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CARLEMAN ESTIMATES FOR A CLASS OF DEGENERATE PARABOLIC OPERATORS*

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Abstract. Given $\alpha \in [0,2)$ and $f \in L^2((0,T)\times(0,1))$, we derive new Carleman estimates for the degenerate parabolic problem $w_t + (x^\alpha w_x)_x = f$, where $(t,x) \in (0,T)\times(0,1)$, associated to the boundary conditions $w(t,1) = 0$ and $w(t,0) = 0$ if $0 \le \alpha < 1$ or $(x^\alpha w_x)(t,0) = 0$ if $1 \le \alpha < 2$. The proof is based on the choice of suitable weighted functions and Hardy-type inequalities. As a consequence, for all $0 \le \alpha < 2$ and $\omega \subset\subset (0,1)$, we deduce null controllability results for the degenerate one-dimensional heat equation $u_t - (x^\alpha u_x)_x = h\chi_\omega$ with the same boundary conditions as above.

Key words. degenerate parabolic equation, null controllability, Carleman estimates, Hardy-type inequality

AMS subject classifications. 35K65, 93B05, 93B07

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1. Introduction.

The study of controllability for nondegenerate parabolic equations has attracted the interest of several authors in the past few decades. After the pioneering works [14, 19, 20, 37, 38], there has been substantial progress in understanding the controllability properties of nondegenerate parabolic equations with variable coefficients. In [30], local Carleman estimates for elliptic equations were used to study the null controllability of the heat equation on a manifold. Finally, a powerful new approach, based on global estimates of Carleman type, was developed in [26].

The theory has also been extended to semilinear problems (see, for example, [2, 3, 12, 15, 21, 24, 25]) and to equations in unbounded domains (see, for example, [13, 33, 34]; see also [31, 41]). For the Stokes and Navier–Stokes equations we also refer the reader to [4, 10, 11, 18, 22, 23, 26, 27, 28].

On the contrary, few results are known for degenerate equations, even though many problems that are relevant for applications are described by parabolic equations degenerating at the boundary of the space domain. For instance, in [5, 32, 8, 9] the reader will find a motivating example of a Crocco-type equation coming from the study of the velocity field of a laminar flow on a flat plate.

The goal of this paper is to study the controllability of a simple model of degenerate parabolic equation, namely,

$$u_t - (x^\alpha u_x)_x = h\chi_\omega, \quad x \in (0,1), t \in (0,T),$$

where the control h acts on a nonempty subinterval ω of $(0,1)$.

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2. Results.

2.1. Statement of the controllability problem. Given $0 \le \alpha < 2$, define

$$\forall x \in [0,1], \qquad a(x) := x^\alpha,$$

and let ω be a nonempty subinterval of $(0,1)$. For $T > 0$, set

$$Q_T = (0,T) \times (0,1)\,,$$

and consider the initial-boundary value problem

$$\begin{cases} u_t - (au_x)_x = h\,\chi_\omega, & (t,x) \in Q_T, \\ u(t,1) = 0, & t \in (0,T), \\ \text{and } \begin{cases} u(t,0) = 0 & \text{for } 0 \le \alpha < 1, \\ (au_x)(t,0) = 0 & \text{for } 1 \le \alpha < 2, \end{cases} & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{2.1}$$

where u_0 is given in $L^2(0,1)$ and $h \in L^2(Q_T)$.

2.2. Well-posedness. Let us recall that the above problem is well-posed in appropriate weighted spaces. For $0 \le \alpha < 1$, define the Hilbert space $H^1_a(0,1)$ as

$$H^1_a(0,1) := \{u \in L^2(0,1) \ \mid \ u \text{ absolutely continuous in } [0,1], \\ \sqrt{a}u_x \in L^2(0,1) \text{ and } u(0) = u(1) = 0\},$$

and the unbounded operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ by

$$\begin{cases} \forall u \in D(A), \quad Au := (au_x)_x, \\ D(A) := \{u \in H^1_a(0,1) \ \mid \ au_x \in H^1(0,1)\}. \end{cases}$$

Notice that if $u \in D(A)$ (or even $u \in H^1_a(0,1)$), then u satisfies the Dirichlet boundary conditions $u(0) = u(1) = 0$.

For $1 \le \alpha < 2$, let us change the definition of $H^1_a(0,1)$ to

$$H^1_a(0,1) := \{u \in L^2(0,1) \ \mid \ u \text{ locally absolutely continuous in } (0,1], \\ \sqrt{a}u_x \in L^2(0,1) \text{ and } u(1) = 0\}\,.$$

Then the operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ will be defined by

$$\begin{cases} \forall u \in D(A), \quad Au := (au_x)_x, \\ D(A) := \{u \in H^1_a(0,1) \ \mid \ au_x \in H^1(0,1)\} \\ \qquad = \{u \in L^2(0,1) \ \mid \ u \text{ locally absolutely continuous in } (0,1], \\ \qquad\qquad au \in H^1_0(0,1),\ au_x \in H^1(0,1), \text{ and } (au_x)(0) = 0\}. \end{cases}$$

Notice that if $u \in D(A)$, then u satisfies the Neumann boundary condition $(au_x)(0) = 0$ at $x = 0$ and the Dirichlet boundary condition $u(1) = 0$ at $x = 1$.

In both cases, the following results hold (see, e.g., [7] and [9]).

PROPOSITION 2.1. *$A : D(A) \subset L^2(0,1) \to L^2(0,1)$ is a closed self-adjoint negative operator with dense domain.*

Hence, A is the infinitesimal generator of a strongly continuous semigroup e^{tA} on $L^2(0,1)$. Consequently, we have the following well-posedness result.

THEOREM 2.1. *Let h be given in $L^2(Q_T)$. For all $u_0 \in L^2(0,1)$, problem* (2.1) *has a unique solution*

$$u \in \mathcal{C}^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_a(0,1)). \tag{2.2}$$

Moreover, if $u_0 \in D(A)$, then

$$u \in \mathcal{C}^0([0,T]; H^1_a(0,1)) \cap L^2(0,T; D(A)) \cap H^1(0,T; L^2(0,1)). \tag{2.3}$$

Remark 2.1. Most of the results of this paper hold and will be stated for solutions in the above class (2.2). However, in the proofs, we will assume—often without further notice—that solutions belong to the stronger class (2.3). This can be done without loss of generality, since the general result can always be recovered by a standard density argument.

2.3. Carleman estimates for degenerate problems. In order to study the controllability properties of (2.1), we need to derive a Carleman estimate for the adjoint problem. Keeping the notation

$$a(x) := x^\alpha, \quad \text{with } 0 \le \alpha < 2\,, \quad \text{and } Q_T = (0,T) \times (0,1) \quad \text{for } T > 0,$$

let us consider the parabolic problem

$$\begin{cases} w_t + (a w_x)_x = f, & (t,x) \in Q_T, \\ w(t,1) = 0, & t \in (0,T), \\ \text{and } \begin{cases} w(t,0) = 0 & \text{for } 0 \le \alpha < 1, \\ (a w_x)(t,0) = 0 & \text{for } 1 \le \alpha < 2, \end{cases} & t \in (0,T), \\ w(T,x) = w_T(x), & x \in (0,1), \end{cases} \tag{2.4}$$

where $w_T \in L^2(0,1)$ and $f \in L^2(Q_T)$. Our main result is the following.

THEOREM 2.2. *Let $0 \le \alpha < 2$ and $T > 0$ be given. Then there exists $\sigma : (0,T) \times [0,1] \to \mathbb{R}^*_+$ of the form $\sigma(t,x) = \theta(t)p(x)$, with*

$$p(x) > 0 \quad \forall x \in [0,1] \qquad \textit{and} \qquad \theta(t) \to \infty \ \textit{as } t \to 0^+, T^-,$$

and two positive constants, C and R_0, such that, for all $w_T \in L^2(0,1)$ and $f \in L^2(Q_T)$, the solution w of (2.4) *satisfies, for all $R \ge R_0$,*

$$\iint_{Q_T} \Big(R\theta x^\alpha w_x^2 + R^3 \theta^3 x^{2-\alpha} w^2 \Big) e^{-2R\sigma}\, dxdt \\ \le C \iint_{Q_T} e^{-2R\sigma} f^2 \, dxdt + C \int_0^T \Big\{ R\theta e^{-2R\sigma} w_x^2 \Big\}_{|x=1}.$$

Remark 2.2. The functions p and θ will be explicitly constructed in the proof. As we shall see, the choice of θ will be

$$\forall t \in (0,T), \qquad \theta(t) = \left(\frac{1}{t(T-t)} \right)^4 .$$

This weight function satisfies the following essential properties:

$$\theta(t) \to +\infty \text{ as } t \to 0^+ \text{ or } T^- \quad \text{ and } \quad |\theta_t| \le c\theta^{5/4}, \quad |\theta_{tt}| \le c\theta^{3/2}$$

for some constant $c > 0$ depending on T. Moreover, we will take

$$p(x) := \frac{2 - x^{2-\alpha}}{(2-\alpha)^2} \qquad \forall x \in [0,1].$$

2.4. Observability inequalities. As it is well known, very useful tools for studying controllability are provided by observability inequalities for the adjoint problem

$$\begin{cases} v_t + (av_x)_x = 0, & (t,x) \in Q_T, \\ v(t,1) = 0, & t \in (0,T), \\ \text{and } \begin{cases} v(t,0) = 0 & \text{for } 0 \le \alpha < 1, \\ (av_x)(t,0) = 0 & \text{for } 1 \le \alpha < 2, \end{cases} & t \in (0,T), \\ v(T,x) = v_T(x), & x \in (0,1), \end{cases} \tag{2.5}$$

where v_T is given in $L^2(0,1)$. From the Carleman estimate of Theorem 2.2, we obtain the following observability inequalities for (2.5).

THEOREM 2.3. *Let $0 \le \alpha < 2$ and $T > 0$ be given, and let ω be a nonempty subinterval of $(0,1)$. Then there exists $C > 0$ such that, for all $v_T \in L^2(0,1)$, the solution v of (2.5) satisfies*

$$\int_0^1 x^\alpha v_x(0,x)^2 \, dx \le C \int_0^T \int_\omega v(t,x)^2 \, dxdt. \tag{2.6}$$

2.5. Application to controllability. For any $0 \le \alpha < 2$, the following observability inequality follows from Theorem 2.3 and Hardy's inequalities (see the proof in section 5):

$$\int_0^1 v(0,x)^2 \, dx \le C \int_0^T \int_\omega v(t,x)^2 \, dxdt. \tag{2.7}$$

The above inequality is well known in the nondegenerate case ($\alpha = 0$) since it follows, for instance, from classical Carleman estimates for nondegenerate parabolic equations.

For $\alpha \in [0,1/2) \cup [5/4,2)$, inequality (2.7) was proved in [9] by means of a different Carleman estimate that had been obtained using a different weight function p but gave no information for $\alpha \in [1/2, 5/4)$.

Therefore, inequality (2.7) above fills the gap between $1/2$ and $5/4$ which was left open in [9]. Thus, we obtain, by standard arguments (see, e.g., [14, 26]), a null controllability result for degenerate heat equations with initial data in $L^2(0,1)$.

THEOREM 2.4. *Let $0 \le \alpha < 2$ and $T > 0$ be given, and let ω be a nonempty subinterval of $(0,1)$. Then, for all $u_0 \in L^2(0,1)$, there exists $h \in L^2((0,T) \times \omega)$ such that the solution of the degenerate problem (2.1) satisfies $u(T) \equiv 0$ in $(0,1)$.*

Remark 2.3. Let us recall that the above result is optimal since, for $\alpha \ge 2$, problem (2.1) fails to be null controllable (see [8]). Indeed, a standard change of variable transforms problem (2.1) into the heat equation in the unbounded domain $]0, +\infty[$, whereas control supports are still bounded. Then a result by Escauriaza, Seregin, and Šverák [16, 17], which generalizes a result by Micu and Zuazua [33], ensures that null controllability fails for such an equation.

Remark 2.4. In [9], inequality (2.7) was applied to a Crocco-type equation to obtain a null controllability result for $\alpha \in [0,1/2) \cup [5/4,2)$. Thus, the results of the present paper also show the null controllability of this equation for all values of $\alpha \in [0,2)$.

2.6. Hardy-type inequalities. A major ingredient for the proofs of Theorems 2.2 and 2.3 is the following well-known lemma (see, for example, [35]; for the reader's convenience, we recall the proof in section 6).

LEMMA 2.1 (Hardy-type inequalities).

(i) *Let* $0 \leq \alpha^\star < 1$. *Then, for all locally absolutely continuous functions* z *on* $(0,1)$ *satisfying*

$$\underset{x \to 0^+}{z(x) \to 0} \quad \textit{and} \quad \int_0^1 x^{\alpha^\star} z_x^2 < \infty,$$

the following inequality holds:

$$\int_0^1 x^{\alpha^\star - 2} z^2 \leq \frac{4}{(1-\alpha^\star)^2} \int_0^1 x^{\alpha^\star} z_x^2. \tag{2.8}$$

(ii) *Let* $1 < \alpha^\star < 2$. *Then the above inequality* (2.8) *still holds for all locally absolutely continuous functions* z *on* $(0,1)$ *satisfying*

$$\underset{x \to 1^-}{z(x) \to 0} \quad \textit{and} \quad \int_0^1 x^{\alpha^\star} z_x^2 < +\infty.$$

Remark 2.5. Notice that (2.8) is false for $\alpha^\star = 1$.

2.7. Further remarks. In the present paper, we study the case of a degenerate operator of the form $-(x^\alpha u_x)_x$ with the boundary condition $u(x=0)=0$ when $0 \leq \alpha < 1$ or $(x^\alpha u_x)(x=0) = 0$ when $1 \leq \alpha < 2$. The choice of such an operator in divergence form probably simplifies parts of the computations arising in the proof of Carleman estimates. Of course, it would be interesting to study, in a next step, other operators like $-x^\alpha u_{xx}$. On the other hand, the choice of the boundary condition at $x=0$ ensures a relatively simple framework for the statement of well-posedness. Here again, it would be interesting to study the cases of other boundary conditions. For example, an interesting problem would be the case of Wentzell boundary conditions; see, e.g., [6, 39]. The techniques developed here may be useful to treat such problems. However, both the form of the operator and the boundary conditions play an important role in the computations of the proof of Carleman estimates. For this reason, these other problems have yet to be studied.

On the other hand, let us mention that the ideas of the present paper allow us to prove similar null controllability results for degenerate semilinear problems using a classical fixed point method (see [1]).

Next, instead of a distributed control on $\omega \subset (0,1)$, one could consider a boundary control acting at one extreme point of the domain $(0,1)$. Theorem 2.2 readily implies a boundary null controllability result if the control acts at $x=1$. The case of a boundary control at $x=0$ has not yet been studied.

Finally, another interesting question would be the study of degenerate operators in higher dimensions. Of course, this opens a lot of perspectives since the study will depend on the domain where the operator degenerates and the way it degenerates. This question will be the subject of a forthcoming paper.

3. Proof of Theorem 2.2 (Carleman estimates).

3.1. Notation and reformulation of the problem. We recall that $a(x) = x^\alpha$ for all $x \in [0,1]$ with $\alpha \in [0,2)$ given. Let $\sigma(t,x) = \theta(t)p(x)$, where

$$p(x) > 0 \ \ \forall x \in [0,1] \quad \text{and} \quad \theta(t) \to \infty \text{ as } t \to 0^+, T^-.$$

For $R > 0$, define

$$z(t,x) = e^{-R\sigma(t,x)} w(t,x), \tag{3.1}$$

where w is a solution of (2.4). Notice that,

$$\forall n \in \mathbb{N}, \ \ \theta^n z = 0 \ \text{ and } z_x = 0 \quad \text{at time } t = 0 \text{ and } t = T. \tag{3.2}$$

Moreover z satisfies

$$\begin{cases} (e^{R\sigma} z)_t + (a(e^{R\sigma} z)_x)_x = f, & (t,x) \in Q_T, \\ z(t,1) = 0, & t \in (0,T), \\ \text{and} \ \begin{cases} z(t,0) = 0 & \text{for } 0 \le \alpha < 1, \\ (az_x)(t,0) = -R(a\sigma_x z)(t,0) & \text{for } 1 \le \alpha < 2, \end{cases} & t \in (0,T). \end{cases} \tag{3.3}$$

This equation may be recast as follows:

$$P_R z = P_R^+ z + P_R^- z = f e^{-R\sigma},$$

where

$$\begin{aligned} P_R^+ z :=&\ R\sigma_t z + R^2 a \sigma_x^2 z + (az_x)_x, \\ P_R^- z :=&\ z_t + R(a\sigma_x z)_x + Ra\sigma_x z_x \\ =&\ z_t + R(a\sigma_x)_x z + 2Ra\sigma_x z_x. \end{aligned}$$

Moreover, we have

$$\|fe^{-R\sigma}\|^2 \ \ \ge \ \ \|P_R^+ z\|^2 + \|P_R^- z\|^2 + 2\langle P_R^+ z, P_R^- z\rangle \ \ \ge \ \ 2\langle P_R^+ z, P_R^- z\rangle, \tag{3.4}$$

where $\|\cdot\|$ and $\langle \cdot,\cdot\rangle$ denote the usual norm and scalar product in $L^2(Q_T)$.

3.2. Computation of the scalar product. We now want to compute the scalar product in $L^2(Q_T)$ of $P_R^+ z$ and $P_R^- z$. This will be done in two steps.

LEMMA 3.1. *The following identity holds:*

$$\left.\begin{aligned} \langle P_R^+ z, P_R^- z\rangle = \int_0^T \Big[& a z_x z_t + R^2 a \sigma_t \sigma_x z^2 + R^3 a^2 \sigma_x^3 z^2 \\ & + Ra(a\sigma_x)_x z z_x + Ra^2 \sigma_x z_x^2 \Big]_0^1 \end{aligned}\right\} \quad (b.t.)$$

$$\left.\begin{aligned} & + \iint_{Q_T} \Big(-\frac{1}{2} R\sigma_{tt} - 2R^2 a\sigma_x \sigma_{xt} - R^3 a\sigma_x (a\sigma_x^2)_x \Big) z^2 \\ & + \iint_{Q_T} -R\frac{a}{\sigma_x}(a\sigma_x^2)_x z_x^2 - Ra(a\sigma_x)_{xx} z z_x. \end{aligned}\right\} \quad (d.t.)$$

Then, using the fact that $a(x) = x^\alpha$ and $\sigma(t,x) = \theta(t)p(x)$, we compute the distributed and boundary terms as follows.

LEMMA 3.2. *For all $0 \le \alpha < 2$, we have*

$$
\begin{aligned}
(d.t.) = &-\frac{1}{2}R\iint_{Q_T}\theta_{tt}pz^2 - 2R^2\iint_{Q_T}\theta\theta_t x^\alpha p_x^2 z^2 \\
&- R^3\iint_{Q_T}\theta^3 x^{2\alpha-1}(2xp_{xx}+\alpha p_x)p_x^2 z^2 \\
&- R\iint_{Q_T}\theta x^{2\alpha-1}(2xp_{xx}+\alpha p_x)z_x^2 - R\iint_{Q_T}\theta x^\alpha (x^\alpha p_x)_{xx} z z_x.
\end{aligned}
$$

Moreover, for $0 \le \alpha < 1$, the boundary terms $(b.t.)$ are given by

$$
(b.t.)_{\ for\ 0 \le \alpha < 1} = \int_0^T \Big\{R\theta a^2 p_x z_x^2\Big\}_{|x=1} - \int_0^T \Big\{R\theta a^2 p_x z_x^2\Big\}_{|x=0}.
$$

For $1 \le \alpha < 2$, the boundary terms $(b.t.)$ become

$$
\begin{aligned}
(b.t.)_{\ for\ 1 \le \alpha < 2} = &\int_0^T \Big\{R\theta a^2 p_x z_x^2\Big\}_{|x=1} \\
+ \int_0^T \Big\{ -&\frac{R}{2}\theta_t a p_x z^2 - R^2\theta_t\theta a p p_x z^2 - 2R^3\theta^3 a^2 p_x^3 z^2 + R^2\theta^2 a p_x (a p_x)_x z^2\Big\}_{|x=0}.
\end{aligned}
$$

Proof of Lemma 3.1. We have

$$
\langle P_R^+ z, P_R^- z\rangle = Q_1 + Q_2 + Q_3 + Q_4,
$$

where

$$
\begin{aligned}
Q_1 &:= \langle R\sigma_t z + R^2 a\sigma_x^2 z + (az_x)_x, z_t\rangle, \\
Q_2 &:= R^2\langle \sigma_t z, (a\sigma_x)_x z + 2a\sigma_x z_x\rangle, \\
Q_3 &:= R^3\langle a\sigma_x^2 z, (a\sigma_x)_x z + 2a\sigma_x z_x\rangle, \\
Q_4 &:= R\langle (az_x)_x, (a\sigma_x)_x z + 2a\sigma_x z_x\rangle.
\end{aligned}
$$

First term: Q_1.

$$
\begin{aligned}
Q_1 &= \iint_{Q_T}\Big(R\sigma_t z + R^2 a\sigma_x^2 z + (az_x)_x\Big)z_t \\
&= \iint_{Q_T}(R\sigma_t + R^2 a\sigma_x^2)\Big(\frac{z^2}{2}\Big)_t + \iint_{Q_T}(az_x)_x z_t \\
&= \Big[\int_0^1 \frac{1}{2}\Big(R\sigma_t + R^2 a\sigma_x^2\Big)z^2\Big]_0^T - \iint_{Q_T}\frac{1}{2}\Big(R\sigma_t + R^2 a\sigma_x^2\Big)_t z^2 \\
&\qquad + \int_0^T \Big[az_x z_t\Big]_0^1 - \iint_{Q_T} a z_x z_{xt} \\
&= \Big[\int_0^1 \Big(R\sigma_t + R^2 a\sigma_x^2\Big)\frac{1}{2}z^2 - \frac{1}{2}az_x^2\Big]_0^T \\
&\qquad - \iint_{Q_T}\frac{1}{2}\Big(R\sigma_t + R^2 a\sigma_x^2\Big)_t z^2 + \int_0^T \Big[az_x z_t\Big]_0^1.
\end{aligned}
$$

By (3.2), the terms integrated in time are equal to zero. Hence,

$$Q_1 = \int_0^T \Big[a z_x z_t \Big]_0^1 + \iint_{Q_T} \Big(-\frac{1}{2} R\sigma_{tt} - R^2 a\sigma_x \sigma_{xt} \Big) z^2. \tag{3.5}$$

Second term: Q_2.

$$\begin{aligned} Q_2 &= R^2 \iint_{Q_T} \sigma_t z \Big((a\sigma_x)_x z + 2a\sigma_x z_x \Big) = R^2 \iint_{Q_T} \sigma_t (a\sigma_x)_x z^2 + a\sigma_t\sigma_x (z^2)_x \\ &= R^2 \iint_{Q_T} \sigma_t (a\sigma_x)_x z^2 + R^2 \int_0^T \Big[a\sigma_t\sigma_x z^2 \Big]_0^1 - R^2 \iint_{Q_T} (a\sigma_t\sigma_x)_x z^2. \end{aligned}$$

Therefore,

$$Q_2 = R^2 \int_0^T \Big[a\sigma_t\sigma_x z^2 \Big]_0^1 - R^2 \iint_{Q_T} a\sigma_x \sigma_{xt} z^2. \tag{3.6}$$

Third term: Q_3.

$$\begin{aligned} Q_3 &= R^3 \iint_{Q_T} a\sigma_x^2 z \Big((a\sigma_x)_x z + 2a\sigma_x z_x \Big) = R^3 \iint_{Q_T} a\sigma_x^2 z \Big((a\sigma_x z)_x + a\sigma_x z_x \Big) \\ &= R^3 \int_0^T \Big[a^2\sigma_x^3 z^2 \Big]_0^1 - R^3 \iint_{Q_T} (a\sigma_x^2 z)_x a\sigma_x z + R^3 \iint_{Q_T} a^2\sigma_x^3 z z_x. \end{aligned}$$

Thus,

$$Q_3 = R^3 \int_0^T \Big[a^2\sigma_x^3 z^2 \Big]_0^1 - R^3 \iint_{Q_T} a\sigma_x (a\sigma_x^2)_x z^2. \tag{3.7}$$

Last term: Q_4.

$$\begin{aligned} Q_4 &= R \iint_{Q_T} (az_x)_x \Big((a\sigma_x)_x z + 2a\sigma_x z_x \Big) \\ &= R \int_0^T \Big[a z_x (a\sigma_x)_x z \Big]_0^1 - R \iint_{Q_T} a z_x \Big((a\sigma_x)_x z \Big)_x + R \iint_{Q_T} \sigma_x \Big((az_x)^2 \Big)_x \\ &= R \int_0^T \Big[a (a\sigma_x)_x z z_x \Big]_0^1 - R \iint_{Q_T} a (a\sigma_x)_x z_x^2 + a (a\sigma_x)_{xx} z z_x \\ &\qquad + R \int_0^T \Big[\sigma_x a^2 z_x^2 \Big]_0^1 - R \iint_{Q_T} \sigma_{xx} a^2 z_x^2. \end{aligned}$$

Consequently,

$$\begin{aligned} Q_4 = R \int_0^T & \Big[a (a\sigma_x)_x z z_x + a^2 \sigma_x z_x^2 \Big]_0^1 \\ & - R \iint_{Q_T} \frac{a}{\sigma_x} (a\sigma_x^2)_x z_x^2 - R \iint_{Q_T} a (a\sigma_x)_{xx} z z_x. \end{aligned} \tag{3.8}$$

Finally, Lemma 3.1 follows from (3.5)–(3.8). $\square$

Proof of Lemma 3.2. With $a(x) = x^\alpha$ and $\sigma(t,x) = \theta(t)p(x)$, the distributed terms (d.t.) can be computed as follows:

$$
\begin{aligned}
(d.t.) = &-\frac{1}{2}R\iint_{Q_T}\theta_{tt}pz^2 - 2R^2\iint_{Q_T}\theta\theta_t x^\alpha p_x^2 z^2 - R^3\iint_{Q_T}\theta^3 x^\alpha p_x (x^\alpha p_x^2)_x z^2\\
&- R\iint_{Q_T}\theta\frac{x^\alpha}{p_x}(x^\alpha p_x^2)_x z_x^2 - R\iint_{Q_T}\theta x^\alpha (x^\alpha p_x)_{xx} z z_x\\
= &-\frac{1}{2}R\iint_{Q_T}\theta_{tt}pz^2 - 2R^2\iint_{Q_T}\theta\theta_t x^\alpha p_x^2 z^2\\
&- R^3\iint_{Q_T}\theta^3 x^{2\alpha-1}(2xp_{xx}+\alpha p_x)p_x^2 z^2\\
&- R\iint_{Q_T}\theta x^{2\alpha-1}(2xp_{xx}+\alpha p_x)z_x^2 - R\iint_{Q_T}\theta x^\alpha (x^\alpha p_x)_{xx} z z_x.
\end{aligned}
$$

On the other hand, also taking into account the fact that $z(t,1) = 0$, the boundary terms (b.t.) become

$$
\begin{aligned}
(b.t.) = \int_0^T \left\{R\theta a^2 p_x z_x^2\right\}_{|x=1} - \int_0^T \Big\{&a z_x z_t + R^2\theta_t\theta a p p_x z^2 + R^3\theta^3 a^2 p_x^3 z^2\\
&+ R\theta a (a p_x)_x z z_x + R\theta a^2 p_x z_x^2\Big\}_{|x=0}.
\end{aligned}
$$

Now, for $0 \le \alpha < 1$, use the fact that $z(t,0) = 0$ to obtain

$$
(b.t.)_{\text{ for } 0 \le \alpha < 1} = \int_0^T \left\{R\theta a^2 p_x z_x^2\right\}_{|x=1} - \int_0^T \left\{R\theta a^2 p_x z_x^2\right\}_{|x=0}.
$$

Similarly, for $1 \le \alpha < 2$, recall that $(az_x)(t,0) = -R\theta(t)(ap_x z)(t,0)$ to conclude that

$$
\begin{aligned}
(b.t.)_{\text{ for } 1 \le \alpha < 2} = &\int_0^T \left\{R\theta a^2 p_x z_x^2\right\}_{|x=1} + \int_0^T \Big\{R\theta a p_x \left(\frac{z^2}{2}\right)_t\\
&- R^2\theta_t\theta a p p_x z^2 - R^3\theta^3 a^2 p_x^3 z^2 + R^2\theta^2 a p_x (a p_x)_x z^2 - R^3\theta^3 a^2 p_x^3 z^2\Big\}_{|x=0}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(b.t.)_{\text{ for } 1 \le \alpha < 2} = &\int_0^T \left\{R\theta a^2 p_x z_x^2\right\}_{|x=1} + \int_0^T \Big\{-\frac{R}{2}\theta_t a p_x z^2\\
&- R^2\theta_t\theta a p p_x z^2 - 2R^3\theta^3 a^2 p_x^3 z^2 + R^2\theta^2 a p_x (a p_x)_x z^2\Big\}_{|x=0}.
\end{aligned}
$$
□

3.3. Bounds from below. Let us first define

$$
\forall t \in (0,T), \qquad \theta(t) := \left(\frac{1}{t(T-t)}\right)^4.
$$

Observe that θ satisfies the following properties:

$$
|\theta_t| \le c\theta^{5/4} \le c\theta^2 \quad \text{and} \quad |\theta_{tt}| \le c\theta^{3/2} \le c\theta^3.
$$

Next, let us recall that $\alpha \in [0,2)$ and let us choose

$$\forall x \in [0,1], \qquad p(x) := \frac{2 - x^{2-\alpha}}{(2-\alpha)^2}.$$

Then

$$p_x(x) = \frac{-x^{1-\alpha}}{2-\alpha}, \qquad p_{xx}(x) = \frac{-(1-\alpha)}{2-\alpha} x^{-\alpha}.$$

Hence

$$2xp_{xx} + \alpha p_x = -x^{1-\alpha}$$

and

$$(x^\alpha p_x)_x = \frac{-1}{2-\alpha}; \quad \text{thus} \quad (x^\alpha p_x)_{xx} = 0.$$

With this choice of θ and p, the distributed and boundary terms can be first computed and then estimated as follows.

LEMMA 3.3. *For all $\alpha \in [0,2)$, the distributed terms (d.t.) become*

$$\begin{aligned}(d.t.) = &-\frac{R}{(2-\alpha)^2}\iint_{Q_T} \theta_{tt} z^2 + \frac{R}{2(2-\alpha)^2}\iint_{Q_T} \theta_{tt} x^{2-\alpha} z^2 \\ &- \frac{2R^2}{(2-\alpha)^2}\iint_{Q_T} \theta\theta_t x^{2-\alpha} z^2 + \frac{R^3}{(2-\alpha)^2}\iint_{Q_T} \theta^3 x^{2-\alpha} z^2 + R\iint_{Q_T} \theta x^\alpha z_x^2.\end{aligned}$$

For $0 \le \alpha < 1$, the boundary terms (b.t.) become

$$(b.t.)_{\ for\ 0 \le \alpha < 1} = -\frac{1}{2-\alpha}\int_0^T \Big\{R\theta z_x^2\Big\}_{|x=1} + \frac{1}{2-\alpha}\int_0^T \Big\{R\theta x^{1+\alpha} z_x^2\Big\}_{|x=0}.$$

For $1 \le \alpha < 2$, the boundary terms (b.t.) become

$$\begin{aligned}(b.t.)_{\ for\ 1 \le \alpha < 2} = &-\frac{1}{2-\alpha}\int_0^T \Big\{R\theta z_x^2\Big\}_{|x=1} + \int_0^T \Big\{\frac{R\theta_t}{2(2-\alpha)} x z^2 \\ &+ \frac{2R^2\theta_t\theta}{(2-\alpha)^3} x z^2 - \frac{R^2\theta_t\theta}{(2-\alpha)^3} x^{3-\alpha} z^2 + \frac{2R^3\theta^3}{(2-\alpha)^3} x^{3-\alpha} z^2 + \frac{R^2\theta^2}{(2-\alpha)^2} x z^2\Big\}_{|x=0}.\end{aligned}$$

LEMMA 3.4. *For all $\alpha \in [0,2)$, the distributed terms (d.t.) and the boundary terms (b.t.) satisfy, for R large enough (depending on α and T),*

$$(d.t.) \ge \frac{1}{4}\frac{R^3}{(2-\alpha)^2}\iint_{Q_T} \theta^3 x^{2-\alpha} z^2 + \frac{3}{4} R \iint_{Q_T} \theta x^\alpha z_x^2,$$

$$(b.t.) \ge -\frac{1}{2-\alpha}\int_0^T \Big\{R\theta z_x^2\Big\}_{|x=1}.$$

Proof of Lemma 3.3. The conclusion follows from the above choice of p and the expressions of (d.t.) and (b.t.) given in Lemma 3.2. □

Proof of Lemma 3.4. Let us first analyze the distributed terms. Recall that, owing to Lemma 3.3,

$$\begin{aligned}(d.t.) \;=\; & -\frac{R}{(2-\alpha)^2}\iint_{Q_T}\theta_{tt}z^2 + \frac{R}{2(2-\alpha)^2}\iint_{Q_T}\theta_{tt}x^{2-\alpha}z^2 \\ & -\frac{2R^2}{(2-\alpha)^2}\iint_{Q_T}\theta\theta_t x^{2-\alpha}z^2 + \frac{R^3}{(2-\alpha)^2}\iint_{Q_T}\theta^3x^{2-\alpha}z^2 + R\iint_{Q_T}\theta x^\alpha z_x^2.\end{aligned}$$

Since the two last terms are nonnegative, we only need to estimate the three other terms. We begin with the second term: since $|\theta_{tt}| \le c\theta^{3/2} \le c\theta^3$, we have

$$\begin{aligned}\left|\frac{R}{2(2-\alpha)^2}\iint_{Q_T}\theta_{tt}x^{2-\alpha}z^2\right| &\le \frac{cR}{2(2-\alpha)^2}\iint_{Q_T}\theta^3x^{2-\alpha}z^2 \\ &\le \frac{1}{4}\frac{R^3}{(2-\alpha)^2}\iint_{Q_T}\theta^3x^{2-\alpha}z^2\end{aligned}$$

for R large enough. Next, using $|\theta\theta_t| \le c\theta^{9/4} \le c\theta^3$, we also obtain a bound of the third term for R large enough:

$$\begin{aligned}\left|\frac{2R^2}{(2-\alpha)^2}\iint_{Q_T}\theta\theta_t x^{2-\alpha}z^2\right| &\le \frac{2cR^2}{(2-\alpha)^2}\iint_{Q_T}\theta^3x^{2-\alpha}z^2 \\ &\le \frac{1}{4}\frac{R^3}{(2-\alpha)^2}\iint_{Q_T}\theta^3x^{2-\alpha}z^2.\end{aligned}$$

Therefore,

$$(d.t.) \ge -\frac{R}{(2-\alpha)^2}\iint_{Q_T}\theta_{tt}z^2 + \frac{1}{2}\frac{R^3}{(2-\alpha)^2}\iint_{Q_T}\theta^3x^{2-\alpha}z^2 + R\iint_{Q_T}\theta x^\alpha z_x^2. \tag{3.9}$$

It remains to bound the first term on the right-hand side above. First let us observe that the solution w of (2.4) belongs to $L^2(0,T;H^1_a(0,1))$ by Theorem 2.1. Since $z = e^{-R\sigma}w$, some direct computations imply that z also belongs to $L^2(0,T;H^1_a(0,1))$. Next we write

$$\begin{aligned}\left|\frac{R}{(2-\alpha)^2}\iint_{Q_T}\theta_{tt}z^2\right| &\le \frac{cR}{(2-\alpha)^2}\iint_{Q_T}\theta^{3/2}z^2 \\ &= \frac{cR}{(2-\alpha)^2}\iint_{Q_T}\left(\theta x^{(\alpha-2)/3}z^2\right)^{3/4}\left(\theta^3x^{2-\alpha}z^2\right)^{1/4} \\ &\le \frac{3\varepsilon cR}{4(2-\alpha)^2}\iint_{Q_T}\theta x^{(\alpha-2)/3}z^2 + \frac{cR}{4\varepsilon^3(2-\alpha)^2}\iint_{Q_T}\theta^3x^{2-\alpha}z^2.\end{aligned} \tag{3.10}$$

As this point, we separate the case $\alpha = 1$ from the other ones. This case is peculiar since Hardy's inequality (Lemma 2.1) does not hold for $\alpha^\star = 1$.

In the case $\alpha \neq 1$, we observe that $x^{(\alpha-2)/3} \le x^{\alpha-2}$ (since $\alpha < 2$), and we apply Lemma 2.1 with $\alpha^\star = \alpha \neq 1$ (z satisfies the assumptions of Lemma 2.1 for almost every t since it belongs to $L^2(0,T;H^1_a(0,1))$) to obtain

$$\iint_{Q_T}\theta x^{(\alpha-2)/3}z^2 \le \iint_{Q_T}\theta x^{\alpha-2}z^2 \le \frac{4}{(\alpha-1)^2}\iint_{Q_T}\theta x^\alpha z_x^2. \tag{3.11}$$

In the case $\alpha = 1$, we apply Lemma 2.1 with $\alpha^\star = 5/3$ and then use the fact that $x^{5/3} \leq x$ to arrive at a similar conclusion:

$$\iint_{Q_T} \theta x^{(\alpha-2)/3} z^2 = \iint_{Q_T} \theta x^{-1/3} z^2 \leq \frac{4}{(\alpha^\star - 1)^2} \iint_{Q_T} \theta x^{5/3} z_x^2 \leq 9 \iint_{Q_T} \theta x z_x^2 = 9 \iint_{Q_T} \theta x^\alpha z_x^2. \tag{3.12}$$

In both cases, combining (3.10) with (3.11) or (3.12), we deduce

$$\left| \frac{R}{(2-\alpha)^2} \iint_{Q_T} \theta_{tt} z^2 \right| \leq \varepsilon c' R \iint_{Q_T} \theta x^\alpha z_x^2 + \frac{cR}{4\varepsilon^3 (2-\alpha)^2} \iint_{Q_T} \theta^3 x^{2-\alpha} z^2$$

for some constant $c' > 0$. Then, for ε small enough and R large enough, we have

$$\left| \frac{R}{(2-\alpha)^2} \iint_{Q_T} \theta_{tt} z^2 \right| \leq \frac{1}{4} R \iint_{Q_T} \theta x^\alpha z_x^2 + \frac{1}{4} \frac{R^3}{(2-\alpha)^2} \iint_{Q_T} \theta^3 x^{2-\alpha} z^2. \tag{3.13}$$

Summing up, we obtain by (3.9) and (3.13)

$$(d.t.) \geq \frac{1}{4} \frac{R^3}{(2-\alpha)^2} \iint_{Q_T} \theta^3 x^{2-\alpha} z^2 + \frac{3}{4} R \iint_{Q_T} \theta x^\alpha z_x^2 \geq 0.$$

We now turn to the boundary terms. In the case $0 \leq \alpha < 1$, there is nothing else to do since, by Lemma 3.3,

$$\begin{aligned} (b.t.)_{\text{ for } 0 \leq \alpha < 1} &= -\frac{1}{2-\alpha} \int_0^T \left\{ R\theta z_x^2 \right\}_{|x=1} + \frac{1}{2-\alpha} \int_0^T \left\{ R\theta x^{1+\alpha} z_x^2 \right\}_{|x=0} \\ &\geq -\frac{1}{2-\alpha} \int_0^T \left\{ R\theta z_x^2 \right\}_{|x=1}. \end{aligned}$$

In the case $1 \leq \alpha < 2$, we recall that, by Lemma 3.3,

$$\begin{aligned} (b.t.)_{\text{ for } 1 \leq \alpha < 2} = &-\frac{1}{2-\alpha} \int_0^T \left\{ R\theta z_x^2 \right\}_{|x=1} + \int_0^T \left\{ \left(\frac{R\theta_t}{2(2-\alpha)} \right.\right. \\ &\left.\left. + \frac{2R^2\theta_t\theta}{(2-\alpha)^3} - \frac{R^2\theta_t\theta}{(2-\alpha)^3} x^{2-\alpha} + \frac{2R^3\theta^3}{(2-\alpha)^3} x^{2-\alpha} + \frac{R^2\theta^2}{(2-\alpha)^2} \right) x z^2 \right\}_{|x=0}. \end{aligned}$$

Thus, applying Lemma 3.5 below (since $z \in H^1_a(0,1)$ for almost every t), it follows that, for almost every $t \in (0,T)$,

$$x z^2(t,x) \to 0 \quad \text{as } x \to 0\,.$$

Hence,

$$(b.t.)_{\text{ for } 1 \leq \alpha < 2} = -\frac{1}{2-\alpha} \int_0^T \left\{ R\theta z_x^2 \right\}_{|x=1}. \qquad \square$$

LEMMA 3.5. *Let $\alpha \in [1,2)$ be given. Then, for all $v \in H^1_a(0,1)$,*

$$x v^2(x) \to 0 \quad \textit{as } x \to 0^+. \tag{3.14}$$

Proof. Let v be given in $H^1_a(0,1)$. By the definition of $H^1_a(0,1)$ in the case $1 \leq \alpha < 2$, we know that $v \in L^2(0,1)$ and $\sqrt{a}v_x = x^{\alpha/2}v_x \in L^2(0,1)$. Then $xv^2 \in L^1(0,1)$. Moreover,

$$(xv^2)_x = v^2 + 2xvv_x,$$

with $v^2 \in L^1(0,1)$ and with $xvv_x = (x^{1-\alpha/2}v)\,(x^{\alpha/2}v_x) \in L^1(0,1)$. Hence, $xv^2 \in W^{1,1}(0,1)$. Thus, $xv^2 \to L \geq 0$ as $x \to 0^+$. Finally, $L = 0$ since $L \neq 0$ would imply $v \notin L^2(0,1)$. This completes the proof. ∎

3.4. Conclusion. From Lemmas 3.1 and 3.4 we obtain, for all $0 \leq \alpha < 2$,

$$\begin{aligned}\langle P_R^+ z, P_R^- z\rangle &= (d.t.) + (b.t.)\\ &\geq cR^3 \iint_{Q_T} \theta^3 x^{2-\alpha} z^2 + cR \iint_{Q_T} \theta x^\alpha z_x^2 - c' \int_0^T \Big\{R\theta z_x^2\Big\}_{|x=1}\end{aligned}$$

for some constants $c, c' > 0$. By (3.4), we have

$$\begin{aligned}\|fe^{-R\sigma}\|^2 &= \|P_R^+ z\|^2 + \|P_R^- z\|^2 + 2\langle P_R^+ z, P_R^- z\rangle \ \geq\ 2\langle P_R^+ z, P_R^- z\rangle\\ &\geq cR^3 \iint_{Q_T} \theta^3 x^{2-\alpha} z^2 + cR \iint_{Q_T} \theta x^\alpha z_x^2 - c' \int_0^T \Big\{R\theta z_x^2\Big\}_{|x=1}.\end{aligned}$$

We recall that $\sigma(t,x) = \theta(t)p(x)$ and $p_x(x) = -x^{1-\alpha}/(2-\alpha)$. Hence, $x^\alpha \sigma_x^2 = c\theta^2 x^\alpha x^{2-2\alpha} = c\theta^2 x^{2-\alpha}$. Moreover, $w = e^{R\sigma} z$. Thus, $w_x = R\sigma_x e^{R\sigma} z + e^{R\sigma} z_x$. Therefore,

$$\begin{aligned}R^3\theta^3 x^{2-\alpha} w^2 + R\theta x^\alpha w_x^2 &\leq R^3\theta^3 x^{2-\alpha} e^{2R\sigma} z^2 + R\theta x^\alpha \Big(2R^2\sigma_x^2 e^{2R\sigma} z^2 + 2e^{2R\sigma} z_x^2\Big)\\ &\leq c\Big(R^3\theta^3 x^{2-\alpha} e^{2R\sigma} z^2 + R\theta x^\alpha e^{2R\sigma} z_x^2\Big).\end{aligned}$$

So,

$$\iint_{Q_T} \Big(R^3\theta^3 x^{2-\alpha} w^2 + R\theta x^\alpha w_x^2\Big) e^{-2R\sigma} \ \leq\ c \iint_{Q_T} f^2 e^{-2R\sigma} + c\int_0^T \Big\{R\theta z_x^2\Big\}_{|x=1}.$$

Moreover $z_x(x=1) = (e^{-R\sigma} w_x)(x=1)$ since $z(x=1) = 0$. It follows that

$$\begin{aligned}&\iint_{Q_T} \Big(R^3\theta^3 x^{2-\alpha} w^2 + R\theta x^\alpha w_x^2\Big) e^{-2R\sigma}\\ &\qquad\leq\ c \iint_{Q_T} f^2 e^{-2R\sigma} + c\int_0^T \Big\{R\theta e^{-2R\sigma} w_x^2\Big\}_{|x=1}.\end{aligned}$$

∎

4. Proof of Theorem 2.3 (observability inequalities). Theorem 2.2 yields a Carleman estimate for the solutions of (2.5).

LEMMA 4.1. *For all $0 \leq \alpha < 2$ and all $T > 0$, there exist positive constants, $R_0, C, c > 0$, such that, for all $v_T \in L^2(0,1)$, the solution v of (2.5) satisfies, for all $R \geq R_0$,*

$$\int_0^T \int_0^1 \Big(R\theta x^\alpha v_x^2 + R^3\theta^3 x^{2-\alpha} v^2\Big) e^{-2cR\theta}\, dxdt \ \leq\ C \int_0^T \int_\omega v^2 dxdt.$$

Let us put off the proof of the above lemma and proceed with the reasoning. Multiplying the equation in (2.5) by v_t and integrating by parts, we get

$$\begin{aligned} 0 &= \int_0^1 \Big(v_t + (x^\alpha v_x)_x\Big) v_t \, dx \\ &= \int_0^1 v_t^2 \, dx + \Big[x^\alpha v_x v_t\Big]_0^1 - \int_0^1 x^\alpha v_x v_{tx} \, dx \ \geq \ -\frac{1}{2}\frac{d}{dt}\int_0^1 x^\alpha v_x^2 \, dx. \end{aligned}$$

Therefore $t \mapsto \int_0^1 x^\alpha v_x^2 \, dx$ is increasing and

$$\int_0^1 x^\alpha v_x(0,x)^2 \, dx \leq \int_0^1 x^\alpha v_x(t,x)^2 \, dx \qquad \forall t \in [0,T]\,.$$

Integrating over $[T/4, 3T/4]$, we have

$$\begin{aligned} \int_0^1 x^\alpha v_x(0,x)^2 \, dx &\leq \frac{2}{T}\int_{T/4}^{3T/4}\int_0^1 x^\alpha v_x(t,x)^2 \, dxdt \\ &\leq C \int_{T/4}^{3T/4}\int_0^1 \theta x^\alpha v_x(t,x)^2 e^{-2cR\theta} \, dxdt. \end{aligned}$$

Hence, owing to Lemma 4.1,

$$\int_0^1 x^\alpha v_x(0,x)^2 \, dx \leq C \int_0^T \int_\omega v^2 dxdt. \qquad \square \tag{4.1}$$

Proof of Lemma 4.1. Let $\omega = (x_0, x_1)$ with $0 \leq x_0 < x_1 \leq 1$ and consider a smooth cut-off function $\psi : \mathbb{R} \to \mathbb{R}$, such that

$$\begin{cases} 0 \leq \psi(x) \leq 1 & \forall x \in \mathbb{R}, \\ \psi(x) = 1 & \text{for } x \in (0, (2x_0 + x_1)/3), \\ \psi(x) = 0 & \text{for } x \in ((x_0 + 2x_1)/3, 1). \end{cases}$$

We define $w := \psi v$ where v is the solution of (2.5). Then w satisfies

$$\begin{cases} w_t + (a w_x)_x = (a\psi_x v)_x + \psi_x a v_x =: f, & (t,x) \in Q_T, \\ w(t,1) = 0, & t \in (0,T), \\ \text{and } \begin{cases} w(t,0) = 0 & \text{for } 0 \leq \alpha < 1, \\ (a w_x)(t,0) = 0 & \text{for } 1 \leq \alpha < 2, \end{cases} & t \in (0,T). \end{cases} \tag{4.2}$$

Therefore, applying Theorem 2.2 and using the fact that $w \equiv 0$ in a neighborhood of $x = 1$ (hence $w_x(1,t) = 0$), we have, for all $R \geq R_0$,

$$\iint_{Q_T} \Big(R\theta x^\alpha w_x^2 + R^3\theta^3 x^{2-\alpha} w^2\Big) e^{-2R\sigma} \, dxdt \ \leq \ C \iint_{Q_T} e^{-2R\sigma} f^2 dxdt.$$

Then using the definition of ψ and in particular the fact that ψ_x and ψ_{xx} are supported in $\omega' := ((2x_0 + x_1)/3, (x_0 + 2x_1)/3)$, we can write

$$f^2 = \Big((a\psi_x v)_x + \psi_x a v_x\Big)^2 = \Big(a_x \psi_x v + 2a\psi_x v_x + a\psi_{xx} v\Big)^2 \chi_{\omega'} \leq C(v^2 + v_x^2)\chi_{\omega'},$$

since the function a_x is bounded on ω'. Hence

$$(4.3)\quad \iint_{Q_T} \Big(R\theta x^\alpha w_x^2 + R^3\theta^3 x^{2-\alpha} w^2\Big) e^{-2R\sigma}\, dxdt \le C \int_0^T \int_{\omega'} e^{-2R\sigma}(v_x^2+v^2)dxdt,$$

where $\omega' := ((2x_0+x_1)/3, (x_0+2x_1)/3)$. At this point, let us apply the following standard estimate, to be proved later on.

LEMMA 4.2 (Caccioppoli's inequality). *For all $R > 0$,*

$$\int_0^T \int_{\omega'} e^{-2R\sigma} v_x^2 dxdt \;\le\; C(R,T) \int_0^T \int_\omega v^2 dxdt.$$

Let us continue with the proof of Lemma 4.1. The proof of Lemma 4.2 will be given later. By (4.3) and Lemma 4.2, we obtain a bound for v on $(0,(2x_0+x_1)/3)$ of the form

$$\begin{aligned}
&\int_0^T \int_0^{(2x_0+x_1)/3} \Big(R\theta x^\alpha v_x^2 + R^3\theta^3 x^{2-\alpha} v^2\Big) e^{-2R\sigma}\, dxdt \\
&= \int_0^T \int_0^{(2x_0+x_1)/3} \Big(R\theta x^\alpha w_x^2 + R^3\theta^3 x^{2-\alpha} w^2\Big) e^{-2R\sigma}\, dxdt \\
&\le \iint_{Q_T} \Big(R\theta x^\alpha w_x^2 + R^3\theta^3 x^{2-\alpha} w^2\Big) e^{-2R\sigma}\, dxdt \le C_R \int_0^T \int_\omega v^2 dxdt.
\end{aligned}$$

Hence,

$$\int_0^T \int_0^{(2x_0+x_1)/3} \Big(R\theta x^\alpha v_x^2 + R^3\theta^3 x^{2-\alpha} v^2\Big) e^{-2c_0 R\theta}\, dxdt \le\; C_R \int_0^T \int_\omega v^2 dxdt,$$

where $c_0 = \max\{p(x); x \in [0,1]\} = 2/(2-\alpha)^2$.

Now, to complete the reasoning, one has to recover a similar inequality on the interval $((x_0+2x_1)/3, 1)$. But the equation is uniformly parabolic on such a domain. Therefore, the well-known Carleman estimate for the nondegenerate case (see [26]) yields, for R large enough,

$$\int_0^T \int_{(x_0+2x_1)/3}^1 \Big(R\theta v_x^2 + R^3\theta^3 v^2\Big) e^{-2c_1 R\theta}\, dxdt \le\; C_R \int_0^T \int_\omega v^2 dxdt$$

for some constant $c_1 > 0$. Indeed it is sufficient to apply the classical Carleman estimate to the function $\tilde v = \rho v$ in the space interval $((2x_0+x_1)/3, 1)$, where $\rho : \mathbb{R} \to \mathbb{R}$ is some smooth cut-off function, such that

$$\begin{cases} 0 \le \rho(x) \le 1 & \forall x \in \mathbb{R}, \\ \rho(x) = 1 & \text{for } x \in ((x_0+2x_1)/3, 1), \\ \rho(x) = 0 & \text{for } x \in (0, (2x_0+x_1)/3). \end{cases}$$

Hence we obtain

$$\int_0^T \int_{(x_0+2x_1)/3}^1 \Big(R\theta x^\alpha v_x^2 + R^3\theta^3 x^{2-\alpha} v^2\Big) e^{-2c_1 R\theta}\, dxdt \le\; C_R \int_0^T \int_\omega v^2 dxdt.$$

Combining the above estimates and using Lemma 4.2 to bound the integral on the middle interval, we obtain

$$\int_0^T \int_0^1 \Big(R\theta x^\alpha v_x^2 + R^3\theta^3 x^{2-\alpha} v^2 \Big) e^{-2c_2 R\theta}\, dxdt \ \le\ C_R \int_0^T \int_\omega v^2 dxdt,$$

where $c_2 = \max(c_0, c_1)$. ◻

Proof of Lemma 4.2. Consider a smooth function $\xi : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} 0 \le \xi(x) \le 1 & \forall x \in \mathbb{R}, \\ \xi(x) = 1 & \text{for } x \in \omega', \\ \xi(x) = 0 & \text{for } x \notin \omega. \end{cases}$$

Then, for all $R > 0$,

$$\begin{aligned} 0 = \int_0^T \frac{d}{dt} \int_0^1 \xi^2 e^{-2R\sigma} v^2 &= \iint_{Q_T} -2\xi^2 R\sigma_t e^{-2R\sigma} v^2 + 2\xi^2 e^{-2R\sigma} v v_t \\ &= -2 \iint_{Q_T} \xi^2 R\sigma_t e^{-2R\sigma} v^2 - 2\iint_{Q_T} \xi^2 e^{-2R\sigma} v (a v_x)_x \\ &= -2 \iint_{Q_T} \xi^2 R\sigma_t e^{-2R\sigma} v^2 + 2\iint_{Q_T} (\xi^2 e^{-2R\sigma} v)_x a v_x \\ &= -2 \iint_{Q_T} \xi^2 R\sigma_t e^{-2R\sigma} v^2 + 2\iint_{Q_T} a(\xi^2 e^{-2R\sigma})_x v v_x + \xi^2 e^{-2R\sigma} a v_x^2. \end{aligned}$$

Hence,

$$\begin{aligned} 2\iint_{Q_T} \xi^2 e^{-2R\sigma} a v_x^2 &= 2\iint_{Q_T} \xi^2 R\sigma_t e^{-2R\sigma} v^2 - 2\iint_{Q_T} a(\xi^2 e^{-2R\sigma})_x v v_x \\ &= 2\iint_{Q_T} \xi^2 R\sigma_t e^{-2R\sigma} v^2 - 2\iint_{Q_T} \Big(\sqrt{a}\,\xi e^{-R\sigma} v_x\Big)\Big(\sqrt{a}\frac{(\xi^2 e^{-2R\sigma})_x}{\xi e^{-R\sigma}} v\Big) \\ &\le 2\iint_{Q_T} \xi^2 R\sigma_t e^{-2R\sigma} v^2 + \iint_{Q_T} \Big(\sqrt{a}\,\xi e^{-R\sigma} v_x\Big)^2 + \iint_{Q_T} \Big(\sqrt{a}\frac{(\xi^2 e^{-2R\sigma})_x}{\xi e^{-R\sigma}} v\Big)^2 \\ &\le 2\iint_{Q_T} \xi^2 R\sigma_t e^{-2R\sigma} v^2 + \iint_{Q_T} \Big(\sqrt{a}\frac{(\xi^2 e^{-2R\sigma})_x}{\xi e^{-R\sigma}} v\Big)^2 + \iint_{Q_T} \xi^2 e^{-2R\sigma} a v_x^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \iint_{Q_T} \xi^2 e^{-2R\sigma} a v_x^2 &\le 2\iint_{Q_T} \xi^2 R\sigma_t e^{-2R\sigma} v^2 + \iint_{Q_T} \Big(\sqrt{a}\frac{(\xi^2 e^{-2R\sigma})_x}{\xi e^{-R\sigma}} v\Big)^2 \\ &\le C(R,T) \int_0^T \int_\omega v^2. \end{aligned}$$

◻

5. Proof of Theorem 2.4 (controllability result). By standard arguments, the null controllability result stated in Theorem 2.4 follows from (2.7). Hence it remains to prove (2.7). For $\alpha \neq 1$, we apply Hardy's inequality (Lemma 2.1) with $\alpha^\star = \alpha$ to deduce from (2.6) that

$$\int_0^1 x^{\alpha-2} v(0,x)^2\, dx \le C \int_0^1 x^\alpha v_x(0,x)^2\, dx \le C \int_0^T \int_\omega v^2 dxdt.$$

In the case of $\alpha = 1$, from (2.6) we deduce that, for all $0 < \eta < 1$,

$$\int_0^1 x^{1+\eta} v_x(0,x)^2\, dx \le \int_0^1 x v_x(0,x)^2\, dx \le C \int_0^T \int_\omega v^2 dx dt.$$

Now, applying Hardy's inequality (Lemma 2.1) with $\alpha^\star = 1 + \eta$, we obtain

$$\int_0^1 x^{\eta-1} v(0,x)^2\, dx \le C \int_0^1 x^{1+\eta} v_x(0,x)^2\, dx \le C \int_0^T \int_\omega v^2 dx dt.$$

In both cases, (2.7) follows. $\square$

6. Proof of Lemma 2.1 (Hardy's inequalities).

First case. $0 \le \alpha^\star < 1$. Since z is absolutely continuous on $(0,1)$, we have

$$\begin{aligned} |z(x) - z(\varepsilon)|^2 &= \Big(\int_\varepsilon^x z_x(s) s^{(3-\gamma)/4} s^{(-3+\gamma)/4}\, ds \Big)^2 \\ &\le \Big(\int_\varepsilon^x z_x(s)^2 s^{(3-\gamma)/2}\, ds \Big) \Big(\int_\varepsilon^x s^{(-3+\gamma)/2}\, ds \Big), \end{aligned}$$

where we denote $\gamma := 2 - \alpha^\star \in (1,2]$. Letting $\varepsilon \to 0^+$, we get

$$|z(x)|^2 \le \Big(\int_0^x z_x(s)^2 s^{(3-\gamma)/2}\, ds \Big) \Big(\int_0^x s^{(-3+\gamma)/2}\, ds \Big).$$

Therefore

$$\begin{aligned} \int_0^1 x^{\alpha^\star - 2} z(x)^2\, dx &\le \int_0^1 x^{-\gamma} \Big(\int_0^x z_x(s)^2 s^{(3-\gamma)/2}\, ds \Big) \Big(\int_0^x s^{(-3+\gamma)/2}\, ds \Big)\, dx \\ &= \int_0^1 x^{-\gamma} \Big(\int_0^x z_x(s)^2 s^{(3-\gamma)/2}\, ds \Big) \frac{x^{(\gamma-1)/2}}{(\gamma-1)/2}\, dx \\ &= \frac{2}{\gamma - 1} \int_0^1 z_x(s)^2 s^{(3-\gamma)/2} \Big(\int_s^1 x^{(-\gamma-1)/2}\, dx \Big)\, ds \\ \le \frac{2}{\gamma-1} \int_0^1 z_x(s)^2 s^{(3-\gamma)/2} \frac{s^{(1-\gamma)/2}}{(\gamma-1)/2}\, ds \ &= \ \frac{4}{(1-\alpha^\star)^2} \int_0^1 s^{\alpha^\star} z_x(s)^2\, ds. \end{aligned}$$

Second case. $1 < \alpha^\star < 2$. Denoting $\gamma := 2 - \alpha^\star \in (0,1)$, we have

$$\begin{aligned} \int_0^1 x^{\alpha^\star - 2} z(x)^2\, dx &\le \int_0^1 x^{-\gamma} \Big(\int_x^1 z_x(s)^2 s^{(3-\gamma)/2}\, ds \Big) \Big(\int_x^1 s^{(-3+\gamma)/2}\, ds \Big)\, dx \\ &\le \int_0^1 x^{-\gamma} \Big(\int_x^1 z_x(s)^2 s^{(3-\gamma)/2}\, ds \Big) \frac{x^{-(1-\gamma)/2}}{(1-\gamma)/2}\, dx \\ &= \frac{2}{1-\gamma} \int_0^1 z_x(s)^2 s^{(3-\gamma)/2} \Big(\int_0^s x^{(-\gamma-1)/2}\, dx \Big)\, ds \\ \le \frac{4}{(1-\gamma)^2} \int_0^1 z_x(s)^2 s^{2-\gamma}\, ds \ &= \ \frac{4}{(\alpha^\star - 1)^2} \int_0^1 s^{\alpha^\star} z_x(s)^2\, ds. \qquad \square \end{aligned}$$

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