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ON BOLZA OPTIMAL CONTROL PROBLEMS WITH CONSTRAINTS

Piermarco Cannarsa

Dipartimento di Matematica, Università di Roma "Tor Vergata"
Via della Ricerca Scientifica 1
00133 Roma, Italy

Hélène Frankowska

Université Pierre et Marie Curie (Paris 6) case 189 - Combinatoire et Optimisation
4 place Jussieu
75252 Paris cedex 05, France

and

Elsa, Maria Marchini

Dipartimento di Matematica, Politecnico di Milano
Piazza Leonardo Da Vinci 32
20133 Milano, Italy

Abstract. We provide sufficient conditions for the existence and Lipschitz continuity of solutions to the constrained Bolza optimal control problem

$$\text{minimize} \quad \int_0^T L(x(t),u(t))\,dt + \ell(x(T))$$

over all trajectory/control pairs (x,u), subject to the state equation

$$\begin{cases} x'(t) = f(x(t),u(t)) & \text{for a.e. } t \in [0,T] \\ u(t) \in U & \text{for a.e. } t \in [0,T] \\ x(t) \in K & \text{for every } t \in [0,T] \\ x(0) \in Q_0\,. \end{cases}$$

The main feature of our problem is the unboundedness of $f(x,U)$ and the absence of superlinear growth conditions for L. Such classical assumptions are here replaced by conditions on the Hamiltonian that can be satisfied, for instance, by some Lagrangians with no growth. This paper extends our previous results in *Existence and Lipschitz regularity of solutions to Bolza problems in optimal control* to the state constrained case.

1. **Introduction.**

The existence theory for the so-called Bolza problem in optimal control is well-established for Lagrangians that exhibit *superlinear growth* with respect to controls or velocities, see, e.g., [17], [14], [13] and [5]. At the same time, some interesting functionals, such as the one in the brachistocrone problem of the calculus of variations, or in the mathematical economics

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models by Baumol or Knowles (see, e.g. Chiang's mongraph [6]), fail to possess such a property. Consequently, Tonelli's direct method for existence cannot be applied to such problems.

It turns out that a useful complementary property to the existence result is the Lipschitz continuity of optimal trajectories. Indeed, such a property is essential to study further regularity of solutions as well as to construct efficient schemes for numerical approximation.

For problems in the calculus of variations without growth conditions, a general approach to the existence and Lipschitz regularity of solutions was proposed by Clarke [7]. In [4], we extended Clarke's results to optimal control problems using a direct method, coupled with penalization and necessary optimality conditions. However, unlike [7], no state constraints were allowed in our work. The main object of this paper is to show that the very same ideas of [4] can be adapted to investigate constrained control problems. As we shall see, such an extension is highly non trivial mainly because optimality conditions in the constrained case involve vector-valued measures without a priori bounds.

More precisely, let us consider the problem of minimizing the functional

$$J(x,u)=\int_0^T L(x(t),u(t))\,dt+\ell(x(T)) \tag{1}$$

over all trajectory/control pairs (x,u), subject to the state equation

$$\begin{cases} x'(t)=f(x(t),u(t)) & \text{for a.e. } t\in I\\ u(t)\in U & \text{for a.e. } t\in I\\ x(t)\in K & \text{for every } t\in I\\ x(0)\in Q_0\,. \end{cases} \tag{2}$$

Here, $I=[0,T]$ where $T>0$, while $K,Q_0\subset\mathbb{R}^N$ and $U\subset\mathbb{R}^m$. Moreover, $L:\mathbb{R}^N\times\mathbb{R}^m\to\mathbb{R}$ and $\ell:\mathbb{R}^N\to\mathbb{R}$ are nonnegative functions, $f:\mathbb{R}^N\times\mathbb{R}^m\to\mathbb{R}^N$, and $u:I\to U$ is measurable. We would like to underline from the very beginning that no growth condition is imposed on the Lagrangian L.

As noted above, the case of $K=\mathbb{R}^n$ (no state constraints) was investigated in [4] studying the penalized problem for some $\alpha\geq 2$

$$\min\int_0^T\Big[L(x(t),u(t))+\frac{\epsilon}{\alpha}\,|f(x(t),u(t))|^{\alpha}\Big]dt+\ell(x(T)), \tag{3}$$

which, having superlinear growth, admits an optimal solution (x_ϵ,u_ϵ). This is still true when state constraints are present, see [5]. On the other hand, in order for (3) to provide a useful approximation of (1) we need to know that, for any trajectory/control pair (x,u) of (2), $J(x,u)$ can be approximated by $J(x_k,u_k)$ where (x_k,u_k) satisfy (2) and $x_k\in W^{1,\alpha}(I;\mathbb{R}^N)$. Such an approximation result, which is trivial in absence of state constraints, is the first main difficulty we have to overcome in this paper, see Theorem 3.2 and Corollary 1.

Once this step is completed, from normal necessary optimality conditions for constrained problems and the superlinear growth of the penalized Lagrangian it follows that all optimal trajectories x_ϵ of (3) are Lipschitz continuous. Then, imposing a structural assumption on the Hamiltonian as in [4], we show that x'_ϵ are essentially bounded uniformly in ϵ. Notice, however, that, to pass to the limit as $\epsilon\downarrow 0$ in (3), we still need to know that the functions

$$L_\epsilon(x_\epsilon(t),u_\epsilon(t))=L(x_\epsilon(t),u_\epsilon(t))+\frac{\epsilon}{\alpha}\,|f(x_\epsilon(t),u_\epsilon(t))|^{\alpha}$$

are essentially bounded on I uniformly in ϵ. For this purpose, we need to show that the co-states in the constrained maximum principle for the penalized problems are bounded uniformly in ϵ. This fact, which is a straightforward consequence of Gronwall's Lemma in the unconstrained case, is the second major difficulty we have to overcome in presence of state constraints. Indeed, the constrained maximum principle involves *vector-valued measures* related to normals to the constraint set, for which no a priori bounds are available. Nevertheless, we show that, under an inward pointing condition for velocities at the boundary of the constraint set, such a uniform bound can be derived. Thus, we can pass to the limit using classical arguments obtaining the existence of a solution (x^*, u^*) to our original problem, such that $x^*(\cdot)$ is Lipschitz continuous and $L(x^*(\cdot), u^*(\cdot))$ is essentially bounded.

The outline of the paper is as follows. In section 2, we introduce our notations and assumptions, and we recall the maximum principle for constrained problems showing how it can be exploited to deduce the Lipschitz continuity of optimal trajectories for problems satisfying a quadratic growth condition. Then, in section 3, we state our main existence and regularity results as well as an approximation theorem and provide an example. In the same section we prove, as a corollary, that our minimization problem can be restricted to trajectory/control pairs (x, u) of (2) such that $x \in W^{1,\alpha}(I;\mathbb{R}^N)$. Section 4 is devoted to the proof of the existence and regularity results, whereas the proof of the approximation theorem is provided in the Appendix.

2. **Preliminaries.** We begin this section with a list of notations:

- $W^{1,1}(I;\mathbb{R}^N)$ denotes the space of absolutely continuous functions from I to $\mathbb{R}^N$ and $W^{1,\infty}(I;\mathbb{R}^N)$ the space of Lipschitz continuous functions from I to $\mathbb{R}^N$;
- $NBV(I;\mathbb{R}^N)$ denotes the space of normalized functions of bounded variation on I with values in $\mathbb{R}^N$, i.e. the space of functions vanishing at zero, right-continuous on $(0,T)$ and having bounded total variation $\|\cdot\|_{BV}$;
- we define the set of controls
$$\mathcal{U} := \{u : I \to U \text{ is measurable}\};$$
- a pair (x, u) where $x \in W^{1,1}(I;\mathbb{R}^N)$ and $u \in \mathcal{U}$ is called a trajectory/control pair if (x, u) satisfies (2);
- for $a, b \in \mathbb{R}$, we set $a \wedge b := \min\{a, b\}$;
- for a Banach space X and $r > 0$, $B(0, r)$ denotes the open ball of center 0 and radius r;
- given mappings $a : \mathbb{R}^N \to \mathbb{R}$, $b : \mathbb{R}^N \times U \to \mathbb{R}$, $c : \mathbb{R}^N \times U \to \mathbb{R}^N$ such that for every $u \in U$, $a(\cdot)$, $b(\cdot, u)$ and $c(\cdot, u)$ are locally Lipschitz, we denoted by $\partial a(x)$, $\partial_x b(x, u)$ and $\partial_x c(x, u)$ respectively their generalized gradients and generalized Jacobian with respect to x, see [8], and by $(\partial_x c)^*(x, u)$ the set of the adjoint elements from $\partial_x c(x, u)$;
- given a nonempty set $V \subset \mathbb{R}^r$ and $y \in V$, $C_V(y)$ and $N_V(y)$ denote, respectively, Clarke's tangent and normal cones to V at y, see [8];
- given $K \subset \mathbb{R}^N$, we denote by $\mathrm{Int}(K)$ its interior, by ∂K its boundary, by $\overline{K}$ its closure and define the signed distance
$$d_K(x) := \begin{cases} -dist(x, \partial K) & \forall\, x \in K \\ dist(x, \partial K) & \text{otherwise;} \end{cases}$$
we say that ∂K is $C^{1,1}_{loc}$ if $\forall\, R > 0$, d_K is $C^{1,1}$ on a neighborhood of $\partial K \cap B(0, R)$;
- C denotes a generic constant that may differ from line to line;

- $\mathcal{L}(\mathbb{R}^m;\mathbb{R}^N)$ denotes the space of linear operators from $\mathbb{R}^m$ into $\mathbb{R}^N$.

Throughout the whole paper we assume that $L:\mathbb{R}^N\times\mathbb{R}^m\to\mathbb{R}$ and $\ell:\mathbb{R}^N\to\mathbb{R}$ are nonnegative functions, $f:\mathbb{R}^N\times\mathbb{R}^m\to\mathbb{R}^N$, $U\subset\mathbb{R}^m$ is closed, $K,Q_0\subset\mathbb{R}^N$ are closed with $K\cap Q_0\neq\emptyset$, and, for every $x\in\mathbb{R}^N$, $L(x,\cdot)$ is locally Lipschitz and $f(x,\cdot)$ is differentiable.

The following assumptions will be in use.

Assumptions **(H)***:*

i) for some $\alpha\geq 2$ and every $R>0$, $\exists\, C_R>0$ such that, for any $x,y\in B(0,R)\cap K$ and any $u\in U$,

i1) $|\ell(x)-\ell(y)|\leq C_R|x-y|$,

i2) $|L(x,u)-L(y,u)|\leq C_R|x-y|\big[1+L(x,u)\wedge L(y,u)\big]$,

i3) $|f(x,u)-f(y,u)|\leq C_R|x-y|\Big[1+|f(x,u)|\wedge|f(y,u)|+\big(L(x,u)\wedge L(y,u)\big)^{1/\alpha}\Big]$;

ii) $\exists\ \overline{u}\in\mathcal{U}$, $\upsilon\in L^\alpha(I;\mathbb{R})$ such that for a.e $t\in I$, $|f(x,\overline{u}(t))|\leq\upsilon(t)(1+|x|)$ and $\forall\, R>0$, $\exists\, m_R\in L^1(I;\mathbb{R})$ satisfying a.e in I,

$$L(x,\overline{u}(t))\leq m_R(t),\ \forall\, x\in B(0,R)\cap K;$$

iii) ∂K is $C^{1,1}_{loc}$, there exists $\kappa>0$, so that for all $R>0$, $\exists\,\rho_R,\ \eta_R>0$ such that for every $x\in\partial K\cap B(0,R)$ and some $u_x\in U$,

$$\langle\nabla d_K(x),f(x,u_x)\rangle<-\rho_R\,,\quad |f(x,u_x)|\leq\kappa(1+|x|)\quad \textit{and}\quad L(x,u_x)\leq\eta_R;$$

iv) for all $x\in\partial K\cap Q_0$, $Int\big(C_K(x)\big)\cap Int\big(C_{Q_0}(x)\big)\neq\emptyset$;

v) $\forall\, x\in K$, the set $F(x):=\Big\{\big(f(x,u),L(x,u)+v\big):u\in U$ and $v\geq 0\Big\}$ is closed and convex.

Remark 1. By (H) ii), iii), applying a measurable viability theorem from [1], it can be shown that there exists a trajectory/control pair (x,u) of (2) such that $x'\in L^\alpha(I;\mathbb{R}^N)$ and $J(x,u)<\infty$.

We recall next the necessary optimality conditions for constrained problems. A trajectory/control pair (x^*,u^*) satisfies an Autonomous Constrained Maximum Principle if there exist $\lambda\in\{0,1\}$, $p\in W^{1,1}(I;\mathbb{R}^N)$ and $\psi\in NBV(I;\mathbb{R}^N)$, not vanishing simultaneously, such that, for some positive Radon measure μ on I and Borel measurable mapping $\nu:I\to\mathbb{R}^n$ with $\nu(t)\in N_K(x^*(t))\cap B(0,1)$ μ-almost everywhere,

$$\psi(t)=\int_{[0,t]}\nu(s)d\mu(s),\qquad\text{for every } t\in(0,T],\tag{4}$$

and the following three relations hold true:

i) autonomous maximum principle : for some $c\in\mathbb{R}$ and for a.e. $t\in I$

$$\begin{aligned}c&=\langle p(t)+\psi(t),{x^*}'(t)\rangle-\lambda L(x^*(t),u^*(t))\\&=\max_{u\in U}(\langle p(t)+\psi(t),f(x^*(t),u)\rangle-\lambda L(x^*(t),u));\end{aligned}\tag{5}$$

ii) adjoint equation : for some measurable selections $A(t)\in\partial_x f(x^*(t),u^*(t))$, $\pi(t)\in\partial_x L(x^*(t),u^*(t))$

$$-p'(t)=A(t)^*(p(t)+\psi(t))-\lambda\pi(t),\quad\text{a.e. in } I;\tag{6}$$

iii) transversality conditions:

$$-p(T)-\psi(T)\in\lambda\partial\ell(x^*(T))\qquad\text{and}\qquad p(0)\in N_{Q_0}(x^*(0)).\tag{7}$$

The maximum principle is called normal if $\lambda=1$.

We refer to [19] for various results on the above necessary optimality conditions, where the adjoint equation is written as an inclusion

$$-p'(t) \in \big(\partial_x f\big)^*(x^*(t), u^*(t))(p(t) + \psi(t)) - \lambda \partial_x L(x^*(t), u^*(t)), \quad \text{for a.e. } t \in I.$$

Notice that, under the assumptions of [19, Theorem 9.3.1], by a measurable selection theorem there exist measurable mappings $A : I \to \mathcal{L}(\mathbb{R}^N; \mathbb{R}^N)$ and $\pi : I \to \mathbb{R}^N$ such that, for a.e. $t \in I$, $A(t) \in \partial_x f(x^*(t), u^*(t))$, $\pi(t) \in \partial_x L(x^*(t), u^*(t))$, and (6) holds true.

Let us define the Hamiltonian $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ associated to problem (1), (2):

$$H(x, p) = \sup_{u \in U} \big(\langle p, f(x, u)\rangle - L(x, u)\big).$$

Notice that, when $\lambda = 1$, equality (5) can be written as

$$c = H(x^*(t), p(t) + \psi(t)) = \langle p(t) + \psi(t), {x^*}'(t)\rangle - L(x^*(t), u^*(t)), \quad \text{a.e. in } I.$$

In the literature, many papers have been devoted to the validity of the Constrained Maximum Principle, see for instance [19] and the bibliography contained therein. The question of normality was considered recently under an additional assumption that x^* is Lipschitz, see e.g. [3, 10, 12, 15], while the case $x^* \in W^{1,1}(I; \mathbb{R}^N)$ was investigated in [11].

When a quadratic growth condition is satisfied, then optimal trajectories are Lipschitz.

Theorem 2.1. *Assume (H) and that for all $(x, u) \in \mathbb{R}^N \times U$, $L(x, u) \geq a|f(x, u)|^2 - b$, where $a, b > 0$. Then,*

- *a) problem (1), (2) has an optimal solution (x^*, u^*) and every optimal trajectory/control pair is so that x^* is Lipschitzian and $L(x^*(\cdot), u^*(\cdot)) \in L^\infty(I; \mathbb{R})$;*
- *b) (x^*, u^*) satisfies a normal Autonomous Constrained Maximum Principle with $p \in W^{1,\infty}(I; \mathbb{R}^N)$.*

Proof. By the growth condition we get $\lim_{|f(x,u)| \to +\infty} L(x, u)/|f(x, u)| = +\infty$. So, by Cesari [5, chapter 11], an optimal solution does exist. Consider an optimal trajectory/control pair (x^*, u^*). By [19, p.203] it satisfies an Autonomous Constrained Maximum Principle and using exactly the same arguments as in [11, Proof of Theorem 2] and assumptions H) we show that this maximum principle is normal. Let c, p, ψ be as in (4) - (7). Then for a.e. $t \in I$, $L(x^*(t), u^*(t)) = \langle p(t) + \psi(t), f(x^*(t), u^*(t))\rangle - c$ and therefore

$$\frac{L(x^*(t), u^*(t))}{|f(x^*(t), u^*(t))|} \leq |p(t) + \psi(t)| - \frac{c}{|f(x^*(t), u^*(t))|}$$

whenever $f(x^*(t), u^*(t)) \neq 0$. Since $\lim_{r \to +\infty} \frac{c}{r} = 0$, by the growth condition, $|f(x^*(\cdot), u^*(\cdot))|$ is essentially bounded and so x^* is Lipschitz. This also implies that $L(x^*(\cdot), u^*(\cdot))$ is essentially bounded and ends the proof of a). To prove b) it is enough to use the adjoint equation and assumption H) i). □

3. **Main results.** Our main theorem concerns existence and Lipschitzianity of solutions for a class of Bolza problems (1), (2).

For $(x, u, p) \in \mathbb{R}^N \times U \times \mathbb{R}^N$, set

$$P(x, u) = \Big\{p \in \mathbb{R}^N : \frac{\partial f}{\partial u}(x, u)^* p \in \partial_u L(x, u) + N_U(u)\Big\},$$
$$\mathcal{H}(x, u, p) = \langle p, f(x, u)\rangle - L(x, u).$$

The key assumption is a separation property 2) below of the pre-Hamiltonian $\mathcal{H}$.

Theorem 3.1. *Assume (H) and suppose that there exists a trajectory/control pair (x_1, u_1) satisfying $x_1' \in L^\alpha(I;\mathbb{R}^N)$ and*

$$\inf\{J(x,u) : (x,u) \text{ is a trajectory/control pair of (2)}\} < J(x_1,u_1) < \infty.$$

Moreover, assume that there exists $k > 0$ such that:

1) *for any trajectory/control pair (x,u) of (2) such that $J(x,u) < J(x_1,u_1)$, we have*
$$\|x\|_\infty \le k \quad \text{and} \quad \operatorname{ess\,inf}_{t\in I}|f(x(t),u(t))| \le k;$$
2) $$\sup_{\substack{|x|\,\le\,k,\,|f(x,u)|\,\le\,k\\ x\,\in\,K,\ p\,\in\,P(x,u)}} \mathcal{H}(x,u,p) < \liminf_{c\to+\infty} \inf_{\substack{|x|\,\le\,k,\,|f(x,u)|\,\ge\,c\\ x\,\in\,K,\ p\,\in\,P(x,u)}} \mathcal{H}(x,u,p).$$

Then,

a) *problem (1), (2) has an optimal solution (x^*,u^*) such that x^* is Lipschitzian and $L(x^*(\cdot),u^*(\cdot)) \in L^\infty(I;\mathbb{R})$;*
b) *(x^*,u^*) satisfies a normal Autonomous Constrained Maximum Principle with a costate $p \in W^{1,\infty}(I;\mathbb{R}^N)$.*

The above theorem is proved in Section 4. For this aim we need the two approximation results below.

Theorem 3.2. *Assume (H) i), iii), iv). Let $\epsilon > 0$ and let (x,u) be a trajectory/control pair of* (2)*. Then, there exists a trajectory/control pair (x_ϵ,u_ϵ) of* (2) *satisfying*

$$x_\epsilon(I) \subset Int(K); \quad \|x_\epsilon - x\|_\infty \le \epsilon \quad \text{and} \quad J(x_\epsilon,u_\epsilon) < J(x,u) + \epsilon. \tag{8}$$

The proof of Theorem 3.2 is postponed to the appendix. Its consequence is that the infimum of the functional (1) evaluated along tajectory/control pairs with absolutely continuous trajectories is equal to the infimum over trajectory/control pairs with more regular trajectories, as stated in the next corollary.

Corollary 1. *Assume (H). Then*

$$\inf\Big\{J(x,u) : (x,u) \text{ is a trajectory/control pair of (2)}\Big\}$$
$$= \inf\Big\{J(x,u) : (x,u) \text{ is a trajectory/control pair of (2) and } x \in W^{1,\alpha}(I;\mathbb{R}^N)\Big\},$$

Proof. Consider a trajectory/control pair (x,u) of (2) with $J(x,u) < \infty$. We claim that, for every $\epsilon > 0$, there exists a trajectory/control pair (x_ϵ,u_ϵ) of (2) such that

$$x_\epsilon \in W^{1,\alpha}(I;\mathbb{R}^N) \quad \text{and} \quad J(x_\epsilon,u_\epsilon) < J(x,u) + \epsilon. \tag{9}$$

By Theorem 3.2, there exists $(\hat{x}_\epsilon,\hat{u}_\epsilon)$ satisfying (2) such that

$$\hat{x}_\epsilon(I) \subset \mathrm{Int}(K) \quad \text{and} \quad J(\hat{x}_\epsilon,\hat{u}_\epsilon) < J(x,u) + \frac{\epsilon}{2}\,.$$

Let $R = \|\hat{x}_\epsilon\|_\infty + 1$ and

$$\delta_\epsilon = -\sup_{t\in I} d_K(\hat{x}_\epsilon(t)). \tag{10}$$

Set, for every $n \in \mathbb{N}$,

$$u_n(t) = \begin{cases} \hat{u}_\epsilon(t) & \text{if } |f(\hat{x}_\epsilon(t),\hat{u}_\epsilon(t))| \le n \\ \overline{u}(t) & \text{otherwise,} \end{cases} \tag{11}$$

where $\overline{u}$ is as in (H) ii), and consider the system

$$\begin{cases} x'(t) = f(x(t), u_n(t)) & \text{for a.e. } t \in I \\ x(0) = \hat{x}_\epsilon(0). \end{cases} \tag{12}$$

We claim that, for every n sufficiently large, (12) admits a solution x_n satisfying

$$x_n(I) \subset \operatorname{Int}(K), \quad x_n' \in L^\alpha(I;\mathbb{R}^N), \quad \text{and} \quad J(x_n,u_n) < J(\hat{x}_\epsilon, \hat{u}_\epsilon) + \frac{\epsilon}{2}, \tag{13}$$

implying (9). Define an absolutely continuous function $z_n(t) := x(0) + \int_0^t f(\hat{x}_\epsilon(s), u_n(s))\, ds$. Then

$$\|\hat{x}_\epsilon - z_n\|_\infty \le \int_{\{s:|f(\hat{x}_\epsilon(s),\hat{u}_\epsilon(s))|>n\}} |f(\hat{x}_\epsilon(s), \overline{u}(s)) - f(\hat{x}_\epsilon(s), \hat{u}_\epsilon(s))|\, ds.$$

As $f(\hat{x}_\epsilon(\cdot), \overline{u}(\cdot))$, $f(\hat{x}_\epsilon(\cdot), \hat{u}_\epsilon(\cdot)) \in L^1(I;\mathbb{R}^N)$, it follows that $z_n \to \hat{x}_\epsilon$ uniformly in I. So, for all large n, $\|\hat{x}_\epsilon - z_n\|_\infty \le 1$. Set $\gamma(t) := |z_n'(t) - f(z_n(t), u_n(t))|$. From (H) i) we deduce

$$\gamma(t) \le C_R |\hat{x}_\epsilon(t) - z_n(t)| \Big[1 + |f(\hat{x}_\epsilon(t), u_n(t))| + L(\hat{x}_\epsilon(t), u_n(t))^{1/\alpha}\Big].$$

From (11), for some positive constant C independent of n,

$$\Big\| 1 + |f(\hat{x}_\epsilon(\cdot), u_n(\cdot))| + L(\hat{x}_\epsilon(\cdot), u_n(\cdot))^{1/\alpha} \Big\|_{L^1} \le C. \tag{14}$$

As $z_n \to \hat{x}_\epsilon$ uniformly in I, for all n large enough,

$$\begin{aligned} &\int_0^T \gamma(t) e^{\left(\int_t^T C_R \left[1 + |f(\hat{x}_\epsilon(s), u_n(s))| + L(\hat{x}_\epsilon(s), u_n(s))^{1/\alpha}\right] ds\right)} dt \\ &\le \int_0^T C_R |\hat{x}_\epsilon(t) - z_n(t)| \Big[1 + |f(\hat{x}_\epsilon(t), u_n(t))| + L(\hat{x}_\epsilon(t), u_n(t))^{1/\alpha}\Big] e^{C_R C}\, dt \le 1. \end{aligned}$$

Applying Filippov's Theorem (see for instance [19]), we deduce that for n sufficiently large, there exists a solution x_n to (12) such that $\|x_n - z_n\|_\infty \le 1$ and, for a.e. $t \in I$,

$$\begin{aligned} |x_n'(t) - \hat{x}_\epsilon'(t)| &\le |f(x_n(t), u_n(t)) - f(\hat{x}_\epsilon(t), u_n(t))| + |f(\hat{x}_\epsilon(t), u_n(t)) - f(\hat{x}_\epsilon(t), \hat{u}_\epsilon(t))| \\ &\le C_{R+2} |x_n(t) - \hat{x}_\epsilon(t)| \Big[1 + |f(\hat{x}_\epsilon(t), u_n(t))| + L(\hat{x}_\epsilon(t), u_n(t))^{1/\alpha}\Big] \\ &\quad + |f(\hat{x}_\epsilon(t), u_n(t)) - f(\hat{x}_\epsilon(t), \hat{u}_\epsilon(t))|. \end{aligned}$$

Set $R_n = \|\hat{x}_\epsilon\|_\infty + \|x_n\|_\infty$. From (11), (H) i), ii), for a.e. $t \in I$,

$$|f(x_n(t), u_n(t))| \le 2\, C_{R_n} \Big[1 + n + L(\hat{x}_\epsilon(t), \hat{u}_\epsilon(t))^{1/\alpha}\Big] + \upsilon(t)(1 + R_n),$$

implying that $x_n' \in L^\alpha(I;\mathbb{R}^N)$. By (H) i) and ii), using Gronwall's Lemma, we obtain

$$\begin{aligned} \|x_n - \hat{x}_\epsilon\|_\infty &\le C \int_0^T |f(\hat{x}_\epsilon(s), u_n(s)) - f(\hat{x}_\epsilon(s), \hat{u}_\epsilon(s))|\, ds \\ &= C \int_{\{s:|f(\hat{x}_\epsilon(s),\hat{u}_\epsilon(s))|>n\}} |f(\hat{x}_\epsilon(s), \overline{u}(s)) - f(\hat{x}_\epsilon(s), \hat{u}_\epsilon(s))|\, ds. \end{aligned}$$

Therefore, as $n \to \infty$, $x_n \to \hat{x}_\epsilon$ uniformly in I. So, for all n large enough, $\|x_n - \hat{x}_\epsilon\|_\infty < \min\left\{\delta_\epsilon, \frac{\epsilon}{2}\right\}$, with δ_ϵ as in (10), implying $x_n(I) \subset \mathrm{Int}(K)$ and $x_n(I) \subset B(0,R)$. Further,

$$L(x_n(t), u_n(t)) - L(\hat{x}_\epsilon(t), u_n(t)) \le C_R|x_n(t) - \hat{x}_\epsilon(t)|\big[1 + L(\hat{x}_\epsilon(t), u_n(t))\big].$$

So, owing to (H) i), for all n sufficiently large

$$J(x_n, u_n) \le J(\hat{x}_\epsilon, \hat{u}_\epsilon) + C_R|x_n(T) - \hat{x}_\epsilon(T)| + C_R\|x_n - \hat{x}_\epsilon\|_\infty \int_0^T \big[1 + L(\hat{x}_\epsilon(t), u_n(t))\big]\,dt +$$

$$+ \int_{\{t \in I : |f(\hat{x}_\epsilon(t), \hat{u}_\epsilon(t))| > n\}} \big[L(\hat{x}_\epsilon(t), \overline{u}(t)) - L(\hat{x}_\epsilon(t), \hat{u}_\epsilon(t))\big]\,dt < J(\hat{x}_\epsilon, \hat{u}_\epsilon) + \frac{\epsilon}{2}$$

and (13) follows. □

Example 1. Let $T > 0$, $U = \mathbb{R}^m_+$, $\ell : \mathbb{R}^N \to \mathbb{R}_+$, $L : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}_+$ with $L(x, \cdot)$ convex for all x, $g : \mathbb{R}^N \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^N)$, $K = \{x : x_N^2 \ge (x_1^2 + ... + x_{N-1}^2)/2\}$, $Q_0 \subset \mathbb{R}^N$ be compact and $Q_0 \cap K \ne \emptyset$. Consider the problem of minimizing the functional

$$J(x,u) = \int_0^T L(x(t), u(t))dt + \int_0^T \sum_{i=1}^m u_i(s)ds + \ell(x(T)) \tag{15}$$

over all pairs (x, u) satisfying

$$\begin{cases} x'(t) = g(x(t))u(t), \quad u(t) \in U & \text{for a.e. } t \in I \\ x(t) \in K & \text{for every } t \in I \\ x(0) \in Q_0. \end{cases} \tag{16}$$

Assume that for all $x \in \partial K$, $\quad g(x)^*(x_1, ..., x_{N-1}, -1) \notin \mathbb{R}^m_+$, $\quad Q_0$ satisfies $iv)$ of assumption **(H)**, L satisfies $i2)$, ℓ is locally Lipschitz and for every $R > 0$, $\exists C_R > 0$ such that

$$|g(x)u - g(y)u| \le C_R|x - y|\Big[1 + |g(x)u| \wedge |g(y)u| + \big(L(x,u) \wedge L(y,u)\big)^{1/2}\Big].$$

Then it is not difficult to verify that all the assumptions **(H)** are satisfied.

We further assume that $L(x, \cdot)$ is differentiable and that for all $k > 0$ there exists $\alpha_k > 0$ such that

$$\begin{cases} \forall\, x \in K, \forall\, u \in U, \ \ |x| + |g(x)u| \le k \implies \big\langle \frac{\partial L}{\partial u}(x,u), u\big\rangle - L(x,u) \le -\alpha_k \\ \\ \liminf\limits_{c \to +\infty} \quad \inf\limits_{\substack{x \in K,\, |x| \le k, \\ u \in U,\ |g(x)u| \ge c}} \left(\left\langle \dfrac{\partial L}{\partial u}(x,u), u \right\rangle - L(x,u)\right) = 0. \end{cases} \tag{17}$$

We claim that then there exists an optimal solution to our problem satisfying the normal autonomous maximum principle.

Indeed observe first that the set $P(x,u)$ defined in section 3 is as follows

$$P(x,u) = \big\{p \in \mathbb{R}^N : g(x)^*p \in \frac{\partial L}{\partial u}(x,u) + (1, ..., 1) + N_U(u)\big\}.$$

Since U is a closed convex cone, for all $u \in U$ and $n \in N_U(u)$ we have $\langle n, u\rangle = 0$. Therefore for all $p \in P(x,u)$,

$$\mathcal{H}(x,u,p) = \langle g(x)^* p, u\rangle - L(x,u) - \sum_{i=1}^{m} u_i = \left\langle \frac{\partial L}{\partial u}(x,u), u) \right\rangle - L(x,u).$$

Consequently assumption 2) of Theorem 2 is satisfied.

Pick $x_0 \in Q_0 \cap K$ and consider the trajectory/control pair $(\overline{x}, \overline{u})$, where $\overline{u} \equiv 0, x(0) = x_0$. If $J(\overline{x}, \overline{u}) = \inf \big\{ J(x,u) : (x,u) \text{ solves } (16) \big\}$, then $(\overline{x}, \overline{u})$ is the optimal solution we are looking for: by the assumptions on L, $\overline{x} \equiv x_0$ is Lipschitzian, $L(\overline{x}(\cdot), \overline{u}(\cdot))$ is essentially bounded, and by [19] an autonomous maximum principle holds true. It follows from [10] that this maximum principle is normal. It remains to consider the case $J(\bar{x}, \bar{u}) > \inf\{J(x,u) : (x,u) \text{ solves } (16)\}$.

Observe that if for some trajectory/control pair (x,u), $J(x,u) < J(x_0, 0)$, then $\|u\|_{L^1} \le J(x_0,0)$. Since Q_0 is bounded, from the assumption on g and the Gronwall lemma we deduce that for a constant $k > 0$ independent of $(x,u)(\cdot)$ we have $\|x\|_\infty \le k$. Furthermore, using that $L \ge 0$ and $l \ge 0$, we deduce that $ess\,inf_{s\in I} \sum_{i=1}^m u_i(s) < \frac{J(x_0,0)}{T}$. This implies that $ess\,inf_{s\in I}\, |u(s)| < \frac{J(x_0,0)}{T}$ and therefore for a constant $k' > 0$ independent of $(x,u)(\cdot)$ we have $ess\,inf_{s\in I}\, |g(x(s))u(s)| \le k'$. Consequently assumption 1) of Theorem 2 is satisfied.

If $g : \mathbb{R}^2 \to \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ is given by

$$g(x,y) = \begin{bmatrix} -x & y \\ 1 & 1 \end{bmatrix}, \tag{18}$$

$U = \mathbb{R}^2_+$ and $L(x,y,u,v) = \sqrt{1 + u(t)^2 + v(t)^2}$ then our assumption (17) holds true. That is the result applies with the Lagrangian having a linear growth. Similarly it can be also applied with the same g and $L(x,y,u,v) = e^{-[u(t)+2v(t)]}$, that goes to zero when the norm of control does increase.

4. **Proof of Theorem 3.1.**

To prove Theorem 3.1 we need some preliminary results.

Let α be as in (H) i). For every $\epsilon > 0$, consider the penalized problem:

$$\text{minimize} \quad J_\epsilon(x,u) = \int_0^T \left[L(x(t),u(t)) + \frac{\epsilon}{\alpha} |f(x(t),u(t))|^\alpha \right] dt + \ell(x(T)) \tag{19}$$

over all trajectory/control pairs (x,u) of (2). Notice that, by Remark 1, the infimum of problem (19), (2) is finite. For $(x,u) \in \mathbb{R}^N \times \mathbb{R}^m$, define

$$L_\epsilon(x,u) = L(x,u) + \frac{\epsilon}{\alpha} |f(x,u)|^\alpha. \tag{20}$$

Lemma 4.1. *Assume (H) and suppose that there exists a trajectory/control pair (x_1, u_1) satisfying $x_1' \in L^\alpha(I; \mathbb{R}^N)$ and*

$$\inf\{J(x,u) : (x,u) \text{ is a trajectory/control pair of (2)}\} < J(x_1,u_1) < \infty.$$

Moreover, assume that there exists $k > 0$ such that:

$$\|x\|_\infty \le k \quad \text{for any trajectory/control pair } (x,u) \text{ of } (2) \text{ with } J(x,u) < J(x_1,u_1). \tag{21}$$

Then, for any sufficiently small $\epsilon > 0$,

a) there exists an optimal solution (x_ϵ, u_ϵ) to problem (19), (2) and $J(x_\epsilon, u_\epsilon) < J(x_1, u_1)$;
b) (x_ϵ, u_ϵ) satisfies a normal Autonomous Constrained Maximum Principle for some ψ_ϵ, p_ϵ and $c_\epsilon \in \mathbb{R}$.

Proof. By Cesari [5, chapter 11], for every $\epsilon > 0$, problem (19), (2) admits a solution (x_ϵ, u_ϵ). From Corollary 1 and the assumptions on (x_1, u_1), we deduce that there exists a trajectory/control pair $(\tilde{x}, \tilde{u})$ such that $\tilde{x}' \in L^\alpha(I; \mathbb{R}^N)$, $\tilde{x}(0) \in Q_0$, and $J(\tilde{x}, \tilde{u}) < J(x_1, u_1)$. Since $f(\tilde{x}(\cdot), \tilde{u}(\cdot)) \in L^\alpha(I; \mathbb{R}^N)$, for all small enough $\epsilon > 0$

$$J(x_\epsilon, u_\epsilon) \le J_\epsilon(x_\epsilon, u_\epsilon) \le J_\epsilon(\tilde{x}, \tilde{u}) < J(x_1, u_1), \tag{22}$$

implying a).

To prove b), fix ϵ sufficiently small. We claim that:

(h) there exist two Borel measurable functions $l, k : U \to \mathbb{R}_+$ such that $l(u_\epsilon(\cdot)), k(u_\epsilon(\cdot)) \in L^1(I; \mathbb{R})$ and

$$|L_\epsilon(x, u) - L_\epsilon(y, u)| \le l(u)|x - y| \quad \text{ and } \quad |f(x, u) - f(y, u)| \le k(u)|x - y|,$$

for every $u \in U$ and every $x, y \in x_\epsilon(I) + B(0, 1)$.

Indeed, let $R_\epsilon := \|x_\epsilon\|_\infty + 1$. From (H) i), for some $C > 0$, any $x \in B(0, R_\epsilon)$, and any $u \in U$,

$$|f(x, u)| \le C\big[1 + |f(0, u)| + L(0, u)^{1/\alpha}\big] \quad \text{ and } \quad L(x, u)^{1/\alpha} \le C[1 + L(0, u)^{1/\alpha}]. \tag{23}$$

Let $x, y \in x_\epsilon(I) + B(0, 1) \subset B(0, R_\epsilon)$. From (23) and (H) i), we obtain that, for all $u \in U$,

$$|f(x, u) - f(y, u)| \le C_{R_\epsilon}|x - y|\big[1 + |f(x, u)| + L(x, u)^{1/\alpha}\big] \le k(u)|x - y|,$$

where $k(u) = C_f\big[1 + |f(0, u)| + L(0, u)^{1/\alpha}\big]$, for some $C_f > 0$.

Moreover, from (23),

$$\begin{aligned}\Big||f(x, u)|^\alpha - |f(y, u)|^\alpha\Big| &\le \alpha|f(x, u) - f(y, u)|\big(|f(x, u)| + |f(y, u)|\big)^{\alpha - 1} \\ &\le \alpha k(u)|x - y|\big[2C(1 + |f(0, u)| + L(0, u)^{1/\alpha})\big]^{\alpha - 1}.\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}|L_\epsilon(x, u) - L_\epsilon(y, u)| &\le |L(x, u) - L(y, u)| + \frac{\epsilon}{\alpha}\big||f(x, u)|^\alpha - |f(y, u)|^\alpha\big| \\ &\le C_{R_\epsilon}|x - y|(1 + L(x, u)) + \frac{\epsilon}{\alpha}\big||f(x, u)|^\alpha - |f(y, u)|^\alpha\big| \le l(u)|x - y|,\end{aligned}$$

where $l(u) = C_l\Big[1 + L(0, u) + \big[1 + |f(0, u)| + L(0, u)^{1/\alpha}\big]^\alpha\Big]$, for some $C_l > 0$.

We claim that, for this choice of k and l, assumption (h) is satisfied. Indeed, since $|f(x_\epsilon(\cdot), u_\epsilon(\cdot))|$ and $L(x_\epsilon(\cdot), u_\epsilon(\cdot))^{1/\alpha}$ belong to $L^\alpha(I; \mathbb{R})$, an easy computation implies that $k(u_\epsilon(\cdot))$ and $l(u_\epsilon(\cdot))$ are integrable. Then, by [19, p.203], (x_ϵ, u_ϵ) satisfies an Autonomous Constrained Maximum Principle. Finally, from assumptions (H) i), iii), iv), using arguments similar to [11, Proof of Theorem 2], we deduce that this maximum principle is normal. $\square$

Lemma 4.2. *Under all assumptions of Theorem 3.1, let (x_ϵ, u_ϵ) be as in Lemma 4.1. Then for some $\epsilon_0 > 0$ we have*

$$\sup_{\epsilon \in (0, \epsilon_0)} \|f(x_\epsilon(\cdot), u_\epsilon(\cdot))\|_\infty < \infty\,.$$

In particular, there exist a sequence $\epsilon_n \to 0+$ and $x^* \in W^{1,\infty}(I; \mathbb{R}^N)$ with $x^*(0) \in Q_0$ satisfying

$$\begin{cases} x_{\epsilon_n} \to x^* & \textit{uniformly} \\ f(x_{\epsilon_n}(\cdot), u_{\epsilon_n}(\cdot)) \rightharpoonup x^{*\prime} & \textit{weakly}-* \textit{ in } L^\infty, \end{cases}$$

Proof. The proof follows the same ideas as the proof of [4, Theorem 1]. For $\epsilon > 0$ consider the penalized problem (19), (2). Lemma 4.1 and our assumptions ensure that, for any small enough ϵ (say $\epsilon \in (0, \epsilon_0)$), problem (19), (2) admits an optimal solution (x_ϵ, u_ϵ) such that

$$J(x_\epsilon, u_\epsilon) < J(x_1, u_1), \quad \|x_\epsilon\|_\infty \le k, \tag{24}$$

satisfying a normal maximum principle for some $c_\epsilon,\ \psi_\epsilon,\ p_\epsilon$. In particular,

$$\begin{aligned} c_\epsilon &= \langle p_\epsilon(t) + \psi_\epsilon(t), f(x_\epsilon(t), u_\epsilon(t))\rangle - L_\epsilon(x_\epsilon(t), u_\epsilon(t)) \\ &= \max_{u\in U}(\langle p_\epsilon(t) + \psi_\epsilon(t), f(x_\epsilon(t), u)\rangle - L_\epsilon(x_\epsilon(t), u)), \quad \text{ for a.e. } t \in I. \end{aligned} \tag{25}$$

By assumption 1) of Theorem 3.1, the set

$$A_\epsilon = \big\{t \in I : |f(x_\epsilon(t), u_\epsilon(t))| \le k\big\} \tag{26}$$

has positive measure. In view of (25) for a.e. $t \in I$, $\frac{\partial f}{\partial u}(x_\epsilon(t), u_\epsilon(t))^*\big(p_\epsilon(t) + \psi_\epsilon(t)\big)$ belongs to the set

$$\partial_u L(x_\epsilon(t), u_\epsilon(t)) + \epsilon|f(x_\epsilon(t), u_\epsilon(t))|^{\alpha-1}\frac{\partial f}{\partial u}(x_\epsilon(t), u_\epsilon(t))^*\phi(x_\epsilon(t), u_\epsilon(t)) + N_U(u_\epsilon(t)),$$

where ϕ is a measurable function satisfying

$$\begin{cases} \phi(x_\epsilon(t), u_\epsilon(t)) = \dfrac{f(x_\epsilon(t), u_\epsilon(t))}{|f(x_\epsilon(t), u_\epsilon(t))|} & \text{if } f(x_\epsilon(t), u_\epsilon(t)) \ne 0 \\[2ex] \phi(x_\epsilon(t), u_\epsilon(t)) \in B(0, 1) & \text{otherwise}. \end{cases}$$

Hence, for a.e. $t \in I$,

$$\begin{aligned} &\frac{\partial f}{\partial u}(x_\epsilon(t), u_\epsilon(t))^*\big[\big(p_\epsilon(t) + \psi_\epsilon(t)\big) - \epsilon|f(x_\epsilon(t), u_\epsilon(t))|^{\alpha-1}\phi(x_\epsilon(t), u_\epsilon(t))\big] \\ &\qquad\qquad \in \partial_u L(x_\epsilon(t), u_\epsilon(t)) + N_U(u_\epsilon(t)), \end{aligned}$$

or equivalently,

$$p_\epsilon(t) + \psi_\epsilon(t) - \epsilon|f(x_\epsilon(t), u_\epsilon(t))|^{\alpha-1}\phi(x_\epsilon(t), u_\epsilon(t)) \in P(x_\epsilon(t), u_\epsilon(t)). \tag{27}$$

Using assumption 2) of Theorem 3.1 we deduce that, for some $c(k) \ge k$,

$$\sup_{\substack{|x| \le k,\, |f(x,u)| \le k \\ p \in P(x,u)}} \mathcal{H}(x, u, p) < \inf_{\substack{|x| \le k,\, |f(x,u)| \ge c(k) \\ p \in P(x,u)}} \mathcal{H}(x, u, p). \tag{28}$$

Set

$$B_\epsilon = \big\{t \in I : |f(x_\epsilon(t), u_\epsilon(t))| > c(k)\big\}.$$

We claim that $\mu(B_\epsilon) = 0$, from which the first conclusion of Lemma 4.2 follows. Indeed, suppose for a moment that $\mu(B_\epsilon) > 0$. Let $a \in A_\epsilon$ and $b \in B_\epsilon$ be such that (25) and (27) hold true.

Then (25) yields

$$\begin{aligned} c_\epsilon &= \langle p_\epsilon(a) + \psi_\epsilon(a), f(x_\epsilon(a), u_\epsilon(a))\rangle - L(x_\epsilon(a), u_\epsilon(a)) - \frac{\epsilon}{\alpha}|f(x_\epsilon(a), u_\epsilon(a))|^\alpha \\ &= \langle p_\epsilon(b) + \psi_\epsilon(b), f(x_\epsilon(b), u_\epsilon(b))\rangle - L(x_\epsilon(b), u_\epsilon(b)) - \frac{\epsilon}{\alpha}|f(x_\epsilon(b), u_\epsilon(b))|^\alpha. \end{aligned}$$

If $|f(x_\epsilon(a), u_\epsilon(a))| > 0$, then, by (27) and (28),

$$\langle p_\epsilon(a) + \psi_\epsilon(a) - \epsilon |f(x_\epsilon(a), u_\epsilon(a))|^{\alpha-2} f(x_\epsilon(a), u_\epsilon(a)), f(x_\epsilon(a), u_\epsilon(a))\rangle - L(x_\epsilon(a), u_\epsilon(a))$$

$$< \langle p_\epsilon(b) + \psi_\epsilon(b) - \epsilon |f(x_\epsilon(b), u_\epsilon(b))|^{\alpha-2} f(x_\epsilon(b), u_\epsilon(b)), f(x_\epsilon(b), u_\epsilon(b))\rangle - L(x_\epsilon(b), u_\epsilon(b)).$$

Hence we obtain

$$\begin{aligned} c_\epsilon &= \langle p_\epsilon(a) + \psi_\epsilon(a) - \epsilon |f(x_\epsilon(a), u_\epsilon(a))|^{\alpha-2} f(x_\epsilon(a), u_\epsilon(a)), f(x_\epsilon(a), u_\epsilon(a))\rangle \\ &\quad - L(x_\epsilon(a), u_\epsilon(a)) + \epsilon\Big(1 - \frac{1}{\alpha}\Big)|f(x_\epsilon(a), u_\epsilon(a))|^\alpha \\ &< \langle p_\epsilon(b) + \psi_\epsilon(b) - \epsilon |f(x_\epsilon(b), u_\epsilon(b))|^{\alpha-2} f(x_\epsilon(b), u_\epsilon(b)), f(x_\epsilon(b), u_\epsilon(b))\rangle \\ &\quad - L(x_\epsilon(b), u_\epsilon(b)) + \epsilon\Big(1 - \frac{1}{\alpha}\Big)|f(x_\epsilon(a), u_\epsilon(a))|^\alpha \\ &= c_\epsilon - \epsilon\Big(1 - \frac{1}{\alpha}\Big)|f(x_\epsilon(b), u_\epsilon(b))|^\alpha + \epsilon\Big(1 - \frac{1}{\alpha}\Big)|f(x_\epsilon(a), u_\epsilon(a))|^\alpha. \end{aligned}$$

Therefore,

$$c(k)^\alpha < |f(x_\epsilon(b), u_\epsilon(b))|^\alpha < |f(x_\epsilon(a), u_\epsilon(a))|^\alpha \le k^\alpha,$$

contradicting the choice of $c(k)$.

If $f(x_\epsilon(a), u_\epsilon(a)) = 0$, then $p_\epsilon(a) + \psi_\epsilon(a) \in P(x_\epsilon(a), u_\epsilon(a))$. Arguing as above we derive the contradiction $c(k)^\alpha < |f(x_\epsilon(b), u_\epsilon(b))|^\alpha < 0$.

This implies that $\mu(B_\epsilon) = 0$. The last statement of Lemma 4.2 follows by applying Ascoli's and Alaoglu's theorems. □

The following lemma is needed to prove the boundedness of $\{\psi_{\epsilon_n}\}_{n\in\mathbb{N}}$ with ϵ_n as in Lemma 4.2.

Lemma 4.3. *Under all assumptions of Theorem 3.1, consider a sequence $(x_{\epsilon_n}, u_{\epsilon_n})$ as in the claim of Lemma 4.2. For any n, let A_n, π_n be measurable mappings such that $A_n(t) \in \partial_x f(x_{\epsilon_n}(t), u_{\epsilon_n}(t))$, $\pi_n(t) \in \partial_x L_{\epsilon_n}(x_{\epsilon_n}(t), u_{\epsilon_n}(t))$ for a.e. $t \in I$. Then there exist $M > 0$, $\tilde\rho > 0$, a function $\gamma \in L^\infty(I;\mathbb{R}_+)$ and $u_n \in \mathcal{U}$ such that for all large n the two systems below*

$$\begin{cases} w'(t) = A_n(t)w(t) + \gamma(t)\big(f(x_{\epsilon_n}(t), u_n(t)) - x'_{\epsilon_n}(t)\big) & a.e.\ in\ I \\ w(0) \in Int\big(C_K(x_{\epsilon_n}(0))\big) \cap Int\big(C_{Q_0}(x_{\epsilon_n}(0))\big) \end{cases} \tag{29}$$

$$\begin{cases} \xi'(t) = \langle \pi_n(t), w(t)\rangle + \gamma(t)\big(L_{\epsilon_n}(x_{\epsilon_n}(t), u_n(t)) - L_{\epsilon_n}(x_{\epsilon_n}(t), u_{\epsilon_n}(t)\big) & a.e.\ in\ I \\ \xi(0) = 0, \end{cases} \tag{30}$$

admit a solution (w_n, ξ_n) satisfying

$$|w_n(t)| \le M, \quad \textit{for any } t \in I \tag{31}$$

$$|\xi_n(t)| \le M, \quad \textit{for any } t \in I \tag{32}$$

$$\langle \nabla d_K(x_{\epsilon_n}(t)), w_n(t)\rangle \le -\tilde\rho, \quad \textit{for any } t \textit{ such that } x_{\epsilon_n}(t) \in \partial K. \tag{33}$$

Proof. From Lemmas 4.1, 4.2 there exist $\tilde M > 0$, $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$,

$$\|f(x_\epsilon(\cdot), u_\epsilon(\cdot))\|_\infty \le \tilde M, \quad \|L(x_\epsilon(\cdot), u_\epsilon(\cdot))\|_{L^1(I;\mathbb{R}^N)} \le \tilde M \quad \text{and} \quad \|x_\epsilon\|_\infty \le k. \tag{34}$$

It is not restrictive to assume that $\epsilon_n \in (0, \epsilon_0)$ for all $n \ge 1$.

By (34) and assumption (H) i), there exists $M' > 0$ such that for all n

$$\|A_n\|_{L^\alpha(I;\mathbb{R}^N\times\mathbb{R}^N)} \leq M' \quad \text{and} \quad \|\pi_n\|_{L^1(I;\mathbb{R}^N)} \leq M'. \tag{35}$$

Assumptions (H) i), iii) and Lipschitz continuity of x_{ϵ_n} imply that there exist $\tau, \delta, \rho, \eta > 0$ (independent from n) and $u_n \in \mathcal{U}$ such that d_K is $C^{1,1}$ on $(\partial K \cap B(0,k)) + B(0,\delta)$

$$\|f(x_{\epsilon_n}(\cdot), u_n(\cdot))\|_\infty \leq \eta, \quad \|L(x_{\epsilon_n}(\cdot), u_n(\cdot))\|_\infty \leq \eta \tag{36}$$

and for all $s, t \in I$ satisfying $|s-t| \leq \tau$ and $x_{\epsilon_n}(t), x_{\epsilon_n}(s) \in \partial K + B(0,\delta)$

$$\langle \nabla d_K(x_{\epsilon_n}(s)), f(x_{\epsilon_n}(s), u_n(s))\rangle < -\rho, \quad |\nabla d_K(x_{\epsilon_n}(t)) - \nabla d_K(x_{\epsilon_n}(s))| \leq \frac{\rho}{4(\tilde{M}+\eta+1)}. \tag{37}$$

Let x^* be as in Lemma 4.2. Fix any $0 \neq \bar{w}_0 \in Int\big(C_K(x^*(0))\big) \cap Int\big(C_{Q_0}(x^*(0))\big)$ and set $v_n(\cdot) := f(x_{\epsilon_n}(\cdot), u_n(\cdot))$, $l_n(\cdot) := L(x_{\epsilon_n}(\cdot), u_n(\cdot))$. Since $x_{\epsilon_n}(0) \to x^*(0)$ and the set-valued map $K \cap Q_0 \ni x \rightsquigarrow Int\big(C_K(x)\big) \cap Int\big(C_{Q_0}(x)\big)$ is lower semicontinuous, see [2], for all large enough n, say $n \geq \tilde{n}_0$, $\bar{w}_0 \in Int\big(C_K(x_{\epsilon_n}(0))\big) \cap Int\big(C_{Q_0}(x_{\epsilon_n}(0))\big)$.

Set

$$\mathcal{A} = \big\{t \in I : x^*(t) \in \partial K + B(0,\delta/2)\big\}, \quad \mathcal{B} = \big\{t \in I : x^*(t) \in \partial K + \overline{B(0,\delta/4)}\big\},$$

and, for any n,

$$\mathcal{A}_n = \big\{t \in I : x_{\epsilon_n}(t) \in \partial K + B(0,\delta)\big\}, \quad \mathcal{B}_n = \big\{t \in I : x_{\epsilon_n}(t) \in \partial K + \overline{B(0,\delta/8)}\big\}.$$

Since $x_{\epsilon_n} \to x^*$ uniformly, for any large enough n (say $n \geq \bar{n}_0 \geq \tilde{n}_0$)

$$\mathcal{B}_n \subset \mathcal{B} \subset \mathcal{A} \subset \mathcal{A}_n.$$

Moreover, as $\mathcal{A}$ is open in I and $\mathcal{B}$ is compact, we deduce that

$$\mathcal{A} = \bigcup_{j\in\mathbb{N}} I_j \supseteq \bigcup_{i=1}^{\nu} I_i \supseteq \mathcal{B},$$

for some disjoint relatively open subintervals I_j of $[0,T]$. Then, it is not difficult to check that for some $m \leq \nu$, $0 \leq t_0^1 < t_f^1 < \cdots < t_0^m < t_f^m \leq t_0^{m+1} = T$ and for any $n \geq n_0$,

$$\mathcal{B}_n \subset \bigcup_{j=1}^{m} [t_0^j, t_f^j] = \bigcup_{i=1}^{\nu} \bar{I}_i$$

To obtain a function w_n as in the claim (of the lemma), we solve the system

$$w'(t) = A_n(t)w(t) + \gamma(t)\big(v_n(t) - x'_{\epsilon_n}(t)\big), \tag{38}$$

with a piecewise constant function γ independent of n and $w_n(0) = \bar{w}_0$. Since $\{v_n - x'_{\epsilon_n}\}_{n\in\mathbb{N}}$ is bounded in L^∞, estimate (35) will imply (31) for some $M > 0$ independent of n.

Let $t_0 \in I$ and, for $n \in \mathbb{N}$, let $W_n(\cdot, t_0)$ be the matrix solution to

$$\begin{cases} \dot{W}_n(t;t_0) = A_n(t)W_n(t;t_0) & \text{for a.e. } t \in [t_0, T] \\ W_n(t_0;t_0) = \mathbb{I}_{\mathbb{R}^N}. \end{cases}$$

From (35) and Gronwall's Lemma, it follows that there exist $\zeta, \sigma > 0$ independent of n such that for any t_0 and any n,

$$\|W_n(\cdot;t_0)\|_\infty \leq \zeta, \tag{39}$$

and for all $s, t \in I$ satisfying $|t-s| \leq \sigma$

$$\left\| W_n(t;t_0)W_n(s;t_0)^{-1} - \mathbb{I}_{\mathbb{R}^m} \right\| \leq \frac{\rho}{4\big(\tilde{M}+\eta+1\big)}. \tag{40}$$

The construction of γ and of a solution w_n to (29) is structured in the following way:

In *Step 1:* we define w_n in $[0, t_f^1]$. For this aim we consider two cases : $t_0^1 > 0$ and $t_0^1 = 0$;
In *Step 2:* we extend w_n on $[0,T]$ iterating the construction of step 1.

Step 1: Let $t_0^1 = t_0 < t_1 < \cdots < t_{m_1} = t_f^1$ be such that

$$t_i - t_{i-1} \leq \min\{\tau, \sigma\}, \qquad \text{for } i = 1, \dots, m_1.$$

Case 1: $t_0^1 > 0$. By the definition of t_0^1, $x^*([0, t_0^1]) \subset Int(K)$. Since $x_{\epsilon_n} \to x^*$ uniformly, for all n large enough, $x_{\epsilon_n}([0, t_0^1]) \subset Int(K)$. So, for some $M_0 > 0$, $n_0 \geq \bar{n}_0$, and all $n \geq n_0$, the solution w_n to (38) with $\gamma \equiv 0$ and $w_n(0) = \bar{w}_0$ satisfies

$$|w_n(t)| \leq M_0, \quad \forall\, t \in [0, t_0^1]. \tag{41}$$

Consider the time interval $[t_0, t_1]$.

If $x^*([t_0, t_1]) \subset Int(K)$, then for any n large enough also $x_{\epsilon_n}([t_0, t_1]) \subset Int(K)$ and we extend w_n on $[t_0, t_1]$, by taking the solution to (38) with $\gamma \equiv 0$ and starting at $w_n(t_0^1)$. As in (41), we obtain that for some $M_1 > 0$, $n_1 > n_0$ and all $n \geq n_1$,

$$|w_n(t)| \leq M_1, \quad \forall\, t \in [t_0, t_1]. \tag{42}$$

If $x^*([t_0, t_1]) \cap \partial K \neq \emptyset$, then define

$$s_0 = \min\{t \in [t_0, t_1] : x^*(t) \in \partial K\}, s_f = \max\{t \in [t_0, t_1] : d_K(x^*(t)) \geq \frac{d_K(x^*(t_0))}{4}\}. \tag{43}$$

Since $x^*(t_0) \notin \partial K$, we deduce that $s_0 > t_0$ and that $d_K(x^*(t_0)) < 0$. As $x_{\epsilon_n} \to x^*$, there exists $\bar{n}_1 \geq n_0$ such that

$$\|x_{\epsilon_n} - x^*\|_\infty < -\frac{d_K(x^*(t_0))}{4}\,, \qquad \text{for any } n \geq \bar{n}_1. \tag{44}$$

Hence, using that x^* is Lipschitz and that $x_{\epsilon_n} \to x^*$ uniformly, for some $\sigma_0 > 0$, $n_1 \geq \bar{n}_1$ and for all $n \geq n_1$,

$$\min\left\{t \in [t_0, t_1] : d_K\big(x_{\epsilon_n}(t)\big) \geq \frac{d_K(x^*(t_0))}{2}\right\} \geq t_0 + \sigma_0.$$

Let $n \geq n_1$ and w_n solve (38) in $[t_0, s_f]$ with the initial condition $w_n(t_0)$, $\gamma \equiv \frac{4\zeta M_0}{\rho\sigma_0}$, where ζ is as in (39), ρ as in (37), and M_0 as in (41). Then,

$$w_n(t) = W_n(t;t_0)w_n(t_0) + \frac{4\zeta M_0}{\rho\sigma_0}\int_{t_0}^{t} W_n(t;t_0)W_n(s;t_0)^{-1}\big(v_n(s) - x'_{\epsilon_n}(s)\big)ds. \tag{45}$$

For any $t \in [t_0, t_0+\sigma_0)$, $x_{\epsilon_n}(t) \in Int(K)$. Let $t \in [t_0+\sigma_0, s_f]$ be such that $x_{\epsilon_n}(t) \in \partial K + B\big(0, -d_K(x^*(t_0))/2\big)$. From (40) and (37), for all $s \in [t_0+\sigma_0, s_f]$,

$$\langle \nabla d_K(x_{\epsilon_n}(t)), W_n(t;t_0)W_n(s;t_0)^{-1}\big(v_n(s) - x'_{\epsilon_n}(s)\big)\big\rangle \leq \big\langle \nabla d_K(x_{\epsilon_n}(s)), v_n(s) - x'_{\epsilon_n}(s)\big\rangle + \frac{\rho}{2}.$$

So, by (37) and (44), for all $t \in [t_0, s_f]$ such that $d_K(x_{\epsilon_n}(t)) \geq d_K(x^*(t_0))/2$

$$\begin{aligned}\int_{t_0}^{t} &\big\langle \nabla d_K(x_{\epsilon_n}(t)), W_n(t;t_0)W_n(s;t_0)^{-1}\big(v_n(s) - x'_{\epsilon_n}(s)\big)\big\rangle ds \\ &\leq -\rho(t-t_0) - d_K(x_{\epsilon_n}(t)) + d_K(x_{\epsilon_n}(t_0)) + \frac{\rho}{2}(t-t_0) \leq -\frac{\rho}{2}(t-t_0).\end{aligned} \tag{46}$$

Hence, from (45), for all $t \in [t_0, s_f]$ satisfying $d_K(x_{\epsilon_n}(t)) \geq d_K(x^*(t_0))/2$

$$\langle \nabla d_K(x_{\epsilon_n}(t)), w_n(t)\rangle \leq \langle \nabla d_K(x_{\epsilon_n}(t)), W_n(t;t_0)w_n(t_0)\rangle - 2\zeta M_0 \leq -\zeta M_0 := -\rho_1. \tag{47}$$

The above inequality implies that

$$\langle \nabla d_K(x_{\epsilon_n}(s_f)), w_n(s_f)\rangle \leq -\rho_1, \tag{48}$$

and so $w_n(s_f) \neq 0$. Further, by (34), (35), (36), (38), (41), and Gronwall's Lemma, for some $\overline{M}_1 > 0$, $|w_n(t)| \leq \overline{M}_1$, for any $n \geq n_1$ and any $t \in [t_1, s_f]$.

If $s_f < t_1$, then we take the solution w_n to (38) in $[s_f, t_1]$ with $\gamma \equiv 0$ and starting at $w_n(s_f)$, so that $w_n(t_1) \neq 0$.

So, we proved that for some $M_1 > 0$, $n_1 \geq 1$ and for all $n \geq n_1$ there exists a solution w_n to (38) in $[t_0, t_1]$ satisfying (33), $w_n(t_1) \neq 0$, and

$$|w_n(t)| \leq M_1, \quad \text{for any } t \in [t_0, t_1].$$

Consider, next, the time interval $[t_1, t_2]$. Then either $s_f < t_1$ or $s_f = t_1$.

a) If $s_f < t_1$, then $x^*(t_1) \in Int(K)$. Arguing as in $[t_0, t_1]$ we deduce that there exist $M_2 > 0$, $\rho_2 > 0$, a piecewise constant function γ independent of n and $n_2 \geq n_1$ such that, for all $n \geq n_2$, the solution w_n to (38) defined in $[t_1, t_2]$ starting at $w_n(t_1)$ satisfies

$$\begin{aligned}&|w_n(t)| \leq M_2, \quad \forall\, t \in [t_1, t_2], \\ &\langle \nabla d_K(x_{\epsilon_n}(t)), w_n(t)\rangle \leq -\rho_2, \quad \forall\, t \in I \text{ such that } x_{\epsilon_n}(t) \in \partial K + B(0, -d_K(x^*(t_1))/2).\end{aligned} \tag{49}$$

b) If $s_f = t_1$, then from (48) and (H) iii), there exist $n_2 \geq n_1$ and $0 < \sigma_1 < t_2 - t_1$ such that, for any $t \in (t_1, t_1 + \sigma_1)$ and any $n \geq n_2$,

$$|\nabla d_K(x_{\epsilon_n}(t)) - \nabla d_K(x_{\epsilon_n}(t_1))| \leq \frac{\rho_1}{8M_1} \quad \text{and} \quad \|W_n(t;t_1) - \mathbb{I}_{\mathbb{R}^m}\| \leq \frac{\rho_1}{8M_1}. \tag{50}$$

We may also assume that, for any $n \geq n_2$,

$$\|x_{\epsilon_n} - x^*\|_\infty < \delta_1 := \min\Big\{\frac{\rho\sigma_1}{8}, \frac{\rho_1 \wedge \zeta M_1}{8}\Big(\frac{4\zeta M_1}{\rho\sigma_1}\Big)^{-1}, \frac{\delta}{2}\Big\}, \tag{51}$$

Define

$$s_f^1 = \max\big\{t \in [t_1, t_2] : d_K(x^*(t)) \geq -\delta_1\big\}$$

and consider the solution w_n to (38) on $[t_1, s_f^1]$ starting at $w_n(t_1)$, with $\gamma \equiv \frac{4\zeta M_1}{\rho\sigma_1}$. Then,

$$w_n(t) = W_n(t;t_1)w_n(t_1) + \frac{4\zeta M_1}{\rho\sigma_1}\int_{t_1}^{t} W_n(t;t_1)W_n(s;t_1)^{-1}\big(v_n(s) - x'_{\epsilon_n}(s)\big)ds.$$

Let t be such that $x_{\epsilon_n}(t) \in \partial K + B(0, 2\delta_1)$.

If $t \le t_1 + \sigma_1$, arguing as in (46), from (50) we deduce that

$$\langle \nabla d_K(x_{\epsilon_n}(t)), w_n(t)\rangle \le \langle \nabla d_K(x_{\epsilon_n}(t_1)), w_n(t_1)\rangle + \big|\nabla d_K(x_{\epsilon_n}(t)) - \nabla d_K(x_{\epsilon_n}(t_1))\big| M_1$$
$$+ \big\|W_n(t;t_1) - \mathbb{I}_{\mathbb{R}^m}\big\| M_1 - \frac{2\zeta M_1}{\sigma_1}(t-t_1) + \frac{4\zeta M_1}{\rho\sigma_1}\big[d_K(x_{\epsilon_n}(t_1)) - d_K(x_{\epsilon_n}(t))\big] \le -\frac{\rho_1}{2}.$$

If $t \ge t_1 + \sigma_1$, then, as in (46)–(47), from the definition of δ_1 we obtain that

$$\langle \nabla d_K(x_{\epsilon_n}(t)), w_n(t)\rangle \le \langle \nabla d_K(x_{\epsilon_n}(t)), W_n(t;t_1) w_n(t_1)\rangle - 2\zeta M_1 \le -\frac{1}{2}\,\zeta M_1 := -\rho_2.$$

The above inequality implies that (33) is satisfied in $[t_1, s_f^1]$ and

$$\langle \nabla d_K(x_{\epsilon_n}(s_f^1)), w_n(s_f^1)\rangle \le -\rho_2.$$

So $w_n(s_f^1) \neq 0$. Further, by (34), (35), (36), (38), (42), and Gronwall's Lemma, for some $M_2 > 0$ and all large enough n, w_n satisfies (49) in $[t_1, s_f^1]$. If $s_f^1 < t_2$, then we extend w_n on $[s_f^1, t_2]$ by a solution to (38) with $\gamma \equiv 0$ starting at $w_n(s_f^1)$.

Iterating the procedure described above, for all n large enough we construct on $[0, t_f^1]$ a measurable bounded function γ and a solution w_n to (38) which satisfies $w_n(0) = \bar{w}_0$, $w_n(t_f^1) \neq 0$, and

$$|w_n(t)| \le \tilde{M}_1, \quad \text{for any } t \in [0, t_f^1],$$
$$\langle \nabla d_K(x_{\epsilon_n}(t)), w_n(t)\rangle \le -\tilde{\rho}_1, \quad \text{for any } t \in [0, t_f^1] \text{ with } x_{\epsilon_n}(t) \in \partial K,$$

where $\tilde{\rho}_1$, $\tilde{M}_1$ are positive constants independent of n.

Case 2: $t_0^1 = 0$. If $x^*(0) \notin \partial K$, then we obtain a solution w_n in $[t_0, t_1]$ arguing exactly as in Case 1. It remains to consider the case $x^*(0) \in \partial K$. Since $x_{\epsilon_n}(0) \to x^*(0)$ and ∂K is $C^{1,1}_{loc}$, there exists $\delta_0 > 0$ such that

$$\big\langle \nabla d_K(x_{\epsilon_n}(0)), \bar{w}_0 \big\rangle \le -\delta_0, \quad \text{for all } n \text{ large enough.}$$

We can define a solution to (38), arguing as in **b)**. The construction of w_n in $[t_0^1, t_f^1]$ follows exactly as in case 1.

Step 2. We extend w_n to $[t_f^1, T]$, as we did in case 1, by setting $\gamma \equiv 0$ in any interval $[t_f^{j-1}, t_0^j]$, for $j = 2, \dots, m+1$, and taking γ piecewise constant in $[t_0^j, t_f^j]$ and independent of n. Notice that by the definition of t_0^j, $x^*(t_0^j) \in Int(K)$. As the number of steps is finite for all n large enough we obtain a solution w_n to (29) satisfying

$$|w_n(t)| \le \tilde{M}_j, \quad \forall\, t \in [t_0^j, t_f^j],$$
$$\langle \nabla d_K(x_{\epsilon_n}(t)), w_n(t)\rangle \le -\tilde{\rho}_j, \quad \forall\, t \in [t_0^j, t_f^j] \quad \text{such that} \quad x_{\epsilon_n}(t) \in \partial K.$$

Estimates (31) and (33) follow.

Consider next, the solution ξ_n to (30) for n large enough. Since $\{\pi_n\}_{n\in\mathbb{N}}$, $\{\ell_n\}_{n\in\mathbb{N}}$ and $\{L(x_{\epsilon_n}(\cdot), u_{\epsilon_n}(\cdot)\}_{n\in\mathbb{N}}$ are bounded in L^1, and γ is essentially bounded and independent of n, the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is bounded in $C(I;\mathbb{R})$ and (32) follows. □

Lemma 4.4. *Under all assumptions of Theorem 3.1, consider a sequence* $(x_{\epsilon_n}, u_{\epsilon_n})$ *as in Lemma 4.2 and let* ψ_{ϵ_n}*,* p_{ϵ_n} *be as in Lemma 4.1. Then*

$$\sup_{n\in\mathbb{N}} \|\psi_{\epsilon_n}\|_{BV} < \infty. \tag{52}$$

Proof. From Lemma 4.3, there exist $\gamma \in L^\infty(I;\mathbb{R}_+)$, $\tilde{\rho}$, $M > 0$ and $u_n \in \mathcal{U}$ such that, for all n large enough, (29), (30) admit a solution (w_n, ξ_n) satisfying

$$w_n(t) + B(0,\tilde{\rho}) \in Int\big(C_K(x_{\epsilon_n}(t))\big), \ \ \forall\, t \in I \ \ \& \quad \|w_n\|_\infty + \|\xi_n\|_\infty \le M. \tag{53}$$

Set $v_n(\cdot) = f(x_{\epsilon_n}(\cdot), u_n(\cdot))$ and $l_n(\cdot) = L(x_{\epsilon_n}(\cdot), u_n(\cdot))$. Using (29) and (30), it is not difficult to prove that

$$\begin{aligned}
\langle p_{\epsilon_n}(T), &w_n(T)\rangle - \langle p_{\epsilon_n}(0), w_n(0)\rangle - \xi_n(T) \\
&= \int_0^T \big(\langle p'_{\epsilon_n}(t), w_n(t)\rangle + \langle p_{\epsilon_n}(t), w'_n(t)\rangle - \xi'_n(t)\big)\, dt \\
&= \int_0^T -\langle \psi_{\epsilon_n}(t), w'_n(t)\rangle\, dt \\
&\quad + \int_0^T \gamma(t)\big(\langle p_{\epsilon_n}(t) + \psi_{\epsilon_n}(t), v_n(t) - x'_{\epsilon_n}(t)\rangle - \big(l_n(t) - L_{\epsilon_n}(x_{\epsilon_n}(t), u_{\epsilon_n}(t))\big)\big)\, dt.
\end{aligned}$$

This and (5) yield

$$\langle p_{\epsilon_n}(T), w_n(T)\rangle - \langle p_{\epsilon_n}(0), w_n(0)\rangle - \xi_n(T) \le \int_0^T -\langle \psi_{\epsilon_n}(t), w'_n(t)\rangle\, dt.$$

Integrating by parts and using (53), we obtain

$$\langle p_{\epsilon_n}(T), w_n(T)\rangle \le \langle p_{\epsilon_n}(0), w_n(0)\rangle + M - \langle \psi_{\epsilon_n}(T), w_n(T)\rangle + \int_0^T w_n(t)\, d\psi_{\epsilon_n}(t).$$

Thus, by the C_k Lipschitzianity of ℓ, and by (7),

$$-\int_0^T w_n(t)\, d\psi_{\epsilon_n}(t) \le |p_{\epsilon_n}(T) + \psi_{\epsilon_n}(T)| M + M \le (C_k + 1)M, \tag{54}$$

with M as in (53).

Let $w \in C(I;\mathbb{R}^N)$ satisfy $\|w\|_\infty \le \tilde{\rho}$. By (53)

$$w_n(t) + w(t) \in Int\big(C_K(x_{\epsilon_n}(t))\big), \quad \forall\, t \in I.$$

So, from (54) and (4),

$$\int_0^T w(t)\, d\psi_{\epsilon_n}(t) = \int_0^T (w_n(t) + w(t))\, d\psi_{\epsilon_n}(t) - \int_0^T w_n(t)\, d\psi_{\epsilon_n}(t) \le (C_k + 1)M,$$

implying (52). □

Lemma 4.5. *Under all assumptions of Theorem 3.1, consider a sequence* $(x_{\epsilon_n}, u_{\epsilon_n})$ *as in Lemma 4.2 and* ψ_{ϵ_n}*,* p_{ϵ_n} *as in Lemma 4.1. Then* $\sup_{n\in\mathbb{N}} \|p_{\epsilon_n}\|_\infty < \infty$*.*

Proof. From Lemmas 4.1, 4.2 and 4.4 we deduce that for some $C>0$ and all n large enough

$$\|L(x_{\epsilon_n}(\cdot),u_{\epsilon_n}(\cdot))\|_{L^1}\le C,\ \ \|x_{\epsilon_n}\|_\infty\le k,\ \ \|f(x_{\epsilon_n}(\cdot),u_{\epsilon_n}(\cdot))\|_\infty\le C,\ \ \|\psi_{\epsilon_n}\|_{BV}\le C. \tag{55}$$

Since the optimal pair $(x_{\epsilon_n},u_{\epsilon_n})$ satisfies the normal maximum principle, from (H) i) and (6) with L replaced by L_{ϵ_n} we deduce that, for a.e. $t\in I$,

$$\begin{aligned}|p'_{\epsilon_n}(t)|\le{}& C_k\big[1+|f(x_{\epsilon_n}(t),u_{\epsilon_n}(t))|+L(x_{\epsilon_n}(t),u_{\epsilon_n}(t))^{1/\alpha}\big]\big(|p_{\epsilon_n}(t)|+|\psi_{\epsilon_n}(t)|\big) \\ &+C_k\big[1+L(x_{\epsilon_n}(t),u_{\epsilon_n}(t))\big] \\ &+\epsilon_nC_k\big[1+|f(x_{\epsilon_n}(t),u_{\epsilon_n}(t))|+L(x_{\epsilon_n}(t),u_{\epsilon_n}(t))^{1/\alpha}\big]|f(x_{\epsilon_n}(t),u_{\epsilon_n}(t))|^{\alpha-1}.\end{aligned} \tag{56}$$

As $\alpha\ge 2$, (56) and (55) imply that for some $\tilde C>0$ and for all n large enough

$$|p'_{\epsilon_n}(t)|\le\tilde C\big[1+L(x_{\epsilon_n}(t),u_{\epsilon_n}(t))\big]\big(|p_{\epsilon_n}(t)|+1\big)\quad\text{a.e. in } I.$$

By the C_k-Lipschitzianity of ℓ on K, (7) and (55), $|p_{\epsilon_n}(T)|\le C+C_k$. Hence, applying estimates (55) and Gronwall's Lemma, for some $C_1>0$ and all n large enough, $\|p_{\epsilon_n}\|_\infty\le C_1$.

Proof of Theorem 3.1. Lemma 4.1 ensures that, for any $\epsilon\in(0,\epsilon_0)$, problem (19), (2) admits an optimal solution (x_ϵ,u_ϵ) such that

$$J(x_\epsilon,u_\epsilon)<J(x_1,u_1),\quad \|x_\epsilon\|_\infty\le k, \tag{57}$$

and the normal Autonomous Constrained Maximum Principle holds true with some $c_\epsilon,\ \psi_\epsilon,\ p_\epsilon$ satisfying (25).

Step 1. Consider a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ as in Lemma 4.2. We claim that there exist a subsequence $\{\epsilon_{n_i}\}_{i\in\mathbb{N}}$ and $\xi\in L^\infty(I;\mathbb{R})$ such that

$$L(x_{\epsilon_{n_i}}(\cdot),u_{\epsilon_{n_i}}(\cdot))\rightharpoonup\xi\quad\text{ weakly}-*\text{ in } L^\infty. \tag{58}$$

Applying Lemmas 4.4 and 4.5, we deduce that for some $M>0$

$$\sup_{n\in\mathbb{N}}\big\{\|p_{\epsilon_n}\|_\infty+\|\psi_{\epsilon_n}\|_{BV}\big\}\le M. \tag{59}$$

Define A_{ϵ_n} as in (26) and fix any $u_0\in U$. Let $t\in A_{\epsilon_n}$ be such that (25) holds true. Since $f(\cdot,u_0)$ and $L(\cdot,u_0)$ are continuous, from (57) and (59) it follows that, for some $m>0$ independent of $n\in\mathbb{N}$,

$$c_{\epsilon_n}\ge\langle p_{\epsilon_n}(t)+\psi_{\epsilon_n}(t),f(x_{\epsilon_n}(t),u_0)\rangle-L(x_{\epsilon_n}(t),u_0)-\frac{\epsilon_n}{\alpha}|f(x_{\epsilon_n}(t),u_0)|^\alpha\ge-m.$$

Equality (25) and the above estimate imply that, for a.e. $t\in I$,

$$\langle p_{\epsilon_n}(t)+\psi_{\epsilon_n}(t),f(x_{\epsilon_n}(t),u_{\epsilon_n}(t))\rangle-L(x_{\epsilon_n}(t),u_{\epsilon_n}(t))-\frac{\epsilon_n}{\alpha}|f(x_{\epsilon_n}(t),u_{\epsilon_n}(t))|^\alpha=c_{\epsilon_n}\ge-m. \tag{60}$$

Hence, by Lemmas 4.2, 4.4 and 4.5 for some $C>0$ and all $n\in\mathbb{N}$ large enough,

$$L(x_{\epsilon_n}(t),u_{\epsilon_n}(t))\le\big(|p_{\epsilon_n}(t)|+|\psi_\epsilon(t)|\big)|f(x_{\epsilon_n}(t),u_{\epsilon_n}(t))|+m\le C.$$

This implies that the sequence $\{L(x_{\epsilon_n}(\cdot),u_{\epsilon_n}(\cdot))\}$ is bounded in $L^\infty(I;\mathbb{R}_+)$. Then, (58) follows from the Alaoglu theorem.

Step 2. From Lemma 4.2 and Step 1 it follows that there exist $x^* \in W^{1,\infty}(I;\mathbb{R}^N)$ with $x^*(0) \in Q_0$, $\xi \in L^\infty(I;\mathbb{R}^N)$, and $\epsilon_n \downarrow 0$ such that

$$\begin{cases} x_{\epsilon_n} \to x^* & \text{uniformly} \\ x'_{\epsilon_n} \rightharpoonup x^{*\prime} & \text{weakly}-* \text{ in } L^\infty \\ L(x_{\epsilon_n}(\cdot), u_{\epsilon_n}(\cdot)) \rightharpoonup \xi & \text{weakly}-* \text{ in } L^\infty. \end{cases} \tag{61}$$

To complete the proof of the existence of an optimal trajectory/control pair for problem (1), (2), we use standard arguments based on Mazur's Theorem. Assumptions (H) i), v) imply that

$$(x^{*\prime}(t), \xi(t)) \in \Big\{\big(f(t, x^*(t), u), L(t, x^*(t), u) + v\big) : u \in U \text{ and } v \geq 0\Big\}, \qquad \text{for a.e. } t \in I.$$

Applying a measurable selection lemma, see [2], we obtain the existence of two measurable functions u^*, v^* satisfying, for a.e. $t \in I$,

$$u^*(t) \in U, \quad v^*(t) \geq 0, \qquad \text{and} \qquad \begin{cases} x^{*\prime}(t) = f(x^*(t), u^*(t)) \\ \xi(t) = L(x^*(t), u^*(t)) + v^*(t). \end{cases} \tag{62}$$

Now, let (x, u) be a trajectory/control pair of (2) with $x' \in L^\alpha(I;\mathbb{R})$. Then we get

$$\liminf_{n\to+\infty} J(x_{\epsilon_n}, u_{\epsilon_n}) \leq \liminf_{n\to+\infty} J_{\epsilon_n}(x_{\epsilon_n}, u_{\epsilon_n}) \leq \liminf_{n\to+\infty} J_{\epsilon_n}(x, u) = J(x, u).$$

From (61) and (62) we deduce that

$$\int_0^T \big[L(x^*(t), u^*(t)) + v^*(t)\big]\, dt + \ell(x^*(T)) \leq \liminf_{n\to+\infty} J(x_{\epsilon_n}, u_{\epsilon_n})$$

$$\leq \inf\Big\{J(x, u) : (x, u) \text{ is a trajectory/control pair of (2) and } x' \in L^\alpha(I;\mathbb{R}^N)\Big\}.$$

In particular, $v^*(t) = 0$, for a.e. $t \in I$. Applying Corollary 1 we obtain that (x^*, u^*) minimizes (1) over all trajectory/control pairs (x, u) of (2). Moreover, from (61) and (62),

$$x^{*\prime} \quad \text{and} \quad L(x^*(\cdot), u^*(\cdot)) \quad \text{are essentially bounded,} \tag{63}$$

implying conclusions a).

Arguing as in the proof of Lemma 4.1, we deduce that the optimal trajectory/control pair (x^*, u^*) satisfies an Autonomous Constrained Maximum Principle for some ψ, p, λ as in (4)-(7). Also, from (63), applying [10, Corollary 6.3] we obtain that the maximum principle is normal. Hence, by (6), (H) i) for a.e. $t \in I$,

$$|p'(t)| \leq C_k\big[1 + |f(x^*(t), u^*(t))| + L(x^*(t), u^*(t))^{1/\alpha}\big]|p(t) + \psi(t)| + C_k\big[1 + L(x^*(t), u^*(t))\big],$$

and b) follows from (63) and Gronwall's lemma.

□

Appendix A. **Proof of the approximation result.**

The two lemmas below are instrumental for the proof of Theorem 3.2.

Lemma A.1. *Assume (H) i), iii). Then, $\forall\, R > 0$, $\exists\, \tilde{\delta}_R, \tilde{\rho}_R, \tilde{\eta}_R > 0$ such that, for any $x \in \big(\partial K \cap B(0, R) + B(0, \tilde{\delta}_R)\big) \cap K$ and some $u_x \in U$,*

$$\langle \nabla d_K(x), f(x, u_x)\rangle < -\tilde{\rho}_R\,, \quad |f(x, u_x)| \leq \tilde{\eta}_R\,, \quad \text{and} \quad L(x, u_x) \leq \tilde{\eta}_R. \tag{64}$$

Proof. Let $0 < \delta_R < 1$ such that the signed distance d_K is $C^{1,1}$ in $\partial K \cap B(0,R) + B(0,\delta_R)$, and let $C_{R+1}, \rho_{R+1}, \eta_{R+1}$ be as in assumptions (H) i), iii). We may assume that $\eta_{R+1} \geq \kappa(2+R)$. Then, there exists

$$0 < \tilde{\delta}_R < \min\left\{\delta_R, \tfrac{\rho_{R+1}}{2C_{R+1}(1+\eta_{R+1}+\eta_{R+1}^{1/\alpha})}\right\}$$

such that, for any $x \in \big(\partial K \cap B(0,R) + B(0,\tilde{\delta}_R)\big) \cap K$, the projection of x on ∂K, $\pi(x)$, belongs to $\partial K \cap B(0,R+1)$. We associate to $\pi(x)$ the control $u_{\pi(x)}$ as in assumption (H) iii), and set $u_x := u_{\pi(x)}$. By the Lipschitz continuity of f and by the definitions of $\tilde{\delta}_R$

$$\begin{aligned}\langle \nabla d_K(x), f(x,u_x)\rangle &= \langle \nabla d_K(\pi(x)), f(x,u_{\pi(x)})\rangle \\ &\leq \langle \nabla d_K(\pi(x)), f(\pi(x),u_{\pi(x)})\rangle + C_{R+1}(1+\eta_{R+1}+\eta_{R+1}^{1/\alpha})|x-\pi(x)| < -\frac{\rho_{R+1}}{2}.\end{aligned}$$

Assumptions (H) i), iii) imply that

$$|f(x,u_x)| \leq \eta_{R+1} + \frac{\rho_{R+1}}{2} \quad \text{and} \quad L(x,u_x) \leq \eta_{R+1} + \frac{\rho_{R+1}}{2}.$$

Setting $\tilde{\rho}_R = \frac{\rho_{R+1}}{2}$ and $\tilde{\eta}_R = \eta_{R+1} + \frac{\rho_{R+1}}{2}$, (64) follows. □

Lemma A.2. *Assume (H) i), iii). Let $R > 1$ be fixed, $\tilde{\delta}_R, \tilde{\rho}_R, \tilde{\eta}_R > 0$ be as in Lemma A.1 and (x,u) be a trajectory/control pair of (2) satisfying $x([0,T]) \subset K \cap B(0,R-1)$. Then for some $\sigma > 0$ the following property holds true:*

for every $\epsilon > 0$ and $t_0 \in [0,T)$, one can find $r > 0$ such that $\forall\, x_0 \in Int(K) \cap B(x(t_0),r)$ there exists a trajectory/control pair (x_ϵ, u_ϵ) of

$$\begin{cases} x_\epsilon'(t) = f(x_\epsilon(t),u_\epsilon(t)), \ \ u_\epsilon(t) \in U & a.e. \ in \ [t_0,t_1] \\ x_\epsilon(t) \in Int(K) & for \ all \ t \in [t_0,t_1] \\ x_\epsilon(t_0) = x_0, \end{cases} \tag{65}$$

where $t_1 = (t_0+\sigma) \wedge T$, which satisfies

$$\max_{t\in[t_0,t_1]} |x_\epsilon(t) - x(t)| \leq \epsilon, \qquad \int_{t_0}^{t_1} L(x_\epsilon(t),u_\epsilon(t))\,dt \leq \int_{t_0}^{t_1} L(x(t),u(t))\,dt + \epsilon. \tag{66}$$

Proof. We shall abbreviate $\delta = \tilde{\delta}_R,\ \rho = \tilde{\rho}_R,\ \eta = \tilde{\eta}_R$. Without loss of generality, we may assume that $0 < \delta < 1,\ 0 < \rho < 1,\ C_R \geq 1$ and that d_K is $C^{1,1}$ on $\partial K \cap B(0,R) + B(0,\delta)$. Define

$$\Phi(t) = C_R\Big(1 + |f(x(t),u(t))| + L(x(t),u(t))^{1/\alpha}\Big)\,, \ \ \forall\, t \in I, \tag{67}$$

$$C_\Phi = e^{\int_0^T \Phi(t)\,dt}, \quad C_\eta = C_R(1+\eta+\eta^{1/\alpha})$$

and let l_d be a Lipschitz constant of ∇d_K on $\partial K \cap B(0,R) + B(0,\delta)$. Take $0 < \sigma < 1$ satisfying, for all $t \in I$,

$$\int_t^{(t+\sigma)\wedge T} \Phi(s)\,ds \leq \frac{\delta}{4} \wedge \frac{\rho}{8C_\Phi(\eta C_\Phi + 1)(l_d+1)}. \tag{68}$$

Fix $\epsilon > 0$ and let $0 < \tau < \frac{\sigma}{2}$ be so that for every $t \in I$

$$\begin{cases} \int_t^{(t+\tau)\wedge T} L(x(s),u(s))\,ds \leq \frac{\epsilon}{8}, \ \ \tau C_\Phi^2(\eta C_\Phi + 1) + \int_t^{(t+\tau)\wedge T} \Phi(s)ds \leq \frac{\epsilon}{4} \\ \\ \tau\eta(\eta l_d + C_\eta)C_\Phi < \rho, \ \ \tau\eta(1 + C_R(1+\eta)C_\Phi) \leq \frac{\epsilon}{2}. \end{cases} \tag{69}$$

Fix any $t_0 \in [0,T)$.

Case 1. $d_K(x(t_0)) < -\delta/2$. Then, by (68), for all $t \in [t_0, (t_0+\sigma)\wedge T]$, $d_K(x(t)) < -\delta/4$. Define $r = (\epsilon \wedge \frac{\delta}{8})/C_\Phi$ and pick any $x_0 \in Int(K) \cap B(x(t_0), r)$. Set $u_\epsilon(\cdot) = u(\cdot)$ and let x_ϵ be the trajectory of

$$\begin{cases} y' = f(y, u_\epsilon(t)) & \text{a.e. in } [t_0, t_1] \\ y(t_0) = x_0 \end{cases}$$

Then, for all $t \in [t_0, t_1]$, $|x_\epsilon(t) - x(t)| \le |x_0 - x(t_0)| + \int_{t_0}^t \Phi(s)|x_\epsilon(s) - x(s)|\, ds$. Hence, by Gronwall's lemma, $|x_\epsilon(t) - x(t)| \le C_\Phi |x_0 - x(t_0)| \le \epsilon \wedge \frac{\delta}{8}$ for all $t \in [t_0, t_1]$ implying that $d_K(x_\epsilon(t)) < -\frac{\delta}{4} + \frac{\delta}{8} < 0$. Furthermore

$$\int_{t_0}^{t_1} L(x_\epsilon(s), u_\epsilon(s))\, ds - \int_{t_0}^{t_1} L(x(s), u(s))\, ds \le \int_{t_0}^{t_1} \Phi(s)|x_\epsilon(s) - x(s)|\, ds < \epsilon \int_{t_0}^{t_1} \Phi(s)\, ds < \epsilon.$$

Case 2. $d_K(x(t_0)) \ge -\delta/2$. Set $r = \rho\tau/8 < \tau$ and pick any $x_0 \in Int(K) \cap B(x(t_0), r)$. Let $u_0 := u_{x_0}$ be given by Lemma A.1 and define

$$u_\epsilon(t) = \begin{cases} u_0 & \text{if } t \in [t_0, (t_0+\tau)\wedge T] \\ u(t-\tau) & \text{if } t \in [(t_0+\tau)\wedge T, (t_0+\sigma)\wedge T]. \end{cases}$$

Then for all $t \in [t_0, (t_0+\tau)\wedge T]$, $|x_\epsilon(t) - x_0| \le \int_{t_0}^t |f(x_0, u_0)|\, ds + \int_{t_0}^t \Phi(s)|x_\epsilon(s) - x_0|\, ds$. So $|x_\epsilon(t) - x_0| \le \eta C_\Phi (t - t_0)$ and, using (69), we obtain

$$\begin{aligned} |x_\epsilon(t) - x(t)| &\le |x_\epsilon(t) - x_0| + |x_0 - x(t_0)| + |x(t) - x(t_0)| \\ &\le \eta\, C_\Phi (t - t_0) + r + \int_{t_0}^t \Phi(s)\, ds \le \frac{\epsilon}{2}\,. \end{aligned}$$

Furthermore, for all $t \in [t_0, (t_0+\tau)\wedge T]$,

$$\begin{aligned} d_K(x_\epsilon(t)) &= d_K(x_0) + \int_{t_0}^t \langle \nabla d_K(x_\epsilon(s)), x_\epsilon'(s) \rangle\, ds \\ &\le d_K(x_0) + \int_{t_0}^t \langle \nabla d_K(x_0), f(x_0, u_0) \rangle\, ds + \big(l_d \eta + C_\eta\big) \int_{t_0}^t |x_\epsilon(s) - x_0|\, ds \\ &\le d_K(x_0) - \rho(t - t_0) + \big(l_d \eta + C_\eta\big)\eta\, C_\Phi \frac{(t - t_0)^2}{2} \le d_K(x_0) - \frac{\rho}{2}(t - t_0) < 0. \end{aligned} \tag{70}$$

Consider, next, any $t \in ((t_0+\tau)\wedge T, (t_0+\sigma)\wedge T]$. Then

$$\begin{aligned} |x_\epsilon(t) - x(t-\tau)| &\le |x_\epsilon(t_0+\tau) - x(t_0)| + \int_{t_0+\tau}^t |x_\epsilon(s) - x(s-\tau)|\Phi(s-\tau)\, ds \\ &\le \eta\, C_\Phi \tau + \tau + \int_{t_0+\tau}^t |x_\epsilon(s) - x(s-\tau)|\Phi(s-\tau)\, ds. \end{aligned}$$

So, applying Gronwall's Lemma,

$$|x_\epsilon(t) - x(t-\tau)| \le \tau(\eta\, C_\Phi + 1) \exp\Big(\int_{t_0+\tau}^t \Phi(s-\tau)\, dr \Big). \tag{71}$$

Moreover, by (69), for all $t \in ((t_0+\tau)\wedge T, (t_0+\sigma)\wedge T]$,

$$|x_\epsilon(t)-x(t)| \le |x_\epsilon(t)-x(t-\tau)| + |x(t-\tau)-x(t)| \le \tau C_\Phi(\eta\, C_\Phi+1) + \int_{t-\tau}^{t}\Phi(s)\,ds \le \frac{\epsilon}{2}.$$

We estimate next $d_K(x_\epsilon(t))$. For this aim observe that

$$\begin{aligned}\langle \nabla d_K(x_\epsilon(t)), x_\epsilon'(t)\rangle &= \langle \nabla d_K(x_\epsilon(t)), f(x_\epsilon(t), u(t-\tau))\rangle \\ &\le \langle \nabla d_K(x(t-\tau)), f(x(t-\tau), u(t-\tau))\rangle + (l_d+1)\Phi(t-\tau)|x_\epsilon(t)-x(t-\tau)|\,. \\ &= \frac{d}{dt}d_K(x(t-\tau)) + (l_d+1)\Phi(t-\tau)|x_\epsilon(t)-x(t-\tau)|.\end{aligned}$$

Therefore, by (70) and (68), for all $t \in ((t_0+\tau)\wedge T, (t_0+\sigma)\wedge T]$,

$$\begin{aligned}&d_K(x_\epsilon(t)) = d_K(x_\epsilon(t_0+\tau)) + \int_{t_0+\tau}^{t}\langle \nabla d_K(x_\epsilon(s)), x_\epsilon'(s)\rangle\,ds \\ &\le d_K(x_0) - \frac{\rho}{2}\tau + d_K(x(t-\tau)) - d_K(x(t_0)) + (l_d+1)\int_{t_0+\tau}^{t}\Phi(s-\tau)|x_\epsilon(s)-x(s-\tau)|ds \\ &\le |d_K(x_0)-d_K(x(t_0))| - \frac{\rho}{2}\tau + \tau C_\Phi(l_d+1)(\eta C_\Phi+1)\int_{t_0+\tau}^{t}\Phi(s-\tau)\,ds \le -\frac{\rho}{2}\tau + \frac{\rho}{4}\tau < 0.\end{aligned}$$

Hence (x_ϵ, u_ϵ) is as in (65) and $\|x_\epsilon - x\|_\infty < \epsilon$.

To prove the last statement it is not restrictive to assume that $\int_{t_0}^{t_1} L(x(s),u(s))\,ds$ is finite. Then, by (69) and (71),

$$\begin{aligned}&\int_{t_0}^{(t_0+\tau)\wedge T}(L(x_\epsilon(s),u_\epsilon(s)) - L(x(s),u(s)))\,ds \le \int_{t_0}^{(t_0+\tau)\wedge T} L(x_\epsilon(s),u_0)\,ds \\ &\qquad = \int_{t_0}^{(t_0+\tau)\wedge T} L(x_0,u_0)ds + \int_{t_0}^{(t_0+\tau)\wedge T}(L(x_\epsilon(s),u_0) - L(x_0,u_0))\,ds \\ &\qquad \le \eta\tau + C_R(1+L(x_0,u_0))\int_{t_0}^{(t_0+\tau)\wedge T}|x_\epsilon(s)-x_0|\,ds \le \tau\eta(1+C_R(1+\eta)C_\Phi) \le \frac{\epsilon}{2}\end{aligned}$$

and whenever $t_0+\tau < T$

$$\begin{aligned}&\int_{t_0+\tau}^{(t_0+\sigma)\wedge T}(L(x_\epsilon(s),u_\epsilon(s)) - L(x(s),u(s)))\,ds = \\ &\int_{t_0+\tau}^{(t_0+\sigma)\wedge T}(L(x_\epsilon(s),u(s-\tau)) - L(x(s-\tau),u(s-\tau)))\,ds + \\ &\int_{t_0+\tau}^{(t_0+\sigma)\wedge T}(L(x(s-\tau),u(s-\tau)) - L(x(s),u(s)))\,ds \le \int_{t_0+\tau}^{(t_0+\sigma)\wedge T}\Phi(s-\tau)|x_\epsilon(s)-x(s-\tau)|\,ds \\ &+ \int_{t_0+\tau}^{(t_0+\sigma)\wedge T-\tau} L(x(s),u(s))\,ds + \int_{t_0}^{t_0+\tau} L(x(s),u(s))\,ds - \int_{t_0+\tau}^{(t_0+\sigma)\wedge T-\tau} L(x(s),u(s))\,ds - \\ &\int_{(t_0+\sigma)\wedge T-\tau}^{(t_0+\sigma)\wedge T} L(x(s),u(s))\,ds \le \tau(\eta C_\Phi+1)C_\Phi^2 + \frac{\epsilon}{4} \le \frac{\epsilon}{2}\,.\end{aligned}$$

Therefore,

$$\int_{t_0}^{(t_0+\sigma)\wedge T} (L(x_\epsilon(s),u_\epsilon(s)) - L(x(s),u(s)))\,ds = \int_{t_0}^{t_0+\tau} (L(x_\epsilon(s),u_\epsilon(s)) - L(x(s),u(s)))\,ds +$$
$$\int_{t_0+\tau}^{(t_0+\sigma)\wedge T} (L(x_\epsilon(s),u_\epsilon(s)) - L(x(s),u(s)))\,ds \le \epsilon.$$

The proof is complete. □

Proof of Theorem 3.2. First of all, thanks to the local Lipschitz continuity of ℓ, it suffices to prove that for every $\epsilon > 0$ there is a trajectory/control pair (x_ϵ, u_ϵ) of (2) such that

$$x_\epsilon(I) \subset \mathrm{Int}\,(K)\,; \quad \|x_\epsilon - x\|_\infty \le \epsilon\,; \quad \int_0^T L(x_\epsilon(s),u_\epsilon(s))\,ds \le \int_0^T L(x(s),u(s))\,ds + \epsilon\,.$$

Now, let $\sigma > 0$ be as in Lemma A.2. If $\sigma \ge T$, then the conclusion follows from Lemma A.2 and assumption (H) iv). Indeed, let $v \in \mathrm{Int}\big(C_K(x(0))\big) \cap \mathrm{Int}\big(C_{Q_0}(x(0))\big)$. Then, for $h > 0$ sufficiently small, $x(0) + hv \in \mathrm{Int}\,(K) \cap Q_0$ (see, for instance, [2]). So, we can apply Lemma A.2 with $x_0 = x(0) + hv$ and $h > 0$ small enough to obtain the conclusion.

Next, suppose $0 < \sigma < T$ and let $n \in \mathbb{N}$ be such that $(n-1)\sigma < T \le n\sigma$. We claim that for any integer $k \in \{1, \dots, n\}$ there is a trajectory/control pair (x_ϵ, u_ϵ) of (65) with $t_0 = 0$, $t_1 = (k\sigma) \wedge T$ and $x_0 \in \mathrm{Int}\,(K)$ such that (66) holds true. Let us show our claim by an induction argument: for $k = 1$ the claim is guaranteed by Lemma A.2. Suppose now it holds true for some $k < n$. Let $\epsilon > 0$ be fixed and set $t_0 = k\sigma$, $t_1 = \big((k+1)\sigma\big) \wedge T$. By Lemma A.2 for some positive number $r < \epsilon$ and $\forall\, x_0 \in Int(K) \cap B(x(k\sigma), r)$ there exists a trajectory/control pair (x_ϵ, u_ϵ) of (65) satisfying (66). We already know, by the induction hypothesis, that there is a trajectory/control pair $(\widetilde{x}_\epsilon, \widetilde{u}_\epsilon)$ of (65) with $t_0 = 0$, $t_1 = k\sigma$ satisfying

$$\max_{t\in[0,k\sigma]} |\widetilde{x}_\epsilon(t) - x(t)| \le r\,; \quad \int_0^{k\sigma} L(\widetilde{x}_\epsilon(s), \widetilde{u}_\epsilon(s))\,ds \le \int_0^{k\sigma} L(x(s),u(s))\,ds + \frac{\epsilon}{2}\,.$$

Applying Lemma A.2 with $t_0 = k\sigma$, $t_1 = \big((k+1)\sigma\big) \wedge T$ and $x_0 = \widetilde{x}_\epsilon(k\sigma)$ we obtain (x_ϵ, u_ϵ) as in (65) satisfying

$$\max_{t\in[k\sigma,((k+1)\sigma)\wedge T]} |x_\epsilon(t) - x(t)| \le \epsilon$$
$$\int_{k\sigma}^{((k+1)\sigma)\wedge T} L(x_\epsilon(s),u_\epsilon(s))\,ds \le \int_{k\sigma}^{((k+1)\sigma)\wedge T} L(x(s),u(s))\,ds + \frac{\epsilon}{2}\,.$$

Finally, extend (x_ϵ, u_ϵ) to $\big[0, \big((k+1)\sigma\big) \wedge T\big]$ by setting $x_\epsilon(s) = \widetilde{x}_\epsilon(s)$ and $u_\epsilon(s) = \widetilde{u}_\epsilon(s)$ for all $s \in [0, k\sigma)$ to complete the proof of our claim. □

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E-mail address: `cannarsa@axp.uniroma2.it`
E-mail address: `frankowska@math.jussieu.fr`
E-mail address: `elsa.marchini@polimi.it`