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Optimal control for evolution equations with memory

P. CANNARSA, H. FRANKOWSKA AND E. M. MARCHINI

Abstract. In this paper, we investigate the existence and regularity of solutions for Bolza optimal control problems in infinite dimension governed by a class of semilinear evolution equations. Our results apply to systems exhibiting hereditary properties, as heat propagation in real conductors and isothermal viscoelasticity, described by equations with memory terms which account for the past history of the variables in play.

1. Introduction

Many phenomena, such as diffusion in physical and biological systems, vibration of strings, membranes, plates, fluid dynamics, and so on, can be modeled by infinite-dimensional equations of evolution. In most of these situations, it is natural to try to control the evolution of the system in order to optimize some measure of best performance. For this reason, infinite-dimensional optimal control theory is a research area of great interest in many applications.

The purpose of this paper is to study some optimal control problems governed by a class of semilinear evolution equations including the so-called *equations with memory*. Such equations, influenced by the past values of the variables in play, account for the delay effects experimentally observed and provide a faithful description of reality in many situations like heat propagation in real conductors or isothermal viscoelasticity.

The literature dealing with infinite-dimensional optimal control theory is very rich, see, for instance, [41] for linear quadratic models, and, for more general settings, [2, 4, 5, 34, 43] and the references therein. On the other hand, for optimal control problems involving equations with memory, just a few isolated results are available at the moment (see, e.g., [11] for a finite dimensional problem, and [48] for a more general case), a systematic study being completely missing.

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The optimal control problem we are concerned with consists in minimizing a cost functional of the form

$$\mathcal{J}(u, \alpha) = \int_0^1 L(t, u(t), \alpha(t)) dt + g(u(1)) \quad (1.1)$$

over all trajectory/control pairs (u, α) satisfying a suitable state equation. The functions $L : I \times H \times Z \rightarrow [0, \infty)$ and $g : H \rightarrow [0, \infty)$ are given, $I = [0, 1]$, the state space H is an infinite-dimensional separable real Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$, and Z is a complete separable metric space modeling the control space.

The state equations governing the optimal control problem analyzed in this work are of three types:

(i) *abstract systems*

$$\begin{cases} u'(t) = \mathbb{A}u(t) + f(t, u(t), \alpha(t)) & \text{for } t \in I \\ \alpha(t) \in U(t) & \text{for } t \in I; \end{cases}$$

(ii) *hyperbolic equations with memory*

$$\begin{cases} \ddot{u}(t) + A[u(t) - \int_0^\infty \mu(s)u(t-s)ds] = f(t, u(t), \alpha(t)) & \text{for } t \in I \\ \alpha(t) \in U(t) & \text{for } t \in I; \end{cases} \quad (1.2)$$

(iii) *parabolic equations with memory*

$$\begin{cases} \dot{u}(t) + A[u(t) + \int_0^\infty \beta(s)u(t-s)ds] = f(t, u(t), \alpha(t)) & \text{for } t \in I \\ \alpha(t) \in U(t) & \text{for } t \in I. \end{cases} \quad (1.3)$$

Here, \mathbb{A} is the infinitesimal generator of a strongly continuous semigroup $S(t) : H \rightarrow H$, $f : I \times H \times Z \rightarrow H$, and $U : I \rightsquigarrow Z$ is a measurable set valued map with closed non-empty images. In (1.2) and (1.3), the linear operator A , with domain $\text{dom}(A) \subset H$, is selfadjoint and strictly positive with compact inverse, and the memory kernels μ and β account for the hereditary feature of systems (1.2) and (1.3).

The main result of the paper is an approximation theorem ensuring that any minimizing sequence of controls for the above Bolza problems can be slightly modified to get a new minimizing sequence with nice boundedness properties. As a consequence, the non-occurrence of Lavrentiev type phenomena [42] for infinite-dimensional problems is obtained. In the literature, this problem was investigated by many authors in finite dimension, see, for example, [12, 36]. Such a result is quite interesting for applications, since it allows to compute the true optimal trajectory by numerical methods. The main hypothesis we employ comes from the direct method in the calculus of variations theory. It is a Tonelli type growth condition: for a.e. $t \in I$, any $u \in H$, and $\alpha \in U(t)$,

$$L(t, u, \alpha) \geq \Phi(\|f(t, u, \alpha)\|_H), \quad \text{with } \Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ satisfying } \lim_{r \rightarrow \infty} \frac{\Phi(r)}{r} = \infty. \quad (1.4)$$

We note that an analogous assumption has been used for finite dimensional Bolza optimal control problems in [13, Chapter 11] for existence, and in [36] for both existence and Lipschitz continuity of optimal solutions. As corollaries of our main approximation result, we get some existence theorems for the above-mentioned Bolza problems.

To perform our proofs, the key idea is the application of a functional tool usually employed to prove necessary conditions for optimality: Ekeland's variational principle. In [3], an existence result for an infinite-dimensional optimal control problem is obtained via Ekeland's theorem, applying Pontryagin's maximum principle to get a minimizing sequence. On the contrary, our approach, which follows the method developed in [36] in the finite dimensional context, requires no use of necessary conditions.

Concerning the optimal control problems analyzed in this paper, the main difficulty lies in their infinite-dimensional setting. If we try to apply the classical approach for finite dimensional problems by taking minimizing sequences of trajectory/control pairs, we are immediately confronted by the lack of compactness of infinite-dimensional spaces. Consequently, to minimize (1.1) subject to the state equation

$$\begin{cases} u'(t) = Au(t) + f(t, u(t), \alpha(t)) & \text{for } t \in I \\ \alpha(t) \in U(t) & \text{for } t \in I \\ u(0) = u_0 \in H \end{cases} \quad (1.5)$$

we suppose that, for a.e. $t \in I$,

$$\begin{cases} G(t) = \{(u, f(t, u, \alpha), L(t, u, \alpha) + v) : u \in H, \alpha \in U(t), v \geq 0\} \\ \text{is a closed convex set.} \end{cases} \quad (1.6)$$

However, when the semigroup $S(t)$ is compact for $t > 0$, we need to only assume the weaker Cesari condition: for a.e. $t \in I$ and any $u \in H$,

$$\begin{cases} G(t, u) = \{(f(t, u, \alpha), L(t, u, \alpha) + v) : \alpha \in U(t), v \geq 0\} \\ \text{is a closed convex set.} \end{cases} \quad (1.7)$$

The above dichotomy explains why we use two different approaches to study optimal control problems involving the equations with memory (1.2) and (1.3): the Dafermos *history approach* for (1.2), and the *resolvent operator* approach for (1.3).

Let us begin by analyzing the semilinear hyperbolic Eq. (1.2) which is a model for viscoelasticity of Boltzmann type [33, 53], see Sect. 4 for more details. The memory kernel $\mu : \mathbb{R}^+ \rightarrow [0, \infty)$, accounting for the viscoelasticity behavior, is decreasing and summable, in accordance with the physical *fading memory principle* introduced by Coleman and Mizel [18, 19].

Following a consolidated tradition which dates back to Dafermos [28], it is possible to translate integrodifferential problems into equations generating dynamical systems on some abstract space, see the related papers [14, 15, 21–25, 37, 49, 50] just to mention some recent contributions. This strategy, called the *history approach*, allows to circumvent the intrinsic difficulties lying in the nonlocal character of integrodifferential

equations. Thanks to Dafermos' method, we obtain the existence of an optimal solution for Bolza problems involving hyperbolic semilinear equations with memory of type (1.2), as a direct application of the results proved for the abstract problem (1.1)(1.5) under assumption (1.6). Observe that the evolution of (1.2) requires the knowledge of the past history of u on the time interval $(-\infty, 0]$, where u needs not solve the equation. Hence, the past history of u , in this framework, has to be considered as an initial datum, and a correct choice of the *initial values* for system (1.2) is then

$$u(0) = w_0, \quad \dot{u}(0) = v_0, \quad \text{and} \quad u(-s)|_{s>0} = \varphi_0(s), \quad (1.8)$$

where w_0 , v_0 , and φ_0 , defined on \mathbb{R}^+ are prescribed data.

The same approach can be used to obtain existence of an optimal solution to Bolza problems governed by the parabolic equation with memory (1.3), which is a model for the heat propagation in real conductors. Indeed, also in this case, it is possible to translate the integrodifferential problem into an abstract problem in order to apply directly the results proved for (1.1)(1.5). On the other hand, in this case one would like to relax the convexity assumption (1.6) by fully exploiting the parabolic character of (1.3). For this reason, this model is treated by a different approach: the *resolvent operator* technique that was developed, for example, in the works [10,26,27,45,52]. The compactness of the resolvent family associated to (1.3) allows to prove an existence theorem for problem (1.1)(1.3) subject to the conditions

$$u(0) = u_0, \quad u(-s)|_{s>0} = g_0(s), \quad (1.9)$$

under the weaker assumption (1.7). The proof follows the same lines as in the abstract existence case, replacing semigroup techniques by resolvent operator techniques.

Let us conclude this introduction pointing out another difference between the two approaches employed in this paper. For parabolic problems, the past history of u is treated as a forcing term included into the function f ; hence, it is fixed in some sense. On the other hand, in the hyperbolic case, the past history is considered to all intents as an initial datum associated with the system.

1.1. Outline of the paper

In Sect. 2, we list the notations, definitions, and assumptions in use. Our main results are stated in Sect. 3, while Sect. 4 is devoted to applications to problems involving hereditary systems. Proofs are given in Sect. 5.

2. Preliminaries

2.1. Notations

Given a Banach space X and $R \geq 0$, define the ball

$$B_X(R) = \{x \in X : \|x\|_X \leq R\},$$

and let $\mathbb{L}(X)$ be the space of bounded linear operators from X into X . We denote by m the Lebesgue measure on the line, and the set of admissible controls by

$$\mathcal{U} = \{\alpha : I \rightarrow Z \mid \alpha \text{ measurable and } \alpha(t) \in U(t)\},$$

with Z as in Sect. 1.

For the state equations mentioned in the introduction, we will use the following notions of solution: *mild* solutions for system (1.5), and *weak* solution for problems (1.2)(1.8), and (1.3)(1.9).

DEFINITION 2.1. Let $u_0 \in H$. A function $u \in \mathcal{C}(I, H)$ is a (mild) solution of (1.5) corresponding to some $\alpha \in \mathcal{U}$ if u solves the integral equation, see [51],

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s, u(s), \alpha(s))ds, \quad \forall t \in I. \quad (2.1)$$

Setting

$$F_0(t) = \int_0^\infty \mu(t+s)\varphi_0(s)ds \quad \text{and} \quad G_0(t) = \int_0^\infty \beta(t+s)g_0(s)ds,$$

we can rewrite system (1.2) as

$$\begin{cases} \ddot{u}(t) + A[u(t) - \int_0^t \mu(s)u(t-s)ds - F_0(t)] = f(t, u(t), \alpha(t)) & \text{for } t \in I \\ \alpha(t) \in U(t) & \text{for } t \in I \end{cases}$$

and (1.3) as

$$\begin{cases} \dot{u}(t) + A[u(t) + \int_0^t \beta(s)u(t-s)ds + G_0(t)] = f(t, u(t), \alpha(t)) & \text{for } t \in I \\ \alpha(t) \in U(t) & \text{for } t \in I. \end{cases}$$

Hence, we employ the following definitions of solutions for the above integrodifferential systems.

DEFINITION 2.2. Let $(w_0, v_0) \in \text{dom}(A^{1/2}) \times H$ and φ_0 be such that $F_0(t)$ belongs to $\text{dom}(A^{1/2})$ for a.e. $t \in I$. A function

$$u \in \mathcal{C}(I, \text{dom}(A^{1/2})) \cap \mathcal{C}^1(I, H)$$

is a (weak) solution of the Cauchy problem (1.2)(1.8) corresponding to some $\alpha \in \mathcal{U}$ if u fulfills initial conditions (1.8) and, for any $w \in \text{dom}(A^{1/2})$ and a.e. $t \in I$,

$$\begin{aligned} \frac{d}{dt} \langle \dot{u}(t), w \rangle + \langle A^{1/2}u(t), A^{1/2}w \rangle_H - \int_0^t \mu(s) \langle A^{1/2}u(t-s), A^{1/2}w \rangle_H ds \\ - \langle A^{1/2}F_0(t), A^{1/2}w \rangle_H = \langle f(t, u(t), \alpha(t)), w \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\text{dom}(A^{-1/2})$ and $\text{dom}(A^{1/2})$.

DEFINITION 2.3. Let $u_0 \in H$ and g_0 be such that $G_0(t)$ belongs to $\text{dom}(A^{1/2})$ for a.e. $t \in I$. A function

$$u \in \mathcal{C}(I, H) \cap L^2(I; \text{dom}(A^{1/2}))$$

is a (weak) solution of the Cauchy problem (1.3)(1.9) corresponding to some $\alpha \in \mathcal{U}$ if u fulfills initial conditions (1.9) and, for any $w \in \text{dom}(A^{1/2})$ and a.e. $t \in I$,

$$\begin{aligned} \frac{d}{dt} \langle u(t), w \rangle + \langle A^{1/2}u(t), A^{1/2}w \rangle_H + \int_0^t \beta(s) \langle A^{1/2}u(t-s), A^{1/2}w \rangle_H ds \\ + \langle A^{1/2}G_0(t), A^{1/2}w \rangle_H = \langle f(t, u(t), \alpha(t)), w \rangle. \end{aligned}$$

Again, the symbol $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\text{dom}(A^{-1/2})$ and $\text{dom}(A^{1/2})$.

DEFINITION 2.4. A trajectory/control pair (u, α) is an *admissible pair* for problem (1.5) (or (1.2)(1.8), or (1.3)(1.9)) if $\alpha \in \mathcal{U}$, and u is a solution of (1.5) (or (1.2)(1.8), or (1.3)(1.9), respectively) corresponding to α .

2.2. Assumptions on f, L, g

The main assumptions on the data are the following:

$$f(\cdot, u, \cdot), L(\cdot, u, \cdot) \text{ are Lebesgue-Borel measurable, for any } u \in H; \quad (2.2)$$

$$g \text{ is convex}; \quad (2.3)$$

$f(t, \cdot, \alpha)$, $L(t, \cdot, \alpha)$, and g are locally Lipschitz in the following sense: for any $R > 0$, there exist $k_R, l_R \in L^1(I; \mathbb{R}^+)$, $g_R > 0$ such that, for a.e. $t \in I$, $\alpha \in U(t)$, $u, v \in B_H(R)$,

$$\|f(t, u, \alpha) - f(t, v, \alpha)\|_H \leq k_R(t) \|u - v\|_H, \quad (2.4)$$

$$|L(t, u, \alpha) - L(t, v, \alpha)| \leq l_R(t) \|u - v\|_H, \quad |g(u) - g(v)| \leq g_R \|u - v\|_H;$$

$\exists \bar{\alpha} \in \mathcal{U}$ such that, for some $\phi \in L^1(I; \mathbb{R}^+)$, a.e. $t \in I$, any $u \in H$,

$$\|f(t, u, \bar{\alpha}(t))\|_H \leq \phi(t)(1 + \|u\|_H) \quad (2.5)$$

and, for any $R > 0$, there exists $m_R \in L^1(I; \mathbb{R}^+)$ such that, for a.e. $t \in I$ and any $u \in B_H(R)$,

$$L(t, u, \bar{\alpha}(t)) \leq m_R(t). \quad (2.6)$$

2.3. Assumptions on the memory kernel μ

The memory kernel μ is supposed to be a (nonnegative) nonincreasing and summable function on \mathbb{R}^+ , with total mass

$$\kappa = \int_0^\infty \mu(s) ds \in (0, 1),$$

mapping nullsets into nullsets. In particular, μ is piecewise absolutely continuous, and thus differentiable almost everywhere with $\mu' \leq 0$, albeit possibly unbounded about zero.

2.4. Assumptions on the memory kernel β

The kernel β is supposed to be summable on \mathbb{R}^+ , hence Laplace transformable (i.e. $\int_0^\infty e^{-\varpi s} |\beta(s)| ds < \infty$ for some $\varpi \in \mathbb{R}$). The Laplace transform

$$\hat{\beta}(\lambda) = \int_0^\infty e^{-\lambda s} \beta(s) ds$$

is analytically extendable to a sector

$$S_\theta = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } |\arg \lambda| < \theta\}, \quad \text{with } \theta \in (\pi/2, \pi).$$

Moreover,

$$|\lambda^\omega \hat{\beta}(\lambda)| \leq C, \quad \text{for some } \omega \in (0, 1], C > 0 \text{ and any } \lambda \in S_\theta.$$

A typical example of such kernels is given by

$$\beta(s) = e^{-\delta s} s^{\omega-1}, \quad \text{with } \delta > 0 \text{ and } \omega \in (0, 1].$$

See [27, 45] for more details.

2.5. Some functional tools

In order to treat the integrodifferential Eq. (1.2) within Dafermos' framework, we define the *memory space*

$$\mathcal{M} = L_\mu^2(\mathbb{R}^+; \text{dom}(A^{1/2}))$$

endowed with the weighted L^2 -inner product

$$\langle \eta_1, \eta_2 \rangle_{\mathcal{M}} = \int_0^\infty \mu(s) \langle A^{1/2} \eta_1(s), A^{1/2} \eta_2(s) \rangle_H ds,$$

jointly with the infinitesimal generator of the right-translation semigroup T on \mathcal{M} , namely,

$$T\eta = -\eta', \quad \text{with domain } \text{dom}(T) = \{\eta \in \mathcal{M} : \eta' \in \mathcal{M}, \eta(0) = 0\}.$$

The prime standing for the distributional derivative, and $\eta(0) = \lim_{s \rightarrow 0} \eta(s)$ in $\text{dom}(A^{1/2})$.

2.6. Comments on the assumptions

- (1) The admissibility hypotheses (2.5) and (2.6) allow to avoid trivial situations since, jointly with (2.2), (2.3), and (2.4), they guarantee the existence of an admissible pair (u, α) such that $\mathcal{J}(u, \alpha) < \infty$, see Lemma 5.1.
- (2) Assumptions (2.2)–(2.4) ensure that all the systems introduced in Sect. 1 admit at most one solution for any $\alpha \in \mathcal{U}$.
- (3) The convexity condition (1.6) does not imply that the state equation of the problem is linear in u , as illustrated by the following simple example: set

$$U(t) = \{\alpha : \langle q(t), \alpha \rangle \geq 0\} \quad \text{and} \quad f(t, u, \alpha) = p(t)\|u\|_H + p(t)\langle q(t), \alpha \rangle,$$

where $p \in L^1(I; H)$, $q \in L^\infty(I; H)$, and $q(t) \neq 0$, for a.e. t . We assume that L is non-negative, independent of α , convex and locally Lipschitz in u , and such that $\sup_{u \in B_H(R)} L(t, u) \leq m_R(t)$, for a.e. t and some $m_R \in L^1(I; \mathbb{R}^+)$. Then, it is not difficult to prove that the set $G(t)$ in (1.6) is closed and convex.

3. Main results

We first discuss the nonoccurrence of the Lavrentiev phenomenon under Tonelli's growth condition (1.4). In the next two results, the notion of *admissible pair* refers without distinction to system (1.5), or (1.2)(1.8), or (1.3)(1.9).

THEOREM 3.1. *Assume (1.4) and (2.2)–(2.6). Then, there exist $M_1, M_2 > 0$ such that*

$$\inf_{(u, \alpha)} \mathcal{J}(u, \alpha) = \inf_{\substack{(u, \alpha) : \|f(\cdot, u(\cdot), \alpha(\cdot))\|_{L^1(I; H)} \leq M_1, \\ \|L(\cdot, u(\cdot), \alpha(\cdot))\|_{L^1(I; \mathbb{R})} \leq M_2}} \mathcal{J}(u, \alpha),$$

where (u, α) is an admissible pair. Moreover, if ϕ and m_R in (2.5) and (2.6) are essentially bounded, then there exist $M_3, M_4 > 0$ such that

$$\inf_{(u, \alpha)} \mathcal{J}(u, \alpha) = \inf_{\substack{(u, \alpha) : \|f(\cdot, u(\cdot), \alpha(\cdot))\|_{L^\infty(I; H)} \leq M_3, \\ \|L(\cdot, u(\cdot), \alpha(\cdot))\|_{L^\infty(I; \mathbb{R})} \leq M_4}} \mathcal{J}(u, \alpha). \quad (3.1)$$

As outlined in Sect. 5, it is possible to give precise bounds for the constants M_1, M_2, M_3, M_4 in the above theorem, see (5.14), (5.15), (5.22), and (5.23).

Our next result is concerned with minimizing sequences. In this paper, it is used to obtain Theorem 3.1, but it is interesting in its own right for applications.

LEMMA 3.2. *Assume (1.4) and (2.2)–(2.6). Then, for any minimizing sequence of admissible pairs for (1.1), $\{(\bar{u}_n, \bar{\alpha}_n)\}$, there exists a minimizing sequence $\{(u_n, \alpha_n)\}$ with*

$$\lim_{n \rightarrow \infty} m(\{t \in I : \alpha_n(t) \neq \bar{\alpha}_n(t)\}) = 0,$$

such that $\{f(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$ and $\{L(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$ are equi-integrable in L^1 with bounds

$$\|f(\cdot, u_n(\cdot), \alpha_n(\cdot))\|_{L^1(I; H)} \leq M_1 \quad \text{and} \quad \|L(\cdot, u_n(\cdot), \alpha_n(\cdot))\|_{L^1(I; \mathbb{R})} \leq M_2,$$

for any $n \in \mathbb{N}$.

If, in addition, ϕ and m_R are essentially bounded, then the new minimizing sequence satisfies the stronger condition

$$\|f(\cdot, u_n(\cdot), \alpha_n(\cdot))\|_{L^\infty(I; H)} \leq M_3 \quad \text{and} \quad \|L(\cdot, u_n(\cdot), \alpha_n(\cdot))\|_{L^\infty(I; \mathbb{R})} \leq M_4,$$

for any $n \in \mathbb{N}$.

As corollaries of the main result, we provide some existence theorems for the optimal control problems introduced in Sect. 1. We would like to recall that without convexity assumptions optimal solutions may not exist even in the finite dimensional context. Hence, we have to require extra hypotheses to get existence of optimal solutions. The first theorem concerns the abstract optimal control problem of minimizing the cost (1.1) subject to Eq. (1.5).

THEOREM 3.3. *Assume (1.4), (1.6), and (2.2)–(2.6). Then, there exists an optimal solution (u^*, α^*) to problem (1.1)(1.5). Moreover, if ϕ and m_R in (2.5) and (2.6) are essentially bounded, then any optimal solution (u^*, α^*) satisfies $f(\cdot, u^*(\cdot), \alpha(\cdot)) \in L^\infty(I; H)$ and $L(\cdot, u^*(\cdot), \alpha^*(\cdot)) \in L^\infty(I; \mathbb{R})$.*

For problem (1.1)(1.2)(1.8), as a direct consequence of Theorem 3.3, we get the following result applying Dafermos' approach.

COROLLARY 3.4. *Under assumptions (1.4), (1.6), and (2.2)–(2.6), for any $(w_0, v_0, \varphi_0) \in \text{dom}(A^{1/2}) \times H \times \mathcal{M}$, there exists an optimal solution (u^*, α^*) to problem (1.1)(1.2)(1.8). Moreover, if ϕ and m_R in (2.5) and (2.6) are essentially bounded, then any optimal solution (u^*, α^*) satisfies $f(\cdot, u^*(\cdot), \alpha(\cdot)) \in L^\infty(I; H)$ and $L(\cdot, u^*(\cdot), \alpha(\cdot)) \in L^\infty(I; \mathbb{R})$.*

As we observed in the introduction, for compact semigroups the convexity assumption (1.6) can be replaced by the weaker and classical Cesari condition (1.7).

THEOREM 3.5. *Suppose that $S(t)$ is a semigroup of compact operators. Under assumptions (1.4), (1.7), and (2.2)–(2.6), the same conclusions as in Theorem 3.3 hold.*

An analogous result is available for problems involving the equation with memory (1.3), due to its parabolicity. Here, a stronger growth condition on L is needed: for a.e. $t \in I$, any $u \in H$, $\alpha \in U(t)$, some $p > 1$, and $a_1, a_2 > 0$,

$$L(t, u, \alpha) \geq a_1 \|f(t, u, \alpha)\|_H^p - a_2. \quad (3.2)$$

THEOREM 3.6. *Under assumptions (1.7), (2.2)–(2.6), and (3.2), for any $u_0 \in H$ and any g_0 such that $G_0 \in L^1(I; \text{dom}(A))$, there exists an optimal solution (u^*, α^*) to problem (1.1)(1.3)(1.9). Moreover, if ϕ and m_R in (2.5) and (2.6) are essentially bounded, then any optimal solution (u^*, α^*) satisfies*

$$f(\cdot, u^*(\cdot), \alpha(\cdot)) \in L^\infty(I; H) \quad \text{and} \quad L(\cdot, u^*(\cdot), \alpha(\cdot)) \in L^\infty(I; \mathbb{R}).$$

4. Some concrete models

We analyze three optimal control problems involving physical phenomena described by integrodifferential equations.

4.1. Reaction-diffusion with memory

The first model arises in the Coleman–Gurtin theory of heat propagation in real conductors [17], where the temperature evolution exhibits a hyperbolic character due to the conductor's inertia. Here, the classical Fourier law for the heat flux is replaced by a constitutive law based on the key assumption that the evolution of the heat flux is influenced by the past history of the temperature gradient, see also [38, 44, 46, 47]. In this setting, we consider a control integrodifferential equation which governs the evolution of the temperature variation field u in a homogeneous isotropic heat conductor $\Omega \subset \mathbb{R}^3$, a bounded domain with smooth boundary $\partial\Omega$, subject to hereditary memory:

$$\dot{u}(\mathbf{x}, t) - \Delta \left[u(\mathbf{x}, t) + \int_0^t \beta(s) u(\mathbf{x}, t-s) ds \right] = \alpha(\mathbf{x}, t) F(t, u(\mathbf{x}, t)), \quad t > 0, \quad (4.1)$$

where, for a.e. $t \in [0, 1]$,

$$\alpha(\cdot, t) \in U(t) = B_{L^\infty(\Omega)}(1).$$

The convolution kernel β satisfying the hypotheses of Sect. 2.4 introduces delay effects in the model and can be interpreted as a conductivity density. The term at the right hand side of (4.1), depending on a multiplicative control α , represents a heat supply. The temperature is required to satisfy the Dirichlet boundary condition

$$u(\mathbf{x}, t)|_{\mathbf{x} \in \partial\Omega} = 0, \quad (4.2)$$

complying with the physical assumption that the boundary $\partial\Omega$ of the conductor is kept at null (equilibrium) temperature for all times. Equation (4.1) is also supplemented with the initial condition

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \text{for some } u_0 \in L^2(\Omega). \quad (4.3)$$

Our goal is to solve (4.1) in an optimal way, namely, choosing α properly so that the corresponding solution u minimizes the following cost functional

$$\mathcal{J}(u, \alpha) = \int_0^1 \int_{\Omega} \left[|u(\mathbf{x}, t) - \bar{u}(\mathbf{x}, t)|^2 + \mathcal{L}(\alpha(\mathbf{x}, t)) \right] d\mathbf{x} dt + \int_{\Omega} \mathcal{G}(u(\mathbf{x}, 1)) d\mathbf{x}, \quad (4.4)$$

where $\bar{u} \in L^2(0, 1; L^2(\Omega))$ represents a reference temperature. Let \mathcal{F} be the Nemytskii operator associated to a function F . Taking

$$H = L^2(\Omega), \quad A = -\Delta \quad \text{with} \quad \text{dom}(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

and omitting the dependence on \mathbf{x} , we assume that

- for a.e. $t \in [0, 1]$, any $y, z \in \mathbb{R}$, some $C > 0$, $p \geq 2$,

$$|F(t, y)| \leq C(1 + |y|^{2/p}) \quad \text{and} \quad |F(t, y) - F(t, z)| \leq C|y - z|;$$

- \mathcal{L} is non-negative, convex and continuous;
- \mathcal{G} is convex and locally Lipschitz.

The hypotheses of Theorems 3.1 and 3.6 are satisfied, implying that problem (4.1)–(4.4) does not present Lavrentiev type phenomena and admits an optimal solution. Indeed, for a.e. t , any $u \in H$, and $\alpha \in U(t)$, we have that

$$\|\mathcal{F}(t, u)\|_H < \infty \quad \text{and} \quad \|\alpha \mathcal{F}(t, u)\|_H^p \leq c(1 + \|u\|_H^2) \leq c_1 \|u(t) - \bar{u}(t)\|_H^2 + c_2,$$

for some $c, c_1, c_2 > 0$, which yields (3.2). Also, taking $\bar{\alpha} \equiv 0$, inequalities (2.5) and (2.6) hold, and the convexity of \mathcal{L} implies the validity of the last assumption (1.7).

4.2. A Budyko-Sellers type climate model

Let us describe another parabolic equation with memory which gives rise to an interesting optimal control problem. Such a system is a variant of the classical energy balance climate model describing the effect of solar radiation on climate, proposed independently by Budyko [8] and Sellers [54] in 1969.

Given a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$ (a local chart for the Earth surface), we describe the evolution of the mean annual temperature of the Earth $u : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ by means of the integrodifferential equation

$$\begin{aligned} \dot{u}(\mathbf{x}, t) - \Delta \left[u(\mathbf{x}, t) + \int_0^\infty \beta(s) u(\mathbf{x}, t-s) ds \right] \\ + \mathcal{F}(\mathbf{x}, u(\mathbf{x}, t)) = \alpha(\mathbf{x}, t) Q(\mathbf{x}, t) b(\mathbf{x}, u(\mathbf{x}, t)), \end{aligned} \quad (4.5)$$

where, for a.e. $t \in [0, 1]$,

$$\alpha(\cdot, t) \in U(t) = B_{L^\infty(\Omega)}(1).$$

The *memory* kernel β satisfies the assumptions of Sect. 2.4 and it complies with the physical feature of keeping memory of the past heating. The term

$$Q(\mathbf{x}, t) b(\mathbf{x}, u(\mathbf{x}, t))$$

represents the fraction of the solar energy absorbed by the Earth, where the positive number $Q(\mathbf{x}, t)$ is the solar flux, $b(\mathbf{x}, u(\mathbf{x}, t)) = 1 - a(\mathbf{x}, u(\mathbf{x}, t))$, a is the reflecting power of the Earth surface (called *albedo*), and \mathcal{F} represents the emitted energy. As in [29], Eq. (4.5) is complemented by the Neumann boundary condition

$$\partial_n u(\mathbf{x}, t)|_{\mathbf{x} \in \partial\Omega} = 0, \quad \forall t \in \mathbb{R}, \quad (4.6)$$

∂_n being the outward normal derivative, and the initial condition

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}) \quad \text{for some } u_0 \in L^2(\Omega). \quad (4.7)$$

Additive control problems—that is, when control takes the form of a source term—for Budyko–Sellers type models have been studied by several authors (see, e.g. Diaz [29,30]), and positive, as well as negative, controllability results have been obtained. On the other hand, when referred to climate models, additive control would require huge amounts of energy to influence the system. The introduction of the multiplicative control α aims at allowing for more realistic human interaction with climate—even if it would involve large time scales—somewhat in the spirit of von Neumann [55]: *microscopic layers of colored matter spread on an icy surface, or in the atmosphere above one, could inhibit the reflection–radiation process, melt the ice, and change the local climate.*

Choosing as performance criterion the quadratic functional

$$\mathcal{J}(u, \alpha) = \int_0^1 \int_{\Omega} \left[|u(\mathbf{x}, t) - \bar{u}(\mathbf{x}, t)|^2 + \mathcal{L}(\alpha(\mathbf{x}, t)) \right] d\mathbf{x} dt + \int_{\Omega} \mathcal{G}(u(\mathbf{x}, 1)) d\mathbf{x}, \quad (4.8)$$

with $\bar{u} \in L^2(0, 1; L^2(\Omega))$ a fixed reference temperature, we aim at finding an admissible pair (u, α) for system (4.5)–(4.7) which minimizes (4.8). Let us set

$$H = L^2(\Omega) \quad \text{and} \quad A = -\Delta \quad \text{with} \quad \text{dom}(A) = \{u \in H^2(\Omega) : \partial_n u = 0 \text{ on } \partial\Omega\}.$$

We shall suppose that the functions involved in the problem are Lebesgue measurable in \mathbf{x} , and omit writing explicitly the dependence on \mathbf{x} . Assume moreover:

- b is Lipschitz continuous in u , \mathcal{F} is Locally Lipschitz in u and, for some $C > 0$,

$$|\mathcal{F}(u)| \leq C(1 + |u|)$$

(notice that in this way the choice of Newton linear emission radiation is allowed);

- \mathcal{L} is non-negative, convex and continuous;
- \mathcal{G} is convex and locally Lipschitz.

Then, applying Theorems 3.1 and 3.6, we deduce that problem (4.5)–(4.8) does not present the Lavrentiev phenomenon and admits an optimal solution.

4.3. A model for Boltzmann viscoelasticity

Our last application concerns a model for isothermal viscoelasticity, whose origins trace back to the works of Boltzmann and Volterra [6,7,56,57]. These authors first introduced the notion of memory in connection with the analysis of elastic bodies, whose mechanical behavior is due both to the instantaneous stress and to the past stresses. Given an elastic body occupying a region $\Omega \subset \mathbb{R}^3$, a bounded domain with smooth boundary $\partial\Omega$, we consider the following linear control equation ruling the evolution of the displacement function $u = u(\mathbf{x}, t) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$,

$$\ddot{u}(\mathbf{x}, t) - \Delta \left[u(\mathbf{x}, t) - \int_0^\infty \mu(s) u(\mathbf{x}, t-s) ds \right] = c_1 u(\mathbf{x}, t) + c_2 \alpha(\mathbf{x}, t), \quad t > 0, \quad (4.9)$$

where, for a.e. $t \in [0, 1]$,

$$\alpha(\cdot, t) \in U(t) = L^2(\Omega).$$

The *memory kernel* μ complies with the assumptions in Sect. 2.3, the coefficients c_1, c_2 are non-negative, the term $c_1 u$ represents a displacement-dependent body force density, while $c_2 \alpha$ an external force acting on \mathbf{x} at time t . The body is kept fixed at the boundary of Ω for all times; hence, u is required to satisfy the Dirichlet boundary condition

$$u(\mathbf{x}, t)|_{\mathbf{x} \in \partial\Omega} = 0. \quad (4.10)$$

Also, some initial conditions are needed, namely

$$u(\mathbf{x}, 0) = w_0(\mathbf{x}), \quad \dot{u}(\mathbf{x}, 0) = v_0(\mathbf{x}), \quad u(\mathbf{x}, -s)|_{s>0} = \varphi_0(\mathbf{x}, s), \quad \text{for } \mathbf{x} \in \Omega \quad (4.11)$$

with $w_0 \in H_0^1(\Omega)$, $v_0 \in L^2(\Omega)$, $\varphi_0 \in L_\mu^2(\mathbb{R}^+; H_0^1(\Omega))$ prescribed data.

We are interested in minimizing the cost functional

$$\mathcal{J}(u, \alpha) = \int_0^1 \int_\Omega \mathcal{L}(u(\mathbf{x}, t), \alpha(\mathbf{x}, t)) d\mathbf{x} dt + \int_\Omega \mathcal{G}(u(\mathbf{x}, 1)) d\mathbf{x}, \quad (4.12)$$

over all the pairs (u, α) which are admissible for system (4.9)–(4.11). We take $H = L^2(\Omega)$ and, supposing that the functions involved in the problem are Lebesgue measurable in \mathbf{x} , we omit writing the dependence on \mathbf{x} . We shall assume the following conditions on the data:

- \mathcal{G} is convex and locally Lipschitz;
- \mathcal{L} is the Nemytskii operator associated to a continuous function $L : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ which is locally Lipschitz in the first variable, convex in both variables, and such that

$$L(y, z) \geq c_1(|y| + |z|)^q - c_2, \quad \text{for any } y, z \in \mathbb{R}, \text{ some } b_1, b_2 > 0, \text{ and } q \geq 2.$$

Then, taking $\bar{\alpha} \equiv 0$ in (2.5)–(2.6) and arguing as in Sect. 4.1, it is not difficult to prove that the hypotheses of Theorem 3.1 and Corollary 3.4 are satisfied. Hence, problem (4.9)–(4.12) admits an optimal solution and no Lavrentiev type phenomena can occur.

5. Proofs

For the proof, we will distinguish four cases: the abstract problem (1.1)(1.5), the hyperbolic integrodifferential problem (1.1)(1.2)(1.8), the abstract problem (1.1)(1.5) with $S(t)$ compact for $t > 0$, and the parabolic integrodifferential problem (1.1)(1.3)(1.9).

5.1. The abstract problem (1.1)(1.5)

In order to prove Theorems 3.1 and 3.3, some preliminary results are needed. Since $S(t)$ is a strongly continuous semigroup

$$\|S(t)\|_{\mathbb{L}(H)} \leq M_S \quad \text{for any } t \in [0, 1] \text{ and some } M_S > 0. \quad (5.1)$$

Also, for any $c > 0$, there exists $r_c > 0$ such that

$$\frac{\Phi(r)}{r} \geq c \quad \text{for any } r \geq r_c, \quad (5.2)$$

with Φ as in (1.4).

The following technical lemmas are instrumental for the proof.

LEMMA 5.1. *Assume (2.2)–(2.6). Then, for any $u_0 \in H$, system (1.5) admits a (mild) solution \bar{u} corresponding to the control $\bar{\alpha}$ that, by hypothesis, satisfies (2.5) and (2.6). Moreover,*

$$\|\bar{u}(t)\|_H \leq \bar{R} = (\|u_0\|_H + \|\phi\|_{L^1})M_S e^{M_S \|\phi\|_{L^1}}, \quad \text{for any } t, \quad (5.3)$$

and

$$\mathcal{J}(\bar{u}, \bar{\alpha}) \leq \|m_{\bar{R}}\|_{L^1} + g(0) + g_{\bar{R}} 2\bar{R} \quad (5.4)$$

Proof. Assumptions (2.2)–(2.5) ensure the existence of the solution \bar{u} . Moreover, since

$$\begin{aligned} \|\bar{u}(t)\|_H &= \left\| S(t)u_0 + \int_0^t S(t-\tau)f(\tau, \bar{u}(\tau), \bar{\alpha}(\tau))d\tau \right\|_H \\ &\leq M_S \|u_0\|_H + M_S \int_0^t \phi(\tau)(1 + \|u(\tau)\|_H)d\tau, \end{aligned}$$

(5.3) follows immediately from the Gronwall lemma. Finally, a straightforward application of (2.4) and (2.6) implies (5.4). \square

LEMMA 5.2. *Assume (1.4). Let $\{(u_n, \alpha_n)\}$ be a sequence of admissible pairs for problem (1.5) satisfying, for any $n \in \mathbb{N}$,*

$$\mathcal{J}(u_n, \alpha_n) \leq \overline{C}.$$

Then, the sequence $\{f(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$ is equi-integrable and, for any n ,

$$\|f(\cdot, u_n(\cdot), \alpha_n(\cdot))\|_{L^1(I; H)} \leq r_1 + \overline{C}, \quad \text{with } r_1 \text{ as in (5.2).} \quad (5.5)$$

Consequently,

$$\|u_n(t)\| \leq M_S(\|u_0\|_H + r_1 + \overline{C}), \quad \text{for any } n \text{ and any } t, \quad (5.6)$$

and, up to a subsequence,

$$f(\cdot, u_n(\cdot), \alpha_n(\cdot)) \rightharpoonup \xi \quad \text{weakly in } L^1(I; H), \quad (5.7)$$

for some $\xi \in L^1(I; H)$. Moreover, for any $t \in I$,

$$u_n(t) \rightharpoonup u(t) \quad \text{weakly in } H, \text{ where } u(t) = S(t)u_0 + \int_0^t S(t-\tau)\xi(\tau)d\tau. \quad (5.8)$$

Proof. Bound (5.5) follows immediately from growth condition (1.4), and (5.6) is a consequence of (2.1). To prove the equi-integrability of $\{f(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$, fix $\varepsilon > 0$. Let $r_{1/\varepsilon}$ be as in (5.2). Then, by (1.4),

$$\begin{aligned} & \int_{\{t: \|f(t, u_n(t), \alpha_n(t))\|_H \geq r_{1/\varepsilon}\}} \|f(t, u_n(t), \alpha_n(t))\|_H dt \\ & \leq \varepsilon \int_{\{t: \|f(t, u_n(t), \alpha_n(t))\|_H \geq r_{1/\varepsilon}\}} L(t, u_n(t), \alpha_n(t)) dt \leq \overline{C}\varepsilon. \end{aligned}$$

Hence, the sequence $\{f(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$ is equi-integrable and, as H is reflexive, the Dunford-Pettis theorem, see [31], allows to deduce (5.7). This implies that, for any $t \in I$ and any $\zeta \in H$,

$$\begin{aligned} & \int_0^t \langle S(t-\tau)f(\tau, u_n(\tau), \alpha_n(\tau)), \zeta \rangle_H d\tau = \int_0^t \langle f(\tau, u_n(\tau), \alpha_n(\tau)), S^*(t-\tau)\zeta \rangle_H d\tau \\ & \rightarrow \int_0^t \langle \xi(\tau), S^*(t-\tau)\zeta \rangle_H d\tau = \int_0^t \langle S(t-\tau)\xi(\tau), \zeta \rangle_H d\tau, \end{aligned} \quad (5.9)$$

where $S^*(t)$ is the adjoint semigroup of $S(t)$. The arbitrariness of ζ and representation formula (2.1) yield (5.8). \square

LEMMA 5.3. Assume (1.4) and (2.2)–(2.4). Let $\tilde{\mathcal{J}} : \mathcal{U} \rightarrow \mathbb{R} \cup \{+\infty\}$ and $d : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}^+$ be defined by

$$\tilde{\mathcal{J}}(\alpha) := \begin{cases} \mathcal{J}(u, \alpha) & \text{if there exists } u \text{ such that } (u, \alpha) \text{ is an admissible pair} \\ +\infty & \text{if there is no } u \text{ such that } (u, \alpha) \text{ is an admissible pair} \\ +\infty & \text{if } L(\cdot, u(\cdot), \alpha(\cdot)) \notin L^1(I; \mathbb{R}), \end{cases}$$

and

$$d(\alpha, \beta) := m(\{t \in I : \alpha(t) \neq \beta(t)\}), \quad \forall \alpha, \beta \in \mathcal{U}.$$

Then, d is a distance and (\mathcal{U}, d) is a complete metric space. Moreover, $\tilde{\mathcal{J}}$ is lower semicontinuous with respect to this metric.

Proof. Classical results, see [32], ensure that d is a distance and (\mathcal{U}, d) is a complete metric space. It remains to show the lower semicontinuity of $\tilde{\mathcal{J}}$ in this space, namely, for any $\alpha_n, \alpha \in \mathcal{U}$ satisfying $\lim_{n \rightarrow \infty} d(\alpha_n, \alpha) = 0$, the following inequality holds

$$\tilde{\mathcal{J}}(\alpha) \leq \liminf_{n \rightarrow \infty} \tilde{\mathcal{J}}(\alpha_n). \quad (5.10)$$

If $\liminf_{n \rightarrow \infty} \tilde{\mathcal{J}}(\alpha_n) = \infty$, then (5.10) holds true. Suppose next that $\liminf_{n \rightarrow \infty} \tilde{\mathcal{J}}(\alpha_n) = C$, for some $C > 0$. Up to a subsequence, we may assume that $\lim_{n \rightarrow \infty} \tilde{\mathcal{J}}(\alpha_n) = C$ and $\tilde{\mathcal{J}}(\alpha_n) \leq C + 1$, for any $n \in \mathbb{N}$, implying the existence of $u_n \in \mathcal{C}(I, H)$ such that (u_n, α_n) is an admissible pair.

Step 1: weak convergence of trajectories.

By Lemma 5.2, it is not restrictive to assume that for some $\xi \in L^1(I; H)$ and $u \in \mathcal{C}(I, H)$ defined as in (5.8)

$$f(\cdot, u_n(\cdot), \alpha_n(\cdot)) \rightharpoonup \xi \quad \text{weakly in } L^1(I; H), \quad \text{and} \quad u_n(t) \rightharpoonup u(t) \quad \text{weakly in } H.$$

Step 2: strong convergence of trajectories.

We prove that for any $t \in I$, $\{u_n(t)\}$ is a Cauchy sequence in H . Indeed, since $\{\alpha_n\}$ is a Cauchy sequence in (\mathcal{U}, d) and, by Lemma 5.2, $\{f(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$ is equi-integrable, for any $\varepsilon > 0$, we can find n_ε such that, for any $n, m \geq n_\varepsilon$

$$\int_{\{0 < \tau < t : \alpha_n(\tau) \neq \alpha_m(\tau)\}} \|f(\tau, u_n(\tau), \alpha_n(\tau)) - f(\tau, u_m(\tau), \alpha_m(\tau))\|_H d\tau \leq \varepsilon. \quad (5.11)$$

As $S(t)$ is a strongly continuous semigroup, $u_n(t), u(t) \in B_H(R)$, for some $R > 0$, any $n \in \mathbb{N}, t \in I$. Also, applying (2.1), (2.4), and (5.11), the following estimates hold for any $n, m \geq n_\varepsilon$

$$\begin{aligned} \|u_n(t) - u_m(t)\|_H &\leq M_S \int_0^t \|f(\tau, u_n(\tau), \alpha_n(\tau)) - f(\tau, u_m(\tau), \alpha_m(\tau))\|_H d\tau \\ &\leq M_S \int_{\{0 < \tau < t : \alpha_n(\tau) \neq \alpha_m(\tau)\}} \|f(\tau, u_n(\tau), \alpha_n(\tau)) - f(\tau, u_m(\tau), \alpha_m(\tau))\|_H d\tau \\ &\quad + M_S \int_0^t \|f(\tau, u_n(\tau), \alpha_n(\tau)) - f(\tau, u_m(\tau), \alpha_n(\tau))\|_H d\tau \\ &\leq M_S \varepsilon + M_S \int_0^t k_R(\tau) \|u_n(\tau) - u_m(\tau)\|_H d\tau. \end{aligned}$$

Then, the Gronwall lemma ensures that

$$\|u_n(t) - u_m(t)\|_H \leq M_S \varepsilon e^{M_S \int_0^t k_R(\tau) d\tau} \leq C\varepsilon,$$

for some positive C , implying the convergence $u_n(t) \rightarrow u(t)$ in H .

Step 3: (u, α) is an admissible pair.

Recalling (2.1), we need to prove that, for any t ,

$$\int_0^t S(t-\tau) \xi(\tau) d\tau = \int_0^t S(t-\tau) f(\tau, u(\tau), \alpha(\tau)) d\tau. \quad (5.12)$$

Notice that, since $u_n(t) \rightarrow u(t)$ in H , from (5.7) we deduce that

$$f(\cdot, u(\cdot), \alpha_n(\cdot)) \rightharpoonup \xi \quad \text{weakly in } L^1(I; H). \quad (5.13)$$

Indeed, by (2.4), for any $t \in I, n \in \mathbb{N}$,

$$\|f(t, u(t), \alpha_n(t)) - f(t, u_n(t), \alpha_n(t))\|_H \leq k_R(t) \|u(t) - u_n(t)\|_H.$$

Hence, for any $\zeta \in L^\infty(I; H)$, we get

$$\begin{aligned} \left| \int_0^t \langle f(\tau, u(\tau), \alpha_n(\tau)) - \xi(\tau), \zeta(\tau) \rangle_H d\tau \right| &\leq \|\zeta\|_{L^\infty(I; H)} \int_0^t k_R(\tau) \|u(\tau) \\ &\quad - u_n(\tau)\|_H d\tau + \left| \int_0^t \langle f(\tau, u_n(\tau), \alpha_n(\tau)) - \xi(\tau), \zeta(\tau) \rangle_H d\tau \right|. \end{aligned}$$

By (5.7) and the dominated convergence theorem applied to the last expression, we get (5.13). Now, since $d(\alpha_n, \alpha) \rightarrow 0$, there exists an increasing subsequence $\{n_i\}$ satisfying $d(\alpha_{n_i}, \alpha) \leq 1/2^i$. Setting $D_k = \{\tau \in I : \alpha_{n_i}(\tau) = \alpha(\tau), \text{ for any } i \geq k\}$, by the definition of d we obtain that

$$D_k = I \setminus \left(\bigcup_{i \geq k} \{t \in I : \alpha_{n_i}(t) \neq \alpha(t)\} \right), \quad D_k \subset D_{k+1} \quad \text{and} \quad m(D_k) \geq 1 - 1/2^{k-1}.$$

Fix k and take $i \geq k$. Since, for any $\tau \in D_k$,

$$f(\tau, u(\tau), \alpha_{n_i}(\tau)) = f(\tau, u(\tau), \alpha(\tau)),$$

applying (5.13) and arguing as in (5.9), for any $\zeta \in H$,

$$\begin{aligned} \int_{D_k \cap (0, t)} \langle S(t-\tau) \xi(\tau), \zeta \rangle_H d\tau &= \lim_{i \rightarrow \infty} \int_{D_k \cap (0, t)} \langle S(t-\tau) f(\tau, u(\tau), \alpha_{n_i}(\tau)), \zeta \rangle_H d\tau \\ &= \int_{D_k \cap (0, t)} \langle S(t-\tau) f(\tau, u(\tau), \alpha(\tau)), \zeta \rangle_H d\tau. \end{aligned}$$

As $m(\cup_k D_k) = 1$, $I = (\cup_k D_k) \cup \mathcal{N}$ with $m(\mathcal{N}) = 0$; hence, equality (5.12) holds for any t .

Step 4: lower semicontinuity of the functional.

To end the proof, notice that, taking $\{n_i\}$ and D_k as in Step 3, for any $k \in \mathbb{N}$, $t \in D_k$, $\lim_{i \rightarrow \infty} L(t, u(t), \alpha_{n_i}(t)) = L(t, u(t), \alpha(t))$ implying

$$\lim_{i \rightarrow \infty} L(t, u(t), \alpha_{n_i}(t)) = L(t, u(t), \alpha(t)) \quad \text{for a.e. } t \in I.$$

Moreover, applying (2.4),

$$\tilde{\mathcal{J}}(\alpha_{n_i}) \geq \int_0^1 L(t, u(t), a_{n_i}(t)) dt - \int_0^1 l_R(t) \|u_{n_i}(t) - u(t)\| dt + g(u_{n_i}(1)).$$

Then, since $\tilde{\mathcal{J}}(\alpha_n) \rightarrow C$ as $n \rightarrow \infty$, the dominated convergence theorem, the lower semicontinuity of g , and Fatou's lemma allow to conclude that

$$\liminf_{n \rightarrow \infty} \tilde{\mathcal{J}}(\alpha_n) = \lim_{i \rightarrow \infty} \mathcal{J}(\alpha_{n_i}) \geq \int_0^1 \liminf_{i \rightarrow \infty} L(t, u(t), \alpha_{n_i}(t)) dt + g(u(1)) = \tilde{\mathcal{J}}(\alpha).$$

□

Thanks to the above technical results, we can prove Lemma 3.2. Theorem 3.1 will follow immediately as a corollary.

Proof of Lemma 3.2. Given any minimizing sequence $\{(\bar{u}_n, \bar{\alpha}_n)\}$ for problem (1.1)(1.5), by Lemmas 5.1 and 5.2 we deduce that $\{f(\cdot, \bar{u}_n(\cdot), \bar{\alpha}_n(\cdot))\}$ is equi-integrable and uniformly bounded in L^1 by M_1 , for any n sufficiently large, where

$$M_1 = r_1 + \mathcal{J}(\bar{u}, \bar{\alpha}) + 1 \leq r_1 + \|m_{\bar{R}}\|_{L^1} + g_{\bar{R}} 2\bar{R} + 1, \quad (5.14)$$

with r_1 as in (5.2) and \bar{R} as in (5.3), and, by (5.6),

$$\|u_n(t)\|_H \leq M_S(\|u_0\|_H + M_1).$$

We are now going to apply Ekeland's principle [32] to obtain a new minimizing sequence $\{(u_n, \alpha_n)\}$ whose controls differ from the original one in a set of small measure, and such that, along it, the sequence of the Lagrangians $\{L(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$ is equi-integrable and uniformly bounded in L^1 by M_2 , where

$$M_2 = CM_1 + C\|\phi\|_{L^1}(1 + R) + \|m_R\|_{L^1} + 1, \quad (5.15)$$

$$R = e^{M_S}(\|u_0\|_H + M_1) \quad \text{and} \quad C = M_S e^{M_S \|k_R\|_{L^1}} (\|l_R\|_{L^1} + g_R).$$

Let $\{(\bar{u}_n, \bar{\alpha}_n)\}$ be a minimizing sequence of admissible pairs, namely,

$$\lim_{n \rightarrow \infty} \mathcal{J}(\bar{u}_n, \bar{\alpha}_n) = \inf \{ \mathcal{J}(u, \alpha) : (u, \alpha) \text{ is an admissible pair} \}, \quad (5.16)$$

and let $\varepsilon_n \downarrow 0$ satisfy

$$\mathcal{J}(\bar{u}_n, \bar{\alpha}_n) < \varepsilon_n + \inf \{ \mathcal{J}(u, \alpha) : (u, \alpha) \text{ is an admissible pair} \}.$$

Defining the lower semicontinuous functionals $\mathcal{J}_n : \mathcal{U} \rightarrow \mathcal{U}$ as

$$\mathcal{J}_n(\alpha) = \tilde{\mathcal{J}}(\alpha) + \sqrt{\varepsilon_n} d(\alpha, \bar{\alpha}_n),$$

with $\tilde{\mathcal{J}}$ as in Lemma 5.3 and applying Ekeland's principle, we obtain a sequence of controls $\{\alpha_n\}$ in the domain of $\tilde{\mathcal{J}}$ satisfying

$$d(\bar{\alpha}_n, \alpha_n) \leq \sqrt{\varepsilon_n} \quad \text{and} \quad \mathcal{J}_n(\alpha_n) = \inf \{ \mathcal{J}_n(\alpha) : \alpha \in \mathcal{U} \}. \quad (5.17)$$

By definition of $\tilde{\mathcal{J}}$ and α_n , for any n , there exists $u_n \in \mathcal{C}(I, H)$ such that $\{(u_n, \alpha_n)\}$ is a minimizing sequence of admissible pairs for problem (1.1)(1.5); hence, $\{f(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$ is equi-integrable and bounded in L^1 by M_1 .

Next, we claim that

$$\{L(\cdot, u_n(\cdot), \alpha_n(\cdot))\} \text{ is equi-integrable and, } \forall n, \quad \|L(\cdot, u_n(\cdot), \alpha_n(\cdot))\|_{L^1(I; \mathbb{R})} \leq M_2. \quad (5.18)$$

To prove it, we are going to perturb the controls α_n in a set of small measure with the control $\bar{\alpha}$ as in (2.5), (2.6), in order to get a family of admissible pairs which allow to compare the sequence $\{L(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$ and the equi-integrable sequence $\{f(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$. Fix $n \in \mathbb{N}$ and take a Lebesgue point $t_0 \in I$ of $L(\cdot, u_n(\cdot), \bar{\alpha}(\cdot)) - L(\cdot, u_n(\cdot), \alpha_n(\cdot))$ and $\|f(\cdot, u_n(\cdot), \alpha_n(\cdot)) - f(\cdot, u_n(\cdot), \bar{\alpha}(\cdot))\|_H$. For $h > 0$ set

$$\alpha_n^h(t) = \begin{cases} \bar{\alpha}(t) & \text{if } t_0 - h < t < t_0 \\ \alpha_n(t) & \text{otherwise.} \end{cases}$$

We prove that there exists a solution u_n^h to

$$\begin{cases} u'(t) = \mathbb{A}u(t) + f(t, u(t), \alpha_n^h(t)) \\ u(0) = u_0. \end{cases} \quad (5.19)$$

For this aim, we consider separately the three time intervals $[0, t_0 - h]$, $[t_0 - h, t_0]$ and $[t_0, 1]$. On $[0, t_0 - h]$ define $u_n^h = u_n$. On the time interval $[t_0 - h, t_0]$, the existence of a solution y_n^h to the system

$$\begin{cases} u'(t) = \mathbb{A}u(t) + f(t, u(t), \bar{\alpha}(t)) \\ u(t_0 - h) = u_n(t_0 - h) \end{cases}$$

is ensured by assumptions (2.4) and (2.5). Finally, an application of [35, Theorem 1.2] allows to solve the system

$$\begin{cases} u'(t) = \mathbb{A}u(t) + f(t, u(t), \alpha_n(t)) \\ u(t_0) = y_n^h(t_0) \end{cases} \quad (5.20)$$

on the last interval $[t_0, 1]$. Indeed, set for $t \in [t_0, 1]$,

$$y(t) = S(t - t_0)y_n^h(t_0) + \int_{t_0}^t S(t - \tau)f(\tau, u_n(\tau), \alpha_n(\tau))d\tau.$$

Since $S(t)$ is strongly continuous and $\{f(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$ is bounded in $L^1(I; H)$, there exists $\bar{h} > 0$ such that $u_n(t)$, $y(t) \in B_H(R)$, for any $t \in [t_0, 1]$, $h \leq \bar{h}$, and R as in (5.15). Moreover, using (2.4) and the representation formula (2.1), the function

$$\gamma(t) = \|f(t, u_n(t), \alpha_n(t)) - f(t, y(t), \alpha_n(t))\|_H,$$

satisfies the estimates

$$\gamma(t) \leq k_R(t)\|u_n(t) - y(t)\|_H \leq M_S k_R(t)\|u_n(t_0) - y_n^h(t_0)\|_H.$$

Now, applying again (2.1), the equi-integrability of $\{f(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$ and assumption (2.5) we deduce that,

$$\begin{aligned} \|u_n(t_0) - y_n^h(t_0)\|_H &\leq M_S \int_{t_0-h}^{t_0} \|f(\tau, u_n(\tau), \alpha_n(\tau)) \\ &\quad - f(\tau, y_n^h(\tau), \bar{\alpha}(\tau))\|_H d\tau \rightarrow 0 \quad \text{as } h \rightarrow 0. \end{aligned}$$

Hence, taking h sufficiently small,

$$\begin{aligned} \int_{t_0}^1 \gamma(\tau) \exp\left(\int_{t_0}^{\tau} k_R(s) ds\right) d\tau &\leq M_S \int_{t_0}^1 k_R(\tau) \|u_n(t_0) - y_n^h(t_0)\|_H e^{\|k_R\|_{L^1}} d\tau \\ &\leq M_S e^{\|k_R\|_{L^1}} \|k_R\|_{L^1} \|u_n(t_0) - y_n^h(t_0)\|_H \leq R. \end{aligned}$$

Theorem 1.2 in [35] ensures then the existence of a solution x_n^h to (5.20) on $[t_0, 1]$. Finally, the function

$$u_n^h(t) = \begin{cases} u_n(t) & \text{for } t \in [0, t_0 - h] \\ y_n^h(t) & \text{for } t \in [t_0 - h, t_0] \\ x_n^h(t) & \text{for } t \in [t_0, 1] \end{cases}$$

solves system (5.19). Also, $u_n(t), u_n^h(t) \in B_H(R)$, for any $t \in I$, any $n \in \mathbb{N}$, and

$$\begin{aligned} \|u_n(t) - u_n^h(t)\|_H &\leq M_S \int_0^t \|f(\tau, u_n(\tau), \alpha_n(\tau)) - f(\tau, u_n^h(\tau), \alpha_n^h(\tau))\|_H d\tau \\ &\leq M_S \int_0^t \left(\|f(\tau, u_n(\tau), \alpha_n(\tau)) - f(\tau, u_n(\tau), \alpha_n^h(\tau))\|_H \right. \\ &\quad \left. + \|f(\tau, u_n(\tau), \alpha_n^h(\tau)) - f(\tau, u_n^h(\tau), \alpha_n^h(\tau))\|_H \right) d\tau \\ &\leq M_S \int_{t_0-h}^{t_0} \|f(\tau, u_n(\tau), \alpha_n(\tau)) - f(\tau, u_n(\tau), \bar{\alpha}(\tau))\|_H d\tau \\ &\quad + M_S \int_0^t k_R(\tau) \|u_n(\tau) - u_n^h(\tau)\|_H d\tau. \end{aligned}$$

Applying the Gronwall lemma we get, for any $t \in I$,

$$\begin{aligned} \|u_n(t) - u_n^h(t)\|_H &\leq e^{M_S \|k_R\|_{L^1}} M_S \int_{t_0-h}^{t_0} \|f(\tau, u_n(\tau), \alpha_n(\tau)) \\ &\quad - f(\tau, u_n(\tau), \bar{\alpha}(\tau))\|_H d\tau. \end{aligned}$$

Hence, assumption (2.4) yields

$$\begin{aligned} \tilde{\mathcal{J}}(\alpha_n^h) &= \int_0^1 L(t, u_n^h(t), \alpha_n^h(t)) dt + g(u_n^h(1)) \\ &= \tilde{\mathcal{J}}(\alpha_n) + g(u_n^h(1)) - g(u_n(1)) + \int_0^1 \left[L(t, u_n(t), \alpha_n^h(t)) - L(t, u_n(t), \alpha_n(t)) \right] dt \end{aligned}$$

$$\begin{aligned}
 & + \int_0^1 \left[L(t, u_n^h(t), \alpha_n^h(t)) - L(t, u_n(t), \alpha_n^h(t)) \right] dt \\
 & \leq \tilde{\mathcal{J}}(\alpha_n) + \int_0^1 \left[L(t, u_n(t), \alpha_n^h(t)) - L(t, u_n(t), \alpha_n(t)) \right] dt \\
 & \quad + \int_0^1 l_R(t) \|u_n(t) - u_n^h(t)\|_H dt + g_R \|u_n^h(1) - u_n(1)\|_H \\
 & \leq \tilde{\mathcal{J}}(\alpha_n) + \int_{t_0-h}^{t_0} \left[L(t, u_n(t), \bar{\alpha}(t)) - L(t, u_n(t), \alpha_n(t)) \right] dt \\
 & \quad + M_S e^{M_S \|k_R\|_{L^1}} (\|l_R\|_{L^1} + g_R) \int_{t_0-h}^{t_0} \|f(t, u_n(t), \alpha_n(t)) \\
 & \quad - f(t, u_n(t), \bar{\alpha}(t))\|_H dt.
 \end{aligned}$$

So, since (5.17) implies

$$\tilde{\mathcal{J}}(\alpha_n) = \mathcal{J}_n(\alpha_n) \leq \mathcal{J}_n(\alpha_n^h) = \tilde{\mathcal{J}}(\alpha_n^h) + \sqrt{\varepsilon_n} h,$$

we obtain that

$$\begin{aligned}
 \int_{t_0-h}^{t_0} \left[L(t, u_n(t), \alpha_n(t)) - L(t, u_n(t), \bar{\alpha}(t)) \right] dt & \leq C \int_{t_0-h}^{t_0} \|f(t, u_n(t), \alpha_n(t)) \\
 & \quad - f(t, u_n(t), \bar{\alpha}(t))\|_H dt + \sqrt{\varepsilon_n} h,
 \end{aligned}$$

with C as in (5.15). Dividing by h the previous inequality, taking the limit as $h \rightarrow 0$, and recalling that t_0 is a Lebesgue point of

$$L(\cdot, u_n(\cdot), \alpha_n(\cdot)) - L(\cdot, u_n(\cdot), \bar{\alpha}(\cdot)) \quad \text{and} \quad \|f(\cdot, u_n(\cdot), \alpha_n(\cdot)) - f(\cdot, u_n(\cdot), \bar{\alpha}(\cdot))\|_H,$$

we conclude

$$\begin{aligned}
 L(t_0, u_n(t_0), \alpha_n(t_0)) & \leq C \|f(t_0, u_n(t_0), \alpha_n(t_0)) - f(t_0, u_n(t_0), \bar{\alpha}(t_0))\|_H \\
 & \quad + L(t_0, u_n(t_0), \bar{\alpha}(t_0)) + \sqrt{\varepsilon_n} \\
 & \leq C \|f(t_0, u_n(t_0), \alpha_n(t_0))\|_H + C\phi(t_0)(1 + R) + m_R(t_0) + 1,
 \end{aligned} \tag{5.21}$$

with R as in (5.15), and ϕ , m_R as in (2.5) and (2.6). Recalling (2.5), (2.6), and the equi-integrability of $\{f(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$, we conclude that the sequence $\{L(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$ is equi-integrable and bounded in L^1 by M_2 .

Assume now that $\phi, m_R \in L^\infty(I; \mathbb{R}^+)$. Estimate (5.21) give, for a.e. t , any n , C as in (5.15), and r_{2C} as in (5.2),

$$\begin{aligned}
 L(t, u_n(t), \alpha_n(t)) & \leq C \|f(t, u_n(t), \alpha_n(t))\|_H + C\phi(t)(1 + R) + m_R(t) + 1 \\
 & \leq \frac{L(t, u_n(t), \alpha_n(t))}{2} + Cr_{2C} + C\phi(t)(1 + R) + m_R(t) + 1
 \end{aligned}$$

implying

$$\|L(\cdot, u_n(\cdot), \alpha_n(\cdot))\|_{L^\infty(I; \mathbb{R})} \leq M_4 = 2Cr_{2C} + 2C\|\phi\|_{L^\infty}(1 + R) + 2\|m_R\|_{L^\infty} + 2. \quad (5.22)$$

The estimates

$$\|f(\cdot, u_n(\cdot), \alpha_n(\cdot))\|_{L^\infty(I; H)} \leq M_3 = r_1 + M_4 \quad (5.23)$$

follow immediately by (5.2). \square

Proof of Theorem 3.3. The proof of the existence of an optimal solution to (1.1)(1.5) is based on a direct method. Applying Lemmas 3.2 and 5.2, we deduce that there exists a minimizing sequence $\{(u_n, \alpha_n)\}$ such that, up to a subsequence, for some $\xi \in L^1(I; H)$ and $\psi \in L^1(I; \mathbb{R})$,

$$f(\cdot, u_n(\cdot), \alpha_n(\cdot)) \rightharpoonup \xi \quad \text{weakly in } L^1(I; H), \quad (5.24)$$

$$L(\cdot, u_n(\cdot), \alpha_n(\cdot)) \rightharpoonup \psi \quad \text{weakly in } L^1(I; \mathbb{R}), \quad (5.25)$$

and

$$u_n(t) \rightharpoonup u^*(t) \quad \text{weakly in } H, \quad \text{with} \quad u^*(t) = S(t)z_0 + \int_0^t S(t-\tau)\xi(\tau)d\tau. \quad (5.26)$$

Nothing can be said about the convergence of controls, since they are not assumed to be regular or bounded. Assumption (1.6) and Mazur's theorem imply that, for a.e. t ,

$$(u^*(t), \xi(t), \psi(t)) \in G(t).$$

Hence, applying a measurable selection result, see e.g. [1], we find two measurable functions α^*, v , such that $\alpha^* \in \mathcal{U}$, $v(t) \geq 0$, and

$$\begin{cases} \xi(t) = f(t, u^*(t), \alpha^*(t)) \\ \psi(t) = L(t, u^*(t), \alpha^*(t)) + v(t). \end{cases}$$

The weak-lower semicontinuity of g , which follows from assumptions (2.3), (2.4) and (5.25), yield

$$\begin{aligned} & \int_0^1 [L(t, u^*(t), \alpha^*(t)) + v(t)]dt + g(u^*(1)) \\ & \leq \lim_{n \rightarrow \infty} \left(\int_0^1 L(t, u_n(t), \alpha_n(t))dt + g(u_n(1)) \right) \\ & = \inf \{ \mathcal{J}(u, \alpha) : (u, \alpha) \text{ is an admissible pair} \}. \end{aligned}$$

So, the function v is identically zero and (u^*, α^*) is an optimal solution, proving the first claim of the theorem.

Finally, the fact that $f(\cdot, u^*(\cdot), \alpha(\cdot)) \in L^\infty(I; H)$ and $L(\cdot, u^*(\cdot), \alpha(\cdot)) \in L^\infty(I; \mathbb{R})$ follows easily from equality (3.1). \square

5.2. The hyperbolic integrodifferential problem

Theorem 3.1 and Corollary 3.4 are obtained as immediate consequences of the corresponding results for the abstract problem (1.1)(1.5). Indeed, applying Dafermos' *history approach*, we can translate the integrodifferential control problem (1.2)(1.8) into a semilinear equation of type (1.5), generating a dynamical system on a suitable abstract space. The key idea is the introduction of an auxiliary variable, ruled by its own equation, which contains all the information about the unknown function up to the actual time:

$$\eta^t(s) = u(t) - u(t-s), \quad t \geq 0, \quad s > 0.$$

Then, problem (1.2)(1.8) can be recast as the system of two variables $u = u(t)$ and $\eta = \eta^t(s)$

$$\begin{cases} \ddot{u}(t) + A \left[(1-\kappa)u(t) + \int_0^\infty \mu(s)\eta^t(s)ds \right] = f(t, u(t), \alpha(t)), \\ \dot{\eta}^t = T\eta^t + \dot{u}(t), \\ \alpha(t) \in U(t), \end{cases} \quad (5.27)$$

with T as in Sect. 2.5, and initial conditions

$$u(0) = w_0, \quad \dot{u}(0) = v_0, \quad \eta^0(s) = w_0 - \varphi_0(s). \quad (5.28)$$

Now, we consider the product space

$$\mathcal{H} = H \times \text{dom}(A^{1/2}) \times \mathcal{M},$$

with \mathcal{M} as in Sect. 2.5, and the linear operator \mathbb{A} on \mathcal{H} , acting as

$$\mathbb{A}(u, v, \eta) = \left(v, -A \left[(1-\kappa)u + \int_0^\infty \mu(s)\eta(s)ds \right], T\eta + v \right)$$

with domain

$$\text{dom}(\mathbb{A}) = \left\{ (u, v, \eta) \in \mathcal{H} \mid \begin{array}{l} v \in \text{dom}(A^{1/2}), \quad \eta \in \text{dom}(T) \\ (1-\kappa)u + \int_0^\infty \mu(s)\eta(s)ds \in \text{dom}(A) \end{array} \right\}.$$

Introducing the vectors

$$z(t) = (u(t), v(t), \eta(t)), \quad z_0 = (w_0, v_0, \eta_0), \quad F(t, z(t), \alpha(t)) = (0, f(t, u(t), \alpha(t)), 0),$$

we view (5.27) and (5.28) as the following Cauchy problem in \mathcal{H} :

$$\begin{cases} z'(t) = \mathbb{A}z(t) + F(t, z(t), \alpha(t)) & \text{for } t \in I \\ \alpha(t) \in U(t) & \text{for } t \in I \\ z(0) = z_0. \end{cases} \quad (5.29)$$

Applying the Lumer–Phillips theorem, see [51], we can prove that \mathbb{A} generates a strongly continuous semigroup of contractions $S(t) : \mathcal{H} \rightarrow \mathcal{H}$. Moreover, it is possible to prove that, if $z(t) = (u(t), v(t), \eta(t))$ is a solution to (5.29) related to some $\alpha \in \mathcal{U}$, the first component $u(t)$ is a solution of (1.2)(1.8) in the sense of Definition 2.2, related to the same α , see, for example, [49]. Notice that, if $\eta_0 \in \mathcal{M}$, the function F_0 appearing in Definition 2.2 takes values in $\text{dom}(A^{1/2})$ a.e. in I .

In this setting, for $z = (u, v, \eta) \in \mathcal{H}$ and $\alpha \in \mathcal{U}$, we consider the problem of minimizing the functional

$$\mathfrak{J}(z, \alpha) = \int_0^1 L(t, u(t), \alpha(t)) dt + g(u(1)), \quad (5.30)$$

over all trajectory/control pairs (z, α) solutions to (5.29), with L and g as in the original functional (1.1). It is not difficult to prove that the data of problem (5.29)(5.30) satisfy the assumptions of Theorems 3.1 and 3.3. Moreover, due to the correspondence between the first component of the solutions of (5.29) and the solutions of (1.2)(1.8), we obtain that all the claims of Theorem 3.1 and Corollary 3.4 hold true.

5.3. The case of compact semigroups

It remains to prove Theorem 3.5. Here, a stronger version of Lemma 5.2 ensuring norm convergence of the trajectories is available. This allows to obtain an existence result also with the classical Cesari condition (1.7).

LEMMA 5.4. *Let $S(t)$ be a strongly continuous semigroup of compact operators (for all $t > 0$) and let $\{(u_n, \alpha_n)\}$ be a sequence of admissible pairs for problem (1.5) satisfying, for any $n \in \mathbb{N}$,*

$$\mathcal{J}(u_n, \alpha_n) \leq \overline{C}.$$

Then, the sequence $\{f(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$ is equi-integrable and bounded in $L^1(I; H)$. Consequently, $\{u_n\}$ is bounded in $\mathcal{C}(I, H)$, and, up to a subsequence,

$$f(\cdot, u_n(\cdot), \alpha_n(\cdot)) \rightharpoonup \xi \quad \text{weakly in } L^1(I; H),$$

for some $\xi \in L^1(I; H)$. Moreover, for any $t \in I$,

$$u_n(t) \rightarrow u(t) \quad \text{in } H, \quad \text{where } u(t) = S(t)u_0 + \int_0^t S(t-s)\xi(s)ds. \quad (5.31)$$

Proof. Working exactly as in the proof of Lemma 5.2, we can prove all the claims of Lemma 5.4, except the strong convergence of the trajectories (5.31) which follows from the compactness of the semigroup $S(t)$, see [9]. Indeed, for any $n, k \in \mathbb{N}$, $t \in I$, set $t_k = \max\{0, t - 1/k\}$,

$$y_n^k(t) = S(t)u_0 + S(1/k) \int_0^{t_k} S(t - 1/k - s)f(s, u_n(s), \alpha_n(s))ds$$

and

$$z_n^k(t) = \int_{t_k}^t S(t-s) f(s, u_n(s), \alpha_n(s)) ds.$$

Fix $k \in \mathbb{N}$. Since $S(1/k)$ is compact and $f(\cdot, u_n(\cdot), \alpha_n(\cdot)) \rightharpoonup \xi$ weakly in $L^1(I; H)$, we obtain the strong convergence in H

$$\lim_{n \rightarrow \infty} y_n^k(t) = S(t)u_0 + \int_0^{t_k} S(t-s)\xi(s)ds \quad \text{for any } t \in I. \quad (5.32)$$

On the other hand, the equi-integrability of $\{f(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$ ensures that

$$\lim_{k \rightarrow \infty} z_n^k(t) = 0 \quad \text{uniformly in } n \text{ and } t. \quad (5.33)$$

Now, let $t \in I$ be fixed and let $\varepsilon > 0$. We prove that for some n_ε and any $n \geq n_\varepsilon$

$$\|u_n(t) - u(t)\|_H < \varepsilon.$$

As ξ is summable, by (5.33) there exists \bar{k} such that

$$M_S \int_{t_{\bar{k}}}^t \|\xi(s)\|_H ds + \|z_n^{\bar{k}}(t)\|_H < \varepsilon/2, \quad \text{for any } n \in \mathbb{N},$$

with M_S as in (5.1), while from (5.32) we deduce that there exists n_ε such that, for any $n \geq n_\varepsilon$,

$$\left\| y_n^{\bar{k}}(t) - S(t)u_0 - \int_0^{t_{\bar{k}}} S(t-s)\xi(s)ds \right\|_H < \varepsilon/2.$$

Then, for any $n \geq n_\varepsilon$,

$$\begin{aligned} \|u_n(t) - u(t)\|_H &= \|z_n^{\bar{k}}(t) + y_n^{\bar{k}}(t) - u(t)\|_H \leq M_S \int_{t_{\bar{k}}}^t \|\xi(s)\|_H ds \\ &\quad + \|z_n^{\bar{k}}(t)\|_H + \left\| y_n^{\bar{k}}(t) - S(t)u_0 - \int_0^{t_{\bar{k}}} S(t-s)\xi(s)ds \right\|_H < \varepsilon, \end{aligned}$$

implying (5.31). □

Following the same steps as in the proof of Lemma 3.1, we obtain the limits (5.24), (5.25). By Mazur's lemma, for a.e. t ,

$$(\xi(t), \psi(t)) \in \bigcap_{n \geq 1} \overline{\text{co}}\{(f(t, u_k(t), \alpha_k(t)), L(t, u_k(t), \alpha_k(t))) : k \geq n\},$$

where $\overline{\text{co}}$ denotes the closed convex hull. Also, applying Lemma 5.4 instead of Lemma 5.2, we get the convergence

$$u_n(t) \rightarrow u^*(t) \quad \text{in } H, \quad \text{with} \quad u^*(t) = S(t)u_0 + \int_0^t S(t-s)\xi(s)ds,$$

which is stronger than (5.26) in the proof of Lemma 3.2. This allows to deduce that, for a.e. t ,

$$(\xi(t), \psi(t)) \in \bigcap_{n \geq 1} \overline{\text{co}} \left\{ G(t, u^*(t)) + k_R(t) \sup_{k \geq n} \|u_n(t) - u(t)\|_H B_{H \times \mathbb{R}}(1) \right\},$$

with $R > 0$ such that $u_n(t), u^*(t) \in B_H(R)$ for any t , and k_R as in (2.4). Hence, since the sets $G(t, u^*(t)) + k_R(t) \sup_{k \geq n} \|u_n(t) - u(t)\|_H B_{H \times \mathbb{R}}(1)$ are closed and convex for a.e. t and any $n \in \mathbb{N}$,

$$(\xi(t), \psi(t)) \in \bigcap_{n \geq 1} \left\{ G(t, u^*(t)) + k_R(t) \sup_{k \geq n} \|u_n(t) - u(t)\|_H B_{H \times \mathbb{R}}(1) \right\} = G(t, u^*(t)).$$

A measurable selection theorem, see [1], ensures the existence of two measurable functions α^*, v , such that $\alpha^* \in \mathcal{U}$, $v(t) \geq 0$, and

$$\begin{cases} \xi(t) = f(t, u^*(t), \alpha^*(t)) \\ \psi(t) = L(t, u^*(t), \alpha^*(t)) + v(t). \end{cases}$$

Now, arguing as in the proof of Theorem 3.3, one can obtain the remaining conclusions of Theorem 3.5.

5.4. The parabolic integrodifferential problem

The analysis of (1.1)(1.3)(1.9) is performed following the same lines as in Sect. 5.1. Here, we replace semigroup techniques by resolvent operators techniques, which allow to exploit completely the compactness properties of Eq. (1.3). In the remaining part of the section, H and A are understood to be the complexifications of the original space and generator, respectively.

The assumptions on A and β ensure the existence of a resolvent operator $R(t) : H \rightarrow H$, for $t \geq 0$, for the integrodifferential equation

$$\begin{cases} \dot{u}(t) + A[u(t) + \int_0^t \beta(s)u(t-s)ds] = \Psi(t) & \text{for } t \in I \\ u(0) = u_0, \end{cases} \quad (5.34)$$

with $\Psi : I \rightarrow H$. In particular, for any $u_0 \in H$ and $\Psi \in L^1(I; H)$, there exists a unique mild solution $u \in C([0, 1], H)$ of (5.34) represented by the formula

$$u(t) = R(t)u_0 + \int_0^t R(t-s)\Psi(s)ds. \quad (5.35)$$

A regularity result proved in [10] allows then to conclude that $u \in L^2(I; \text{dom}(A^{1/2}))$. Moreover, the following inequality holds (see e.g. [40])

$$\left(\int_0^1 \left\| \int_0^t \beta(s)A^{1/2}u(t-s)ds \right\|^2 dt \right)^{1/2} \leq \|\beta\|_{L^1(\mathbb{R}^+)} \|u\|_{L^2(0,1;\text{dom}(A^{1/2}))}$$

implying that the function

$$t \rightarrow \int_0^t \beta(s) Au(t-s) ds$$

belongs to $L^2(0, 1; \text{dom}(A^{-1/2}))$. Hence, by classical results obtained by the Faedo–Galerkin method, system (5.34) admits a solution in the sense of Definition 2.3.

Using analyticity as shown, for example, in [26, 27, 45, 52], one can represent $R(t)$ as

$$R(t) = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} F(\lambda) d\lambda, \quad \forall t \geq 0, \quad (5.36)$$

where

$$F(\lambda) = (\lambda + A + \hat{\beta}(\lambda)A)^{-1}$$

is formally the Laplace transform of the resolvent of problem (1.3), and γ is a suitable path defined below.

Notice that to obtain the existence of a resolvent operator for problem (1.3)(1.9), we could weaken the assumptions on β , see, for example, [10, 20, 52] and the bibliographies contained therein. As for the inversion of the Laplace transform of R , we refer the reader to [16, 39] where such a result is proved under weaker assumptions on the data.

We now recall the properties of F we will need in the proofs: fixed $\theta_0 \in (\pi/2, \theta)$, with θ as in Sect. 2.4, there exists $r_0 > 0$ such that, for any λ satisfying $|\lambda| \geq r_0$ and $|\arg \lambda| \leq \theta_0$,

$$\begin{aligned} F(\lambda) \in \mathbb{L}(H) \quad & \text{is a compact operator,} \\ \|F(\lambda)\|_{\mathbb{L}(H)} & \leq \frac{C}{|\lambda|}, \quad \text{for some } C > 0. \end{aligned} \quad (5.37)$$

The path $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, with

$$\begin{aligned} \gamma_1 &= \{\lambda \in \mathbb{C} : |\lambda| \geq r_0, |\arg \lambda| = -\theta_0\}, \\ \gamma_2 &= \{\lambda \in \mathbb{C} : |\lambda| = r_0, |\arg \lambda| \leq \theta_0\}, \\ \gamma_3 &= \{\lambda \in \mathbb{C} : |\lambda| \geq r_0, |\arg \lambda| = \theta_0\} \end{aligned}$$

is oriented counterclockwise.

Following the same lines as in the proof of Theorem 3.1 for the abstract problem (1.1)(1.5) and using the representation formula (5.35) instead of (2.1) the conclusions of Theorem 3.1 hold also in the case of problem (1.1)(1.3)(1.9). To this hand, we have to notice that [35, Theorem 1.2] is valid also in the case of (1.3), replacing again (2.1) by (5.35). Moreover, we can still evaluate exactly the constants M_1, M_2, M_3, M_4 in the theorem, as in the abstract case. Here, M_S satisfies $\|R(t)\|_{\mathbb{L}(H)} \leq M_S$, for any t .

Concerning Theorem 3.6, as in the case of the compact semigroup of Theorem 3.5, the parabolicity of Eq. (1.3) allows to obtain a stronger version of Lemma 5.2.

LEMMA 5.5. *Let $\{(u_n, \alpha_n)\}$ be a sequence of admissible pairs for problem (1.3)(1.9) satisfying, for any $n \in \mathbb{N}$,*

$$\mathcal{J}(u_n, \alpha_n) \leq \overline{C}. \quad (5.38)$$

Then, the sequence $\{f(\cdot, u_n(\cdot), \alpha_n(\cdot))\}$ is equi-integrable and bounded in $L^1(I; H)$. Consequently, $\{u_n\}$ is bounded in $\mathcal{C}(I, H)$, and, up to a subsequence,

$$f(\cdot, u_n(\cdot), \alpha_n(\cdot)) \rightharpoonup \xi \quad \text{weakly in } L^1(I; H),$$

for some $\xi \in L^1(I; H)$. Moreover, for any $t \in I$,

$$u_n(t) \rightarrow u(t) \quad \text{in } H, \quad \text{where } u(t) = R(t)u_0 + \int_0^t R(t-s)[AG_0(s) + \xi(s)]ds. \quad (5.39)$$

Proof. Working exactly as in the proof of Lemma 5.2, replacing formula (2.1) with the analogue (5.35), we can prove all the claims of the lemma, except the strong convergence of the trajectories (5.39). To this aim, we need to apply the properties of the resolvent family $R(t)$ listed above. By (5.35) and (5.36), exchanging the order of integration we get

$$\begin{aligned} u_n(t) &= R(t)u_0 + \int_0^t R(t-s)AG_0(s)ds + \int_0^t R(t-s)f(s, u_n(s), \alpha_n(s))ds \\ &= R(t)u_0 + \int_0^t R(t-s)AG_0(s)ds \\ &\quad + \frac{1}{2\pi i} \int_{\gamma} F(\lambda) \left(\int_0^t e^{\lambda(t-s)} f(s, u_n(s), \alpha_n(s))ds \right) d\lambda. \end{aligned} \quad (5.40)$$

Moreover, since $f(\cdot, u_n(\cdot), \alpha_n(\cdot)) \rightharpoonup \xi$ in $L^1(I; H)$ and $F(\lambda)$ is compact for any λ , we obtain, for any $t > 0$, the strong convergence in H

$$F(\lambda) \int_0^t e^{\lambda(t-s)} f(s, u_n(s), \alpha_n(s))ds \rightarrow F(\lambda) \int_0^t e^{\lambda(t-s)} \xi(s)ds.$$

Also, the functions $\lambda \rightarrow F(\lambda) \int_0^t e^{\lambda(t-s)} f(s, u_n(s), \alpha_n(s))ds$ admit a dominating summable function on γ . Indeed, from (3.2) and (5.38),

$$\int_0^1 \|f(s, u_n(s), \alpha_n(s))\|_H^p ds \leq \frac{\mathcal{J}(u_n, \alpha_n) + b}{a} \leq c,$$

for some $c > 0$, and, setting $p' = p/(p-1)$, from (5.37), for some $M > 0$,

$$\begin{aligned} \|F(\lambda) \int_0^t e^{\lambda(t-s)} f(s, u_n(s), \alpha_n(s))ds\|_H &\leq \frac{M}{|\lambda|} \int_0^t e^{Re(\lambda)(t-s)} \|f(s, u_n(s), \alpha_n(s))\|_H ds \\ &\leq \frac{M}{|\lambda|} \left(\int_0^t e^{p' Re(\lambda)(t-s)} ds \right)^{1/p'} \left(\int_0^1 \|f(s, u_n(s), \alpha_n(s))\|_H^p ds \right)^{1/p} \\ &\leq \frac{Mc^{1/p}}{|\lambda|^{1/p'} Re(\lambda)^{1/p'}} = \Gamma(\lambda). \end{aligned}$$

Since Γ is summable over γ , the dominated convergence theorem allows to deduce the desired limit (5.39). \square

Following the same arguments as in the proof of Theorem 3.5 and applying Lemma 5.5 instead of Lemma 5.4, we obtain Theorem 3.6.

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