > +2(< C_{0,ii}C_{0,ij} >)^2 = \frac{5}{3}\xi^2_{\tilde{\theta}}(0). \tag{5.43f}$$

The first moment is zero:

$$< \tilde{\theta} >=< (C_{0,ii})^2 > h_c+ < C_{0,ij}C_{0,ij} > h_d+ < C_{0,i}C_{0,jji} > h_f = 0, \tag{5.44}$$

and the skewness is

$$\begin{aligned}< \tilde{\theta}^3 >= \quad & 3 < \tilde{\theta}^2_{(1)}\frac{1}{2}\tilde{\theta}_{(2)} >= 3\left[(3h_c + \frac{5}{3}h_d - h_f)\right]\xi^2_{(\tilde{\theta})}(0)\\ = \quad & \left(-\frac{26}{7}y - \frac{16}{105} - \frac{16}{35}[\ln(\sqrt{1+y}+1) - \ln 2] - \frac{8}{35}\frac{\frac{2}{3}+y}{y+\sqrt{1+y}+1}\right.\\ & \left.+\frac{8}{35\sqrt{1+y}}\right)\xi^2_{\tilde{\theta}}(0).\end{aligned} \tag{5.45}$$

Defining the ratio

$$T_3 = \frac{< \tilde{\theta}^3 >}{< \tilde{\theta}^2 >^2} \tag{5.46}$$

we have $T_3 \to -\frac{26}{7} y$ in the matter era (Figure 5.2), [9].

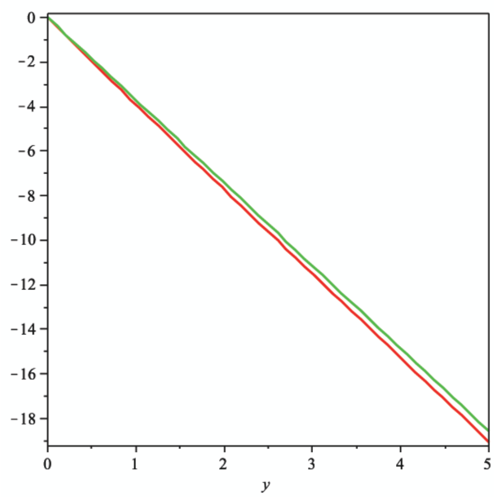

Figure 5.2: Skewness of $\tilde{\theta}$. Comparison with the matter era result (green line)

We end this section with a further relation between the second moments of density contrast, velocity divergence and shear; from (5.43b) and the definition of shear:

$$< \sigma^2 >=< \frac{1}{2} C_{0,ij} C_{0,ij} - \frac{1}{6} C_{0,ii} C_{0,ii} >= \frac{1}{3} < C_{0,ii} C_{0,ii} >= \frac{1}{3} < \theta^2 >, \tag{5.47}$$

using (5.13)

$$< \theta^2 >=< (\theta_{(1)} + \frac{1}{2} \theta_{(2)})^2 >=< \delta_1^2 > (\frac{2}{3} + y)^{-2} + < \theta_{(1)} \theta_{(2)} > . \tag{5.48}$$

Substituting (5.34), the last term can be written as a function of y time $< \theta_{(1)}^3 >$, that is zero for the Gaussianity condition, so the final relation is

$$< \sigma^2 >= \frac{< \delta^2 >}{(\frac{2}{3} + y)^2}. \tag{5.49}$$

The matter era result is recovered neglecting 2/3.

5.4 Zel'dovich approximation in the epoch of equality

As we saw in chapter 2, the Zel'dovich approximation obtains non-linear solutions in matter era from some first order results.
Now we use the same approach for the dust radiation system (5.1). The Zel'dovich ansatz is that the linear result for the velocity field and the gravitational potential remains true in the non-linear regime, such that (5.7) holds:

$$\frac{3y+2}{3y}\tilde{\theta} = -\nabla^2\tilde{\phi}, \qquad E_{ij} = -\left(1+\frac{2}{3}\frac{1}{y}\right)\tilde{\sigma}_{ij}. \tag{5.50}$$

Putting this in (5.1) we obtain a local system with vanishing of the square brackets.

$$\delta' + (1+\delta)\tilde{\theta} = 0 \tag{5.51a}$$

$$\tilde{\theta}' + \frac{1}{3}\tilde{\theta}^2 + 2\sigma^2 = 0 \tag{5.51b}$$

$$\sigma'_{ij} + \frac{2}{3}\tilde{\theta}\sigma_{ij} + \sigma_{il}\sigma_{lj} - \frac{2}{3}\delta_{ij}\sigma^2 = 0 \tag{5.51c}$$

Formally, this system is exactly the same of the matter era in the Zel'dovich approximation, so we have the same solutions (2.54):

$$\delta^{(Zel)} = \frac{(\delta_I+1)}{\prod_{i=1}^3(1+\lambda_i(y-y_I))} - 1, \qquad \tilde{\theta}^{(Zel)} = \sum_{i=1}^{3}\frac{\lambda_i}{1+\lambda_i(y-y_I)},$$
$$\tilde{\sigma}_i^{(Zel)} = \frac{\lambda_i}{1+\lambda_i(y-y_I)} - \frac{1}{3}\tilde{\theta}^{(Zel)}, \tag{5.52}$$

where λ_i are the three eigenvalues of the tidal field and the index I means the value of the variable at the initial time.
The difference between the matter era solution is enclosed in the relation between the initial values of θ and δ which have to be consistent with the linear solution:

$$\delta_I = -\tilde{\theta}_I(y_I + \frac{2}{3}). \tag{5.53}$$

From the second of (5.52) we have

$$\theta_I = \sum_{i=1}^{3}\lambda_i. \tag{5.54}$$

Therefore the limit of Meszaros equation from $\delta^{(Zel)}$ is recovered for small values of y and using (5.53) and (5.54).

In a different way we can find a dynamic solution for δ, using the Poisson equation instead of the continuity equation. In general these two solutions do not coincide because they are both approximated solutions. Nevertheless in the one-dimensional case, the Zel'dovich solution becomes exact and it coincides with the dynamic solution.
The dynamic solution is easily obtained from the θ expression (the first of (5.52)), using the relation

$$\delta^{(dyn)} = -(y + \frac{2}{3})\tilde{\theta}^{(Zel)} \tag{5.55}$$

which follows from the Poisson equation.
In the one-dimensional case we obtain

$$\delta^{(dyn)} = -\frac{\lambda(y + \frac{2}{3})}{1 + \lambda(y - y_I)} \tag{5.56}$$

which coincides with $\delta^{(Zel)}$ when only one eigeinvalue is different to zero and considering (5.53).

Chapter 6

Post Newtonian Cosmology

Introduction

In the previous chapters we studied cosmological perturbation theory both in the Newtonian and in the General Relativistic approach. We naturally aspect the first treatment to be correct in the small scales case, considering all the relativistic effects acting just at large scales. However the experimental data assure that the perturbations at scale larger than the horizon are so small that the General Relativistic theory can be treat linearly. Instead at sub-horizon scales, as we pointed out more times, the perturbations assume a non-linear character and their evolution is commonly studied within the full non-linear Newtonian theory (see [12] for a review in N-body numerical simulations).

Now we want to investigate in a systematic way the connection between the two theories, finding the correct limit procedure for passing from General Relativity to Newtonian gravity. This connection necessarily requires some approximation scheme, in which the relativistic effects are seen as small corrections. The aim of this study will be to find a consistent set of equations describing the evolution of cosmic inhomogeneities from the large scales to the small scales, including also the intermediate range of distances, where the relativistic effects can not be ignored but the physics mainly keeps its Newtonian aspect and the non-linearity starts to be relevant. Therefore, the final equations will include both the linear General Relativistic theory and the fully non-linear Newtonian theory, allowing to study also the intermediate regime between them. In addition, even if the approximation relegates the general relativistic effects as small corrections so that just terms quadratic in the geometric fields are allowed, this does not happen for the proprieties of matter (in particular the density field), which are left fully non-linear.

The connection between the Einstein theory of gravitation and the Newtonian laws has an intrinsic wide relevance and a historical account apart from a cosmological contest. Starting from the pioneering studies of Landau [63], Einstein [38] and Fock [43], some famous applications in this sense include the precession of Mercury's perihelion and various solar system tests of Einsteins gravity theory like the light deflection [99].

The Post-Newtonian theory is the most common and natural way to link General Relativity with Newtonian gravity. It consists in expanding the Einstein equations with respect to a small parameter representing the ratio of the velocity of matter to the fundamental speed of gravity; this parameter indicates the relativistic amount of each terms and is essentially the inverse of the speed of light [84]. The Newtonian equations are formally obtained in the limit $c \to \infty$; in this case the multi-component metric tensor representing the gravitational field, can be expressed with a single scalar field, the Newtonian potential. In this framework, keeping just the $O(1/c^4)$ terms of the General Relativistic expanded equations means to consider the first order correction terms and the approximation is called first Post-Newtonian (1PN) (see [44] for a basic work, the series of papers of Chandasekhar [25][26][27][28] and [72][96] as textbook works).
The validity of such approximation is restricted at physical cases of slow motion and weak gravitational field regime. The most successful recent applications of this method are the generation of gravitational waves from compact binary objects [14], the weakly relativistic evolution stages of isolated systems of celestial bodies [2] and other important applications in relativistic celestial mechanics regarding precise measurements of the solar system bodies [16][98]. All these investigation are based on the PN approximation of isolated systems assuming a Minkowski background space-time.

In our work we have applied the 1PN procedure in the cosmological contest of perturbation theory focusing on the density contrast evolution of a cold dark matter component. Related to this topic we mention same early and more recent works where different cosmological aspects are analyzed in the Post-Newtonian scheme: in [91] and [92] the PN equation of motion for particles in the expanding universe are derived, in [58] the authors clarify the Newtonian role of the Electric and Magnetic part of the Weyl tensor, in [90] and [70] perturbation theory is given in Lagrangian coordinates using respectively the 3+1 and 1+3 framework, [20] and [53] derive a complete set of field and hydrodynamic equations, other remarkable works presenting a different PN method are [89][88].
Differently from these works, our way to proceed in the 1PN scheme does not use the standard iterative approach of all the perturbative theories, this leading to a different final set of equations with respect to the mentioned previous works. Therefore our procedure can be seen as a resumming up to the 1PN order.

6.1 Newtonian and Post Newtonian variables

We start considering the metric tensor. We want a homogeneous and isotropic expanding background where two kinds of perturbation terms are added, both representing the gravitational content of the theory but at two different levels of accuracy: the first will give the Newtonian theory (0PN approximation), the second will give the first relativistic corrections (1PN). As we have already noted, the expanding parameter is given by $1/c$, so that we can write

$$
\begin{aligned}
g_{00} &= -\left[1 - \frac{2U}{c^2} + \frac{1}{c^4}(2U^2 - 4\Phi)\right] \\
g_{0i} &= -\frac{a}{c^3}P_i \\
g_{ij} &= a^2\left[\left(1 + \frac{2V}{c^2} + \frac{1}{c^4}(2V^2 + 4\Psi)\right)\delta_{ij} + \frac{1}{c^4}h_{ij})\right].
\end{aligned}
\tag{6.1}
$$

The space coordinates are here understood as an Eulerian Cartesian system of reference and, as in all the work, the c factors are written, so that the time coordinate is given by ct. This particular form of the metric is a generalization of Chandrasekhar metric [25] (see also [53]). The two scalar variables U and V, appearing in the lower power of $1/c$ have a Newtonian character, in fact as we will see, they are both needed in order to obtain the Newtonian limit. Obviously the fact that in the Newtonian theory there is only one gravitational potential tell us that at this level U and V have to be equal.
The 1PN corrections are written in terms of second order Newtonian quantities and intrinsic 1PN variables; we have included also tensorial modes, but as we will see and it is already known, gravitational waves show up starting from the 2.5 order of the PN expansion (i.e. considering up to $O(c^{-5})$ terms). For this reason the tensorial variables at 1PN approximation can not be interpreted as gravitational waves, as we will see they satisfy a constraint equation and do not have a dynamic equation. Note the scale factor $a(t)$ factorized in the spatial metric, this gives the flat FRW as fundamental background.
In writing the metric components (6.1) we have decided to consider the Poisson gauge so that the three-vector P_i is divergencefree and h_{ij} is transverse and tracefree (TT). We recall the six degrees of freedom of Poisson gauge: U and Φ (1 d.o.f.), V and Ψ (1d.o.f.), P_i (2 d.o. f) and h_{ij} (for the TT condition only 2 d.o.f).

6.2 Matter variables

Having defined the basic geometrical quantities and their weight with respect to the expansion parameter c^{-1}, we shall look at the matter components.
Using the unitarity of the 4-velocity

$$
g_{\mu\nu}u^\mu u^\nu = -1, \tag{6.2}
$$

and the definition of peculiar velocity (2.3) so that

$$
u^i = \frac{dx^i}{cd\tau} = \frac{dx^i}{cdt}\frac{dt}{d\tau} = \frac{v^i}{ca}u^0, \tag{6.3}
$$

we obtain the following 4-velocity components:

$$u^i = \frac{1}{c}\frac{v^i}{a}u^0 \tag{6.4a}$$

$$u^0 = 1 + \frac{1}{c^2}\left(U + \frac{1}{2}v^2\right) + \frac{1}{c^4}\left[\frac{1}{2}U^2 + 2\Phi + v^2 V + \frac{3}{2}v^2 U + \frac{3}{8}v^4 - P_i v^i\right] \tag{6.4b}$$

$$u_0 = -1 + \frac{1}{c^2}\left(U - \frac{1}{2}v^2\right) + \frac{1}{c^4}\left[2\Phi - \frac{1}{2}U^2 - \frac{1}{2}v^2 U - v^2 V - \frac{3}{8}v^4\right] \tag{6.4c}$$

$$u_i = \frac{av_i}{c} + \frac{a}{c^3}\left[-P_i + v_i U + 2v_i V + \frac{1}{2}v_i v^2)\right] \tag{6.4d}$$

The peculiar velocity vector is a perturbed quantity with respect to the FRW universe, its index is based on the Euclidean three-metric.

We consider the universe filled by a single cold dark matter component. Therefore the energy-momentum tensor reads

$$T_{\mu\nu} = \rho c^2 u_\mu u_\nu. \tag{6.5}$$

Using the previous relations (6.4) we obtain:

$$T_{00} = \rho c^2 + \rho(v^2 - 2U) + \frac{\rho}{c^2}\left[v^4 - 4\Phi + 2v^2 V + 2U^2\right], \tag{6.6a}$$

$$T_{0i} = -\rho a c v_i + \frac{\rho a}{c}[P_i - v_i(v^2 + 2V)], \tag{6.6b}$$

$$T_{ij} = \rho a^2 v_i v_j + \frac{a^2 \rho}{c^2}\left[(4V + 2U + v^2)v_i v_j - 2P_{(i}v_{j)}\right], \tag{6.6c}$$

$$T = -\rho c^2. \tag{6.6d}$$

All these quantities are written with the purpose of clearly identifying the two orders, the Newtonian and the Post-Newtonian.

6.3 Einstein equations

We take the field equations from the standard form of Einstein's equations:

$$G^\mu_\nu = R^\mu_\nu - \frac{1}{2}R g^\mu_\nu = \frac{8\pi G}{c^4}T^\mu_\nu - \Lambda g^\mu_\nu \tag{6.7}$$

having included also the cosmological constant.
Expanding in powers of $1/c$ we retain all the terms up to $O(1/c^4)$ obtaining the following equations:

time-time component G^0_0

$$\frac{1}{c^2}\left[3\left(\frac{\dot a}{a}\right)^2-2\frac{\nabla^2 V}{a^2}\right]+\frac{1}{c^4}\left[6\frac{\dot a}{a}\dot V+6\left(\frac{\dot a}{a}\right)^2U-4\frac{\nabla^2\Psi}{a^2}+\frac{2}{a^2}\nabla^2(V^2)-\frac{5}{a^2}V_{,i}V_{,i}\right]$$
$$=\frac{1}{c^2}8\pi G\rho+\frac{1}{c^4}8\pi G\rho v^2+\Lambda; \quad (6.8)$$

spatial component G^j_i

$$\frac{1}{c^2}\left[\frac{(V-U)^{,j}_{,i}}{a^2}+\delta^j_i\left(\frac{\nabla^2(U-V)}{a^2}+\left(\frac{\dot a}{a}\right)^2+2\frac{\ddot a}{a}\right)\right]+\frac{1}{c^4}\left[-\frac{\dot a}{a^2}\left(P_i{}^{,j}+P^j{}_{,i}\right)\right.$$
$$-\frac{1}{2a}\left(\dot P^j{}_{,i}+\dot P_i{}^{,j}\right)-\frac{2}{a^2}\Phi_{,i}{}^{,j}+\frac{2}{a^2}\Psi_{,i}{}^{,j}+\frac{1}{a^2}U_{,i}U^{,j}-\frac{1}{a^2}V_{,i}V^{,j}+\frac{1}{a^2}\left(U_{,i}V^{,j}\right.$$
$$\left.+U^{,j}V_{,i}\right)+\frac{2}{a^2}V(U-V)_{,i}{}^{,j}+\delta^j_i\left(2\frac{\dot a}{a}\dot U+4\frac{\ddot a}{a}U+2\left(\frac{\dot a}{a}\right)^2U+6\frac{\dot a}{a}\dot V+2\ddot V\right.$$
$$\left.\left.+\frac{2}{a^2}\nabla^2\Phi-\frac{2}{a^2}\nabla^2\Psi-\frac{1}{a^2}U_{,k}U^{,k}+\frac{2}{a^2}V\nabla^2(V-U)\right)+\frac{1}{2a^2}\nabla^2h^j_i\right]$$
$$=\Lambda\delta^j_i-\frac{8\pi G}{c^4}\rho v_i v^j\ , \quad (6.9)$$

time-space component G^0_i

$$\frac{1}{c^3}\left[-\frac{1}{2a^2}\nabla^2P_i+2\frac{\dot a}{a^2}U_{,i}+\frac{2}{a}\dot V_{,i}\right]=\frac{8\pi G}{c^3}\rho v_i\ . \quad (6.10)$$

From (6.8) and (6.9) we can easily derive the background equations neglecting Newtonian and post-Newtonian terms :

$$\frac{1}{c^2}\left(\frac{\dot a}{a}\right)^2=\frac{1}{c^2}\frac{8}{3}\pi G\rho_b+\frac{\Lambda}{3}, \quad (6.11)$$

$$\frac{1}{c^2}\left[\left(\frac{\dot a}{a}\right)^2+2\frac{\ddot a}{a}\right]=\Lambda, \quad (6.12)$$

where ρ_b denotes the background matter density.
Subtracting these equations from (6.8), (6.9) we obtain the field equations involving only perturbative quantities. The temporal component gives the first Post Newtonian generalization of Poisson equation:

$$\frac{1}{c^2}\frac{\nabla^2 V}{3a^2}-\frac{1}{c^4}\left[\frac{\dot a}{a}\dot V+\left(\frac{\dot a}{a}\right)^2U+\frac{1}{3a^2}\nabla^2(V^2)-\frac{5}{6a^2}V_{,i}V_{,i}-2\frac{\nabla^2\Psi}{3a^2}\right]$$
$$=-\frac{4}{3c^2}\pi G\rho_b\delta-\frac{4}{3c^4}\pi G\rho_b(1+\delta)v^2, \quad (6.13)$$

where the density contrast is defined as in the previous chapters $\delta = (\rho - \rho_b)/\rho_b$. Note that the cosmological constant has disappeared with the subtracting procedure but it is included in the Hubble expansion $\dot{a}/a$.
The trace of the space-space component (6.9) reads

$$\frac{1}{c^2}\left[\frac{2}{a^2}\nabla^2(U-V)\right] + \frac{1}{c^4a^2}\Bigg[4\nabla^2(\Phi-\Psi) - 2U_{,k}U^{,k} - V_{,k}V^{,k} + 2U_{,k}V^{,k}$$
$$+4V\nabla^2(V-U) + 6a^2\left(\frac{\dot{a}}{a}(\dot{U}+3\dot{V}) + 2\frac{\ddot{a}}{a}U + \left(\frac{\dot{a}}{a}\right)^2 U + \ddot{V}\right)\Bigg]$$
$$= -\frac{8\pi G}{c^4}\rho_b(1+\delta)v^2, \tag{6.14}$$

while the trace-free part becomes

$$\frac{1}{c^2a^2}\left[(V-U)_{,i}{}^{,j} + \frac{1}{3}\nabla^2(U-V)\delta_i^j\right] + \frac{1}{c^4a^2}\Bigg\{-\dot{a}\left(P_i{}^{,j} + P^j{}_{,i}\right) - \frac{a}{2}\left(\dot{P}_i{}^{,j} + \dot{P}^j{}_{,i}\right)$$
$$+2(\Psi-\Phi)_{,i}{}^{,j} + U_{,i}U^{,j} - V_{,i}V^{,j} + U_{,i}V^{,j} + U^{,j}V_{,i} + 2V(U-V)_{,i}{}^{,j} + \frac{1}{2}\nabla^2 h_i^j$$
$$+\delta_i^j\left[\frac{2}{3}\nabla^2(\Phi-\Psi) - \frac{1}{3}(U_{,k}U^{,k} - V_{,k}V^{,k}) - \frac{2}{3}U_{,k}V^{,k} + \frac{2}{3}V\nabla^2(V-U)\right]\Bigg\}$$
$$= -\frac{8\pi G}{c^4}\rho_b(1+\delta)\left(v_iv^j - \frac{1}{3}\delta_i^j v^2\right). \tag{6.15}$$

Now it is useful to recast the previous equations in order to isolate the scalar, vector and tensor modes. For the scalar sector let us apply the divergence operator to equation (6.10):

$$\frac{1}{c^3}\nabla^2\left(\frac{\dot{a}}{a}U + \dot{V}\right) = \frac{1}{c^3}4\pi G a\rho_b(v_i(1+\delta))^{,i}\ . \tag{6.16}$$

Replacing this equation in (6.13), we obtain the following constraint equation

$$\frac{1}{c^2}\frac{\nabla^2\nabla^2 V}{3a^2} - \frac{1}{c^4}\left[\frac{1}{3a^2}\nabla^2\nabla^2(V^2) - \frac{5}{6a^2}\nabla^2(V_{,i}V^{,i}) - 2\frac{\nabla^2\nabla^2\Psi}{3a^2}\right]$$
$$= \frac{4\pi G}{3}\rho_b\left\{-\frac{1}{c^2}\nabla^2\delta - \frac{1}{c^4}\left[\nabla^2((1+\delta)v^2) + 3\dot{a}(v_i(1+\delta))^{,i}\right]\right\}\ . \tag{6.17}$$

Now, let us apply the operator $\partial_i\partial^j$ at both sides of (6.15). We find a second constraint equation, i.e.

$$\frac{4}{3}\frac{1}{c^4a^2}\nabla^2\nabla^2(\Psi-\Phi) = -\frac{1}{c^2a^2}\frac{2}{3}\nabla^2\nabla^2(V-U) - \frac{1}{c^4a^2}\Big\{\left(U_{,k}U^{,l}\right)_{,l}^{\ ,k} - \left(V_{,k}V^{,l}\right)_{,l}^{\ ,k}$$
$$+\left(U_{,k}V^{,l}\right)_{,l}^{\ ,k} + \left(V_{,k}U^{,l}\right)_{,l}^{\ ,k} - \frac{1}{3}\nabla^2\left(U_{,k}U^{,k}\right) + \frac{1}{3}\nabla^2\left(V_{,k}V^{,k}\right) - \frac{2}{3}\nabla^2\left(U_{,k}V^{,k}\right)$$
$$+2\left[V(V-U)_{,k}^{\ ,l}\right]_{,l}^{\ ,k} + \frac{2}{3}\nabla^2\left[V\nabla^2(V-U)\right]\Big\} - \frac{8\pi G}{c^4}\rho_b\left[(1+\delta)\left(v_iv^j - \frac{1}{3}\delta_i^jv^2\right)\right]_{,j}^{\ ,i} . \tag{6.18}$$

For the vector sector a constraint equation for P_i can be obtained applying the operator ∇^2 in both sides of eq. (6.10) and using (6.16), i.e.

$$\nabla^2\nabla^2\frac{P_i}{a} = 16\pi Ga\rho_b[((1+\delta)v_k)_{,i}^{\ ,k} - \nabla^2((1+\delta)v_i)] . \tag{6.19}$$

A dynamic equation for P_i can be obtained applying the operators ∇^2 in both sides of eq. (6.18) and the divergence and the spatial gradient in both sides of eq. (6.15). We get

$$\frac{1}{c^4}\nabla^2\nabla^2\left(\frac{1}{2a}\dot{P}_i + \frac{\dot{a}}{a^2}P_i\right) = -\frac{1}{c^4}\frac{1}{a^2}\Big\{\nabla^2\left(UU_{,i}^{\ ,l}\right)_{,l} - \left(UU_{,k}^{\ ,l}\right)_{,li}^{\ ,k} - \nabla^2\left(VV_{,i}^{\ ,l}\right)_{,l}$$
$$+\left(VV_{,k}^{\ ,l}\right)_{,li}^{\ ,k} + \nabla^2\left(UV_{,i}^{\ ,l}\right)_{,l} - \left(UV_{,k}^{\ ,l}\right)_{,li}^{\ ,k} + \nabla^2\left(VU_{,i}^{\ ,l}\right)_{,l} - \left(VU_{,k}^{\ ,l}\right)_{,li}^{\ ,k}$$
$$+2\nabla^2\left[(U-V)^{,l}V_{,i}\right]_{,l} - 2\nabla^2\left[(U-V)^{,l}V_{,k}\right]_{,li}^{\ ,k}\Big\} + \frac{8\pi G}{c^2}\rho_b\Big\{\nabla^2\left[(1+\delta)(v_iv^k\right.$$
$$\left.-\frac{1}{3}\delta_i^kv^2)\right]_{,k} - \left[(1+\delta)(v_kv^l - \frac{1}{3}\delta_k^lv^2)\right]_{,li}^{\ ,k}\Big\} \tag{6.20}$$

In order to isolate the tensorial modes h_{ij}, let us apply the operator $\nabla^2\nabla^2$ in both sides of (6.15), the operator $\partial_i\partial^j$ in both sides of (6.18) and the spatial gradient in both sides of eq. (6.20). We have

$$\frac{1}{c^4}\nabla^2\nabla^2\nabla^2h_i^j = \frac{1}{c^4}\left[\frac{1}{a^2}\left(A_{k,li}^{l,kj} + \nabla^2A_{k,l}^{l,k}\delta_i^j + 2\nabla^2A_{i,k}^{k,j} + 2\nabla^2A_{k,i}^{j,k} - 2\nabla^2\nabla^2A_i^j\right)\right.$$
$$\left.+8\pi G\left(S_{k,li}^{l,kj} + \nabla^2S_{k,l}^{l,k}\delta_i^j + 2\nabla^2S_{i,k}^{k,j}j + 2\nabla^2S_{k,i}^{j,k} - 2\nabla^2\nabla^2S_i^j\right)\right] , \tag{6.21}$$

where

$$A_i^j = U_{,i}U^{,j} - V_{,i}V^{,j} + U_{,i}V^{,j} + U^{,j}V_{,i} + 2V(U-V)_{,i}^{\ ,j} + \delta_i^j\left(-\frac{1}{3}(U_{,k}U_{,k}\right.$$

$$-V_{,k}V_{,k}) - \frac{2}{3}(U_{,k}V_{,k}) + \frac{2}{3}(V\nabla^2(V-U))\Bigg) \ ,$$

$$S_i^j = \rho_b \left((1+\delta)(v_i v^j - \frac{1}{3}\delta_i^j v^2)) \right) \ .$$

Alternatively a simpler equation for tensorial modes can be obtained using in the tracefree equation (6.15) the operator

$$\mathcal{P}_j^l \mathcal{P}_m^i - \frac{1}{2}\mathcal{P}_m^l \mathcal{P}_j^i, \tag{6.22}$$

where the projection tensor is defined as $\mathcal{P}_j^i = \delta_j^i - (\nabla^2)^{-1}\partial^i\partial_j$, [72].
The resulting equation is

$$\nabla^2 h_i^j = -2\left(\mathcal{P}_j^l \mathcal{P}_m^i - \frac{1}{2}\mathcal{P}_m^l \mathcal{P}_j^i\right)\mathcal{R}_j^l \tag{6.23}$$

with

$$\mathcal{R}_j^l = 8\pi G a^2 \rho_b (1+\delta)(v_j v^l - \frac{1}{3}\delta_j^l v^2) + U_{,j}U^{,l} - V_{,j}V^{,l} + U_{,j}V^{,l} + U^{,l}V_{,j}$$
$$+2V(U-V)_{,j}{}^{,l} + \delta_j^l\left[-\frac{1}{3}(U_{,k}U_{,k} - V_{,k}V_{,k}) - \frac{2}{3}U_{,k}V_{,k} + \frac{2}{3}V\nabla^2(V-U)\right] . \tag{6.24}$$

Therefore, we get eqs. (6.13), (6.14), (6.16), (6.17) and (6.18) for the scalar sector (dependent equations), eqs. (6.19) and (6.20), for the pure vector part, and, finally, eq. (6.21) or (6.23) for the tensorial modes. These equations allow to determine the metric terms in the first Post-Newtonian approximation.
Let us note that the equation for the transverse traceless component h_{ij}, (6.21) or (6.23), is a constraint; we do not have a dynamic equation for h_{ij} because of the approximation procedure.

6.4 Conservation equations

The conservation equations in general relativity are given by the contracted Bianchi identities

$$T^{\nu}_{\mu;\nu} = 0 \ . \tag{6.25}$$

Therefore, considering the zero component, i.e. when $\mu = 0$, and keeping all terms up to $1/c^2$ order, we get

$$\frac{(a^3\rho)^{\cdot}}{a^3} + \frac{(v^i\rho)_{,i}}{a} + \frac{1}{c^2}\left[\frac{(a^4\rho v^2)^{\cdot}}{a^4} + 3\dot{V}\rho + \frac{(\rho v^2 v^i)_{,i}}{a} + \rho v^i \frac{(3V-U)_{,i}}{a}\right] = 0 \ . \tag{6.26}$$

Obviously, setting $v_i = U = V = 0$, we obtain that $\rho = \rho_b$ and eq. (6.26) reduces to the continuity equation of cold dark matter in the cosmic background, i.e. $\dot{\rho}_b/\rho_b = -3(\dot{a}/a)$. Subtracting this equation in (6.26) and defining the total derivative as $dA/dt = \dot{A} + v^i A_{,i}/a$, eq. (6.26) becomes

$$\frac{d\delta}{dt} + \frac{v^i_{\ ,i}}{a}(\delta+1) + \frac{1}{c^2}\left\{(\delta+1)\left[v^2\left(\frac{\dot{a}}{a} + \frac{v^i_{\ ,i}}{a}\right) + 3\frac{dV}{dt} - \frac{v^i U_{,i}}{a}\right] + \frac{d}{dt}\left[(\delta+1)v^2\right]\right\} = 0\ . \tag{6.27}$$

The spatial component of the Bianchi identity gives:

$$\frac{(a^4\rho v_i)^{\cdot}}{a^4} - \frac{\rho U_{,i}}{a} + \frac{(v^j\rho v_i)_{,j}}{a} + \frac{1}{c^2}\left\{\rho v_i v^j\frac{(3V-U)_{,j}}{a} - 2\frac{\rho\Phi_{,i}}{a} + \rho v_i(3V-U)^{\cdot} + \frac{(\rho v_i v^j v^2)_{,j}}{a} + \frac{(a^4\rho v^2 v_i)^{\cdot}}{a^4} + + \frac{\left[2a^4\rho v_i(V+U)\right]^{\cdot}}{a^4} + 2\frac{\left[\rho v^j v_i(V+U)\right]_{,j}}{a} - \rho v^2\frac{(V+U)_{,i}}{a} - \frac{(a^4 P_i\rho)^{\cdot}}{a^4} - \frac{(P_i\rho v^j)_{,j}}{a} + \frac{P_{j,i}}{a}\rho v^j\right\} = 0. \tag{6.28}$$

Using eq. (6.27) and the continuity equation for ρ_b we can derive the final Euler equation up to 1PN order:

$$\frac{dv_i}{dt} + \frac{\dot{a}}{a}v_i - \frac{U_{,i}}{a} + \frac{1}{c^2}\left\{-\frac{\dot{a}}{a}v^2 v_i + v^2\frac{U_{,i}}{a} - v_i\frac{dU}{dt} + \frac{v^j v_i U_{,j}}{a} + \frac{2}{a}\frac{d}{dt}\left[a v_i(V+U)\right] - v^2\frac{(V+U)_{,i}}{a} - 2\frac{\Phi_{,i}}{a} - \frac{1}{a}\frac{d}{dt}(aP_i) + \frac{P_{j,i}v^j}{a}\right\} = 0. \tag{6.29}$$

Moreover, considering the $O(1/c^0)$ of the previous equation, we can write the continuity equation in the final form:

$$\frac{d\delta}{dt} + \frac{v^i_{\ ,i}}{a}(\delta+1) + \frac{1}{c^2}\left\{(\delta+1)\left[v^2\left(\frac{\dot{a}}{a} + \frac{v^i_{\ ,i}}{a}\right) + 3\frac{dV}{dt} - \frac{v^i U_{,i}}{a}\right] + v^2\frac{d}{dt}\delta - 2v^i\left(\frac{\dot{a}}{a}v_i - \frac{V_{,i}}{a}\right)\right\} = 0\ . \tag{6.30}$$

Equations (6.30) and (6.29) are the hydrodynamic equations of motion of a dust component in the Post Newtonian approximation. Let us note that in these equations the potential V does not appear in the 0-order part. Moreover the vector modes P_i contained in the Euler equation can not be decoupled from the scalar modes as is usual in the non-linear theory.

At this point, after having obtained the field equations and the dynamic equations, it is crucial to analyze both the limit on small scales and that on large scales. In fact it is interesting to understand if this approach, so far formulated, is capable of describing the physics of Newton on non-linear but small scales, and linear relativistic perturbation theory on scales of the order of the cosmological horizon. In the next two sections, we will study these cases limits.

6.5 Newtonian limit

Retaining just the first order in the c^{-1} expansion, we naturally recover the Newtonian theory. From the Einstein's equations (6.13), (6.14), (6.15), we have

$$\frac{1}{c^2}\frac{1}{a^2}\nabla^2 V = -\frac{4\pi G}{c^2}\rho_b\delta \ , \tag{6.31a}$$

$$\frac{1}{c^2}\frac{2}{a^2}\nabla^2(U-V) = 0 \ , \tag{6.31b}$$

$$\frac{1}{c^2}\frac{1}{a^2}\left[(V-U)_{,i}{}^{,j} \quad + \quad \frac{1}{3}\nabla^2(U-V)\delta_i^j\right] = 0 \ . \tag{6.31c}$$

Equation (6.31a) is the Poisson equation of chapter 2 (2.15c), from which we obviously deduce the Newtonian character of V, that is identified with minus the gravitational potential, $V = -\phi_G$. Equations (6.31b) and (6.31c) are the constraint equations provided by general relativity, they give the relation between the scalar fields of the metric to be consistent in a GR contest: as it is known U must be equal to V, the metric tensor generating Newtonian Gravity is definitively the weak field metric. Now, from the hydrodynamic equations (6.27), (6.29), we find

$$\frac{d\delta}{dt} + \frac{v^i{}_{,i}}{a}(\delta+1) = 0, \tag{6.32}$$

$$\frac{dv_i}{dt} + \frac{\dot{a}}{a}v_i = \frac{1}{a}U_{,i} \ , \tag{6.33}$$

in complete agreement with the Newtonian continuity and Eulero equation (2.15a)(2.15b), once we consider $U = V = -\phi_G$.

6.5.1 Passive and active approach

The usual way to obtain the Newtonian limit from General Relativity is demanding that the space components of the geodetic equation agree with the

Newtonian equation of motion. This naturally gives that $g_{00} = -(1 - 2U/c^2)$ identifying U with minus the gravitational potential. In fact, as we can see from (6.32) and (6.33), the continuity and the Euler equations do not need V so that it could be interpreted as a Post-Newtonian variable.
This way of think considers the "passive" aspect of gravitation, namely the response of matter to gravity [72]: one wants to determine the equation of motion of a particle in a given gravitational potential.
In this spirit the only needed metric variable is U and the Poisson equation is obtained considering only the time-time component of the following form of the Einstein's equation

$$R^\mu_\nu = \frac{8\pi G}{c^4}(T^\mu_\nu - \frac{1}{2}T\delta^\mu_\nu) + \Lambda\delta^\mu_\nu \ . \tag{6.34}$$

In fact, at $O(c^{-2})$, it gives the Poisson equation for U

$$\frac{1}{c^2}\frac{1}{3a^2}\nabla^2 U = -\frac{4\pi G}{3c^2}\rho_b\delta. \tag{6.35}$$

In our approach instead we consider the "active" aspect of gravitation, i.e. the generation of gravity by matter. In fact the cosmological application requires a self-gravitating fluid so that the field equations become as important as the equation of motion. Differently from the passive Newtonian limit, where the geodetic equation determines the order of the metric variables (U as Newtonian and Vas Post-Newtonian variable), now the field equations deriving from all the components of the Einstein's equation establish the metric order giving that both U and V have a Newtonian character. This emerges from (6.31a), (6.31b) and (6.31c) and naturally also from the spatial components of (6.34), whose trace is

$$\frac{1}{c^2}\frac{1}{a^2}\nabla^2(U - 4V) = \frac{12\pi G}{c^2}\rho_b\delta \ . \tag{6.36}$$

Clearly imposing $V = 0$ would be not consistent with (6.35) unless $U = 0$.
Therefore, if we want to obtain the Newtonian theory from GR, in the active picture and so keeping the consistency of all the components of the Einstein's equations, we do need to consider both the two scalar potentials U and V as 0PN terms, with the result

$$U = V = -\phi_G. \tag{6.37}$$

6.6 Linear limit

In order to develop a theoretical formalism that allows the combined study of the universe from large scales, where the relativistic effects are not negligible, down to small scales, it is fundamental to obtain the linear relativistic cosmological perturbation theory. In fact, the tiny anisotropies that we observe in the

Cosmic Microwave Background (CMB) and the matter power spectrum related to the density small inhomogeneity are well explained by the linear relativistic theory.
Dealing with the first order perturbative equations, as we well know, the scalar, vector and tensorial sectors are decoupled so that we can separately look at the scalar and vector type equations of section 6.3, (as already noted, the 1PN order is not able to describe the dynamics of tensorial modes that can not be associated with gravitational waves).
Therefore, starting from the scalar sector and considering only the linear terms, we obtain from equations (6.13), (6.14), (6.16), and (6.18):

$$\frac{1}{c^2}\frac{\nabla^2 V}{3a^2} - \frac{1}{c^4}\left[\frac{\dot{a}}{a}\dot{V} + \left(\frac{\dot{a}}{a}\right)^2 U - 2\frac{\nabla^2\Psi}{3a^2}\right] = -\frac{4\pi G}{3c^2}\rho_b\delta \;, \tag{6.38a}$$

$$\frac{1}{c^2}\left[\frac{2}{a^2}\nabla^2(U-V)\right] + \frac{1}{c^4a^2}\left\{4\nabla^2\left(\Phi-\Psi\right) + 6a^2\left[\frac{\dot{a}}{a}\left(\dot{U}+3\dot{V}\right)\right.\right.$$
$$\left.\left. +2\frac{\ddot{a}}{a}U + \left(\frac{\dot{a}}{a}\right)^2 U + \ddot{V}\right]\right\} = 0 \;, \tag{6.38b}$$

$$\frac{1}{c^3}\nabla^2\left(\frac{\dot{a}}{a}U + \dot{V}\right) = \frac{1}{c^3}4\pi G a\rho_b\theta \;, \tag{6.38c}$$

$$\frac{4}{3}\frac{1}{c^4a^2}\nabla^2\nabla^2(\Psi-\Phi) = -\frac{1}{c^2a^2}\frac{2}{3}\nabla^2\nabla^2(V-U) \tag{6.38d}$$

The continuity and the divergence of Euler equation from (6.30) and (6.29), after linearization, are

$$\dot{\delta} + \frac{\theta}{a} + \frac{3}{c^2}\dot{V} = 0 \tag{6.39}$$

$$\dot{\theta} + \frac{\dot{a}}{a}\theta - \frac{\nabla^2 U}{a} - \frac{2}{c^2}\frac{\nabla^2\Phi}{a} = 0, \tag{6.40}$$

where the convective term in the total time derivative is neglected and partial and total derivative coincide.
Now, defining the scalar gravitational fields in the following way

$$\phi_P = -(U + \frac{2}{c^2}\Phi), \tag{6.41}$$

$$\psi_P = -(V + \frac{2}{c^2}\Psi), \tag{6.42}$$

the previous equations become

$$\nabla^2\psi_P - \frac{3}{c^2}a^2\left[\frac{\dot{a}}{a}\dot{\psi}_P + \left(\frac{\dot{a}}{a}\right)^2\phi_P\right] = 4\pi G\rho_b a^2\delta \;, \tag{6.43a}$$

$$3a^2\left[\frac{\dot a}{a}(\dot\phi_P+3\dot\psi_P)+2\frac{\ddot a}{a}\phi_P+\left(\frac{\dot a}{a}\right)^2\phi_P+\ddot\psi_P\right]-\nabla^2(\psi_P-\phi_P)=0\ , \tag{6.43b}$$

$$\nabla^2\left(\frac{\dot a}{a}\phi_P+\dot\psi_P\right)=-4\pi Ga\rho_b\theta\ , \tag{6.43c}$$

$$\frac{1}{c^2a^2}\frac{2}{3}\nabla^2\nabla^2(\phi_P-\psi_P)=0\ , \tag{6.43d}$$

$$\dot\delta+\frac{\theta}{a}-\frac{3}{c^2}\dot\psi_P=0\ , \tag{6.43e}$$

$$\dot\theta+\frac{\dot a}{a}\theta+\frac{\nabla^2\phi_P}{a}=0\ . \tag{6.43f}$$

Note from eq. (6.43d) that $\psi_P=\phi_P$, this is given by the perfect fluid matter model we have chosen (null anisotropic stress).
The system above coincides with the set of equations of standard cosmological perturbations theory in the Newtonian gauge (3.32)[11].
The conclusion of this analysis is important: the 1PN approximation includes the first order full relativistic perturbation theory so that all the linear terms coming from General Relativity are just 1PN order; naturally, on the other hand, the 1PN order terms do not involve just linear terms, there are terms of second and third order in the perturbations.
This result is due to our active approach to the Post-Newtonian theory, with the assumption of U and V as both Newtonian variables, and the simple derivation of 1PN equations retaining the first two orders in the $1/c$ expansion, without any iterative scheme. Indeed, in the other similar Post-Newtonian cosmological analysis [53], the zero order (0PN) implies that the potential U satisfies the Poisson equation and so eq.(6.43a) can not be recovered in the linear limit.
In conclusion, let us consider briefly the linear vector sector. Immediately, from (6.19), and (6.20), we get

$$\nabla^2\nabla^2\frac{P_i}{a}=16\pi Ga\rho_b[v_{k,i}{}^{,k}-\nabla^2v_i], \tag{6.44}$$

$$\frac{1}{c^4}\nabla^2\nabla^2\left(\frac{1}{2a}\dot P_i+\frac{\dot a}{a^2}P_i\right)=0. \tag{6.45}$$

Also in the vector sector, the 1PN is able to describe the full dynamics of the equations of the relativistic linear perturbations theory. From 6.45 one can easily verify that P_i decays very quickly, allowing to conclude that the linear vector P_i can be safely disregarded.

6.7 Non linear set of 1PN equations.

In section 6.3 we found the non linear system of field and matter equations including the Newtonian terms and the first relativistic corrections. We demonstrated that, under a redefinition of field variables ϕ_P and ψ_P, and keeping just

the first order in the perturbations with respect to the FRW background, we obtain the linear general relativistic perturbation theory. We could ask if the same correspondence applies also at the second perturbative order. Naturally we do not use an iterative scheme, so the comparison has to be done with the resummation of GR perturbative theory up to second order.
Let us write Einstein equations (6.13), (6.14), (6.15) and (6.10), using (6.42) and (6.41),

$$\frac{1}{c^2}\left[\frac{-\nabla^2\psi_P}{3a^2}\right]+\frac{1}{c^4}\left[\frac{\dot a}{a}\dot\psi_P+\left(\frac{\dot a}{a}\right)^2\phi_P-\frac{1}{3a^2}\nabla^2\psi_P^2+\frac{5}{6a^2}\psi_{P,i}\psi_P{}^{,i}\right]$$
$$=-\frac{4\pi G}{3}\left[\frac{1}{c^2}\rho_b\delta+\frac{1}{c^4}\rho_b(1+\delta)v^2\right]\ , \tag{6.46a}$$
$$\frac{1}{c^2}\left[\frac{2}{a^2}\nabla^2(\psi_P-\phi_P)\right]+\frac{1}{c^4a^2}\Bigg\{-2\phi_{P,k}\phi_P{}^{,k}-\psi_{P,k}\psi_P{}^{,k}+2\phi_{P,k}\psi_P{}^{,k}$$
$$+4\psi_P\nabla^2(\psi_P-\phi_P)\ -6a^2\left[\frac{\dot a}{a}\left(\dot\phi_P+3\dot\psi_P\right)+2\frac{\ddot a}{a}\phi_P+\left(\frac{\dot a}{a}\right)^2\phi_P+\ddot\psi_P\right]\Bigg\}$$
$$=-\frac{8\pi G}{c^4}\rho_b(1+\delta)v^2 \tag{6.46b}$$
$$\frac{1}{c^2}\Big[\frac{1}{a^2}(\phi_P-\psi_P)_{,i}{}^{,j}-\frac{1}{3a^2}\nabla^2(\phi_P-\psi_P)\delta_i^j\Big]+\frac{1}{c^4}\Bigg\{-\frac{\dot a}{a^2}\left(P_i{}^{,j}+P^j{}_{,i}\right)$$
$$-\frac{1}{2a}\left(\dot P^j{}_{,i}+\dot P_i{}^{,j}\right)+\frac{1}{a^2}\phi_{P,i}\phi_P{}^{,j}-\frac{1}{a^2}\psi_{P,i}\psi_P{}^{,j}+\frac{1}{a^2}\left(\phi_{P,i}\psi_P{}^{,j}+\phi_P{}^{,j}\psi_{P,i}\right)$$
$$+\frac{2}{a^2}\psi_P(\phi_P-\psi_P)_{,i}{}^{,j}+\frac{1}{a^2}\delta_i{}^j\left[-\frac{1}{3}(\phi_{P,k}\phi_P{}^{,k}-\psi_{P,k}\psi_P{}^{,k})-\frac{2}{3}\phi_{P,k}\psi_P{}^{,k}\right.$$
$$\left.+\frac{2}{3}\psi_P\nabla^2(\psi_P-\phi_P)\right]+\frac{1}{2a^2}\nabla^2h_i^j\Bigg\}=-\frac{8\pi G}{c^4}\rho_b(1+\delta)(v_iv^j-\frac{1}{3}\delta_i^jv^2) \tag{6.46c}$$
$$\frac{1}{c^3}\left[-\frac{1}{2a^2}\nabla^2P_i-2\frac{\dot a}{a^2}\phi_{P,i}-\frac{2}{a}\dot\psi_{P,i}\right]=\frac{8\pi G}{c^3}\rho_b(1+\delta)v_i. \tag{6.46d}$$

We realize that, neglecting the third order matter terms in the right hand sides of the equations, we can not obtain the second order perturbation theory [69] because of the lack of some terms; these terms are made of two potential fields and one or two derivatives respect to the time, they are $O(c^{-5})$ or $O(c^{-6})$ respectively. In this we can see one of the most meaningful differences between the Post-Newtonian approximation and the perturbative approach: it concerns the role of time. In the PN scheme the time has a dominant importance among the other coordinates, in fact, following the Newtonian spirit, the absolute time concept has been restored and the temporal coordinate contributes to alter the approximation order because of the c factor; in the standard perturbative theory instead the time is considered as a space-time coordinate like the others, in agreement with the relativistic concept of equivalence between time and space.

It follows that in our scheme the temporal derivatives always appears with an extra c^{-1} factor increasing the perturbative order. For this reason the dynamic equations are at least one order higher than the constraint equations so that at Newtonian level (0PN) we do not have the evolution equation for the gravitational potential. Increasing the approximation accuracy and passing at 1PN order we get the dynamic equation for the gravitational potential but others variables appear for which we have only the constraint type equation (for example the pure tensorial components h_{ij}, only the 2.5PN order provides the second-order differential equations for such modes).

We can say that, neglecting the third order terms, the system (6.46a)-(6.46d) is a reduced form of the resummed second order GR equations, in which the missed terms are those whose effect is mainly relativistic. This comparison is a new result, due to our particular not standard approach of the Post-Newtonian approximation.

As we aspect our equations are no more valid at very large scales, where the general relativistic effects have to be considered; but it is also true that at such scales the non-linear fluctuations are so small that the linear theory , that we can correctly reproduce, is the right large scale limit to be considered. On the other hand our equations better describe the physics at small scales where the non-linear effects start to be relevant and the second order perturbation theory is not satisfactory.

As we have seen, both in the Newtonian and the linear limit, we found $\phi_P = \psi_P$ (this reduces to $U = V$ at Newtonian order). We say that in the general case this is not more valid because of some non linear and post-Newtonian expressions , so we can define:

$$\frac{D_P}{c^2} = \frac{1}{2}(\psi_P - \phi_P); \tag{6.47}$$

the new variable D encloses all the post-Newtonian non linearity.
Similarly we introduce the variable

$$\phi_G = \frac{1}{2}(\psi_P + \phi_P), \tag{6.48}$$

that generalizes the definition of the gravitational potential, given in section 6.5. As we will see later, it is strictly related with the tidal force.
From the analysis of the Euler equation (6.29) we can define a new velocity field

$$v_i^* = v_i - \frac{1}{c^2}P_i \ , \tag{6.49}$$

which allows to write the equation of motion in a simpler form. Moreover, thanks to the gauge condition, we have $v^i{}_{,i} = v^{*i}{}_{,i}$ and the new velocity can be used in all the other equations.
With the new variables the time-time component (6.13), the trace (6.14), the tracefree part (6.15) and the time-space component(6.10) of Einstein equations,

become:

$$-\frac{1}{c^2}\frac{1}{3a^2}\nabla^2\phi_G+\frac{1}{c^4}\left[-\frac{1}{3a^2}\nabla^2 D_{PN}+\frac{\dot a}{a}\dot\phi_G+\left(\frac{\dot a}{a}\right)^2\phi_G-\frac{1}{3a^2}\nabla^2\phi_G^2\right.$$
$$\left.+\frac{5}{6a^2}\phi_{G,i}\phi_G{}^{,i}\right]=-\frac{4\pi G}{3}\rho_b\left[\frac{1}{c^2}\delta+\frac{1}{c^4}(1+\delta)v^{*2}\right], \tag{6.50a}$$
$$\frac{1}{c^4}\left[\frac{4}{a^2}\nabla^2 D_{PN}-\frac{1}{a^2}\phi_{G,k}\phi_G{}^{,k}-6\left(4\frac{\dot a}{a}\dot\phi_G+2\frac{\ddot a}{a}\phi_G+\left(\frac{\dot a}{a}\right)^2\phi_G+\ddot\phi_G\right)\right]$$
$$=-\frac{8\pi G}{c^4}\rho_b(1+\delta)v^{*2}, \tag{6.50b}$$
$$\frac{1}{c^4}\left[-\frac{2}{a^2}\left(D_{PN,i}{}^{,j}-\frac{1}{3}\nabla^2 D_{PN}\,\delta_i^j\right)+\frac{2}{a^2}\phi_{G,i}\phi_G{}^{,j}-\frac{2}{3a^2}\phi_{G,k}\phi_G{}^{,k}\,\delta_i^j\right.$$
$$\left.-\frac{\dot a}{a^2}\left(P_i{}^{,j}+P^j{}_{,i}\right)-\frac{1}{2a}\left(\dot P_i{}^{,j}+\dot P^j{}_{,i}\right)+\frac{1}{2a^2}\nabla^2 h_i^j\right]=-\frac{8\pi G}{c^4}\rho_b(1+\delta)$$
$$\left(v_i^* v^{*j}-\frac{1}{3}v^{*2}\,\delta_i^j\right), \tag{6.50c}$$
$$\frac{1}{c^3}\left(\frac{1}{2a}\nabla^2 P_i+2\frac{\dot a}{a}\phi_{G,i}+2\dot\phi_{G,i}\right)=-\frac{8\pi G}{c^3}a\rho_b(1+\delta)v_i^*. \tag{6.50d}$$

The continuity equation and the Euler equation, up to 1PN order, are

$$\frac{d\delta}{dt}+\frac{v^{*i}{}_{,i}}{a}(\delta+1)+\frac{1}{c^2}\left\{(\delta+1)\left[v^{*2}\left(\frac{\dot a}{a}+\frac{v^{*i}{}_{,i}}{a}\right)-3\frac{d\phi_G}{dt}+\frac{v^{*i}}{a}\phi_{G,i}\right]\right.$$
$$\left.+v^{*2}\left(\frac{d\delta}{dt}-2\frac{\dot a}{a}\right)-2\frac{v^{*i}}{a}\phi_{G,i}\right\}=0\ , \tag{6.51}$$
$$\frac{dv_i^*}{dt}+\frac{\dot a}{a}v_i^*+\frac{1}{a}\phi_{G,i}+\frac{1}{c^2}\left[-\frac{\dot a}{a}v^{*2}v_i^*+\frac{v^{*2}}{a}\phi_{G,i}+v_i^*\frac{d\phi_G}{dt}-\frac{v_i^* v^{*j}}{a}\phi_{G,j}\right.$$
$$\left.-\frac{4}{a}\frac{d}{dt}\left(av_i^*\phi_G\right)+\frac{v^{*k}}{a}P_{k,i}\right]=0\ . \tag{6.52}$$

This two equations give the matter dynamics and its connection with the metric fields, that have to be found through eq. (6.50a)-(6.50d). In order to determine P_i, it is more convenient to use the purely vectorial equation (instead of eq (6.50d)), from (6.19) we have

$$\nabla^2\nabla^2\frac{P_i}{a}=16\pi Ga\rho_b[((1+\delta)v_k^*)_{,ik}-\nabla^2((1+\delta)v_i^*)]. \tag{6.53}$$

Eqs. (6.50a) and (6.50b) are the equations for the two scalars ϕ_G and D_P, these can be arranged in order to have an evolution equations for ϕ_G, and a constraint equation for D_P. The latter has already been computed, it is eq. (6.17), that we rewrite with the new variables:

$$\frac{1}{c^4}\frac{\nabla^2\nabla^2 D_P}{3a^2} = -\frac{1}{c^2}\frac{1}{3a^2}\nabla^2\nabla^2(\phi_G) - \frac{1}{c^4}\left[\frac{1}{3a^2}\nabla^2\nabla^2(\phi_G) - \frac{5}{6a^2}\nabla^2(\phi_{G,i}\phi_{G,i})\right]$$
$$+\frac{4}{3c^2}\pi G\rho_b\nabla^2\delta - \frac{4}{3c^4}\pi G\rho_b\left[\nabla^2((1+\delta)v^{*2}) + \dot{a}(v_i^*(1+\delta))_{,i}\right]. \quad (6.54)$$

Substituting the expression for $\nabla^2 D_P$ from (6.50a), in (6.50b), we obtain the evolution equation for ϕ_G.

$$\frac{1}{c^4}\left[\ddot{\phi}_G + 2\frac{\dot{a}}{a}\dot{\phi}_G + 2\frac{\ddot{a}}{a}\phi_G - \left(\frac{\dot{a}}{a}\right)^2\phi_G + \frac{2}{3a^2}\nabla^2(\phi_G^2) - \frac{3}{2a^2}\phi_{G,i}\phi_{G,i} + \frac{c^2}{a^2}\nabla^2\phi_G\right]$$
$$= \frac{16\pi G}{c^2}\rho_b\delta + \frac{24\pi G}{c^4}\rho_b(1+\delta)v^{*2} \quad (6.55)$$

Finally eq. (6.50c), relates the tensorial modes h_{ij} with the scalar sector of the metric. The convenient tensorial equation to be used is eq.(6.23) with:

$$\mathcal{R}_j^l = 8\pi G a^2\rho_b(1+\delta)(v_j v^l - \frac{1}{3}\delta_j^l v^2) + 2\phi_{G,j}\phi_G{}^{,l} - \delta_j^l\frac{2}{3}\phi_{G,k}\phi_G{}^{,k}. \quad (6.56)$$

At this order the post-Newtonian TT tensor and the non-linear post-Newtonian variable D_P do not have an evolution equations. So we can conclude that the first correction to Newtonian gravity introduces three not dynamic degree of freedom (two d.o.f. in h_{ij} and 1 d.o.f. in D_P)and provides the evolution equation for the gravitational potential.

We end this chapter concluding that the final set of equations (6.50)-(6.52) provides a good description of the gravitational instability in the quasi non-linear regime, in particular for fluctuations of intermediate scales. Considering fluctuations of larger characteristic size the non-linear contributes become always smaller and our set of equations are almost equivalent to the second-order equations of perturbation theory. With the growing of the size of the perturbations the difference between these two sets of equations grows, in fact the regime is becoming relativistic and the standard second-order treatment better describes the geometrical and matter variables.

Therefore we can think of the cosmological theory of structure formation as a series of approximated equations describing, in the chosen scale regime, the clustering process. In the very small scales the non linearity is important and the correct theory is given by the Newtonian gravity. Considering larger scales the relativistic effects become relevant and one should use the 1PN theory, which preserves the full non-linearity of the matter terms (in the density contrast) and introduces the first relativistic terms. Approaching the horizon scale the non-linearity loses importance and the relativistic effects become more relevant so that one can use the second-order perturbation theory and than, at very large scales, the first-order perturbation theory.

How much important is the introduction of the 1PN treatment will require a quantitative analysis that can be done relatively to the accuracy level of the experimental data and the current capability of the N-body simulations.

Chapter 7

1PN covariant approach: an outline

7.1 Introduction

In this chapter we analyze the Post-Newtonian expressions of the kinematic quantities connected with the velocity covariant derivative defined in paragraph 3.1; the Newtonian limit will give the definitions and the results of paragraphs 2.1.2 and 2.1.3.

As we have seen, such quantities are used in the covariant approach to General Relativity [40], leading to a set of perturbation equations that do not depend on any defined system [93]. On the other hand, we are studying the Post-Newtonian approximation in the metric approach point of view [11], starting from a specific set of coordinates giving the metric components (6.1). Therefore our purpose is to identify the post-Newtonian terms deriving from that particular choice of coordinates system. So we will use the covariant relativistic equations describing the fluid motion through its shear, its vorticity and its expansion; their Newtonian-like form reflects the compactness of the 1+3 approach and the advantage of not using a defined coordinates system. Having already chosen the metric components, we will not preserve the covariance of the equations, unveiling the convective and geometric terms up to 1PN order [22][23][24].

7.2 1PN effective time derivative

We want to explicitly find the Newtonian part and the first Post-Newtonian corrections of the relativistic expansion scalar, the shear, the vorticity and the acceleration 4-vector. As we defined in (3.4), such quantities are strictly related with the covariant derivative operator associated with a projection procedure; the projection into the 4-velocity direction (contraction with u^{μ}) defines the co-

variant time derivative that in section 3.1 we indicated with a dot. Now, keeping the Newtonian formalism so that the dot denotes the time partial derivative, we indicate the covariant time derivative operator with D_t.
Let us start finding the Newtonian and post-Newtonian parts of this effective time derivative, acting on a generic tensor $\alpha^{\mu\ldots\nu}_{\alpha\ldots\beta}$

$$D_t\alpha^{\mu\ldots\nu}_{\alpha\ldots\beta} = \alpha^{\mu\ldots\nu}_{\alpha\ldots\beta;\gamma}u^\gamma. \tag{7.1}$$

For a scalar field, composed by a Newtonian and a 1PN component in the form[1]

$$\alpha = \frac{\alpha^N}{c} + \frac{\alpha^{PN}}{c^3}, \tag{7.2}$$

from (7.1) and (6.4), the expansion procedure gives:

$$D_t\alpha = \frac{1}{c^2}\frac{d}{dt}\alpha^N + \frac{1}{c^4}\left[\frac{d}{dt}\alpha^{PN} + \frac{d}{dt}\alpha^N\left(U + \frac{1}{2}v^2\right)\right]; \tag{7.3}$$

therefore the 0 order of the effective time derivative coincides with the convective derivative [82]. In a higher rank tensor the Christoffel symbols are involved, this reflecting also the Newtonian part in which an expansion term and a geometric term arise; for a vector with Newtonian and post-Newtonian components

$$\begin{aligned}\alpha_0 &= \alpha_0^N + \frac{1}{c^2}\alpha_0^{PN}\\ \alpha_i &= \frac{1}{c}\alpha_i + \frac{1}{c^3}\alpha_i^{PN}\ ,\end{aligned}$$

we have

$$\begin{aligned}D_t\alpha_i = &\frac{1}{c^2}\left[\frac{d}{dt}\alpha_i^N - \frac{\dot a}{a}\alpha_i^N + U_{,i}\alpha_0^N - \dot a v_i\alpha_0^N\right] + \frac{1}{c^4}\left[\frac{d}{dt}\alpha_i^{PN} - \frac{\dot a}{a}\alpha_i^{PN} + U_{,i}\alpha_0^{PN}\right.\\ &-\dot a v_i\alpha_0^{PN} + \left(\frac{d}{dt}\alpha_i^N - \frac{\dot a}{a}\alpha_i^N\right)\left(U + \frac{1}{2}v^2\right) - \dot V(\alpha_i + a v_i\alpha_0) + \frac{1}{2a}(P^k{}_{,i} - P_i{}^{,k})\alpha_k^N\\ &-\frac{1}{2}\alpha_0 v_k\left(P_i{}^{,k} + P^k{}_{,i}\right) + \dot a\alpha_0 P_i - (V_{,j}\alpha_i^N - V_{,i}\alpha_j^N + V^{,k}\alpha_k^N\delta_{ij})\frac{v^j}{a} + 2\Phi_{,i}a_0\\ &\left.+\alpha_0 U U_{,i} - \alpha_0\dot a v_i(3U + 2V) + \frac{1}{2}\alpha_0 v^2 U_{,i}\right]\ ;\end{aligned} \tag{7.4}$$

[1]The choice of the power of c^{-1} in the decomposition of this general scalar and of the following vector and tensor components is in agreement with the 4-velocity vector and the next results for the scalar expansion, the shear and the vorticity (see (6.4), (7.9)).

similarly, for a second rank tensor decomposed as

$$\alpha_{ij} = \frac{\alpha_{ij}^{N}}{c} + \frac{\alpha_{ij}^{PN}}{c^3},$$
$$\alpha_{00} = \frac{\alpha_{00}^{PN}}{c^3},$$
$$\alpha_{0i} = \frac{\alpha_{0i}^{PN}}{c^2},$$

we obtain

$$D_t\alpha_{ij} = \frac{1}{c^2}\left[\frac{d}{dt}\alpha_{ij}^N - 2\frac{\dot{a}}{a}\alpha_{ij}^N\right] + \frac{1}{c^4}\left[\frac{d}{dt}\alpha_{ij}^{PN} - 2\frac{\dot{a}}{a}\alpha_{ij}^{PN} + \left(\frac{d}{dt}\alpha_{ij}^N - 2\frac{\dot{a}}{a}\alpha_{ij}^N\right)\right.$$
$$\left(U + \frac{1}{2}v^2\right) - 2\dot{V}a\alpha_{ij}^N + \alpha_{k(j}^N\left(P_{,i)}^k - P_{i)}^{;k}\right) - 2a\alpha_{l(j}^N(V_{,|k|}\delta_{i)l} - V_{,i)}\delta_l^k - V_{,|l|}\delta_{i)}^k)$$
$$\left.-2\alpha_{0(j}^{PN}U_{,i)} - 2\dot{a}v_{(i}\alpha_{0j)}^{PN}\right]. \tag{7.5}$$

Having split the covariant derivative projected along the 4-velocity, we consider now the orthogonal projection tensor h_{ij}. From (3.1) we obtain

$$h_j^i = \delta_j^i + \frac{1}{c^4}v^iv_j + \frac{1}{c^4}\left[v^iv_j\left(v^2 + 2U + 2V\right) - v^iP_j\right], \tag{7.6a}$$
$$h_i^0 = \frac{a}{c}v_i + \frac{1}{c^3}\left[v_i\left(v^2 + 2U + 2V\right) - P_i\right], \tag{7.6b}$$
$$h_0^0 = -\frac{1}{c^2}v^2 - \frac{1}{c^4}\left[v^2\left(v^2 + 2U + 2V\right) - v^iP_i\right]. \tag{7.6c}$$

We do not compute its action on the covariant derivative of a generic tensor, i.e.

$$D_\rho\alpha^{\mu\nu..}{}_{\gamma\delta..} = h_\rho{}^\tau h_\alpha{}^\mu h_\beta{}^\nu h_\gamma{}^\xi h_\delta{}^\eta \alpha^{\alpha\beta..}{}_{\xi\eta..;\tau}, \tag{7.7}$$

but we can easily note that, in the case of the 4-velocity, h_μ^ν has a fully Post-Newtonian role, so that at Newtonian level just the Kronecker delta in (7.6a) is relevant.

7.3 Kinematic Quantities

We recall the relativistic definitions of the shear, the vorticity and the expansion rate (3.4):

$$\sigma_{\mu\nu} = D_{\langle\nu}u_{\mu\rangle}, \qquad \omega_{\mu\nu} = D_{[\nu}u_{\mu]}, \qquad \Theta = D^\mu u_\mu. \tag{7.8}$$

Expanding in powers of c^{-1} and following our 1PN scheme based on the metric (6.1), we obtain[2]

$$\begin{aligned}\Theta &= 3\frac{\dot a}{ca}+\frac{\theta^N}{ca}+\frac{\theta^{PN}}{c^3 a}=3\frac{\dot a}{ca}+\frac{\theta^N}{ca}+\frac{1}{c^3}\left[3\dot V+\frac{3}{2}\frac{\dot a}{a}v^2+\frac{v_{k,l}v_l v_k}{a}+v_k\dot v_k\right.\\ &\left.+\frac{1}{2a}v_{k,k}v_l v_l+v_{k,k}\frac{U}{a}+3\frac{\dot a}{a}U+\frac{3}{a}v_k V_{,k}\right], \qquad (7.9a)\end{aligned}$$

$$\begin{aligned}\sigma_{ij} &= \frac{a\sigma_{ij}^N}{c}+\frac{a\sigma_{ij}^{PN}}{c^3}=\frac{\sigma_{ij}^N}{c}+\frac{1}{c^3}\left[a^2 v_{(i}\dot v_{j)}+\frac{1}{2}av_{(i,j)}v^2+av_{(i,|k|}v_{j)}v_k\right.\\ &-\frac{a}{3}v_{k,k}v_i v_j+av_{k,(i}v_{j)}v_k+av_{(i,j,)}(U+2V)+\delta_{ij}\left(-\frac{1}{3}a^2 v_k\dot v_k\right.\\ &\left.\left.-\frac{1}{3}(2V+U)av_{k,k}-\frac{a}{6}v_{l,l}v^2-\frac{a}{3}v_{l,k}v_k v_l\right)\right], \qquad (7.9b)\end{aligned}$$

$$\begin{aligned}\omega_{ij} &= \frac{a\omega_{ij}^N}{c}+\frac{a\omega_{ij}^{PN}}{c^3}=\frac{a\omega_{ij}^N}{c}+\frac{1}{c^3}\left[a^2\dot v_{[i}v_{j]}+\frac{1}{2}av_{[i,j]}v^2+aP_{[i,j]}-2av_{[i}U_{,j]}\right.\\ &\left.-av_{[i}V_{,j]}-2av_{[i,j]}V-av_k v_{[i,|k|}v_{j]}\right]. \qquad (7.9c)\end{aligned}$$

The acceleration is the effective time derivative of u_μ so that, from (7.4), we obtain for the spatial component:

$$\begin{aligned}A_i &= \frac{\dot a v_i}{c^2}+\frac{aA_i^N}{c^2}-\frac{U_{,i}}{c^2}+\frac{a_i^{PN}}{c^4}=\frac{\dot a v_i}{c^2}+\frac{aA_i^N}{c^2}-\frac{U_{,i}}{c^2}+\frac{1}{c^4}\left[-2\Phi_{,i}\right.\\ &\left.+(2v_i a(U+V))^{\cdot}-(aP_i)^{\cdot}+a(v_i v^2)^{\cdot}-av_i v_k\dot v_k\right]. \qquad (7.9d)\end{aligned}$$

Note the background part of the expansion rate and the first term of the acceleration; we expect these Newtonian terms because the 4-velocity includes not only the peculiar velocity, but also the scale factor expanding motion. Consequently the Hubble parameter is the unperturbed term of the expanding scalar and the combination $\dot a/av_i + A_i^N$ is nothing else that the peculiar acceleration (2.4).
In our zero pressure case there are not forces other than gravity so that, for the equation of motion (3.18), the 4-acceleration has to vanish. Therefore from (7.9d) we have

$$\frac{\dot v_i}{c^2}+\frac{Hv_i}{c^2}-\frac{U_{,i}}{ac^2}+\frac{1}{ac^4}\left[-2\Phi_{,i}+(2v_i a(U+V))^{\cdot}-(aP_i)^{\cdot}+a(v_i v^2)^{\cdot}\right.$$
$$\left.-av_i v_k\dot v_k\right]=0. \qquad (7.10)$$

This equation is an other way to write the Euler equation, that in the previous chapter we obtained through the Bianchi identity.

[2]Summation of repeated indices is intended also if both are covariant

Besides these spatial components whose zero order coincides with the Newtonian quantities, we have the time-time and time-space components that are intrinsically 1PN order:

$$\sigma_{00} = \frac{\sigma_{00}^{PN}}{c^3} = \frac{1}{c^3 a}\left(v_i v_{j,i} v_j - \frac{1}{3} v_{i,i} v^2\right) \tag{7.11a}$$

$$\sigma_{0i} = \frac{\sigma_{0i}^{PN}}{c^2} = \frac{1}{c^2}\left(-\frac{1}{2} v_{j,i} v_j - \frac{1}{2} v_{i,j} v_j + \frac{1}{3} v_{k,k} v_i\right) \tag{7.11b}$$

$$\omega_{00} = \frac{\omega_{ij}^{PN}}{c^3} = 0 \tag{7.11c}$$

$$\omega_{0i} = \frac{\omega_{0i}^{PN}}{c^2} = \frac{1}{2c^2}\left(v_j v_{j,i} - v_{i,j} v_j\right) \tag{7.11d}$$

$$A_0 = \frac{A_0^{PN}}{c^3} = \frac{1}{c^3}\left[-\frac{\dot{a}}{a} v^2 - v_i \frac{dA_i^N}{dt} + \frac{v_i U_{,i}}{a}\right] \tag{7.11e}$$

7.4 The 1PN Raychauduri equation

We have found the Newtonian and the Post-Newtonian terms of the covariant proper time derivative and of the General Relativistic kinematic variables. Now we are ready to derive the 1PN expressions for the dynamic equations of such quantities, by splitting the equations (3.15).

We focus on the Raychaudury equation (3.15a). Using (7.3) and (7.9a) - (7.9d), we obtain:

$$\begin{aligned}
&\frac{1}{c^2}\left[\frac{d\theta^N}{dt} + \frac{\dot{a}}{a}\theta^N + \frac{1}{3}\frac{(\theta^N)^2}{a} + 3\ddot{a} + \frac{2}{a}\left((\sigma^N)^2 - (\omega^N)^2\right) + 4\pi G\rho a - \Lambda a\right] \\
&+\frac{1}{c^4}\left[\left(\frac{d\theta^N}{dt} - \frac{\dot{a}}{a}\theta^N - 3\frac{(\dot{a})^2}{a} + 3\ddot{a}\right)\left(U + \frac{1}{2}v^2\right) + \frac{d\theta^{PN}}{dt} + \frac{\dot{a}}{a}\theta^{PN}\right. \\
&\left.+\frac{2}{3a}\theta^N\theta^{PN} - \frac{4}{a}\left((\sigma^N)^2 - (\omega^N)^2\right)V + \frac{2}{a}\left(\sigma_{ij}^N\sigma^{PNij} - \omega_{ij}^N\omega^{PNij}\right)\right] = 0.
\end{aligned} \tag{7.12}$$

Let us consider the Newtonian part, proportional to c^{-2}. As we did in chapter 2, we change the time variable from t to a; consequently the velocity vector is (see (2.26))

$$u_i = (Ha^2)^{-1} v_i. \tag{7.13}$$

Remembering that the quantities with the tilde are those referring to u_i, we obtain

$$\frac{1}{c^2}\left[\frac{d\tilde{\theta}^N}{da}+\frac{1}{3}(\tilde{\theta}^N)^2+2\left((\tilde{\sigma}^N)^2-(\tilde{\omega}^N)^2\right)+\frac{\tilde{\theta}^N}{a}\left(\frac{3}{2}+\frac{\Lambda}{2H^2}\right)+\frac{3\ddot{a}}{H^2a^3}\right.$$
$$\left.+\frac{1}{H^2a^3}(4\pi G\rho a-\Lambda a)\right]=0. \tag{7.14}$$

Now we can use the background Friedmann equations (6.11) and (6.12) for which the following expression holds:

$$\frac{3\ddot{a}}{a}+4\pi G\rho-\Lambda=4\pi G\rho_b\delta. \tag{7.15}$$

Finally, if we use the Newtonian Poisson equation (6.31a), and defining the rescaled gravitational potential as [18]

$$\tilde{\phi}=-\frac{2}{3H^2a^3}V, \tag{7.16}$$

we recover the Newtonian expansion equation of chapter 1 (2.34b) without the pressure term:

$$\frac{1}{c^2}\left[\frac{d\tilde{\theta}^N}{da}+\frac{1}{3}(\tilde{\theta}^N)^2+2\left((\tilde{\sigma}^N)^2-(\tilde{\omega}^N)^2\right)+\frac{\tilde{\theta}^N}{a}\left(\frac{3}{2}+\frac{\Lambda}{2H^2}\right)+\nabla^2\tilde{\phi}\right]=0. \tag{7.17}$$

7.5 Tidal field

In chapter 3 we introduced the electric and magnetic components of the Weyl tensor, whose corresponding Newtonian object is the tidal field [37]. Now we explicitly find this correspondence extending the analysis up to the first Post-Newtonian correction.
From (3.10) and (3.11) we get

$$E_{00}=\frac{1}{c^4a^2}\left[\frac{1}{6}v^2\nabla^2(U+V)-\frac{2}{3}v_lv_k(U+V)_{,lk}\right], \tag{7.18}$$

$$E_{0i}=\frac{1}{2c^3}\left[v_k(U+V)_{,ki}-\frac{1}{3}v_i\nabla^2(U+V)\right], \tag{7.19}$$

$$E_{ij}=\frac{1}{c^2}\left[-\frac{1}{2}(V+U)_{,ij}+\frac{\delta_{ij}}{6}\nabla^2(V+U)\right]+\frac{1}{c^4}\left[-(\Psi+\Phi)_{,ij}+\frac{1}{2}U_{,i}U_{,j}\right.$$
$$+\frac{1}{2}V_{,i}V_{,j}+U_{,(i}V_{,j)}+v_{(i}v_{|k|}(V+U)_{,j)k}-\frac{1}{3}v_iv_j\nabla^2(U+V)-v^2(U+V)_{,ij}$$
$$-\frac{1}{2}a\dot{P}_{(i,j)}+v_kP_{k,ij}-v_kP_{(i,j)k}-\frac{1}{2}v_{(i}P_{|k|,j)k}+\delta_{ij}\left(-\frac{1}{3}U_{,k}V_{,k}-\frac{1}{6}U_{,k}U_{,k}\right.$$

$$-\frac{1}{6}V_{,k}V_{,k}-\frac{1}{2}v_kP_{k,ll}-\frac{1}{2}v_kv_l(U+V)_{,kl}+\frac{1}{2}v^2\nabla^2(U+V)+\frac{1}{3}\nabla^2(\Phi+\Psi)\Big)\Big]. \tag{7.20}$$

In this case we get simpler expressions if we use the variable ϕ_G introduced in section 6.7:

$$E_{00}=-\frac{1}{c^4a^2}\left[\frac{1}{3}v^2\nabla^2\phi_G-\frac{4}{3}v_lv_k\phi_{G,lk}\right], \tag{7.21a}$$

$$E_{0i}=-\frac{1}{c^3}\left[v_k\phi_{G,ki}-\frac{1}{3}v_i\nabla^2\phi_G\right], \tag{7.21b}$$

$$\begin{aligned}E_{ij}=&\frac{1}{c^2}\left[\phi_{G,ij}-\frac{\delta_{ij}}{3}\nabla^2\phi_G\right]+\frac{1}{c^4}\Big[2\phi_{G,i}\phi_{G,j}+2v_kv_{(i}\phi_{G,j)k}+\frac{2}{3}v_iv_j\nabla^2\phi_G\\&-2v^2\phi_{G,ij}-\frac{1}{2}a\dot{P}_{(i,j)}+v_kP_{k,ij}-v_kP_{(i,j)k}-\frac{1}{2}v_{(i}P_{|k|,j)k}\\&+\delta_{ij}\left(-\frac{2}{3}\phi_{G,k}\phi_{G,k}-\frac{1}{2}v_k\nabla^2P_k+v_kv_l\phi_{G,kl}-v^2\nabla^2\phi_G\right)\Big].\end{aligned} \tag{7.21c}$$

For the magnetic part we obtain

$$H_{ij}=\frac{1}{2c^3}\left[P_{\mu,\nu(i}\varepsilon_{j)}{}^{\mu\nu}+2v_\mu(U+V)_{,\nu(i}\varepsilon_{j)}{}^{\mu\nu}\right], \tag{7.22a}$$

$$H_{00}=O\left(\frac{1}{c^5}\right), \tag{7.22b}$$

$$H_{i0}=O\left(\frac{1}{c^4}\right). \tag{7.22c}$$

We can see that there is not a Newtonian component in the magnetic part of the Weyl tensor, its lower order is the 1PN term of the spatial component. Nevertheless, in order to obtain the Newtonian equations, in particular the tidal force equation (2.40), we can not simply ignore it. In fact the $1/c$-expansion of the relativistic evolution equation of E_{ij}, eq.(3.20), begins with $1/c^3$ order, therefore we must take into account the magnetic part in the first non vanishing post-Newtonian order $1/c^3$. This term is responsible for the non-locality of the Newtonian theory [58].

With the previous expressions for the electric and magnetic part of the Weyl tensor, it will be easy to compute the Post-Newtonian expression for the evolution of the tidal field. What we have to do is substituting the components (7.21) and (7.22) in the relativistic equations (3.20) and (3.21), then we must use the decomposition (7.5) for the covariant derivative of a second rank tensor. The result will be the 1PN generalization of the Newtonian equation obtained in chapter 2, eq. (2.40). Let us note that in order to do this we need to know the magnetic tensor H_{ij} up to order $O\left(c^{-5}\right)$.

We conclude this chapter summarizing the main results that we have obtained. Starting from the 1PN metric of chapter 6 we have shown the Newtonian and the first Post-Newtonian terms arising from the splitting of the covariant time derivative based on the fundamental fluid-frame four-vector u^μ. The relativistic kinematic quantities have been decomposed recovering the Newtonian definitions at the 0PN order and the first relativistic corrections at 1PN order. The same splitting has been done for the electric and the magnetic part of the Weyl tensor. As it is known the 0PN part of the electric component coincides with the Newtonian tidal field, whereas the magnetic component has not a Newtonian counterpart. With the obtained results we have expanded the Raychauduri equation up to 1PN order recovering the Newtonian expression at 0PN order. While the 1PN fluid-frame kinematic quantities can be found in literature [53], the 1PN Raychauduri equation is here obtained for the first time.

Finally we remark that there are important issues about the 1PN method that will be interesting to investigate, for example the meaning of a gauge transformation in such framework; it is important to understand how the 1PN equations transform changing the gauge condition, relating other results obtained in a different gauge [70]. Also it is interesting to develop a gauge-invariant 1PN formalism.

A further application of the Post-Newtonian theory, connected in particular with the kinematic approach of this chapter, is trying to find the 1PN Zel'dovich approximation. This can be done in analogy with how we did in chapter 2 for the Newtonian case: using the continuity equation (6.51), the Raychauduri equation (7.12) and the 1PN equation for the shear. It is still not clear how to proceed in this sense and this constitutes one of our works in progress.
Extending our model at more realistic cases of a non-perfect fluid or multi-fluid models is a natural continuation of our work, so as considering the radiation-matter equidensity epoch using the Meszaros approximation.
The final purpose is however to implement the 1PN cosmological perturbed equations in a numerical N-body simulations, this asks for a full research on this topic.

Chapter 8

Conclusions

In this thesis we have studied some non-linear aspects of cosmological perturbation theory in view of the hight resolution experimental data from the next generation galaxies survey and from the Planck ESA's CMB mission.

After having introduced in Chapter 1 the standard hot big bang ΛCDM model of the smoothed universe, we summarized the two approach used to explain the structure formation in the universe respectively in the super-horizon scales and in the sub-horizon scales. In Chapter 1 the Newtonian theory of gravitational instability is presented with a particular emphasis on the kinematic approach and the non-local character of the Newtonian theory. Using the velocity divergence tensor and the rescaled variables, the Zel'dovich approximation is obtained in a easy way as a consequence of the assumed first order velocity constantness.

The relativistic theory of cosmological perturbations is presented in Chapter 2. In order to compare with the Newtonian kinematic approach of the previous chapter, we consider the covariant 1+3 theory of General Relativity for which one has not to define a coordinates system. Consequently the equations are given with respect to an observer's instantaneous rest frame, in analogy with the Newtonian Lagrangian picture. We presented also the standard metric point of view of cosmological perturbation theory, both in Poisson gauge and in a gauge-invariant formalism.

Chapter 4 shows an application of the 1+3 gauge-invariant theory in the case of a fluid with a non-null velocity dispersion. The influence of the fluid pressure is analyzed in a series of qualitative plots showing the time evolution of the density contrast with the varying of the barotropic parameter. Two periods of the universe are taken into consideration, the matter period and the radiation-matter equidensity epoch. This second case is treated using the Meszaros approximation, that is neglecting the inhomogeneities in the relativistic component of the universe.

In Chapter 5 we have performed the second-order study of the gravitation instability process in the small scale case, using therefore the Newtonian theory. We considered again the period around the matter-radiation equidensity. This period is extremely important because it is responsible for the density power spectrum shape, the main experimental test for the growing of inhomogeneities in the universe. In fact the different behavior of the matter density contrast before and after the equidensity leads to the turn over of the power spectrum at small scales; such turn over, that correctly agrees with the experimental data, is model dependent so that it is important to investigate the contribution of the non-linear effects besides that of the considered fluid model. In this contest it could be interesting to perform the semi-analytical analysis leading to the theoretical power spectrum both in the second order case and the α-fluid model case of chapter 4.

The main result of Chapter 5 is the statistical analysis related with the second-order computation. We found the normalized skewness of the matter density distribution as a function of the ratio between the generic scale factor and the scale factor evaluated at the equidensity era. This allows to follow the skewness behavior from the late radiation era up to the matter dominated era , when it reaches the known constant value of 34/7. The importance of this result is the fact that at the time of decoupling the skewness is already at 86% of its maximum value; this means that in the CMB we can see a signature of non-Gaussianity due to the second-order effects. The simplest way to parametrize the departure from Gaussianity is usually through the f_{NL} parameter. In our work we set up the first step needed for evaluate such parameter in the case of Gaussian initial conditions. The research of a total f_{NL}, and so including also the primordial non-Gaussianity, will be the next investigation to be briefly concluded.

In Chapter 6 we started a new topic, a more theoretical investigation still related with the non-linear cosmological perturbation theory but with a further intrinsic interest on a specific topic of General Relativity: its Newtonian limit. In a cosmological contest we have applied the Post-Newtonian scheme on the field and matter equations of a perturbed FRW universe, taking into account the first relativistic corrections on the Newtonian approximation (1PN order). Differently from other Post-Newtonian applications where one wants to find the approximated equations of motion for a particle in a given gravitational space-time, we have considered the self-gravitational matter case, namely the case in which the matter is responsible for the geometry of space-time. This requires to focus on the field equations besides the equations of motion and so a different rearrangement of the approximation order for some geometrical variables. Moreover our method does not use the standard iterative procedure of the perturbative analysis where the equations are solved order-by-order, we rather use a resummed perturbation scheme up to the first relativistic corrections.

The final result is a new set of equations that enable one to describe, in a almost-Newtonian regime, the collapsing process leading to the first non-linear structures such as galaxy clusters and galaxy haloes. Moreover at large scale we can recover the standard linear perturbation theory for the scalar and vectorial modes so that we can treat a large variety of scales and phenomena within the same computational technique.

Within the same approximation Chapter 7 is focused on the covariant approach. In this chapter we have expanded all the relativistic variables concerning the covariant gradient of the velocity field in a Newtonian plus corrections form. The Newtonian Raychauduri equation is therefore generalized to the weakly relativistic case.

The utility of such study, that until now has been purely theoretical, is that our equations could be used in the N-body simulations as a valid alternative to the Newtonian equations. This could be important if one thinks that the fluctuations relevant for the large-scale structures have not been always much smaller than horizon scales in the past. For example, the horizon scale at the decoupling time is of the order of $ct_{dec}(1 + z_{dec}) \approx 80h^{-1}$Mpc in the present physical length; considering that the galaxy surveys can cover the region of several hundred megaparsecs we must consider also the large and intermediate scales for which the Newtonian theory is not appropriate.
Even if a numerical implementation of our equations is far from being shortly realized, we have given a theoretical starting point aimed to this scope. Trying to quantify the relativistic effects understanding whether they are significant enough to be detected in future observations and numerical experiments is the scope of a future work.

Conventions

Symbols	*Definition*
	General
a	scale factor of the Universe
c	light velocity
G	gravitational constant
t	time
$\mathbf{r}$	physical position
$\mathbf{x}$	comoving position
V^i	physical velocity
v^i	peculiar velocity
φ	peculiar velocity potential
u^i	rescaled velocity
$\tilde{\varphi}$	rescaled velocity potential
ϕ	peculiar gravitational potential
H	Hubble expansiona scalar
$H_0,\ h$	Hubble constant and its uncertainty
∇	gradient
∇_x	comoving gradient
k	wave number
ρ	density
P	pressure
c_s	sound velocity
δ	density contrast
	Newtonian
θ_{ij}^N	Newtonian deformation tensor of v^i
Θ^N	Newtonian expansion scalar of v^i
σ_{ij}^N	Newtonian shear tensor of v^i
ω_{ij}^N	Newtonian vorticity tensor of v^i
$\tilde{\theta}^N$	Newtonian expansion scalar of u^i
$\tilde{\sigma}_{ij}^N$	Newtonian shear tensor of u^i
$\tilde{\omega}_{ij}^N$	Newtonian vorticity tensor of u^i
$\mathbf{g}$	peculiar acceleration
Ψ^N	Newtonian gravitational potential
ϕ^N	peculiar gravitational potential
E_{ij}^N	tidal force
$\tilde{\phi}^N$	rescaled peculiar gravitational potential
$\tilde{E}_{ij}^N$	rescaled tidal force

General Relativity	
$-+++$	signature
$\mu\nu\delta...$	time-space indices,values 0..3
$ijk...$	space indices, values 1..3
$g_{\mu\nu}$	generic metric tensor
K	spatial curvature
u^{μ}	4-velocity
$h_{\mu\nu}$	projector tensor
$\dot{S}^{\mu\nu..}{}_{\gamma\delta..}$	proper time derivative along u^{μ}
$D_{\rho}S^{\mu\nu..}{}_{\gamma\delta..}$	projected spatial derivative
$\sigma_{\mu\nu},\ \omega_{\mu\nu},\ \Theta$	shear, vorticity and volume expansion
$\omega^{\mu},\ A^{\mu}$	vorticity and acceleration vectors
$R_{\mu\nu\gamma\delta},\ R_{\mu\nu},\ R$	Riemann tensor, Ricci tensor, Ricci scalar
$C_{\mu\nu\gamma\delta},\ E_{\mu\nu},\ H_{\mu\nu}$	Weyl tensor, electric and magnetic part
$\eta_{\mu\nu\gamma\delta},\ \varepsilon_{\mu\nu\gamma}$	4-D and 3-D permutation tensor
$T^{\mu\nu}$	energy-momentum tensor
$\rho,\ p$	energy density and isotropic pressure
$q_{\mu},\ \pi_{\mu\nu}$	energy flux and anisotropic stress
$\Phi_A, \quad \Phi_H$	Bardeen potentials
$Z_{\mu}, \quad Z$	Gauge-invariant expansion variables
$\Delta_{\mu}, \quad \Delta$	Gauge-invariant density variables
$\Sigma_{\mu}, \quad \Sigma$	Gauge-invariant entropy variables
Post-Newtonian approximation	
$U, \quad \Phi$	Newtonian and 1PN time-time scalar fields
$V, \quad \Psi$	Newtonian and 1PN spacial scalar fields
P_i	vectorial 1PN modes
h_{ij}	tensorial 1PN modes
ϕ_P	effective time-time scalar field
ψ_P	effective spacial scalar field
D_p	non linear 1PN scalar field
ϕ_G	1PN gravitational potential

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