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FREE GROUP FACTORS AND HECKE OPERATORS

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ABSTRACT. This is a note on Radulescu's work taken by N. Ozawa. We first review the construction of the classical Hecke operator and recall the Ramanujan–Petersson conjecture about its spectrum. We then relate it to the study of a type II_1 factor via Berezin calculus.

Part 1. Hecke algebras (Ref: [Kr]).

1. DEFINITION OF HECKE ALGEBRAS

For subsets $\alpha, \beta \subset G$ of a group G, we define α^* and $\alpha\beta \subset G$ by

$$\alpha^* = \{g^{-1} : g \in \alpha\} \text{ and } \alpha\beta = \{x : \exists g \in \alpha, \exists h \in \beta \text{ such that } x = gh\}.$$

Let $\Gamma \le G$ be discrete groups, and $\rho\colon G \curvearrowright \ell^2(\Gamma\backslash G)$ be the quasi-regular representation defined by $\rho(g)\xi(x) = \xi(xg)$. What is $\rho(G)''$? It suffices to know $\rho(G)'$. Since δ_Γ is $\rho(G)$-cyclic, it is $\rho(G)'$-separating. Let $T \in \rho(G)'$. Then, $T\delta_\Gamma \in \ell^2(\Gamma\backslash G)$ is $\rho(\Gamma)$-invariant. Hence, $T\delta_\Gamma$ can be regarded as a function on the double coset space $\Gamma\backslash G/\Gamma$, which is supported on those double cosets $\Gamma g\Gamma$ which are finite unions of right cosets.

Lemma 1.1. *For $\Gamma \le G$, TFAE.*

1. *Every double coset is a finite union of right cosets.*
2. *Every double coset is a finite union of left cosets.*
3. *For every $g \in G$, one has $[\Gamma : \Gamma \cap g^{-1}\Gamma g] < +\infty$.*

Moreover, for $s_i \in \Gamma$, one has $\Gamma g\Gamma = \bigsqcup \Gamma g s_i$ iff $\Gamma = \bigsqcup (\Gamma \cap g^{-1}\Gamma g)s_i$.

We assume that $\Gamma \le G$ satisfies the above conditions. Such a pair $\Gamma \le G$ is called a Hecke pair. We define $\mathrm{ind}\colon \Gamma\backslash G/\Gamma \to \mathbb{N}$ by

$$\mathrm{ind}(\alpha) = |\{x \in \Gamma\backslash G : x \subset \alpha\}| = [\Gamma : \Gamma \cap g^{-1}\Gamma g].$$

For each $\alpha \in \Gamma\backslash G/\Gamma$ with $\alpha = \bigsqcup_{i=1}^{\mathrm{ind}(\alpha)} \Gamma g_i$, there is an operator $\lambda(\alpha) \in \rho(G)'$ such that $\lambda(\alpha)\delta_\Gamma = \delta_\alpha = \sum_i \delta_{\Gamma g_i}$. (We freely view a function on $\Gamma\backslash G/\Gamma$ as a right Γ-invariant function on $\Gamma\backslash G$.) For $x = \Gamma h \in \Gamma\backslash G$, one has

$$\lambda(\alpha)\delta_x = \lambda(\alpha)\rho(h)\delta_\Gamma = \rho(h)\sum_i \delta_{\Gamma g_i} x = \sum_i \delta_{\Gamma g_i h} = \sum_{y \in \Gamma\backslash G,\ y \subset \alpha x} \delta_y = \chi_{\alpha x}.$$

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Thus, $\|\lambda(\alpha)\| \le \mathrm{ind}(\alpha)$ and

$$\lambda(\alpha)^*\delta_\Gamma = \sum_{x\in\Gamma\backslash G,\ \Gamma\subset\alpha x} \delta_x = \lambda(\alpha^*)\delta_\Gamma.$$

Let $\beta\in\Gamma\backslash G/\Gamma$ and $\beta = \bigsqcup_j \Gamma h_j$. Then,

$$\lambda(\alpha)\lambda(\beta)\delta_\Gamma = \lambda(\alpha)\sum_{x\subset\beta}\delta_x = \sum_{x\subset\beta}\sum_{y\subset\alpha x}\delta_y = \sum_{z\in\Gamma\backslash G,\ z\subset\alpha\beta} c(\alpha,\beta;z)\delta_z,$$

where for $z = \Gamma u$ one has

$$\begin{aligned} c(\alpha,\beta;z) &= |\{(i,j):\Gamma g_i h_j = z\}| = |\{j : u\in\alpha h_j\}| \\ &= |\{(g,h)\in\alpha\times\beta : gh = u\}/\sim_\Gamma|. \end{aligned}$$

The last one is the number of the Γ-orbits of $\{(g,h)\in\alpha\times\beta : gh = u\}$, where $s\in\Gamma$ acts on (g,h) by (gs^{-1}, sh). We observe that $z\mapsto c(\alpha,\beta;z)$ is constant on each double coset. For $\gamma\in\Gamma\backslash G/\Gamma$, let $c(\alpha,\beta;\gamma) = c(\alpha,\beta;z)$ for $z\subset\gamma$. It follows that

$$\lambda(\alpha)\lambda(\beta) = \sum_{\gamma\in\Gamma\backslash G/\Gamma} c(\alpha,\beta;\gamma)\lambda(\gamma)$$

and

$$\mathrm{ind}(\alpha)\,\mathrm{ind}(\beta) = \sum_z |\{(i,j):\Gamma g_i h_j = z\}| = \sum_{\gamma\in\Gamma\backslash G/\Gamma} c(\alpha,\beta;\gamma)\,\mathrm{ind}(\gamma).$$

These naturally make $\mathbb{C}[\Gamma\backslash G/\Gamma]$ a $*$-algebra, $\mathbb{C}[\Gamma\backslash G]$ a $\mathbb{C}[\Gamma\backslash G/\Gamma]$-module, and ind a homomorphism on $\mathbb{C}[\Gamma\backslash G/\Gamma]$. The trivial double coset Γ becomes the unit of the Hecke algebra $\mathbb{C}[\Gamma\backslash G/\Gamma]$. We note that for $\alpha,\beta\in\Gamma\backslash G/\Gamma$, the product $\alpha\cdot\beta$ in the Hecke algebra $\mathbb{C}[\Gamma\backslash G/\Gamma]$ has the set theoretic product $\alpha\beta$ as its support, but the coefficients of $\alpha\cdot\beta$ are not necessarily 1. We write $\mathscr{H} = \mathbb{C}[\Gamma\backslash G/\Gamma]$ and $\mathrm{vN}(\mathscr{H}) := \rho(G)' = \lambda(\mathscr{H})''$. The proof of the last equality is easy when the $\mathrm{vN}(\mathscr{H})$-separating vector δ_Γ is tracial (see [BC] for a general case). Indeed, $\lambda(\mathscr{H})\delta_\Gamma = \mathbb{C}[\Gamma\backslash G/\Gamma]$ is dense in $\mathrm{vN}(\mathscr{H})\delta_\Gamma$ which consists of $\rho(\Gamma)$-invariant functions in $\ell^2(\Gamma\backslash G)$. Let $T\in\mathrm{vN}(\mathscr{H})$ and $f = T\delta_\Gamma$ on $\Gamma\backslash G/\Gamma$. We write T formally as $\lambda(f)$. We will write ω for the vector state of δ_Γ:

$$\omega(\lambda(f)) = \langle f,\delta_\Gamma\rangle = f(\Gamma).$$

One still has

$$(\lambda(f)\lambda(g))(\gamma) = \sum_{\alpha,\beta\in\Gamma\backslash G/\Gamma} c(\alpha,\beta;\gamma) f(\alpha) g(\beta).$$

Moreover,

$$\|T\delta_\Gamma\|^2 = \sum_\alpha \mathrm{ind}(\alpha)|f(\alpha)|^2 \ \text{ and } \ \|T^*\delta_\Gamma\|^2 = \sum_\alpha \mathrm{ind}(\alpha^*)|f(\alpha)|^2.$$

Thus, δ_Γ is a trace vector for $\mathrm{vN}(\mathscr{H})$ iff ind is symmetric on $\Gamma\backslash G/\Gamma$.

A map $\sigma\colon G\to H$ between groups is called an anti-homomorphism if $\sigma(gh) = \sigma(h)\sigma(g)$ for all $g,h\in G$.

Lemma 1.2. *Let σ be an anti-automorphism on G such that $\sigma(\Gamma) = \Gamma$. Then, σ extends to an anti-automorphism on $\mathscr{H}$. Suppose moreover that $\sigma(\alpha) = \alpha$ for all $\alpha\in\Gamma\backslash G/\Gamma$. Then, $\mathscr{H}$ is commutative and δ_Γ is a separating trace vector for* $\mathrm{vN}(\mathscr{H})$.

Proof. We define $\sigma(\alpha) = \{\sigma(g) : g \in \alpha\} \in \Gamma\backslash G/\Gamma$. Then, for $\alpha,\beta,\gamma \in \Gamma\backslash G/\Gamma$ and $u \in \gamma$, one has

$$\begin{aligned} c(\alpha,\beta;\gamma) &= |\{(g,h) \in \alpha\times\beta : gh = u\}/\sim_\Gamma| \\ &= |\{(h',g') \in \sigma(\beta)\times\sigma(\alpha) : h'g' = \sigma(u)\}/\sim_\Gamma| \\ &= c(\sigma(\beta),\sigma(\alpha);\sigma(\gamma)) \end{aligned}$$

It follows that $\sigma(\alpha\cdot\beta) = \sigma(\beta)\cdot\sigma(\alpha)$ and σ extends to an anti-homomorphism on $\mathscr{H}$ such that $\mathrm{ind}(\sigma(\alpha)) = \mathrm{ind}(\alpha^*)$. If $\sigma(\alpha) = \alpha$ for all $\alpha \in \Gamma\backslash G/\Gamma$, then

$$\alpha\cdot\beta = \sigma(\alpha\cdot\beta) = \sigma(\beta)\cdot\sigma(\alpha) = \beta\cdot\alpha$$

and $\mathrm{ind}(\alpha) = \mathrm{ind}(\sigma(\alpha)) = \mathrm{ind}(\alpha^*)$. □

Lemma 1.3. *Let $g \in G$ be in the normalizer of Γ. Then, for every $h \in G$, one has* $(\Gamma g\Gamma)\cdot(\Gamma h\Gamma) = \Gamma gh\Gamma$ *and* $(\Gamma h\Gamma)\cdot(\Gamma g\Gamma) = \Gamma hg\Gamma$.

Proof. We only prove the fist equality. Since $(\Gamma g\Gamma)(\Gamma h\Gamma) = \Gamma gh\Gamma$ as a set, it remains to show $c(\Gamma g\Gamma,\Gamma h\Gamma;\Gamma gh\Gamma) = 1$. This reduces to show $\mathrm{ind}(\Gamma gh\Gamma) = \mathrm{ind}(\Gamma h\Gamma)$. Observe now that for $h_j \in G$, one has $\Gamma h\Gamma = \bigsqcup \Gamma h_j$ iff $\Gamma gh\Gamma = \bigsqcup \Gamma gh_j$. □

2. Actions of Hecke algebras

Let $\Gamma \le G$ be a Hecke pair and V be a vector space on which G acts. Then, the Hecke algebra $\mathscr{H}$ acts on the space V^Γ of Γ-invariant vectors by

$$\alpha v = \sum g_k v$$

for $v \in V^\Gamma$ and $\alpha = \bigsqcup g_k\Gamma \in \Gamma\backslash G/\Gamma$. Note that we are using left cosets here. It is easy to see that αv is indeed Γ-invariant. Moreover, if $\alpha = \bigsqcup g_k\Gamma$ and $\beta = \bigsqcup h_l\Gamma$ are elements in $\Gamma\backslash G/\Gamma$, then for every $\gamma \in \Gamma\backslash G/\Gamma$ and $u \in \gamma$, one has

$$|\{(k,l) : u \in g_k h_l\Gamma\}| = |\{(g,h) \in \alpha\times\beta : gh = u\}/\sim_\Gamma| = c(\alpha,\beta;\gamma).$$

It follows that $(\alpha, v) \mapsto \alpha v$ is indeed an action of the Hecke algebra $\mathscr{H}$. If V is a Hilbert space and $G \curvearrowright V$ is unitary, then for every $v \in H^\Gamma$ and $g \in G$, one has

$$\alpha v = P_{H^\Gamma} \sum_{k=1}^{\mathrm{ind}(\Gamma g^{-1}\Gamma)} g_k v = \mathrm{ind}(\Gamma g^{-1}\Gamma) P_{H^\Gamma} g v,$$

since $P_{H^\Gamma} s = sP_{H^\Gamma} = P_{H^\Gamma}$ for any $s \in \Gamma$.

Example 2.1. Let V be the space of all functions $\xi\colon G \to \mathbb{C}$ and define the G-action on V by $(g\xi)(h) = \xi(g^{-1}h)$. Then, V^Γ is identified as the space of functions on $\Gamma\backslash G$. Suppose $x \in \Gamma\backslash G$ and view δ_x also as the characteristic function χ_x on G. Then,

$$\alpha\delta_x = \chi_{\bigsqcup g_k x} = \sum_k \delta_{g_k x} = \chi_{\alpha x} = \lambda(\alpha)\delta_x.$$

3. The Hecke algebra associated with $\mathrm{SL}(2,\mathbb{Z}) \le \mathrm{GL}^+(2,\mathbb{Q})$.

Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and $G = \mathrm{GL}^+(2,\mathbb{Q})$. We will show $\Gamma \le G$ is a Hecke pair and studies the structure of its Hecke algebra. Let $\mathbb{M}(l) = \{A \in \mathbb{M}(2,\mathbb{Z}) : \det A = l\}$ and $\mathbb{M} = \bigcup_{l=1}^{\infty} \mathbb{M}(l) \subset G$.

Theorem 3.1. *Let Γ act on $\mathbb{Z}^2$ and on $\mathbb{M}$ from the left. Then,*

$$\mathbb{Z}^2 = \bigsqcup_{\alpha=0}^{\infty} \Gamma \left(\begin{smallmatrix} \alpha \\ 0 \end{smallmatrix}\right) \text{ and } \mathbb{M} = \bigsqcup \{\Gamma \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : a,d \in \mathbb{N},\ b \in \mathbb{N}_0,\ 0 \le b < d\}.$$

Proof. The first assertion is easy: $\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right) \in \Gamma \left(\begin{smallmatrix} \alpha \\ 0 \end{smallmatrix}\right)$ iff $\alpha = \gcd(a,c)$. By the first assertion, every Γ-orbit in $\mathbb{M}$ has a representative of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ where $a,b,d \in \mathbb{N}_0$ and $ad > 0$. By multiplying appropriate $\left(\begin{smallmatrix} 1 & f \\ 0 & 1 \end{smallmatrix}\right)$ from the left, one may assume the representative $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ satisfies $0 \le b < d$ in addition. Now, suppose that $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right)$ belong to the same Γ-orbit. Then,

$$\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} a/\alpha & (\alpha b - a\beta)/\alpha\delta \\ 0 & d/\delta \end{smallmatrix}\right) \in \Gamma,$$

which implies $a = \alpha$, $d = \delta$, and $b = \beta$. □

Corollary 3.2. *The pair $\Gamma \le G$ is a Hecke pair.*

Proof. Let $A \in G$ and choose $m \in \mathbb{N}$ so that $mA \in \mathbb{M}$. Then for $B_i \in G$, one has $\Gamma A \Gamma = \bigsqcup \Gamma B_i$ iff $\Gamma m A \Gamma = \bigsqcup \Gamma m B_i$. Since $\Gamma m A \Gamma \subset \mathbb{M}(l)$ for $l = \det(mA)$ and $\mathbb{M}(l)$ has finitely many Γ-orbits by Theorem 3.1, one has $|\Gamma \backslash \Gamma A \Gamma| < +\infty$. □

For $A \in \mathbb{M}(2,\mathbb{Z})$, let $\delta(A)$ be the g.c.d. of its entries. Then, one has

$$\det(UAV) = \det A \text{ and } \delta(UAV) = \delta(A)$$

for every $A \in \mathbb{M}$ and $U, V \in \Gamma$. The pair of these values is a complete invariant for the double coset in $\mathbb{M}$.

Theorem 3.3. *For $A \in \mathbb{M}$, one has $A \in \Gamma \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \Gamma$ iff $a = \delta(A)$ and $d = (\det A)/\delta(A)$. In particular,*

$$\mathbb{M} = \bigsqcup \{\Gamma \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \Gamma : a,d \in \mathbb{N},\ a \mid d\}.$$

Proof. Since A is a 2-by-2 matrix, $\det A$ is divisible by $\delta(A)^2$ and $a = \delta(A)$ divides $d = (\det A)/\delta(A)$. It remains to show that every $A \in \mathbb{M}$ is equivalent to $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ with $a \mid d$. We may assume that $\delta(A) = 1$. By Theorem 3.1, we may further assume that $A = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. By the Chinese Remainder Theorem, there is $m \in \mathbb{Z}$ such that $b + md \equiv 1 \mod p$ for all prime divisors p of a which do not divides d. Since $\gcd(a,b,d) = 1$, one has $\gcd(a, b + md) = 1$. Replacing A with $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) A$, we may now assume that $\gcd(a,b) = 1$. Then, for $x, y \in \mathbb{Z}$ such that $ax - by = 1$, one has $A \left(\begin{smallmatrix} x & -b \\ -y & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & * \\ * & * \end{smallmatrix}\right)$. By elementary transformation over $\mathbb{Z}$, it is equivalent to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & * \end{smallmatrix}\right)$. □

Corollary 3.4. *The Hecke algebra of $\Gamma \le G$ is commutative.*

Proof. The transpose σ is an anti-automorphism of G such that $\sigma(\Gamma) = \Gamma$. Let $A \in G$ and choose $m \in \mathbb{N}$ so that $mA \in \mathbb{M}$. Then, by Theorem 3.3, one has

$$m(\Gamma A \Gamma) = \Gamma(mA)\Gamma = \Gamma \sigma(mA) \Gamma = m(\Gamma \sigma(A) \Gamma).$$

Hence, $\Gamma A\Gamma = \Gamma\sigma(A)\Gamma$. By Lemma 1.2, we are done. □

By Lemma 1.3, for any $r \in \mathbb{Q}_+$, the element $\Gamma\left(\begin{smallmatrix} r & 0\\ 0 & r\end{smallmatrix}\right)\Gamma$ is in the center of $\mathscr{H}$ and satisfies $(\Gamma\left(\begin{smallmatrix} r & 0\\ 0 & r\end{smallmatrix}\right)\Gamma)\cdot(\Gamma A\Gamma) = \Gamma(rA)\Gamma$. for every $A \in G$.

Lemma 3.5. *If* $\gcd(k,l) = 1$, *then for every* $A \in \mathbb{M}(k)$ *and* $B \in \mathbb{M}(l)$, *one has* $\delta(AB) = \delta(A)\delta(B)$ *and* $(\Gamma A\Gamma)\cdot(\Gamma B\Gamma) = \Gamma AB\Gamma$.

Proof. By the remark preceding this lemma, we may assume that $\delta(A) = 1 = \delta(B)$. We first prove that $\delta(AB) = 1$. In light of Theorem 3.3, it suffices to show that for every $U = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma$ one has $\delta(\left(\begin{smallmatrix} 1 & 0\\ 0 & k\end{smallmatrix}\right)U\left(\begin{smallmatrix} 1 & 0\\ 0 & l\end{smallmatrix}\right)) = 1$. Suppose that a prime number p divides the entries of $\left(\begin{smallmatrix} 1 & 0\\ 0 & k\end{smallmatrix}\right)U\left(\begin{smallmatrix} 1 & 0\\ 0 & l\end{smallmatrix}\right) = \left(\begin{smallmatrix} a & bl\\ kc & kdl\end{smallmatrix}\right)$. Then $p \mid a$ and $p \mid kc$, but since $U \in \Gamma$, this implies $p \mid k$. Likewise $p \mid l$. A contradiction. This proves $\delta(AB) = 1$. Now, Theorem 3.3 implies that $(\Gamma A\Gamma)(\Gamma B\Gamma) = \Gamma AB\Gamma$ as a set. It remains to show $c(\Gamma A\Gamma, \Gamma B\Gamma; \Gamma AB\Gamma) = 1$. Since this value is independent of the representatives, we may assume that A and B are of the forms $A = \left(\begin{smallmatrix} 1 & 0\\ 0 & k\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 0\\ 0 & l\end{smallmatrix}\right)$. According to Theorems 3.1 and 3.3, one has

$$\Gamma B\Gamma = \bigsqcup\{\left(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right) : ad = l,\ 0 \le b < d \text{ and } \gcd(a,b,d) = 1\} = \bigsqcup \Gamma B_j$$

It follows that

$$c(\Gamma A\Gamma, \Gamma B\Gamma; \Gamma AB\Gamma) = |\{j : AB \in \Gamma A\Gamma B_j\}|.$$

But, the matrix $ABB_j^{-1} = \left(\begin{smallmatrix} 1/a & -b/ad\\ 0 & kl/d\end{smallmatrix}\right)$ is integral only if $a = 1$, $b = 0$, or equivalently only if $B_j = B$. Therefore, $c(\Gamma A\Gamma, \Gamma B\Gamma; \Gamma AB\Gamma) = 1$. □

4. The Hecke algebra associated with $\mathrm{PSL}(2,\mathbb{Z}) \le \mathrm{PGL}^+(2,\mathbb{Q})$.

The structure of the Hecke algebra of $\mathrm{PSL}(2,\mathbb{Z}) \le \mathrm{PGL}^+(2,\mathbb{Q})$ is easier than that of $\mathrm{SL}(2,\mathbb{Z}) \le \mathrm{GL}^+(2,\mathbb{Q})$.

Proposition 4.1. *The pair* $\mathrm{PSL}(2,\mathbb{Z}) \le \mathrm{PGL}^+(2,\mathbb{Q})$ *is a Hecke pair and the quotient map from* $\mathrm{GL}^+(2,\mathbb{Q})$ *onto* $\mathrm{PGL}^+(2,\mathbb{Q})$ *induces a surjective homomorphism between their Hecke algebras.*

Proof. Let $G = \mathrm{GL}^+(2,\mathbb{Q})$, $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, $G' = \mathrm{PGL}^+(2,\mathbb{Q})$ and $\Gamma' = \mathrm{PSL}(2,\mathbb{Z})$. The quotient map $\pi\colon G \to G'$ extends to a surjection from $\Gamma\backslash G/\Gamma$ onto $\Gamma'\backslash G'/\Gamma'$.

Claim 4.2. *If* $\Gamma A\Gamma = \bigsqcup \Gamma A_i$, *then* $\Gamma'\pi(A)\Gamma' = \bigsqcup \Gamma'\pi(A_i)$.

Proof. We only prove that $\Gamma'\pi(A_i)$'s are mutually disjoint. Suppose the contrary. It follows that there are $i \ne j$ and $B \in \Gamma$ such that $A_i^{-1}BA_j \in \ker\pi$. But, since $\det(A_i^{-1}BA_j) = 1$ and $\ker\pi = \{\left(\begin{smallmatrix} r & 0\\ 0 & r\end{smallmatrix}\right)\}$, one has $A_i^{-1}BA_j \in \Gamma$. A contradiction. □

We proceed to prove that π is a homomorphism between Hecke algebras. Let $\alpha = \Gamma A\Gamma = \bigsqcup \Gamma A_i$, $\beta = \Gamma B\Gamma = \bigsqcup \Gamma B_j$ and $\gamma = \Gamma C\Gamma \subset \alpha\beta$. To see

$$\begin{aligned} c(\alpha,\beta;\gamma) &= |\{(i,j) : C \in \Gamma A_i B_j\}| \\ &= |\{(i,j) : \pi(C) \in \Gamma'\pi(A_i B_j)\}| = c(\pi(\alpha),\pi(\beta);\pi(\gamma)), \end{aligned}$$

it suffices to show $\pi(C) \in \Gamma'\pi(A_iB_j)$ implies $C \in \Gamma A_iB_j$. If $\pi(C) \in \Gamma'\pi(A_iB_j)$, then there exists $D \in \Gamma$ such that $DA_iB_jC^{-1} \in \ker\pi$. But $\det(DA_iB_jC^{-1}) = 1$, this implies $DA_iB_jC^{-1} \in \Gamma$. This completes the proof. □

Theorem 4.3. *Let* $A \in \mathrm{GL}^+(2,\mathbb{Q})$ *and* $r \in \mathbb{Q}$ *be such that* $rA \in \mathbb{M}(2,\mathbb{Z})$ *and* $\delta(rA) = 1$*; and let* $k = \det(rA) \in \mathbb{N}$*. Then,* A *belongs to the same double coset as* $\left(\begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix}\right)$ *in* $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathrm{PGL}^+(2,\mathbb{Q})/\mathrm{PSL}(2,\mathbb{Z})$*. Moreover,*

$$\mathrm{PSL}(2,\mathbb{Z})\backslash\mathrm{PGL}^+(2,\mathbb{Q})/\mathrm{PSL}(2,\mathbb{Z}) = \bigsqcup_{k=1}^{\infty} \mathrm{PSL}(2,\mathbb{Z}) \left(\begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix}\right) \mathrm{PSL}(2,\mathbb{Z}).$$

Proof. Let A, r and k be as in the statement. Then, $A = rA$ in G' and A is equivalent in $\Gamma'\backslash G'/\Gamma'$ to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix}\right)$ by Theorem 3.3. Moreover, if $\left(\begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & l \end{smallmatrix}\right)$ belong to the same double coset, then there are $A, B \in \Gamma$ such that $A\left(\begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix}\right)B\left(\begin{smallmatrix} 1 & 0 \\ 0 & l \end{smallmatrix}\right)^{-1} \in \ker\pi$. Hence, there is $r \in \mathbb{Q}$ such that $A\left(\begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix}\right)B = \left(\begin{smallmatrix} r & 0 \\ 0 & lr \end{smallmatrix}\right)$. It follows that $r \in \mathbb{Z}$ and, by Theorem 3.3, $r = 1$ and $k = l$. □

Let $\mathscr{H}$ be the Hecke algebra of $\mathrm{PSL}(2,\mathbb{Z}) \le \mathrm{PGL}^+(2,\mathbb{Q})$ and, for every prime number p, $\mathscr{H}_p$ be the subalgebra generated by $\{\begin{pmatrix} 1 & 0 \\ 0 & p^l \end{pmatrix} : l \in \mathbb{N}_0\}$. By Theorem 4.3 and Lemma 3.5, one has

$$\mathscr{H} \cong \bigotimes_{p \text{ prime}} \mathscr{H}_p.$$

We investigate the structure of $\mathscr{H}_p$. We write T_p for the element in $\mathscr{H}$ represented by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. We note that $T_p^* = T_p$.

Theorem 4.4. *One has* $T_p^2 = T_{p^2} + (p+1)$ *and* $T_p \cdot T_{p^k} = T_{p^{k+1}} + pT_{p^{k-1}}$ *for* $k \ge 2$*. In particular,* $\mathscr{H}_p$ *is generated by* T_p*.*

Proof. By Theorem 3.3, one has $(\Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma)(\Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma) = \Gamma\begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}\Gamma \sqcup \Gamma\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\Gamma$ as a subset of $\mathrm{GL}^+(2,\mathbb{Q})$. By Theorem 3.1, one has

$$\Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma = \Gamma\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \sqcup \bigsqcup_{b=0}^{p-1} \Gamma\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}.$$

We note that

$$\begin{aligned}
\Gamma\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) &= \Gamma\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}, \\
\Gamma\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\begin{pmatrix} 1 & b' \\ 0 & p \end{pmatrix} &= \Gamma\begin{pmatrix} p & pb' \\ 0 & p \end{pmatrix} = \Gamma\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \text{ for all } b', \\
\Gamma\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) &= \Gamma\begin{pmatrix} p & b \\ 0 & p \end{pmatrix} \text{ for all } b, \\
\Gamma\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & b' \\ 0 & p \end{pmatrix} &= \Gamma\begin{pmatrix} 1 & bp+b' \\ 0 & p^2 \end{pmatrix} \text{ for all } b \text{ and } b'.
\end{aligned}$$

It follows that $T_p^2 = T_{p^2} + (p+1)\Gamma\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\Gamma$ in $\Gamma\backslash G/\Gamma$. Passing to the quotient, one sees $T_p^2 = T_{p^2} + (p+1)$ in $\Gamma'\backslash G'/\Gamma'$. Next, observe that

$$(\Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma)(\Gamma\begin{pmatrix} 1 & 0 \\ 0 & p^k \end{pmatrix}\Gamma) = \Gamma\begin{pmatrix} 1 & 0 \\ 0 & p^{k+1} \end{pmatrix}\Gamma \sqcup \Gamma\begin{pmatrix} p & 0 \\ 0 & p^k \end{pmatrix}\Gamma$$

as a subset of $\mathrm{GL}^+(2,\mathbb{Q})$. Indeed, if $U = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, then $\gcd(a,c) = 1$ and the value

$$\delta(\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)U\begin{pmatrix} 1 & 0 \\ 0 & p^k \end{pmatrix}) = \delta(\begin{pmatrix} a & bp^k \\ cp & dp^{k+1} \end{pmatrix})$$

can only be 1 or p. We note that

$$\Gamma\begin{pmatrix} 1 & 0 \\ 0 & p^k \end{pmatrix}\Gamma = \bigsqcup\{\Gamma\begin{pmatrix} p^{k-i} & b' \\ 0 & p^i \end{pmatrix} : 0 \le i \le k, 0 \le b' < p^i, \delta(\begin{pmatrix} p^{k-i} & b' \\ 0 & p^i \end{pmatrix}) = 1\}$$

and

$$\Gamma\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^{k-i} & b' \\ 0 & p^i \end{pmatrix} = \Gamma\begin{pmatrix} p^{k-i+1} & [pb'] \\ 0 & p^i \end{pmatrix}, \text{ where } [pb'] \equiv pb' \mod p^i,$$

$$\Gamma\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}\begin{pmatrix} p^{k-i} & b' \\ 0 & p^i \end{pmatrix} = \Gamma\begin{pmatrix} p^{k-i} & bp^i+b' \\ 0 & p^{i+1} \end{pmatrix}.$$

Observe that $\Gamma\begin{pmatrix} p & 0 \\ 0 & p^k \end{pmatrix}$ appears only in the first case and only if $i = k$ and $p^{k-1} \mid b'$, because $\delta(\begin{pmatrix} p & 0 \\ 0 & p^{k-1} \end{pmatrix}) \neq 1$. It follows that $T_p \cdot T_{p^k} = T_{p^{k+1}} + p\Gamma\begin{pmatrix} p & 0 \\ 0 & p^k \end{pmatrix}\Gamma$ in $\Gamma\backslash G/\Gamma$. Passing to the quotient, one sees $T_p \cdot T_{p^k} = T_{p^{k+1}} + pT_{p^{k-1}}$ in $\Gamma'\backslash G'/\Gamma'$. □

Let $N = (p+1)/2$ and $\chi_k = \sum_{|g|=k} \lambda(g) \in \mathrm{vN}(\mathbb{F}_N)$. Then, $\{T_{p^k} : k \in \mathbb{N}_0\}$ satisfies the same relations as $\{\chi_k : k \in \mathbb{N}_0\}$ and $\omega(T_{p^k}) = \delta_{k,0} = \tau(\chi_k)$. Hence, the von Neumann subalgebra $\mathrm{vN}(\mathscr{H}_p)$ of $\mathrm{vN}(\mathscr{H})$ is naturally $*$-isomorphic to $\mathrm{vN}(\chi_1)$. In particular, the spectrum of $\lambda(T_p)$ is $[-2\sqrt{p}, 2\sqrt{p}]$ by Kesten's theorem.

Lemma 4.5. *Let p be a prime number and consider the Hecke algebra of* $\mathrm{SL}(2,\mathbb{Z}) \le \mathrm{SL}(2,\mathbb{Q}[\sqrt{p}])$. *Then, the elements $\tilde{T}_{p^k}$ represented by $\begin{pmatrix} p^{-k/2} & 0 \\ 0 & p^{k/2} \end{pmatrix}$ are self-adjoint and satisfy the same recursion formula as T_{p^k} in Theorem 4.4.*

Proof. Since $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the elements $\tilde{T}_{p^k}$ are self-adjoint. Moreover, one has $\tilde{T}_{p^k} = p^{-k/2}T_{p^k}$ in the Hecke algebra of $\mathrm{SL}(2,\mathbb{Z}) \le \mathrm{GL}^+(2,\mathbb{Q}[\sqrt{p}])$. It follows from the proof of Theorem 4.4 that

$$\tilde{T}_p^2 = \frac{1}{p}T_{p^2} = \frac{1}{p}\left(T_{p^2} + (p+1)\Gamma\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\Gamma\right) = \tilde{T}_{p^2} + (p+1)$$

and

$$\tilde{T}_p \cdot \tilde{T}_{p^k} = \frac{1}{\sqrt{p^{k+1}}}T_p \cdot T_{p^k} = \frac{1}{\sqrt{p^{k+1}}}\left(T_{p^{k+1}} + p\Gamma\begin{pmatrix} p & 0 \\ 0 & p^k \end{pmatrix}\Gamma\right) = \tilde{T}_{p^{k+1}} + p\tilde{T}_{p^{k-1}}.$$ □

5. Classical Hecke operators and the Ramanujan–Petersson conjecture

(Ref: [Iw].) Let $\mathbb{H} = \{x + \mathrm{i}y : y > 0\}$ be the upper half plane and $\mathrm{PGL}^+(2,\mathbb{R}) \curvearrowright \mathbb{H}$ be the action given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

Hence, $g \in \mathrm{PGL}^+(2,\mathbb{R})$ also acts on the functions on $\mathbb{H}$ by $(g\xi)(z) = \xi(g^{-1}z)$. We note that

$$\Im\left(\frac{az+b}{cz+d}\right) = \frac{(\Im z)(ad-bc)}{|cz+d|^2}$$

and the measure $d\mu_0 = y^{-2}\,dx\,dy$ is $\mathrm{PGL}^+(2,\mathbb{R})$-invariant. Let $\mathscr{F}$ be a $\mathrm{PSL}(2,\mathbb{Z})$-fundamental domain of $\mathbb{H}$, and view $L^2(\mathscr{F},\mu_0)$ as the space of those functions which are

$\mathrm{PSL}(2,\mathbb{Z})$-invariant and square-integrable on $\mathscr{F}$. Recall the discussion in Section 2 and consider the action of the Hecke algebra of $\mathrm{PSL}(2,\mathbb{Z}) \le \mathrm{PGL}^+(2,\mathbb{Q})$ on $L^2(\mathscr{F},\mu_0)$. Since

$$\Gamma\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\Gamma = \Gamma\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right) \sqcup \bigsqcup_{b=0}^{p-1} \Gamma\begin{pmatrix}1&b\\0&p\end{pmatrix},$$

it is given by

$$(T_p f)(z) = f(\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right) z) + \sum_{b=0}^{p-1} f(\begin{pmatrix}1&b\\0&p\end{pmatrix} z) = f(pz) + \sum_{b=0}^{p-1} f(\frac{z+b}{p}).$$

This is the classical Hecke operator. The constant function $\mathbf{1}$ is an eigenvector with eigenvalue $p+1$. "Hecke's trivial estimate" says $\|T_p\| \le p+1$, and the Ramanujan–Petersson conjecture for Maass forms asserts that $\|T_p|_{L^2_0(\mathscr{F},\mu_0)}\| \le 2\sqrt{p}$. Namely, T_p restricted to the orthogonal complement $L^2_0(\mathscr{F},\mu_0) := (\mathbb{C}\mathbf{1})^\perp$ of the constant functions has norm $\le 2\sqrt{p}$.

Consider the hyperbolic Laplacian

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

on $L^2(\mathscr{F},\mu_0)$. It is an unbounded, (essentially) self-adjoint and positive operator. The spectral resolution of Δ has both discrete and continuous parts:

$$L^2(\mathscr{F},\mu_0) = \mathbb{C}\mathbf{1} \oplus \bigoplus \mathbb{C}u_j \oplus \int_0^{+\infty} E(\cdot,\frac{1}{2}+\mathrm{i}t)\,dt.$$

The constant function $\mathbf{1}$ has eigenvalue 0. The other eigenvectors u_j are called Maass (cusp) forms. Selberg's conjecture asserts that the eigenvalues of Maass forms are all $\ge \frac{1}{4}$. The function $z \mapsto E(z,\frac{1}{2}+\mathrm{i}t)$ belongs to the Eisenstein series. It is an eigenvector for Δ with the eigenvalue $\frac{1}{4}+t^2$, but a care is needed because these functions do not belong to L^2 (and this is why they do not constitute discrete spectra). Since the hyperbolic Laplacian on $\mathbb{H}$ commutes with $\mathrm{PGL}^+(2,\mathbb{R}) \curvearrowright \mathbb{H}$, every Hecke operators on $L^2(\mathscr{F},\mu_0)$ commutes with Δ also. Thus u_j's and Eisenstein series are eigenvectors for T_p as well. The eigenvalues of T_p at the Eisenstein series $E(\cdot,\frac{1}{2}+\mathrm{i}t)$ are all $\le 2\sqrt{p}$ and pose no problem. (The Ramanujan–Petersson conjecture in more general setting only asserts that the eigenvalues of T_p at cusp forms are $\le 2\sqrt{p}$.)

Part 2. **Berezin transform: From the Ramanujan–Petersson conjecture to operator algebras (Ref:** [Ra1]**).**

Throughout this chapter, let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ (or any other lattice in $\mathrm{SL}(2,\mathbb{R})$), and $\mathscr{F}$ be a Γ-fundamental domain of $\mathbb{H}$.

6. Discrete series representations of $\mathrm{SL}(2,\mathbb{R})$

(Ref: Section IX in [La] and Sections 16 and 17 in [Ro].) Throughout this paper, let $m \ge 2$ be an integer.

Let $\mu_m = y^m \mu_0$ and $H_m = H^2(\mathbb{H}, \mu_m)$ be the space of all $L^2(\mu_m)$-holomorphic functions on $\mathbb{H}$. The space H_m is closed in $L^2(\mathbb{H}, \mu_m)$. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} \in \mathrm{SL}(2,\mathbb{R})$,

$$(\pi_m(g)f)(z) = f(g^{-1}z)(cz+d)^{-m}$$

defines a unitary representation π_m on H_m. Indeed, multiplicativity is directly checked and since

$$\Im(g^{-1}z) = \frac{\Im z}{|cz+d|^2},$$

one has

$$\|\pi_m(g)f\|_2^2 = \int_{\mathbb{H}} |f(g^{-1}z)|^2 \frac{|\Im z|^m}{|cz+d|^{2m}}\, d\mu_0(z) = \|f\|_2^2.$$

The representation π_m is irreducible and square integrable, i.e., for every $u, v \in H_m$, the function $c_v^u \colon g \mapsto \langle \pi_m(g)u, v\rangle$ belongs to $L^2(\mathrm{SL}(2,\mathbb{R}))$. (Proofs omitted.) For a fixed nonzero v, the map $H_m \ni u \mapsto c_v^u \in L^2(\mathrm{SL}(2,\mathbb{R}))$ defines a (bounded) intertwiner between π_m and the right regular representation ρ. By irreducibility, H_m can be regarded as a subrepresentation of $\rho_{\mathrm{SL}(2,\mathbb{R})}$.

Lemma 6.1. *There is $d > 0$, called the formal dimension of π_m, which satisfies for every $u, v, u', v' \in H_m$ that*

$$\langle c_v^u, c_{v'}^{u'}\rangle_{L^2(G)} = \frac{1}{d}\langle u, u'\rangle\overline{\langle v, v'\rangle}.$$

Proof. Fix v, v' and let $Tu = c_v^u$ and $T'u' = c_{v'}^{u'}$. Then, T^*T' commutes with $\pi_m(\mathrm{SL}(2,\mathbb{R}))$ and hence is a constant $\alpha_{v,v'}$. Thus $\langle c_v^u, c_{v'}^{u'}\rangle = \gamma_{v,v'}\langle u, u'\rangle$ for all u, u'. Likewise there exists $\beta_{u,u'}$ which satisfies $\langle c_v^u, c_{v'}^{u'}\rangle = \beta_{u,u'}\overline{\langle v, v'\rangle}$ for all u, u'. Therefore, $\gamma_{v,v'}\langle u, u'\rangle = \beta_{u,u'}\overline{\langle v, v'\rangle}$ and $d = \langle u, u'\rangle / \beta_{u,u'}$ is independent of u, u'. □

Every lattice in $\mathrm{SL}(2,\mathbb{R})$ is essentially ICC; namely, there are only finitely many conjugacy classes that are finite. Note that $\pi_m|_\Gamma$ is *stably* unitary equivalent to the subrepresentation ρ^χ of the right regular representation. Here $\chi = (\pi_m|_{Z(\Gamma)})^{-1}$ is a character on the center $Z(\Gamma)$ of Γ and, more generally for any central subgroup Z of Γ and a character χ on Z, we define ρ^χ to be the right regular representation restricted to the subspace

$$\ell_2^\chi\Gamma := \{\xi \in \ell_2\Gamma : \forall s \in \Gamma, \forall z \in Z \quad \xi(z^{-1}s) = \chi(z)\xi(s)\}.$$

Likewise for λ^χ. Note that $\lambda^\chi(z) = \chi(z) \in \mathbb{C}1$ for $z \in Z$. We define

$$\mathscr{A}_m = \pi_m(\Gamma)' \subset \mathbb{B}(H_m).$$

Note that $\mathscr{A}_m$ is stably isomorphic to the factor $\mathscr{L}^\chi\Gamma := \lambda^\chi(\Gamma)'' = p^\chi\mathscr{L}\Gamma$, where $p^\chi = |Z|^{-1}\sum_{g\in Z}\bar{\chi}(g)\lambda(g)$ is a central projection in $\mathscr{L}\Gamma$. It is also isomorphic as the group von Neumann algebra of Γ/Z twisted by the 2-cocycle associated with χ. We view χ as a function on Γ which is supported on Z. Then vector $\hat{1}^\chi = |Z|^{-1/2}\bar{\chi}$ is a cyclic separating tracial vector for $\mathscr{L}^\chi\Gamma$ and

$$\tau^\chi(\lambda^\chi(s)) := \langle \lambda^\chi(s)\hat{1}^\chi, \hat{1}^\chi\rangle = \chi(s)$$

for all $s \in \Gamma$. A calculations shows that the formal dimension of π_m is $\frac{m-1}{4\pi}$, which depends on the choice of a Haar measure of $\mathrm{SL}(2,\mathbb{R})$. We use the standard Haar measure

$d\mu_0 \frac{d\theta}{2\pi}$, and in this case, $\mathrm{vol}(\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})) = \frac{\pi}{6}$. A further calculation ([GHJ, 3.3.d]) shows

$$\dim_{\pi_m(\mathrm{SL}(2,\mathbb{Z}))''} H_m = \frac{m-1}{12}.$$

7. Berezin Calculus

The Hilbert space H_m is a reproducing kernel Hilbert space. Namely, for every $z \in \mathbb{H}$, the map $H_m \ni f \mapsto f(z) \in \mathbb{C}$ is bounded and hence there is $e_z \in H_m$ such that

$$\forall f \in H_m \quad \langle f, e_z \rangle = f(z).$$

The kernel $K(z,\zeta) = \langle e_\zeta, e_z \rangle = e_\zeta(z)$ is called the reproducing kernel. It is known that for some constant $c_m > 0$ (probably $c_m = (m-1)/4$) one has

$$K(z,\zeta) = \frac{c_m}{((z-\bar{\zeta})/2\mathrm{i})^m} \text{ and in particular } K(z,z) = \frac{c_m}{(\Im z)^m}.$$

(See Section 1.1 in [HKZ], where they work with the disk model, but it is interchangeable with the upper half plane model by using formulae in Section IX.3 in [La].) Note that $d\mu_m(z) = c_m \cdot K(z,z)^{-1} d\mu_0(z)$. We define

$$u(z,\zeta) = \frac{|z-\zeta|^2}{4(\Im z)(\Im \zeta)}.$$

Recall the relation between u and the hyperbolic distance $d_{\mathbb{H}}$ on $\mathbb{H}$:

$$\cosh d_{\mathbb{H}}(z,\zeta) = 1 + 2u(z,\zeta) = \frac{|z-\bar{\zeta}|^2}{2(\Im z)(\Im \zeta)} - 1.$$

In particular, u is invariant under the diagonal action of $\mathrm{SL}(2,\mathbb{R})$. We further define

$$\delta(z,\zeta) = \left(\frac{1}{1+u(z,\zeta)}\right)^m = \left(\frac{4(\Im z)(\Im \zeta)}{|z-\bar{\zeta}|^2}\right)^m = \frac{|K(z,\zeta)|^2}{K(z,z)K(\zeta,\zeta)}.$$

Note that $c_m \int_{\mathbb{H}} \delta(z,\zeta)\, d\mu_0(\zeta) = K(z,z)^{-1} \int |e_z(\zeta)|^2 \, d\mu_m(\zeta) = 1$. Also note that every $f \in H_m$ is expressed as the weak integral

$$f = \int_{\mathbb{H}} f(\zeta) e_\zeta \, d\mu_m(\zeta).$$

Indeed, $\langle f, e_z \rangle = \int f(\zeta)\overline{e_z(\zeta)}\, d\mu_m(\zeta) = \int f(\zeta) \langle e_\zeta, e_z \rangle \, d\mu_m(\zeta)$.

Let $A \in \mathbb{B}(H_m)$ be given. Since $\{e_z\}$ is a total subset of H_m, the kernel

$$\hat{A}(z,\zeta) = \langle A e_\zeta, e_z \rangle / K(z,\zeta)$$

on $\mathbb{H} \times \mathbb{H}$ determines A. The kernel $\hat{A}$ is sesqui-holomorphic, i.e., it is holomorphic in the first variable and anti-holomorphic in the second variable. It follows from the Cauchy–Riemann equations that if $F(z,\zeta)$ is a sesqui-holomorphic function and $f(z) = F(z,z)$, then

$$(\frac{\partial f}{\partial z})(z) = (\frac{\partial F}{\partial z})(z,\zeta)\big|_{\zeta=z} \text{ and } (\frac{\partial f}{\partial \bar{z}})(z) = (\frac{\partial F}{\partial \bar{\zeta}})(z,\zeta)\big|_{\zeta=z}.$$

In particular, the restriction $F \mapsto f$ is a one-to-one map for sesqui-holomorphic functions. Thus $\hat{A}(z) := \hat{A}(z,z)$ determines A. This abuse of notation should not cause any confusion. The function $\hat{A}$ is called the *symbol* or the *Berezin transform* of A.

Proposition 7.1. *Let $A, B \in \mathbb{B}(H_m)$. Then,*

(1) $\sup_{z\in\mathbb{H}} |\hat{A}(z)| \le \|A\|$,
(2) $\widehat{A^*}(z,\zeta) = \bar{\hat{A}}(\zeta, z)$,
(3) $\widehat{AB}(z,\zeta) = \int_{\mathbb{H}} \frac{K(z,\eta)K(\eta,\zeta)}{K(z,\zeta)} \hat{A}(z,\eta)\hat{B}(\eta,\zeta)\, d\mu_m(\eta)$.

Proof. We only prove the third identity:

$$\langle ABe_\zeta, e_z\rangle = \langle A\int (Be_\zeta)(\eta)e_\eta\, d\mu_m(\eta), e_z\rangle = \int_{\mathbb{H}} \langle Ae_\eta, e_z\rangle\langle Be_\zeta, e_\eta\rangle\, d\mu_m(\eta). \qquad \square$$

Lemma 7.2. *For $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ and $z \in \mathbb{H}$, one has $\pi_m(g)e_z = (c\bar{z}+d)^{-m}e_{gz}$.*

Proof. $\langle e_\zeta, \pi_m(g)e_z\rangle = (\pi_m(g^{-1})e_\zeta)(z) = e_\zeta(gz)(cz+d)^{-m} = (cz+d)^{-m}\langle e_\zeta, e_{gz}\rangle$. $\square$

Proposition 7.3. *For $A \in \mathbb{B}(H_m)$ and $g \in \mathrm{SL}(2,\mathbb{R})$, the symbol of $\pi_m(g)^{-1}A\pi_m(g)$ is $\hat{A}(gz, g\zeta)$. In particular, $A \in \mathscr{A}_m$ iff $\hat{A}$ is Γ-invariant.*

Proof. We note that

$$K(gz, g\zeta) = \langle e_{gz}, e_{g\zeta}\rangle = (c\bar{z}+d)^m(c\zeta+d)^m K(z,\zeta)$$

It follows that

$$\begin{aligned}\langle \pi_m(g^{-1})A\pi_m(g)e_\zeta, e_z\rangle / K(z,\zeta) &= \langle Ae_{gz}, e_{g\zeta}\rangle(c\bar{z}+d)^{-m}(c\zeta+d)^{-m}/K(z,\zeta)\\ &= \langle Ae_{gz}, e_{g\zeta}\rangle / K(gz, g\zeta).\end{aligned} \qquad \square$$

Let $A \in \mathscr{A}_m$. Then, the symbol $\hat{A}(z)$ is a bounded Γ-invariant function on $\mathbb{H}$ and hence A is determined by $\hat{A}|_{\mathscr{F}}$. Recall that μ_0 is the $\mathrm{SL}(2,\mathbb{R})$-invariant measure on $\mathbb{H}$ given by $d\mu_0 = y^{-2}\,dx\,dy$.

Theorem 7.4. *Let $\tau\colon \mathbb{B}(H_m) \to \mathbb{C}$ be the positive linear functional defined by*

$$\tau(A) = \frac{1}{\mu_0(\mathscr{F})}\int_{\mathscr{F}} \hat{A}(z)\, d\mu_0(z)$$

for $A \in \mathbb{B}(H_m)$. Then, τ is the unique tracial state on $\mathscr{A}_m$.

Proof. Observe that τ is indeed a state on $\mathbb{B}(H_m)$. Let $A \in \mathscr{A}_m$. By Proposition 7.1, one has

$$\begin{aligned}\tau(A^*A) &= \frac{1}{\mu_0(\mathscr{F})}\int_{z\in\mathscr{F}}\int_{\zeta\in\mathbb{H}} \frac{K(z,\zeta)K(\zeta,z)}{K(z,z)}\widehat{A^*}(z,\zeta)\hat{A}(\zeta,z)\, d\mu_m(\zeta)\, d\mu_0(z)\\ &= \frac{c_m}{\mu_0(\mathscr{F})}\int_{z\in\mathscr{F}}\int_{\zeta\in\mathbb{H}} \delta(z,\zeta)|\hat{A}(z,\zeta)|^2\, d\mu_0(\zeta)\, d\mu_0(z).\end{aligned}$$

The last integration is over a Γ-fundamental domain of $\mathbb{H}\times\mathbb{H}$ (w.r.t. the diagonal action) and since the measure $\mu_0\times\mu_0$ is Γ-invariant, we may replace it with the integration over another fundamental domain $(z,\zeta) \in \mathbb{H}\times\mathscr{F}$. Therefore, the above integration is invariant under the swap $z \leftrightarrow \zeta$. This means that $\tau(A^*A) = \tau(AA^*)$ and hence τ is tracial. $\square$

We note another expression of $\tau(A^*A)$:

$$\tau(A^*A) = \frac{1}{\mu_0(\mathscr{F})}\int_{\mathscr{F}} \frac{\langle (A^*A)e_z, e_z\rangle}{K(z,z)}\, d\mu_0(z) = \frac{1}{\mu_0(\mathscr{F})}\int_{\mathscr{F}} \frac{\|Ae_z\|^2}{\|e_z\|^2}\, d\mu_0(z).$$

8. TOEPLITZ OPERATORS

For $f \in L^\infty(\mathbb{H})$, we define the associated Toeplitz operator T_f on H_m by $T_f = PM_f|_{H^2(\mathbb{H},\mu_m)}$, where M_f is the multiplication operator by f and P is the orthogonal projection from $L^2(\mathbb{H},\mu_m)$ onto $H_m = H^2(\mathbb{H},\mu_m)$. One calculates the symbol $\hat{T}_f(z,\zeta)$ of T_f as

$$\hat{T}_f(z,\zeta) = \langle f e_\zeta, e_z\rangle / K(z,\zeta) = \int_{\mathbb{H}} f(w) \frac{e_\zeta(w)\overline{e_z(w)}}{K(z,\zeta)}\, d\mu_m(w)$$
$$= c_m \int_{\mathbb{H}} f(w) \frac{K(w,\zeta)\overline{K(w,z)}}{K(z,\zeta)K(w,w)}\, d\mu_0(w).$$

In particular,

$$\hat{T}_f(z) = c_m \int_{\mathbb{H}} f(w) \frac{|K(w,z)|^2}{K(z,z)K(w,w)}\, d\mu_0(w) = c_m \int_{\mathbb{H}} \delta(z,w) f(w)\, d\mu_0(w).$$

Since δ and μ_0 are $\mathrm{SL}(2,\mathbb{R})$-invariant, Proposition 7.3 implies that for $g \in \mathrm{SL}(2,\mathbb{R})$, $f \in L^\infty(\mathbb{H})$ and $(g\cdot f)(z) = f(g^{-1}z)$, one has

$$T_{g\cdot f} = \pi_m(g) T_f \pi_m(g)^*.$$

If $f \in L^\infty(\mathbb{H})$ is Γ-invariant, then T_f commutes with $\pi_m(\Gamma)$. Therefore, T can be regarded as an operator from $L^\infty(\mathscr{F})$ into $\mathscr{A}_m$. We freely view $L^\infty(\mathscr{F})$ as Γ-invariant functions in $L^\infty(\mathbb{H})$. Let D be the positive function on $\mathscr{F}\times\mathscr{F}$ defined by

$$D(z,\zeta) = c_m \sum_{g\in\Gamma} \delta(z, g\zeta).$$

This function D is called an automorphic kernel. Note that

$$\int_{\mathscr{F}} D(z,\zeta)\, d\mu_0(\zeta) = c_m \int_{\mathbb{H}} \delta(z,\zeta)\, d\mu_0(\zeta) = 1.$$

In particular, the infinite sum in the definition of D is convergent almost everywhere. Since δ is symmetric and Γ-invariant, the kernel D is symmetric. It follows that the integral operator associated with D is contractive on $L^p(\mathscr{F},\mu_0)$ for all $1 \le p \le \infty$. (Indeed, it is easy to check this for $p = 1,\infty$. For the rest of $p \in (1,\infty)$, use interpolation or Hölder's inequality.) We write the operator by B_Δ:

$$(B_\Delta f)(z) = c_m \int_{\mathbb{H}} \delta(z,\zeta) f(\zeta)\, d\mu_0(\zeta) = \int_{\mathscr{F}} D(z,\zeta) f(\zeta)\, d\mu_0(\zeta).$$

One has $\hat{T}_f = B_\Delta f$ for $f \in L^\infty(\mathscr{F})$.

The operator B_Δ on $L^2(\mathscr{F},\mu_0)$ is a function of Δ in the spectral sense (Theorem 7.4 in [Iw]). Indeed, by [Be, (4.17)] (see also Section 2.2 in [HKZ]), one has

$$B_\Delta = \prod_{k=m}^{\infty} \left(1 + \frac{\Delta}{(k+1)(k+2)}\right)^{-1}.$$

It follows that B_Δ is positive and injective on $L^2(\mathscr{F},\mu_0)$. Injectivity of B_Δ also follows from Proposition 2.6 in [HKZ].

Theorem 8.1. *For $A \in \mathscr{A}_m$ and $f \in L^\infty(\mathscr{F})$, one has*

$$\tau(AT_f) = \frac{1}{\mu_0(\mathscr{F})} \int_{\mathscr{F}} \hat{A}(z) f(z)\, d\mu_0(z).$$

Moreover, the operator $L^\infty(\mathscr{F}) \ni f \mapsto T_f \in \mathscr{A}_m$ extends to a bounded linear operator $S\colon L^2(\mathscr{F},\mu_0) \to L^2(\mathscr{A}_m,\tau)$ which satisfies $S^*S = \mu_0(\mathscr{F})^{-1}B_\Delta$ and $S^*A = \mu_0(\mathscr{F})^{-1}\hat{A}$ for $A \in \mathscr{A}_m \subset L^2(\mathscr{A}_m,\tau)$.

Proof. Let $f \in L^\infty(\mathscr{F})$, which is regarded as Γ-invariant function in $L^\infty(\mathbb{H})$. Then, one calculates (all integrals below are absolutely convergent)

$$\begin{aligned}
\tau(AT_f) &= \tfrac{c_m}{\mu_0(\mathscr{F})} \int_{\Gamma\backslash(\mathbb{H}\times\mathbb{H})} \delta(z,\zeta)\hat{A}(z,\zeta)\hat{T}_f(z,\zeta)\, d\mu_0^2(z,\zeta)\\
&= \tfrac{c_m^2}{\mu_0(\mathscr{F})} \int_{\Gamma\backslash(\mathbb{H}\times\mathbb{H}\times\mathbb{H})} \delta(z,\zeta)\hat{A}(z,\zeta) f(w) \tfrac{K(w,z)\overline{K(w,\zeta)}}{K(\zeta,z)K(w,w)}\, d\mu_0^3(z,\zeta,w)\\
&= \tfrac{c_m^2}{\mu_0(\mathscr{F})} \int_{\Gamma\backslash(\mathbb{H}\times\mathbb{H}\times\mathbb{H})} f(w)\hat{A}(z,\zeta) \tfrac{\overline{K(\zeta,z)}K(w,z)\overline{K(w,\zeta)}}{K(z,z)K(\zeta,\zeta)K(w,w)}\, d\mu_0^3(z,\zeta,w)\\
&= \tfrac{1}{\mu_0(\mathscr{F})} \int_{\mathscr{F}} f(w) \int_{\mathbb{H}\times\mathbb{H}} \langle Ae_\zeta, e_z\rangle \tfrac{\overline{e_w(z)}e_w(\zeta)}{K(w,w)}\, d\mu_m^2(z,\zeta)\, d\mu_0(w)\\
&= \tfrac{1}{\mu_0(\mathscr{F})} \int_{\mathscr{F}} f(w) \tfrac{\langle Ae_w, e_w\rangle}{K(w,w)}\, d\mu_0(w)\\
&= \tfrac{1}{\mu_0(\mathscr{F})} \int_{\mathscr{F}} f(z)\hat{A}(z)\, d\mu_0(z).
\end{aligned}$$

Hence by letting $A = T_f^* = T_{\bar{f}}$, one obtains

$$\tau(T_f^*T_f) = \frac{1}{\mu_0(\mathscr{F})} \int_{\mathscr{F}} f(z)\overline{(B_\Delta f)(z)}\, d\mu_0(z) = \frac{1}{\mu_0(\mathscr{F})} \langle f, B_\Delta f\rangle_{L^2(\mathscr{F},\mu_0)}. \qquad \square$$

Corollary 8.2. *The operator S is injective and has dense range.*

Proof. Since $S^*S = B_\Delta$ is injective, so is S. We prove the injectivity of S^*. We essentially prove that the formula $S^*A = \mu_0(\mathscr{F})^{-1}\hat{A}$ is valid on $L^2(\mathscr{A}_m,\tau)$. By the proof of Theorem 7.4, the map $\mathscr{A}_m \ni A \mapsto \hat{A}(z,\zeta)$ extends to a scalar multiple of an isometry R from $L^2(\mathscr{A}_m)$ into $L^2(\Gamma\backslash(\mathbb{H}\times\mathbb{H}), \delta\cdot(\mu_0\times\mu_0))$. We note that $\operatorname{ran} R$ consists of Γ-invariant sesqui-holomorphic functions on $\mathbb{H}\times\mathbb{H}$. By Theorem 8.1, the operator S^*R^* on $R(\mathscr{A}_m)$ is the map $\hat{A}(z,\zeta) \mapsto \hat{A}(z) \in L^2(\mathscr{F},\mu_0)$, and a fortiori, the latter map is bounded and well-defined on $\operatorname{ran} R$. By sesqui-holomorphic property, the operator S^*R^* is injective on $\operatorname{ran} R$. $\square$

9. Hecke operators acting on von Neumann algebras

(Ref: [Ra2].) Sticking to the notation of previous section, let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ (or any other lattice in $\mathrm{SL}(2,\mathbb{R})$). The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{B}(H_m)$ by conjugation Ad_{π_m} and one has $\mathscr{A}_m = \pi_m(\Gamma)' = \mathbb{B}(H_m)^\Gamma$. Since the action Ad_{π_m} is trivial on the center, it can be regarded as an action of $\mathrm{PSL}(2,\mathbb{R})$. We still denote by Γ the image of Γ in $\mathrm{PSL}(2,\mathbb{R})$. Let $G = \mathrm{PGL}^+(2,\mathbb{Q}) \le \mathrm{PSL}(2,\mathbb{R})$ (or any other intermediate subgroup $\Gamma \le G \le \mathrm{PSL}(2,\mathbb{R})$ such that $\Gamma \le G$ is a Hecke pair). By the arguments in Section 2, the Hecke algebra of $\Gamma \le G$ acts on $\mathscr{A}_m$ by completely positive maps:

$$\Psi_\alpha(x) = \sum_k \pi_m(g_k) x \pi_m(g_k)^*,$$

where $\alpha = \bigsqcup g_k\Gamma \in \Gamma\backslash G/\Gamma$.

Theorem 9.1. *Let $\alpha \in \Gamma\backslash G/\Gamma$. Then, the completely positive map Ψ_α, viewed as an operator on $L^2(\mathscr{A}_m, \tau)$, is unitarily equivalent to the classical Hecke operator T_α on $L^2(\mathscr{F}, \mu_0)$.*

Proof. Let $T\colon L^\infty(\mathbb{H}) \to \mathbb{B}(H_m)$ and $S\colon L^2(\mathscr{F}, \mu_0) \to L^2(\mathscr{A}_m, \tau)$ be the operators defined in Section 8 and recall that $T_{g\cdot f} = \pi_m(g) T_f \pi_m(g)^*$. Hence, for every $f \in L^\infty(\mathscr{F}) \subset L^2(\mathscr{F}, \mu_0)$ and $\alpha = \bigsqcup g_k \Gamma \in \Gamma\backslash G/\Gamma$, one has

$$ST_\alpha f = S \sum g_k \cdot f = T_{\sum g_k \cdot f} = \sum \pi_m(g_k) T_f \pi_m(g_k)^* = \Psi_\alpha(T_f).$$

This implies $ST_\alpha = \Psi_\alpha S$ on $L^2(\mathscr{F}, \mu_0)$. Let $S = U|S|$ be the polar decomposition. By Corollary 8.2, U is a unitary operator. Moreover, since $|S| = B_\Delta^{1/2}$ commutes with T_α and is a positive injective operator, one has $UT_\alpha = \Psi_\alpha U$. □

We are inclined to conjecture that the operator $U \Delta U^*$ on $L^2(\mathscr{A}_m, \tau)$ gives rise to a quantum Dirichlet form in the sense of [Sa].

Part 3. **Von Neumann representations for Hecke operators (Ref:** [Ra2]**).**

10. From Hecke algebras to group von Neumann algebras

Let $\Gamma \leq G$ be a Hecke pair and χ be a character on a central subgroup $Z \subset \Gamma \cap Z(G)$. (We have $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, $G = \mathrm{SL}(2, \mathbb{Q}[\sqrt{p}])$ and $Z = Z(\Gamma)$ in mind.) We assume that there is a unitary representation $\pi\colon G \curvearrowright \ell_2^\chi \Gamma$ such that $\pi|_\Gamma = \rho^\chi$. (See Section 6.) Let $\mathscr{H}$ be the Hecke algebra of $\Gamma \leq G$ and $\theta\colon \Gamma\backslash G/\Gamma \to \mathscr{L}^\chi G$ be the map defined by the formal sum

$$\theta(\alpha) = \frac{1}{|Z|} \sum_{g \in \alpha} \langle \pi(g) \hat{1}^\chi, \hat{1}^\chi \rangle \lambda^\chi(g),$$

where $\hat{1}^\chi \in \ell_2^\chi \Gamma$ is the cyclic trace vector for $\mathscr{L}^\chi \Gamma$ such that $\lambda^\chi(s)\hat{1}^\chi = \rho^\chi(s^{-1})\hat{1}^\chi$. Observe that formally $\theta(\alpha^*) = \theta(\alpha)^*$. Moreover, since $\pi(z) = \rho^\chi(z) = \chi(z^{-1})$ and $\lambda^\chi(z) = \chi(z)$ for $z \in Z$, the function $g \mapsto \langle \pi(g)\zeta, \zeta \rangle \lambda^\chi(g)$ is Z-invariant and the map θ factors through $\Gamma'\backslash G'/\Gamma'$, where $G' = G/Z$ and $\Gamma' = \Gamma/Z$. The Zg-th coordinate of $\theta(\alpha)$ is $\langle \pi(g)\hat{1}^\chi, \hat{1}^\chi \rangle \lambda^\chi(g)$.

Lemma 10.1. *For every $\alpha \in \Gamma\backslash G/\Gamma$, one has $\|\theta(\alpha)\|_2^2 = \mathrm{ind}(\alpha)$. In particular,* $\mathrm{ind}(\alpha) = \mathrm{ind}(\alpha^*)$.

Proof. Let $\alpha = \bigsqcup \Gamma g_i$. Then,

$$\|\theta(\alpha)\|_2^2 = \sum_{i=1}^{\mathrm{ind}(\alpha)} \sum_{s \in \Gamma/Z} |\langle \pi(s g_i)\hat{1}^\chi, \hat{1}^\chi \rangle|^2 = \sum_{i=1}^{\mathrm{ind}(\alpha)} \|\pi(g_i)\hat{1}^\chi\|^2 = \mathrm{ind}(\alpha),$$

since $\{\rho^\chi(s)\hat{1}^\chi : s \in \Gamma/Z\}$ forms a complete orthonormal basis for $\ell_2^\chi \Gamma$. □

Because of this lemma, θ is well-defined as a map into $L^2(\mathscr{L}^\chi G)$. We consider $L^2(\mathscr{L}^\chi G)$ as the space of closed operators on $\ell_2^\chi \Gamma$ which are affiliated with $\mathscr{L}^\chi G$. The product $L^2(\mathscr{L}^\chi G) \times L^2(\mathscr{L}^\chi G) \to L^1(\mathscr{L}^\chi G)$ makes sense and can be computed by a formal calculation.

Lemma 10.2. *View $\theta(\alpha)$'s as vectors in $\ell_2^\chi \Gamma$. Then, one has $\theta(\alpha) * \theta(\beta) = \theta(\alpha \cdot \beta)$.*

Proof. Let $\alpha = \bigsqcup g_i$ and $x \in G$. We will compute the Zx-th coordinate of

$$\theta(\alpha) * \theta(\beta) = \frac{1}{|Z|^2} \sum_{g\in\alpha,\, h\in\beta} \langle \pi(g)\hat{1}^\chi, \hat{1}^\chi\rangle \langle \pi(h)\hat{1}^\chi, \hat{1}^\chi\rangle \lambda^\chi(gh).$$

Choose representatives (g_i, h_i) of Γ-orbits in $\{(g,h) : gh = x\}$. Note that there are $c(\alpha,\beta;x)$ orbits. Hence the Zx-th coordinate is

$$\begin{aligned}
&\frac{1}{|Z|^2} \sum_{\substack{1\le i\le c(\alpha,\beta;x)\\ z\in Z}} \sum_{s\in\Gamma} \langle \pi(zg_i s)\hat{1}^\chi, \hat{1}^\chi\rangle \langle \pi(s^{-1}h_i)\hat{1}^\chi, \hat{1}^\chi\rangle \lambda^\chi(zx) \\
&\quad = \frac{1}{|Z|} \sum_{i,z} \langle \pi(h_i)\hat{1}^\chi, \pi(zg_i)^*\hat{1}^\chi\rangle \lambda^\chi(zx) = c(\alpha,\beta;x)\langle \pi(x)\hat{1}^\chi, \hat{1}^\chi\rangle \lambda^\chi(x).
\end{aligned}$$

This shows indeed $\theta(\alpha) * \theta(\beta) = \theta(\alpha\cdot\beta)$. □

Theorem 10.3. *The map θ is well defined and extends to a trace-preserving $*$-isomorphism from* $\mathrm{vN}(\mathscr{H}(\Gamma'\backslash G'/\Gamma'))$ *into* $\mathscr{L}^\chi G$.

Proof. By above lemmas, the map θ is a $*$-homomorphism from the Hecke algebra $\mathscr{H}(\Gamma'\backslash G'/\Gamma')$ into the $*$-algebra of closed operators affiliated with $\mathscr{L}^\chi G$. Moreover, its range is in L^2 and it is trace-preserving. These conditions imply that θ is bounded and extends to a von Neumann algebra isomorphism. □

Recall that since $\mathscr{L}^\chi\Gamma = \mathbb{B}(\ell_2^\chi\Gamma)^\Gamma$ where G acts on $\mathbb{B}(\ell_2^\chi\Gamma)$ by Ad_π, the Hecke algebra of $\Gamma \le G$ acts on $\mathscr{L}^\chi\Gamma$ by

$$\Psi_\alpha(x) = \sum \pi(g_k)x\pi(g_k)^*$$

for $\alpha = \bigsqcup \Gamma g_k$. This action factors through $\Gamma'\backslash G'/\Gamma'$.

Proposition 10.4. *For $\alpha \in \Gamma'\backslash G'/\Gamma'$ and $x \in \mathscr{L}^\chi\Gamma$, one has*

$$\Psi_\alpha(x) = E^{\mathscr{L}^\chi G}_{\mathscr{L}^\chi\Gamma}(\theta(\alpha)x\theta(\alpha)^*).$$

Proof. For $\alpha = \bigsqcup g_k\Gamma$ and $x, y \in \Gamma$, the Zy-th coordinate of $\theta(\alpha)\lambda^\chi(x)\theta(\alpha)^*$ is

$$\begin{aligned}
&\frac{1}{|Z|^2} \sum_{z\in Z} \sum_{\substack{g,h\in\alpha \text{ s.t.}\\ gxh^{-1}=zy}} \langle \pi(g)\hat{1}^\chi, \hat{1}^\chi\rangle \langle \pi(h^{-1})\hat{1}^\chi, \hat{1}^\chi\rangle \lambda^\chi(gxh^{-1}) \\
&\quad = \frac{1}{|Z|^2} \sum_{z\in Z} \sum_{\substack{1\le k\le \mathrm{ind}(\alpha^*)\\ s\in\Gamma}} \langle \pi(g_k s^{-1})\hat{1}^\chi, \hat{1}^\chi\rangle \langle \pi(x^{-1}sg_k^{-1}zy)\hat{1}^\chi, \hat{1}^\chi\rangle \lambda^\chi(zy) \\
&\quad = \frac{1}{|Z|^2} \sum_{z\in Z} \sum_{\substack{1\le k\le \mathrm{ind}(\alpha^*)\\ s\in\Gamma}} \langle \pi(s^{-1})\hat{1}^\chi, \pi(g_k)^*\hat{1}^\chi\rangle \langle \pi(g_k^{-1}zy)\hat{1}^\chi, \pi(s^{-1}x)\hat{1}^\chi\rangle \lambda^\chi(zy) \\
&\quad = \frac{1}{|Z|^2} \sum_{z\in Z} \sum_{\substack{1\le k\le \mathrm{ind}(\alpha^*)\\ s\in\Gamma}} \langle \pi(s^{-1})\hat{1}^\chi, \pi(g_k)^*\hat{1}^\chi\rangle \langle \lambda^\chi(x)\pi(g_k^{-1}zy)\hat{1}^\chi, \pi(s^{-1})\hat{1}^\chi\rangle \lambda^\chi(zy)
\end{aligned}$$

$$= \frac{1}{|Z|}\sum_{z\in Z}\sum_{1\le k\le \mathrm{ind}(\alpha^*)} \langle \lambda^\chi(x)\pi(g_k^{-1}zy)\hat{1}^\chi, \pi(g_k)^*\hat{1}^\chi\rangle \lambda^\chi(zy)$$
$$= \frac{1}{|Z|}\sum_{z\in Z}\langle \Psi_\alpha(\lambda^\chi(x))\lambda^\chi((zy)^{-1})\hat{1}^\chi, \hat{1}^\chi\rangle \lambda^\chi(zy)$$
$$= \tau\big(\Psi_\alpha(\lambda^\chi(x))\lambda^\chi(y)^*\big)\lambda^\chi(y),$$

which coincides with the Zy-th coordinate of $\Psi_\alpha(\lambda^\chi(x))$. □

Corollary 10.5. *Suppose that $\xi\in L^2(\mathscr{L}^\chi\Gamma)$ is a unit eigenvector of Ψ_α with the eigenvalue $\lambda(\alpha)$. Then, one has*

$$E^{\mathscr{L}^\chi G}_{\mathrm{ran}\,\theta}(\xi^*\theta(\alpha)\xi) = \frac{\lambda(\alpha)}{\mathrm{ind}(\alpha)}\theta(\alpha).$$

Proof. For every $\beta\in\Gamma\backslash G/\Gamma$, one has

$$\tau_{\mathscr{L}^\chi G}\big(\xi^*\theta(\alpha)\xi\theta(\beta)^*\big) = \tau_{\mathscr{L}^\chi\Gamma}\big(\xi^* E^{\mathscr{L}^\chi G}_{\mathscr{L}^\chi\Gamma}(\theta(\alpha)\xi\theta(\beta)^*)\big) = \begin{cases} \tau_{\mathscr{L}^\chi\Gamma}\big(\xi^*\Psi_\alpha(\xi)\big) & \text{if } \beta=\alpha \\ 0 & \text{if } \beta\neq\alpha \end{cases}$$

by Proposition 10.4, since $\alpha\Gamma\beta^*\cap\Gamma\neq\emptyset$ iff $\alpha=\beta$. □

11. Symmetric enveloping algebras

We stick to the notation of the previous section. By Proposition 10.4, when viewed as a bounded linear operator on $\ell_2^\chi\Gamma$, the completely positive map Ψ_α has the form

$$P_{\ell_2^\chi\Gamma}\lambda^\chi(\theta(\alpha))\rho^\chi(\bar\theta(\alpha))P_{\ell_2^\chi\Gamma},$$

where λ^χ is the identity representation of $\mathscr{L}^\chi G$, and and we will explain $\bar\theta$. The map θ was constructed from π. Out of $\bar\pi$, we may construct $\bar\theta$. Since $\bar\pi|_\Gamma = \overline{\rho^\chi} = \rho^{\bar\chi}$, the map $\bar\theta$ maps into $\mathscr{L}^{\bar\chi}\Gamma$, which is $*$-isomorphic to

$$\overline{(\mathscr{L}^{\bar\chi}\Gamma)'} \cong \overline{\mathscr{R}^{\bar\chi}\Gamma} \cong \mathscr{R}^\chi\Gamma \overset{\rho^\chi}{\hookrightarrow} \mathbb{B}(\ell_2^\chi\Gamma).$$

Since $\alpha\mapsto\Psi_\alpha$ is a representation of the Hecke algebra, the same is true for

$$\Psi^{(2)}\colon \alpha\mapsto P_{\ell_2^\chi\Gamma}\lambda^\chi(\theta(\alpha))\rho^\chi(\bar\theta(\alpha))P_{\ell_2^\chi\Gamma}.$$

Let $C\subset\ell^\infty G$ be the C*-subalgebra generated by $\{e_{g\Lambda h} : \Lambda\le_{\mathrm f}\Gamma,\ g,h\in G\}$, where $\Lambda\le_{\mathrm f}\Gamma$ means that Λ is a finite index subgroup of Γ which contains Z, and $e_{g\Lambda h}$ is the characteristic function on $g\Lambda h$. We note that $g\Gamma h\cap\Gamma$ is a right Λ-coset of Γ for some $\Lambda\le_{\mathrm f}\Gamma$ if $h\in\Gamma g^{-1}\Gamma$; else $g\Gamma h\cap\Gamma=\emptyset$. Since $\Lambda_1 s_1\cap\Lambda_2 s_2$ is a union of right $(\Lambda_1\cap\Lambda_2)$-cosets, C is the closure of the directed union $\bigcup_{\Lambda\le_{\mathrm f}\Gamma} c_{00}(\Lambda\backslash G)$, where c_{00} is the space of finitely supported functions. There is a unbounded positive linear functional τ such that $\tau(e_{\Lambda h}) = [\Gamma:\Lambda]^{-1}$ for every $\Lambda\le_{\mathrm f}\Gamma$ and $h\in G$. We claim that τ is invariant under the conjugation by G. Let $\Lambda\le_{\mathrm f}\Gamma$ and $g\in G$. Then, one has $\Lambda' := \Gamma\cap g\Lambda g^{-1}\le_{\mathrm f}\Gamma$. Moreover since $[g\Lambda g^{-1}:\Lambda'] = [\Lambda : g^{-1}\Lambda' g]$ and

$$[\Gamma:\Lambda'] = [\Gamma:\Gamma\cap g\Gamma g^{-1}][\Gamma\cap g\Gamma g^{-1}:\Lambda']$$
$$= [\Gamma:\Gamma\cap g^{-1}\Gamma g][g^{-1}\Gamma g\cap\Gamma : g^{-1}\Lambda' g] = [\Gamma : g^{-1}\Lambda' g]$$

(we have used the fact that $\alpha(\Gamma g\Gamma) = \alpha(\Gamma g^{-1}\Gamma)$), one has

$$\tau(e_{g\Lambda g^{-1}}) = \frac{[g\Lambda g^{-1} : \Lambda']}{[\Gamma : \Lambda']} = \frac{[\Lambda : g^{-1}\Lambda' g]}{[\Gamma : g^{-1}\Lambda' g]} = \frac{1}{[\Gamma : \Lambda]} = \tau(e_\Lambda).$$

This proves the claim. The group $G \times G$ acts on C by right and left, leaving τ invariant. Since we assumed $Z \leq \Lambda$ for every $\Lambda \leq_{\mathrm{f}} \Gamma$, the right and left action of Z on C is trivial. In particular, $p^\chi \otimes p^{\bar\chi}$ (defined in Section 6) commutes with C in the crossed product. We define

$$\mathscr{B}_\Gamma^\chi = e_\Gamma (C \rtimes_{\mathrm{red}} (G \times G))(p^\chi \otimes p^{\bar\chi}) e_\Gamma.$$

There is a canonical faithful conditional expectation E from $C \rtimes_{\mathrm{red}} (G \times G)$ onto C, and $|Z|^2 \tau \circ E|_{\mathscr{B}_\Gamma}$ is a faithful tracial state, which will be simply denoted by τ. The map $\lambda^\chi(g) \mapsto e_\Gamma(\lambda^\chi(g) \otimes p^{\bar\chi}) e_\Gamma$ extends to a trace-preserving completely positive map from $\mathscr{L}^\chi G$ into $(\mathscr{B}_\Gamma^\chi)''$.

Proposition 11.1. *The map*

$$\Phi \colon \Gamma' \backslash G' / \Gamma' \ni \alpha \mapsto e_\Gamma\big(\theta(\alpha) \otimes \bar\theta(\alpha)\big) e_\Gamma \in (\mathscr{B}_\Gamma^\chi)''$$

extends to a trace preserving $*$*-isomorphism from* $\mathrm{vN}(\Gamma' \backslash G' / \Gamma')$ *into* $(\mathscr{B}_\Gamma^\chi)''$.

Proof. The map Φ is trace-preserving. Moreover, it is a quotient of $\Psi^{(2)}$ and hence is multiplicative. □

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